ON THE MÖBIUS FUNCTION

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ABSTRACT. We investigate incomplete convolutions of the Möbius function of the form \( \sum_{d|n; d \leq z} \mu(d) \). It is shown that for almost all integers \( n \) one can find \( z \) for which this sum is large.

1. Introduction. The function \( M(n) = \sup_{z \leq n} \left| \sum_{d|n; d \leq z} \mu(d) \right| \) has been studied in various papers [1, 2, 5]. Erdős and Katai [2] proved that

\[
M(n) \leq A \omega(n)
\]

(p.p.)

if \( A > \sqrt{2} \).

(We use (p.p.) to indicate that a property holds on a sequence of asymptotic density 1.)

This result was improved by Hall [5], who showed that \( A > 3/e \) is sufficient.

A recent result of G. Tenenbaum and the author [9] implies almost immediately

THEOREM 1.

\[
M(n) \leq \psi(n) \log \log n,
\]

(p.p.)

where \( \psi(n) \) is any function tending to \( \infty \).

PROOF. Let \( p_1(n) \) be the smallest prime factor of \( n \). Then \( \mu(d) + \mu(p_1(n)d) = 0 \) for all \( d \neq 0 \mod p_1(n) \). Therefore

\[
M(n) \leq \sup_{z \leq n} \left| \sum_{\frac{z}{d} \leq zp_1(n)} 1 \right|.
\]

In [9] it is shown that

\[
\Delta(n) \leq \psi(n) \log \log n,
\]

where \( \Delta \) is Hooley’s function [7], defined by

\[
\Delta(n) = \sup_{z \leq n} \sum_{z \leq d < ez} 1.
\]

(p.p.)

It follows by sieve methods that

\[
p_1(n) \leq \psi(n).
\]

(p.p.)

Received by the editors December 19, 1985.
1980 Mathematics Subject Classification (1985 Revision). Primary 11K65; Secondary 11B05.
Research supported by an NSF Grant.
Thus

\[ M(n) \leq \sup_{z \leq n} \left| \sum_{z < d \leq z \psi_1(n)} \frac{1}{d} \right| \leq \Delta(n) \log \psi(n) \]

\[ \leq \psi(n) \log \psi(n) \log \log n. \]  

(p.p.)

Since \( \psi(n) \) was arbitrary Theorem 1 follows.

In [1 and 2] the question for a lower bound for \( M(n) \) was raised. The purpose of this paper is to establish such a lower bound.

We will prove

**Theorem 2.** Let

\[ \gamma < -\frac{\log 2}{\log(1 - (\log 3)^{-1})} = 0.28754 \ldots. \]

Then

\[ M(n) \geq (\log \log n)^{\gamma}. \]  

(p.p.)

Many of the techniques applied will be very similar to those applied in [8], where the same lower bound was obtained for \( \Delta(n) \). However we need also some new devices which bear resemblance to those used in [9].

**2. Notations and preliminary lemmas.** We fix a function \( v(x) \), to be specified later, with \( v(x) \rightarrow \infty (x \rightarrow \infty) \) and also a constant \( \rho > 1 \).

Based on these two parameters we define \( r_k = \rho^k v(x) \) and \( r_{k,l} = \rho^k v(x) + l \) where \( k \) and \( l \) are any nonnegative integers. For any positive integer \( n \leq x \) and a real number \( z > 0 \) we set

\[ n_z = \prod_{\log \log p < z} p, \quad n_z^* = \prod_{\log \log p < z} p^{\nu p}. \]

We use \( n_{(k)} \) for \( n_{r_k} \); \( n_{(k,l)} \) for \( n_{r_{k,l}} \); \( n_{(k)}^* \) for \( n_{r_k}^* \); and \( n_{(k,l)}^* \) for \( n_{r_{k,l}}^* \). We define

\[ \hat{n}_{(k,l)} = \prod_{r_k \leq \log \log p < r_{k,l}} p \quad \text{and} \quad \hat{n}_{(k,l)} = \prod_{r_k \leq \log \log p < r_{k,l}} p^{\nu p}. \]

Assume that

\[ n = n_{(k)}^* p_1^{(k)^{s_1}} \cdots p_t^{(k)^{s_t(n)}} , \quad p_1^{(k)} < \cdots < p_t^{(k)}. \]

Then we set

\[ \hat{n}_{(k)} = \prod_{t \leq s} p_t^{(k)}, \quad \hat{n}_{(k)}^* = \prod_{t \leq s} p_t^{(k)^{s_t}}, \]

\[ n_{(k)} = n_{(k)} \hat{n}_{(k)}, \quad n_{(k)}^* = n_{(k)}^* \hat{n}_{(k)}^*. \]
For any quadruplet \((n, k, l, \eta)\), where \(n \leq x\), \(k, l\) nonnegative integers, and \(\eta > 0\) we denote by \(\mu(n, k, l, \eta)\) the Lebesgue measure of the set

\[ \mathcal{E}(n, k, l, \eta) = \bigcup_{\substack{dd' \mid \mu(n, k, l) \leq 1 \\mu(dd') = 1}} \left( \log \frac{d'}{d} \right) + [-\eta, \eta]. \]

We now proceed with some auxiliary lemmas. They all are either identical or rather similar to the lemmas applied in [8].

**LEMMA 1.** Let \(f\) be a nonnegative multiplicative function such that for all primes \(p\)

\[ 0 \leq f(p^\nu) \leq \lambda_1 \lambda_2^\nu \quad (\nu = 1, 2, \ldots), \]

where \(0 < \lambda_1, 0 < \lambda_2 < 2\). Then for \(x \geq 1\)

\[ \sum_{n \leq x} f(n) \ll \lambda_1 \lambda_2 x \prod_{p \leq x} (1 - p^{-1}) \sum_{\nu=0}^\infty f(p^\nu) p^{-\nu}. \]

This is a weakening of a theorem of Halberstam and Richert [4] generalizing a result of Hall.

**LEMMA 2.** For \(2 \leq u \leq v \leq x\), we have

\[ \text{card} \left\{ n \leq x : \prod_{p \leq u, p^\nu \mid n} p^\nu \geq v \right\} \ll x \exp \left( -c_0 \frac{\log v}{\log u} \right) \]

where \(c_0 > 0\) is an absolute constant.

This is established in [3] and, in a stronger version, in [10].

**LEMMA 3.** Let \(u(x)\) be any function tending to \(\infty\) such that \(u(x) \leq \log \log x\). Let \(\delta_0 > 0\) be a fixed constant. Then for each \(r\) with \(u(x) \leq r \leq \log \log x\) we have uniformly in \(s\), \(u(x) \leq s \leq r\),

\[ |\omega(n_r/n_{r-s}) - s| \leq \delta_0 s \]

for all \(n \leq x\) except a set of cardinality \(\ll \delta_0 x \exp(-c(\delta_0)u(x))\), where \(c(\delta_0) > 0\) depends only on \(\delta_0\).

**PROOF.** We first estimate the number of integers \(n \leq x\) for which \(\omega(n_r/n_{r-s}) \leq s(1 - \delta_0)\) for any integer \(s\). By Lemma 1 this number does not exceed

\[ \sum_{u(x) \leq s \leq r} \sum_{n \leq x} \alpha^{\omega(n_r/n_{r-s}) - \alpha s} \ll x \sum_{s \geq u(x)} e^{-Q(\alpha)s}, \]

where \(\alpha = 1 - \delta_0\), \(Q(\alpha) = \alpha \log \alpha - \alpha + 1 > 0\). The number of integers \(n \leq x\) for which \(\omega(n_r/n_{r-s}) \geq s(1 + \delta_0)\) for any \(s\) does not exceed

\[ \sum_{u(x) \leq s \leq r} \sum_{n \leq x} \beta^{\omega(n_r/n_{r-s}) - \beta s} \ll x \sum_{s \geq u(x)} e^{-Q(\beta)s}, \]

where \(\beta = 1 + \delta_0\), \(Q(\beta) > 0\).
LEMMA 4. Let \( w(x) \) be any function defined for \( x \geq 1 \) such that \( w(x) > 1 \), \( w(x) \to \infty \) for \( x \to \infty \), and \( v(x) \geq w(x) \). Moreover, we assume that
\[
l \geq \rho^k v(x) (\log 3 - 1)^{-1} (1 + \delta_1)
\]
for some \( \delta_1 > 0, r_{k,l} \leq \log \log x \) and \( 1 \geq \eta \geq 1/r_k \).

Then there exists \( c_1 = c_1(\delta_1) > 0 \) such that \( \mu(n, k, l, \eta) \geq \exp(r_{k,l}) w(x)^{-2} \) for all \( n \leq x \) except a set of cardinality \( \ll \delta_1 x w(x)^{-c_1} \).

PROOF. Set
\[
F(z) = F(k, l, z) = \sum_{\substack{dd'|\tilde{n}_{k,l}: \log(d'/d) \leq z \mu(dd') = 1}} 1.
\]
The \( \mu(n, k, l, \eta) \) is the measure of the set of those \( z \) for which \( F(z + \eta) - F(z - \eta) \neq 0 \).

We introduce the exponential sum
\[
S_{k,l}(\theta, n) = \sum_{\substack{dd'|\tilde{n}_{k,l}: \mu(dd') = 1}} (d'/d)^{i\theta}.
\]
We have
\[
F(z + \eta) - F(z - \eta) \leq 2 \int_{-\infty}^{\infty} \left( \frac{\sin((u - z)/2\eta)}{(u - z)/2\eta} \right)^2 d\mu(u)
= 2\eta \int_{-1/\eta}^{1/\eta} e^{i\theta z} (1 - |\theta\eta|) S_{k,l}(\theta, n) d\theta
\]
by Parseval's formula.

A second application of this formula implies
\[
\int_{-\infty}^{\infty} (F(z + \eta) - F(z - \eta))^2 dz \leq 8\pi\eta^2 \int_{-1/\eta}^{1/\eta} (1 - |\theta\eta|)^2 S_{k,l}(\theta, n)^2 d\theta.
\]
This together with
\[
(2\eta 3^{\omega(\tilde{n}_{k,l})})^2 \leq \left( \int_{-\infty}^{\infty} (F(z + \eta) - F(z - \eta)) dz \right)^2
\leq \mu(n, k, l, \eta) \int_{-\infty}^{\infty} (F(z + \eta) - F(z - \eta))^2 dz
\]
gives
\[
\mu(n, k, l, \eta) \geq 3^{2\omega(\tilde{n}_{k,l})} - 2 \left( 2\pi \int_{-1/\eta}^{1/\eta} S_{k,l}(\theta, n)^2 d\theta \right)^{-1}
\]
Thus to establish Lemma 4 it suffices to prove
\[
\int_{-1/\eta}^{1/\eta} S_{k,l}(\theta, n)^2 d\theta \leq 3^{2\omega(\tilde{n}_{k,l})} - 2 e^{-r_{k,l} w(x)^2} (2\pi)^{-1}
\]
for all \( n \leq x \) except a set of cardinality \( \ll \delta_1 x w(x)^{-c_1} \), where \( c_1 = c_1(\delta_1) \) is a suitable constant. For this purpose we decompose
\[
S_{k,l}(\theta, n) = \frac{1}{2} (S_{k,l}^{(1)}(\theta, n) + S_{k,l}^{(2)}(\theta, n))
\]
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where

\[
S_{k,l}^{(1)}(\theta, n) = \sum_{dd'|\hat{n}(k,l)} (d'/d)^i \theta = \prod_{p|\hat{n}(k,l)} \left(1 + 2 \cos(\theta \log p)\right),
\]
\[
S_{k,l}^{(2)}(\theta, n) = \sum_{dd'|\hat{n}(k,l)} \mu(dd')(d'/d)^i \theta = \prod_{p|\hat{n}(k,l)} \left(1 - 2 \cos(\theta \log p)\right).
\]

Since \(S_{k,l}^{2} \leq \frac{1}{2}(S_{k,l}^{(1)2} + S_{k,l}^{(2)2})\) it is sufficient to show that

\[
(2.1) \quad \int_{-\frac{1}{\eta}}^{\frac{1}{\eta}} S_{k,l}^{(1)}(\theta, n)^2 d\theta \leq (2\pi)^{-1} 3^{2\omega(\hat{n}(k,l))} e^{-\tau_{k,l} w(x)},
\]

for all \(n \leq x\) except a set of cardinality

\[
\ll \delta \cdot xw(x)^{-c_1} \quad (i = 1, 2).
\]

We show this only for \(i = 1\), the case \(i = 2\) being analogous.

For the range \(|\theta| \leq \exp(-r_{k,l})w(x)\) we take the trivial estimate

\[
|S_{k,l}^{(1)}(\theta, n)| \leq 3^{\omega(\hat{n}(k,l))}
\]

and obtain

\[
(2.2) \quad \int_{|\theta| \leq \exp(-r_{k,l})w(x)} S_{k,l}^{(1)}(\theta, n)^2 d\theta \leq 2 \cdot 3^{2\omega(\hat{n}(k,l))} \exp(-r_{k,l} w(x)).
\]

For the estimate of the contribution from the remaining range we introduce

\[
f_\theta(n) = S_{k,l}^{(1)}(\theta, n)^2 z^{\omega(\hat{n}(k,l))} y^{\omega(\hat{n}(k,l))}
\]

with

\[
\omega(\theta(r)) = \sum_{\log p \leq 1/|\theta|} 1
\]

and estimate \(\sum_{n \leq x} f_\theta(n)\) by Lemma 1.

We have

\[
f_\theta(n) = \prod_{p|n} f(p),
\]

where

\[
f_\theta(p) = \begin{cases} 
(1 + 2 \cos(\theta \log p))^2 y z, & \text{if } \exp(r_k) \leq \log p \leq \theta^{-1}, \\
(1 + 2 \cos(\theta \log p))^2 z, & \text{if } \theta^{-1} < \log p < \exp(r_{k,l}), \\
1, & \text{otherwise}
\end{cases}
\]

in the range \(\exp(-r_{k,l})w(x) \leq \theta \leq \exp(-r_k)\) and

\[
f_\theta(p) = \begin{cases} 
(1 + 2 \cos(\theta \log p))^2 z, & \text{if } r_k \leq \log \log p < r_{k,l}, \\
1, & \text{otherwise}
\end{cases}
\]

in the range \(\theta > \exp(-r_k)\).

We obtain for the first range

\[
\sum_{n \leq x} f_\theta(n) \ll x \exp \left( \sum_{\exp(r_k) \leq \log p \leq \theta^{-1}} \frac{9yz - 1}{p} \right)
\]

\[
+ \sum_{\theta^{-1} < \log p < \exp(r_{k,l})} \frac{z(1 + 2 \cos(\theta \log p))^2 - 1}{p}
\]

\[
\ll x \exp \left\{ (9yz - 1)(\log^+(|\theta|^{-1}) - r_k + 1) \right. \\
+ (3z - 1)(r_{k,l} - \log^+(|\theta|^{-1})) + O(z) \right\}
\]
the second sum over \( p \) being estimated, using the prime number theorem as explained in [6, Lemma 4].

For the second range we obtain

\[
\sum_{n \leq x} f_\theta(n) \ll x \exp \left( \sum_{r_k \leq \log \log p < r_{k,l}} \frac{z(1 + 2 \cos(\theta \log p))^2 - 1}{p} \right) 
\ll x \exp \{ (3z - 1)l + O_z(\log \log (3 + |\theta|)) \}.
\]

Now we choose \( y = \frac{1}{3}, z = \frac{1}{3} \) and we obtain

\[
(2.3) \sum_{n \leq x} f_\theta(n) \ll \begin{cases} x, & \text{if } \exp(-r_{k,l}) w(x) \leq |\theta| \leq \exp(-r_k), \\ x(\log(3 + |\theta|))^c_2, & \text{if } \exp(-r_k) < |\theta| \leq 1/\eta, \end{cases}
\]

where \( c_2 > 0 \) is an absolute constant.

To get estimates for \( S_{k,l}^{(1)}(\theta, n) \) itself we need estimates for \( \omega(\tilde{n}_{(k,l)}) \) and \( \omega_\theta(\tilde{n}_{(k,l)}) \).

We set \( \delta_2 = \frac{1}{2}(1 - (\log 3)^{-1}) \) and obtain by

**Lemma 3.** \( \omega(\tilde{n}_{(k,l)}) - \omega_\theta(\tilde{n}_{(k,l)}) \geq (1 - \delta_2)(r_{k,l} - \log(|\theta|^{-1})) \) in the range \( \exp(-r_{k,l}) w(x) < |\theta| \leq \exp(-r_k) \) for all \( n \leq x \) except a set of cardinality \( \ll x \exp(-c_1 \log w(x)) \) for an appropriate \( c_1 = c_1(\delta_1) > 0 \).

Together with (2.3) this yields

\[
\sum'_{n \leq x} S_{k,l}^{(1)}(\theta, n)^2 3^{-2\omega(\tilde{n}_{(k,l)})} \ll x 3^{-(1-\delta_2)(r_{k,l} - \log(|\theta|^{-1}))}
\]

for the range \( \exp(-r_{k,l}) w(x) < |\theta| \leq \exp(-r_k) \), where the sum \( \sum' \) is extended over all \( n \leq x \) except a set of cardinality \( \ll xw(x)^{-c_1} \). Thus

\[
(2.4) \sum'_{n \leq x} 3^{-2\omega(\tilde{n}_{(k,l)})} \int_{\exp(-r_{k,l}) w(x) < |\theta| \leq \exp(-r_k)} S_{k,l}^{(1)}(\theta, n)^2 d\theta 
\ll x \exp(-r_{k,l} w(x))^{-1-\delta_2} \log 3+1.
\]

For the estimate of the integral over the second range \( \exp(-r_k) < |\theta| \leq 1/\eta \) we observe that because of \( \lambda \geq r_k (\log 3 - 1)^{-1}(1 + \delta_1) \) we can find \( \delta_3 = \delta_3(\delta_1) > 0 \) such that

\[
l((1 - \delta_3) \log 3 - 1) \geq r_k (1 + \delta_3).
\]

\( |\omega(\tilde{n}_{(k,l)}) - l| \leq \delta_3 l \) for all \( n \leq x \) except a set of cardinality \( \ll_{\delta_1} x \exp(-c_3(\delta_1) w(x)) \), where \( c_3 > 0 \) depends only on \( \delta_1 \). Thus

\[
(2.5) \sum'_{n \leq x} \left( \int_{|\theta| \leq 1/\eta} S_{k,l}^{(1)}(\theta, n)^2 d\theta \right) 3^{-2\omega(\tilde{n}_{(k,l)})} 
\ll \frac{x}{\eta} 3^{-(1-\delta_3)l} \left( \log \left( 3 + \frac{1}{\eta} \right) \right)^{c_2}
\]

where \( \sum' \) means that the sum is extended over all \( n \leq x \) except a set of cardinality \( \ll_{\delta_1} x \exp(-c_3(\delta_1) w(x)) \). But \( 3^{-(1-\delta_3)l} \ll \exp(-r_{k,l}) \exp(-\delta_3 r_k) \). Now (2.2), (2.4), (2.5) give that for \( x \geq x_0 \)

\[
\sum'_{n \leq x} \left( \int_{|\theta| \leq 1/\eta} S_{k,l}^{(1)}(\theta, n)^2 d\theta \right) 3^{-2\omega(\tilde{n}_{(k,l)})} \leq xe^{-r_{k,l} w(x)^2},
\]
where $\sum'$ is extended over all $n \leq x$ except a set of cardinality $\ll \delta_1 x w(x)^{-c_1}$. This proves (2.1) and thus finishes the proof of Lemma 4.

3. Proof of Theorem 2. Given now

$$
\gamma < -\frac{\log 2}{\log(1 - (\log 3)^{-1})}
$$

then we fix $\varepsilon_1 > 0$ such that

$$
(1 - 10\varepsilon_1) \frac{\log 2}{\gamma} > -\log(1 - (\log 3)^{-1}).
$$

Then we set

$$
\rho = \min \left( \exp \left( (1 - 8\varepsilon_1) \frac{\log 2}{\gamma} \right), \frac{\log 3}{\log 3 - 1} + \frac{1}{2} \frac{1 - \log 2}{\log 3 - 1} \right),
$$

$$
(3.2) \quad v(x) = (\log \log x)^{6\varepsilon_1}, \quad w(x) = (\log \log x)^{\varepsilon_1},
$$

$$
K = \left\lfloor \frac{1 + \varepsilon_1}{\log 2} \gamma \log \log x \right\rfloor.
$$

These choices imply that $\rho^K v(x) \leq (\log \log x)^{1 - \varepsilon_1}$ and $2^K > 2(\log \log x)^\gamma$.

In the following considerations we always assume that $x$ is sufficiently large: $x \geq x_0(\gamma)$. We are now asking for blocks of divisors $d_1 < d_2 < \cdots < d_s$ such that $\mu(d_1) = \mu(d_2) = \cdots = \mu(d_s) \neq 0$, which are not interrupted by other divisors.

To make our demands more precise we introduce the two sequences:

$$
\xi_k = \frac{1}{100} \sum_{l=1}^{k} \frac{1}{l^2} \quad \text{and} \quad \zeta_k = \log 2 - \frac{1}{100} \sum_{l=1}^{k} \frac{1}{l^2}, \quad k \geq 0.
$$

Later we will still need

$$
\eta_k = 1/100k^2.
$$

We now introduce the property $B(k)$. We say that an integer $n \leq x$ has property $(B(k))$, if the following is true:

There are $2^k$ divisors of $n(k)$ having the following property $(P(k))$.

$$
d_1 < \cdots < d_{2^k}, \quad \mu(d_1) = \cdots = \mu(d_{2^k}) \neq 0,
$$

$$
|\log d_{2^k} - \log d_1| \leq \xi_k \quad \text{and} \quad d \mid n, \mu(d) \neq 0,
$$

$$
d \notin \{d_1, \ldots, d_{2^k}\} \Rightarrow \log d < \log d_1 - \zeta_k \text{ or } \log d > \log d_2 + \zeta_k.
$$

We will prove by induction in $k$ for $0 \leq k \leq K$ the statement $S(k)$:

All integers $n \leq x$ have property $(B(k))$ except those of a set of cardinality $\leq c_4(\gamma) x w(x)^{-c_5(\gamma)}(k + 1)$. If $k = K$ this means that all integers $n \leq x$ except a set of cardinality $\leq c_4(\gamma) x w(x)^{-c_5(\gamma)}(K + 1)$ have property $(B(k))$, which proves Theorem 2, since $2^K > 2(\log \log x)^\gamma$.

Proof of $S(0)$. $S(0)$ means that there is a single divisor $d_1 \mid n_0 = n_v(x)$ with property

$$(P(0)) \quad \mu(d_1) \neq 0, \quad |\log d_1 - \log d| \geq \log 2 \quad \text{for all } d \mid n, \ d \neq d_1, \mu(d) \neq 0.
$$

We set $z_0 = \frac{1}{2} v(x)$ and write

$$
n = n_{z_0}^* p_1^{(z_0)\alpha_1} \cdots p_r^{(z_0)\alpha_r(n)}.
$$
We claim that for all \( n \leq x \) except a set of cardinality \( \leq xw(x)^{-2c_5(\gamma)} \) the divisor \( p_1^{(z_0)} \) has property \( P(0) \). We denote the exceptional set by \( \mathcal{E}(x) \).

\( n \in \mathcal{E}(x) \) implies that there is a \( d|n_{z_0} \) such that \( |\log d - \log p_1^{(z_0)}| < \log 2 \) or that \( |\log p_2^{(z_0)} - \log p_1^{(z_0)}| < \log 2 \). There are \( \ll xw(x)^{-A} \) integers \( n \leq x \) for which \( n_{z_0}^{*} \geq x^{1/6} \) or \( p_1^{(z_0)} \geq x^{1/6} \) or \( \omega(n_{z_0}^{*}) \geq ((\log 5)/(\log 2) - 1)z_0 \), by Lemmas 2 and 3, where \( A \) is arbitrarily large. Denote by \( m_{z_0}^{*} \) any integer equal to \( n_{z_0}^{*} \) for some \( n \leq x \) and by \( h(r) \) an integer all of whose prime factors are \( > r \).

Then we have

\[
\text{card } \mathcal{E}(x) \ll \sum_{m_{z_0}^{*} : m_{z_0}^{*} < x^{1/6}} \sum_{p_1 \mid m_{z_0}^{*}} h(p_1-1) \leq x/m_{z_0}^{*}p_1 \\
+ \sum_{m_{z_0}^{*} < x^{1/6}} \sum_{p_1 \mid m_{z_0}^{*}} \sum_{p_2 : p_1 \leq p_2 \leq 2p_1} \frac{x}{m_{z_0}^{*}p_1p_2} + xw(x)^{-A}
\]

\[
= \sum_1 + \sum_2 + xw(x)^{-A}, \quad \text{say} .
\]

In \( \sum_2 \) the sum is extended over all \( p_1 \) with \( \exp \exp z_0 \leq p_1 < x^{1/6} \) for which there exists a \( d \mid m_{z_0}^{*} \) with \( |\log d - \log p_1| < \log 2 \).

Since now \( m_{z_0}^{*}p_1 \leq x^{1/3} \), the last sum \( \sum_{h(p_1-1) \leq x/m_{z_0}^{*}p_1} \) is \( \ll x/m_{z_0}^{*}p_1 \log p_1 \) by the sieve. Thus we obtain

\[
\sum_1 \ll x \sum_{m_{z_0}^{*} < x^{1/6}} \frac{1}{m_{z_0}^{*}} \sum_{d \mid m_{z_0}^{*} \mid p_1 : |\log p_1 - \log d| < \log 2} \frac{1}{p_1 \log p_1} .
\]

In the inner sum \( \log \log p_1 \) is contained in an interval of length \( \ll e^{-z_0} \). Thus the \( p_1 \)-sum is \( \ll e^{-2z_0} \).

We get

\[
\sum_1 \ll x \sum_{m_{z_0}^{*} < x^{1/6}} \frac{1}{m_{z_0}^{*}} e^{-z_0} \left( \frac{5}{2} \right)^{x_0} e^{-z_0} .
\]

By the sieve

\[
\text{card } \{ n \leq x : n_{z_0}^{*} = m_{z_0}^{*} \} \sim x e^{-z_0}/m_{z_0}^{*}
\]

such that

\[
\sum_{m_{z_0}^{*} \leq x^{1/6}} x e^{-z_0} \ll x .
\]

Thus \( \sum_1 \ll x (5/2e)^{z_0} \).

For \( \sum_2 \) we get

\[
\sum_2 \ll x \sum_{m_{z_0}^{*} \leq x^{1/6}} \sum_{p_1 \geq \exp \exp z_0} x \frac{x}{m_{z_0}^{*}p_1 \log p_1} \ll x e^{-z_0} .
\]

This concludes the proof of \( S(0) \).

**Induction step** \( S(k) \Rightarrow S(k + 1) \). The induction step is similar to the proof of Theorem 2 in [8] but there are additional difficulties. Since the induction step is rather complicated, we start by giving an outline.
Outline of the induction step. Assume that $n$ has property $(B(k))$ and let the block of $2^k$ divisors of $n(k): d_1, \ldots, d_{2^k}$ be contained in the interval $I_k = [\log d_1 - \zeta_k, \log d_{2^k} + \zeta_k]$. We then consider the translates $I_k + \log d$, where $d$ consists of larger prime factors of $n$. Our aim is to show that almost always two such translates merge into a block of the double size $2^{k+1}$.

That would conclude the induction step $S(k) \Rightarrow S(k + 1)$ if the $d$'s are not too large. The aim, to establish the merger of two translates, is roughly achieved as follows:

We denote by $\mathcal{B}(k, l)$ the exceptional set of integers for which no two translates $I_k + \log d, I_k + \log d', d, d'|n(k, l)$ have merged. We then will show that $\text{card } \mathcal{B}(k, l)$ is exponentially decreasing for increasing $l$. We have already shown in Lemma 4 that the measure of

$$\bigcup_{dd' \mid \tilde{n}(k, l), \mu(dd') = 1} \log(d'/d) + [-\eta, \eta]$$

is fairly large for most $\eta$.

This leaves many possibilities for the subsequent prime divisors $p_1^{(k, l)}$ and $p_2^{(k, l)}$ that the difference $\log p_2^{(k, l)} - \log p_1^{(k, l)}$ is close to a logarithm $\log(d'/d)$. But then $\log d + \log p_1^{(k, l)} + I_k$ contains the block of $2^{k+1}$ divisors:

$$\log d_j + \log d + \log p_1^{(k, l)}, \quad \log d_j + \log d' + \log p_2^{(k, l)} \quad (j = 1, \ldots, 2^k).$$

Thus, if $n(k, l)$ does not have property $(B(k))$ and therefore by definition $n(k, l) \in \mathcal{B}(k, l)$, the conditional probability that for small $j, n(k, l+j)$ still does not have property $(B(k))$ and thus $n(k, l+j) \in \mathcal{B}(k, l+j)$ is not too close to 1. This fact accounts for the shrinking of $\mathcal{B}(k, l)$ with increasing $l$.

There is one additional difficulty to overcome. We have to guarantee that the new larger block of $2^{k+1}$ divisors is not interrupted by other divisors with different $\mu$-values. This is accomplished by only considering translates $I_k + \log d$, which do not contain $\log d$-values other than the translates of the $\log d_j$. We will call such divisors $d$ pure.

Thus instead of the measure of

$$\mathcal{E}(n, k, l, \eta) = \bigcup_{dd' \mid \tilde{n}(k, l), \mu(dd') = 1} \log(d'/d) + [-\eta, \eta]$$

we have to consider the measure of

$$\mathcal{D}(n, k, l, \eta) = \bigcup_{dd' \mid \tilde{n}(k, l), \mu(dd') = 1} \log(d'/d) + [-\eta, \eta].$$

In Lemma 5 we will show that the contribution in $\mathcal{E}(n, k, l, \eta)$ of $d, d'$ that are not pure is very small. Thus $\text{meas } \mathcal{D}(n, k, l, \eta)$ is approximately $\text{meas } \mathcal{E}(n, k, l, \eta)$.

After this outline we now give the details of the induction step.

DEFINITIONS. We denote by $\mathcal{B}(k)$ the set of all $n \leq x$ that possess property $(B(k))$ and by $\mathcal{B}(k, l)$ the set of all integers $n \leq x$ that possess the property $(B(k))$,
but for which there exists no block of $2^{k+1}$ divisors $d_j$, $1 \leq j \leq 2^{k+1}$, $d_j|n_{(k,l)}$ with property $(P(k+1))$.

Given $n \in B(k,l)$ and a block of $2^k$ divisors $d_j|n_{(k)}$, $1 \leq j \leq 2^k$, with property $(P(k))$. We set $I_k(n_{(k)}) = [\log d_1 - \zeta_k, \log d_{2^k} + \zeta_k]$. If there are several blocks we arbitrarily choose one of them to define $I_k(n_k)$. Many of the following definitions will depend on this choice of $I_k(n_{(k)})$.

Given any positive integer $r$, we call $d|n_{(k)}^*$ $r$-pure if $I_k(n_{(k)}) + \log d$ contains no $\log d'$, $d'$ $(n,r)$ other than the translates $\log d' := \log d_j + \log d$ $(1 \leq j \leq 2^k)$. For $\eta > 0$ we denote by $\lambda(n,k,l,\eta)$ the Lebesgue measure of the set

$$\mathcal{D}(n,k,l,\eta) = \bigcup_{dd' \mid I_{(k,l)}, \mu(dd')=1, d,d' n_{(k,l)}-pure} \log(d'/d) + [-\eta, \eta].$$

Let now $\varepsilon_2 > 0$ be a constant to be determined later. Then we define

$$L_k = \rho^k(\rho - 1 - 2\varepsilon_2)v(x) \quad \text{and} \quad M_k = \rho^k(\rho - 1 - \varepsilon_2)v(x).$$

We will prove

**Lemma 5.** For all $n \in B(k)$ except a set of cardinality $\ll \gamma x \exp(-c_6(\gamma)w(x))$ we have $\mu(n,k,l,\eta) - \lambda(n,k,l,\eta) \leq \exp(r_{k,1}(1 - c_7(\gamma)))$, for $L_k < l \leq M_k$, where $c_6(\gamma) > 0$, $c_7(\gamma) > 0$ depend only on $\gamma$.

In preparation for the proof of Lemma 5 we first give some more definitions and prove some auxiliary lemmas.

We set $q_k = r_{k+1} - r_k$ and $s_k = [q_k(1 + \varepsilon_3)]$, where $\varepsilon_3 > 0$ will be determined later.

We denote by $R(k)$ the set of all $n \in B(k)$ with the following properties:

(i) $n_{(k,q_k)} \mid n_{(s_k)}$,
(ii) $\omega(n_{(k)}) \leq r_k(1 + \varepsilon_4),$
(iii) $\log p_{s_k}^{(k)}(1 + \varepsilon_5) + s$ for $1 \leq s \leq s_k,$
(iv) $n_{(s_k)}^* \leq x^{1/6},$

**Lemma 6.** $\text{card}(B(k) \setminus R(k)) \leq C(\gamma, \varepsilon_3, \varepsilon_4, \varepsilon_5)x \exp(-c(\gamma, \varepsilon_3, \varepsilon_4, \varepsilon_5)w(x))$ where the constants $c > 0$ and $C > 0$ depend only on the indicated parameters.

**Proof.** For any of the properties (i)–(iii) we estimate the set of $n \leq x$ not possessing this property by Lemma 3. For the estimate of the set exceptional with respect to (iv) we observe that $\log \log p_{s_k}^{(k)} \leq r_k + s_k(1 + \varepsilon_3)$ for most $n$ and then apply Lemma 2 for the estimate of $n_{(k,s_k(1+\varepsilon_3))}^*$; observing that $k \leq K$ and thus $r_k \leq (\log \log x)^{1-\varepsilon_1}$.

**Definitions.** We introduce the set

$$\mathcal{F}(n,k,l) = \{d \mid n_{(k,l)} : d \text{ not } n_{(k,l)}-pure\}.$$

For $d \mid n_{(k,l)}$ we define $c_{k,l}(d,n) = \text{card}\{dd' \mid n_{(k,l)} : (d,d') = 1\}$ and obtain

$$\mu(n,k,l,\eta) - \lambda(n,k,l,\eta) \leq 2\eta \sum_{d \in \mathcal{F}(n,k,l)} c_{k,l}(d,n) = 2\eta C(n,k,l),$$

say.
Thus
\[ (3.3) \quad \sum_{n \in \mathcal{R}(k)} (\mu(n, k, l, \eta) - \lambda(n, k, l, \eta)) \leq 2\eta \sum_{n \in \mathcal{R}(k)} C(n, k, l). \]

We introduce the set
\[ \mathcal{G}(n, k) = \left\{ d|\mathbb{N}_k : d \text{ not } n_k^{(s_k)}-\text{pure} \right\} \]
and define
\[ b_k(d, n) = \text{card} \left\{ d'|\mathbb{N}_k : (d, d') = 1 \right\}. \]

Since \( L_k < l \leq M_k \), we have for \( n \in \mathcal{R}(k) \): \( \hat{n}(k, l) \mid n_k^{(s_k)} \) and therefore \( c_{k, l}(d, n) \leq b_{k, l}(d, n) \) and \( \mathcal{F}(n, k, l) \subseteq \mathcal{G}(n, k) \). Therefore we have the majorization
\[ C(n, k, l) \leq \sum_{d \in \mathcal{G}(n, k)} b(d, n). \]

We introduce the sequence of sets
\[ \mathcal{H}(n, k, s) = \left\{ d|\mathbb{N}_k : d \text{ not } n_k^{(s)}-\text{pure} \right\}, \quad 1 \leq s \leq s_k, \]
such that
\[ \mathcal{H}(n, k, s_k) \supseteq \mathcal{G}(n, k) \quad \text{for all } n \in \mathcal{R}(k). \]

We set
\[ B(k, s) = \sum_{n \in \mathcal{R}(k)} \sum_{d \in \mathcal{H}(n, k, s)} b_k(d, n) \]
such that
\[ (3.4) \quad B(k, s_k) \geq \sum_{n \in \mathcal{R}(k)} C(n, k, l) \quad \text{for } L_k \leq l \leq M_k. \]

We now prove

**Lemma 7.** For \( 1 \leq s \leq s_k, \varepsilon_6 > 0 \) we have
\[ B(k, s) \leq C(\gamma, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6) \exp(-r_k(1 - \varepsilon_5)2^{r_k(1+\varepsilon_4)s_k} (3/2 + \varepsilon_6)^s). \]

**Proof.** If \( d \in \mathcal{H}(n, k, s) \) we have \( d = d^* \) or \( d = d^* p_s^{(k)} \), where \( d^* \mid n_k^{(s-1)} \). We have
\[
B(k, s) = \sum_{n \in \mathcal{R}(k)} \left\{ \sum_{d^* \in \mathcal{H}(n, k, s-1)} b_k(d^*, n) + b_k(d^* p_s^{(k)}, n) + \sum_{d^* \mid n_k^{(s-1)} : d^* \notin \mathcal{H}(n, k, s-1)} b_k(d^*, p_s^{(k)}) + \sum_{d^* \mid n_k^{(s-1)} : d^* \notin \mathcal{H}(n, k, s)} b_k(d^*, n) \right\}. 
\]
Since $b_k(d^* p_s^{(k)}, n) = \frac{1}{2} b_k(d^*, n)$, we have

\[ B(k, s) = \frac{3}{2} B(k, s - 1) \]

\[ B(k, s) = \frac{3}{2} B(k, s - 1) + E(k, s), \] say.

**Estimate of $E(k, s)$**. We have

\[ E(k, s) \leq \sum_{n \in R(k)} \sum_{d \mid n^{(s-1)}_k} s^{2k - \omega(d)}, \]

where the $\sum'$-sum is extended over all $d \mid n^{(s-1)}_k$ for which there exists a $\tilde{d} \mid n^{(s-1)}_k$ with $\log \tilde{d} \in \log(dp^{(k)}_s) + I_k(n_k)$ or a $d \mid n^{(s-1)}_k$ with $\log d \in \log(dp^{(k)}_s) + I_k(n_k)$.

Denoting the interval $I_k(n_k)$ by $[a_k(n_k), b_k(n_k)]$ we have for $s \geq 2$

\[ E(k, s) \ll \sum_{l \in B(k)} \sum_{\tilde{d} \mid l, \mu(\tilde{d}) \neq 0} \sum_{d \mid (l/l^{(k)}_s)} 2^{s_k - \omega(d)} \]

where the $\sum''$-sum is extended over all $p \geq p^{(k)}_{s-1}$ for which

\[ | \log p + \log d + a_k(l^{(k)}_s) - \log \tilde{d}| < \log 2 \quad \text{or} \quad | \log d + a_k(l^{(k)}_s) - \log \tilde{d} - \log p| < \log 2. \]

We recall that $h(r)$ denotes an integer all of whose prime factors are $> r$. Since $l \cdot p \leq 2x^{1/3}$ the inner sum is $\ll x/(lp \log p)$ by the sieve.

The interval for $\log p$ in $\sum''_p$ has length $\ll 1/\log p^{(k)}_{s-1}$ such that

\[ \sum_{l \in B(k)} \sum_{p \mid l \cdot p \cdot h(p-1) \leq x} 1 \]

Moreover,

\[ \sum_{\tilde{d} \mid l} \sum_{d \mid (l/l^{(k)}_s)} 2^{s_k - \omega(d)} \ll 2^{s_k - 3s/2} \omega(l). \]
Thus,

\[ E(k, s) \ll \sum_{l \leq x^{1/6} : \omega(l/l^*_k) = s-1} \frac{x}{l \log p_{s-1}^{(k)}} 2^{\omega(l)} 2^{s_k - s^3} . \]

Since

\[ \text{card} \{ n \leq x : n_{(k)}^{(s-1)} = l \} \gg \frac{x}{l \log p_{s-1}^{(k)}} \text{ for } l \leq x^{1/6} \]

we obtain

\[ \sum_{l \leq x^{1/6} : \omega(l/l^*_k) = s-1} \ll x . \]

Therefore,

(3.6) \[ E(k, s) \ll x 2^{r_k(1+\varepsilon_4)+s_k} \exp(-r_k(1-\varepsilon_5) - (s-1))3^s \]

for \( s \geq 2 \).

The estimate of \( E(k, 1) \) is accomplished in a similar manner. We omit the condition \( \log p_{s-1}^{(k)} \geq \exp(r_k(1-\varepsilon_5) + (s-1)) \) and observe instead, that \( \log p_{1}^{(k)} \geq \exp r_k \). This leads to the estimate (3.6) also for \( s = 1 \).

Now we prove Lemma 7 by induction in \( s \). We choose the integer \( s_0 = s_0(\varepsilon_6) > 0 \) such that

(3.7) \[ \varepsilon_6(3/2)^{s_0} \geq (3/e)^{s_0} . \]

First it is easily proven by induction, using (3.5) and (3.6), that

\[ B(k, s) \leq x C_0(5/2)^s , \]

where

\[ C_0 = C'(\gamma, \varepsilon_3, \varepsilon_4, \varepsilon_5) \exp(-r_k(1-\varepsilon_5))2^{r_k(1+\varepsilon_4)+s_k} \]

for \( s \leq s_0 \). This gives

\[ B(k, s) \leq x C_0(\frac{5}{3})^{s_0} \frac{3}{2}^{s_0} \leq x C_0(\frac{5}{3})^{s_0} \frac{3}{2} + \varepsilon_6)^{s_0} . \]

For \( s \geq s_0 \) we continue the induction, observing (3.5), (3.6), and (3.7). This concludes the proof of Lemma 7.

**Proof of Lemma 5.** From (3.3), (3.4), and Lemma 7 we obtain that

(3.8) \[ \sum_{n \in \mathbb{N}(k)} (\mu(n, k, l, \eta) - \lambda(n, k, l, \eta)) \leq 2\eta C(\gamma, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6) x \exp(r_k(1-\varepsilon_5))2^{r_k(1+\varepsilon_4)+s_k} (\frac{3}{2} + \varepsilon_6)^{s_k} . \]

We now fix the constants \( c_7(\gamma), \varepsilon_2, \ldots, \varepsilon_6 \) in a manner such that

(3.9) \[ (\rho - 1 - 2\varepsilon_2) > (\log 3 - 1)^{-1}(1 + \varepsilon_2) \]
and
\[-(1 - \varepsilon_5) + \{1 + \varepsilon_4 + (\rho - 1)(1 + \varepsilon_3)\} \log 2
+ (\rho - 1)(1 + \varepsilon_3) \log (\frac{3}{2} + \varepsilon_6)
\leq [1 + (\rho - 1 - 2\varepsilon_2)(1 - 2c_7(\gamma))],\]

which is possible because of (3.1) and (3.2). Then (3.8) gives, that
\[
\sum_{n \in \mathcal{R}(k)} (\mu(n, k, l, \eta) - \lambda(n, k, l, \eta)) \ll_{\gamma} \exp(r_{k,l}(1 - 2c_7(\gamma)))
\]
for \(L_k < l \leq M_k\). This implies that
\[
\mu(n, k, l, \eta) - \lambda(n, k, l, \eta) \leq \exp(r_{k,l}(1 - c_7(\gamma))
\]
for all \(n \in \mathcal{R}(k)\) except a set of cardinality \(\ll_{\gamma} x \exp(-r_{k,l}c_7(\gamma))\). This together with Lemma 6 implies Lemma 5. Because of (3.9) Lemma 4 is applicable. As an immediate corollary of Lemmas 4 and 5 we obtain

**Lemma 8.** We have \(\lambda(n, k, l, \eta) \geq \exp(r_{k,l})w(x)^{-3}\) for all \(n \in \mathcal{B}(k)\) except a set of cardinality \(\ll_{\gamma} x w(x)^{-c_5(\gamma)}\).

**Conclusion of the Proof of Theorem 2.** To complete the induction step and thus the proof of Theorem 2 we want to show that
\[
\text{card } \mathcal{B}(k, l) \leq c_4(\gamma)xw(x)^{-c_5(\gamma)} \quad \text{for some } l \in [L_k, M_k].
\]
We denote by \(\mathcal{C}(k, l)\) the subset of \(\mathcal{B}(k, l)\) of those integers which satisfy the three extra conditions:

(a) \(\log n^*_{(k,l)} \leq \exp(r_{k,l})w(x)\),
(b) \(\omega(n_{(k,l)}) \leq 2r_{k,l}\),
(c) \(\lambda(n, k, l, \eta) \geq \exp(r_{k,l})w(x)^{-3}\).

By Lemma 2, 3, and 8 we have
\[
\text{card}(\mathcal{B}(k, l)/\mathcal{C}(k, l)) \ll_{\gamma} x w(x)^{-c_5(\gamma)}.
\]
Thus to complete the proof of Theorem 2 it suffices to show that
\[
\text{card } \mathcal{C}(k, l) \leq x w(x)^{-2c_5(\gamma)} \quad \text{for some } l \in [L_k, M_k].
\]
Assume that
\[
n = n^*_{(k,l)}p_{1}^{(k,l)} \cdots p_{r}^{(k,l)}, \quad p_{1}^{(k,l)} \leq \cdots \leq p_{r}^{(k,l)}.
\]
We consider the set \(\mathcal{A}(k, l)\) of \(n \in \mathcal{C}(k, l)\), whose prime factors \(p_{1}^{(k,l)}, p_{2}^{(k,l)}, p_{3}^{(k,l)}\) satisfy the following conditions:

(i) \(\exp(r_{k,l})w(x) \leq \log p_{1}^{(k,l)} \leq 2\exp(r_{k,l})w(x)\),
(ii) \(\log p_{2}^{(k,l)} - \log p_{1}^{(k,l)} \in \bigcup_{d'd' | n_{(k,l)}, \text{pure}} \log(d'/d) + [-\eta_{k+1}, \eta_{k+1}]\),
(iii) \(\log p_{3}^{(k,l)} \geq \log(n_{(k,l)}p_{1}^{(k,l)}p_{2}^{(k,l)})\).

These conditions ensure that there exists a block of \(2^{k+1}\) divisors of \(n_{(k,l+j)}, j \leq 2\log w(x)\), satisfying \((P(k + 1))\), namely the divisors \(p_{1}^{(k,l)}d'd_i, p_{2}^{(k,l)}dd_i, (1 \leq i \leq 2^k)\). Condition (iii) ensures that this block is not destroyed by larger prime factors.
Thus \( C(k, l + j) \subseteq C(k, l) / A(k, l) \) such that

\[
\text{(3.12)} \quad \text{card } C(k, l + j) \leq \text{card } C(k, l) - \text{card } A(k, l).
\]

We now give a lower bound for \( \text{card } A(k, l) \). Denote by \( m_{(k, l)}^* \) an integer equal to \( n_{(k, l)}^* \) for some \( n \in C(k, l) \).

We have

\[
\text{card } A(k, l) \gg \sum_{m_{(k, l)}^*, p_1^{(k, l)}, p_2^{(k, l)} : h(m_{(k, l)}^*, p_1^{(k, l)}, p_2^{(k, l)}) \leq x} 1
\]

where * means that \( n_{(k, l)}^* \in B(k) \) and that the \( n_{(k, l)}^*, p_1^{(k, l)} \) satisfy (i)-(iii).

By the sieve we have

\[
\text{card } A(k, l) \gg \sum_{m_{(k, l)}^*, p_1^{(k, l)}, p_2^{(k, l)} : p_2^{(k, l)} \log p_2^{(k, l)} \leq x} \frac{x}{m_{(k, l)}^*}.
\]

For a fixed pair \( (m_{(k, l)}^*, p_1^{(k, l)}) \) the \( p_2^{(k, l)} \) cover a union of at most \( 3^{\omega(m_{(k, l)}^*)} \leq 3^{2r_{k, l}} \) disjoint intervals with total logarithmic length \( \geq \frac{1}{2} \exp(r_{k, l}) w(x)^{-3} \). Moreover all the limit points have logarithm of order \( \exp(r_{k, l}) w(x)^{-5} \). This implies that the \( p_2^{(k, l)} \)-sum is \( \gg \exp(-r_{k, l}) w(x)^{-5} \). The \( p_1^{(k, l)} \)-sum is \( \gg 1 \). Finally,

\[
\text{card } A(k, l) \gg \left( \sum_{m_{(k, l)}^*, \log m_{(k, l)}^* \leq \exp(r_{k, l}) w(x)} \frac{x}{m_{(k, l)}^*} \right) \exp(-r_{k, l}) w(x)^{-5},
\]

\[
\text{card } C(k, l) \leq \sum_{m_{(k, l)}^*, \log m_{(k, l)}^* \leq \exp(r_{k, l}) w(x)} 1
\]

\[
\ll \left( \sum_{m_{(k, l)}^*, \log m_{(k, l)}^* \leq \exp(r_{k, l}) w(x)} \frac{x}{m_{(k, l)}^*} \right) \exp(-r_{k, l}).
\]

Thus

\[
\text{card } A(k, l) \gg \text{card } C(k, l) w(x)^{-5}.
\]

Together with (3.11) this gives

\[
\text{card } C(k, M_k) \leq \text{card } C(k, l)(1 - w(x)^{-5})^{(M_k - L_k)/2j} \ll x \exp(-w(x)^{1/2}),
\]

which is sufficient.

\begin{thebibliography}{9}


\end{thebibliography}

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