

ON THE INVARIANCE OF q -CONVEXITY AND HYPERCONVEXITY UNDER FINITE HOLOMORPHIC SURJECTIONS

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ABSTRACT. In this note we have proved that 0-convexity and hyperconvexity are invariant under finite holomorphic surjections. Invariance of cohomological q -convexity for the case of finite dimension also has been established.

It is known [7] that Steinness is invariant under finite holomorphic surjections. In this note we investigate the invariance property for 0-convexity generally, for cohomological completeness, for cohomological q -convexity of finite dimension, and for hyperconvexity.

1. The invariance of 0-convexity. We recall that a complex space X is called q -convex if there exists an exhaustion function φ on X which is strictly q -pseudoconvex outside some compact $K \subset X$. It is known [2] that X is 0-convex if and only if there exists a proper holomorphic map θ from X onto a Stein space S such that $\theta_*\mathcal{O}_X \simeq \mathcal{O}_S$ and θ is biholomorphic outside some set of the form $\theta^{-1}A$, where A is a finite subset of S . The proper surjection $\theta: X \rightarrow S$ with S Stein, such that $\theta_*\mathcal{O}_X \simeq \mathcal{O}_S$ is said to be the *Remmert reduction* of X . By [2] if X is holomorphically convex then there exists a Remmert reduction. In this section we prove the following theorem.

1.1 THEOREM. *Let $\varphi: X \rightarrow Y$ be a finite holomorphic surjective map. Then Y is 0-convex if and only if X is.*

The proof of Theorem 1.1 is based on the following assertion, essentially as in [9].

1.2 ASSERTION. A complex space X is 0-convex if and only if $\dim H^1(X, \mathcal{S}) < \infty$ for every coherent ideal subsheaf $\mathcal{S} \subset \mathcal{O}_X$.

PROOF. The necessity follows from a theorem of Andreotti-Grauert [1].

Conversely, assume that $\dim H^1(X, \mathcal{S}) < \infty$ for every coherent ideal subsheaf \mathcal{S} of \mathcal{O}_X . We have to prove that X is 0-convex.

First we show that X is holomorphically convex. Let $V = \{x_n\}_{n=1}^\infty$ be a discrete sequence in X and let J_V denote the ideal subsheaf of \mathcal{O}_X associated to V . Consider the exact sequence:

$$(1) \quad 0 \rightarrow J_V \rightarrow \mathcal{O}_X \xrightarrow{\eta} \tilde{\mathcal{O}}V \rightarrow 0.$$

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By hypothesis and by exactness of the cohomology sequence associated to (1) we get

$$(2) \quad \dim \mathcal{O}(V)/\text{Im} \eta = \dim C^\infty/\text{Im} \eta < \infty.$$

Let $l_\infty(V)$ denote the subspace of $\mathcal{O}(V)$ consisting of bounded functions on V . Then $\dim C^\infty/l_\infty(V) = \infty$. Thus by (2) it follows that $\text{Im} \eta \setminus l_\infty(V) \neq \emptyset$. This implies that $\sup|f(x_n)| = \infty$ for some $f \in \mathcal{O}(X)$. Hence X is holomorphically convex.

Let $\theta: X \rightarrow S$ be the Remmert reduction of X . To prove that X is 0-convex it suffices to show that $\theta: X \setminus K \rightarrow S \setminus \theta(K)$ is injective for some compact set $K \subset X$. For a contradiction suppose there is a discrete set $V = \{x_n, y_n\}_{n=1}^\infty$ such that $\theta(x_n) = \theta(y_n)$ for every $n \geq 1$. For each n let $\sigma_n \in \mathcal{O}(V)$ be given by the formula

$$\sigma_n(x_n) = 1, \quad \sigma_n|_{V - x_n} = 0.$$

Since $\theta_*\mathcal{O}_X \simeq \mathcal{O}_S$ it is easy to see that $\{\sigma_n \text{ mod } \eta\}$ ($n = 1, 2, \dots$) is linearly independent in $\mathcal{O}(V)/\text{Im} \eta$. This contradicts (2). Hence Assertion 1.2 is proved.

1.3 ASSERTION. Let $\theta: X \rightarrow Y$ be an n -analytic covering and let Y be normal. Then for every coherent ideal subsheaf \mathcal{S} of \mathcal{O}_Y there exists a morphism $Q: \theta_*\theta^*\mathcal{S} \rightarrow \mathcal{S}$ such that $Qe = \text{id}$, where $e: \mathcal{S} \rightarrow \theta_*\theta^*\mathcal{S}$ is the canonical injection.

PROOF. Let $V \subset Y$ be the branch locus of θ and let $U \subset Y$ be a Stein open subset of Y on which there exists an exact sequence

$$(4) \quad \mathcal{O}_Y^q \rightarrow \mathcal{O}_Y^p \xrightarrow{\eta} \mathcal{S} \rightarrow 0.$$

Then the sequence

$$(5) \quad \theta^*\mathcal{O}_Y^q \rightarrow \theta^*\mathcal{O}_Y^p \xrightarrow{\tilde{\eta}} \theta^*\mathcal{S} \rightarrow 0$$

is also exact. Note that $\theta^*\mathcal{O}_Y^m \simeq \mathcal{O}_X^m$ for every $m \geq 1$.

Consider $\sigma \in H^0(U, \theta_*\theta^*\mathcal{S}) = H^0(\theta^{-1}(U), \mathcal{S})$. Since $\theta^{-1}(U)$ is Stein [7] we can find $\beta \in H^0(\theta^{-1}(U), \mathcal{O}_X^p)$ such that $\tilde{\eta}\beta = \sigma$. Since Y is normal, the formula $P_U(\beta)(z) = (1/n)\sum_{j=1}^n \beta(x_j)$ for $z \in U \setminus V$ where $\theta^{-1}(z) = \{x_1, x_2, \dots, x_n\}$, defines an element $Q_U(\beta) \in \mathcal{O}_Y^q(U)$. Put

$$(6) \quad Q_U(\sigma) = \eta P_U(\beta).$$

It is easy to see that $Q_U(\sigma)$ is independent of choice of $\beta \in H^0(\theta^{-1}(U), \mathcal{O}_X^p)$, $\tilde{\eta}\beta = \sigma$, and $Q_U(\sigma) = \sigma$ for all $\sigma \in H^0(U, \mathcal{S})$.

Assume now that

$$(7) \quad \mathcal{O}_Y^{q'} \rightarrow \mathcal{O}_Y^{p'} \rightarrow \mathcal{S} \rightarrow 0$$

in another exact sequence on U . Then there exists a commutative diagram:

$$(8) \quad \begin{array}{ccccccc} \mathcal{O}_Y^q & \rightarrow & \mathcal{O}_Y^p & \xrightarrow{\eta} & \mathcal{S} & \rightarrow & 0 \\ \downarrow & & \downarrow & & \parallel & & \\ \mathcal{O}_Y^{q'} & \rightarrow & \mathcal{O}_Y^{p'} & \rightarrow & \mathcal{S} & \rightarrow & 0 \end{array}$$

By the commutativity of (8) it follows that Q_U is independent of choice of presentation. Hence $Q = \{Q_U\}$ defines a morphism $Q: \theta_*\theta^*\mathcal{S} \rightarrow \mathcal{S}$ such that $Qe = \text{id}$. Assertion 1.3 is proved.

Now we are able to prove Theorem 1.1. Assume that Y is 0-convex. Since φ is proper it follows that X is holomorphically convex. Considering the commutative diagram

$$(9) \quad \begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \theta_X \downarrow & & \downarrow \theta_Y \\ S_X & \xrightarrow{\tilde{\varphi}} & S_Y \end{array}$$

where θ_X and θ_Y are Remmert reductions of X and Y respectively, it is easy to see that θ_X is biholomorphic outside some compact set K in X . Hence X is 0-convex.

Conversely, assume that X is 0-convex. We prove that Y is also 0-convex.

(a) First we consider the case, where $\dim Y < \infty$.

We assume that the theorem has been proved for all complex spaces Y of dimension $< m$. Now assume that $\dim Y = m$. Consider the commutative diagram:

$$(10) \quad \begin{array}{ccc} (X \times_Y \tilde{Y})_{\text{red}} & \xrightarrow{\tilde{\varphi}} & \tilde{Y} \\ \downarrow \eta & & \downarrow \nu \\ X & \xrightarrow{\varphi} & Y \end{array}$$

of finite holomorphic surjective maps, where \tilde{Y} is the normalization of Y . By the necessary condition already proved, $(X \times_Y \tilde{Y})_{\text{red}}$ is 0-convex. On the other hand, since $\tilde{\varphi}$ is finite and \tilde{Y} normal it follows that $\tilde{\varphi}$ is a finite analytic n -covering for some n [4]. Thus by 1.2 and 1.3 we infer that \tilde{Y} is 0-convex. To prove that Y is 0-convex by 1.2 it suffices to show that $\dim H^1(Y, \mathcal{S}) < \infty$ for every coherent ideal subsheaf \mathcal{S} of \mathcal{O}_Y . Let $\tilde{\mathcal{O}}_Y$ denote the coherent analytic sheaf of germs of weakly holomorphic functions on Y [4]. Put $\mathcal{D} = \mathcal{O}_Y : \tilde{\mathcal{O}}_Y$. Note that $\nu_* \mathcal{O}_{\tilde{Y}} = \tilde{\mathcal{O}}_Y$ and $\text{supp } \theta_Y / \mathcal{D} = N(Y)$ where $N(Y)$ denotes the nonnormal locus of Y . Let \mathcal{V} be the coherent ideal subsheaf of $\mathcal{O}_{\tilde{Y}}$ which is the image of $\nu^* \mathcal{D} \mathcal{S} = \nu^{-1}(\mathcal{D} \mathcal{S}) \otimes_{\nu^{-1} \mathcal{O}_Y} \mathcal{O}_{\tilde{Y}}$ under multiplication. By using the definition of $\tilde{\varphi}$ it follows that $\nu_* \mathcal{V} \subset \mathcal{S}$ and since \tilde{Y} is 0-convex and ν is finite we have [4]

$$(11) \quad \dim H^1(Y, \nu_* \mathcal{V}) = \dim H^1(\tilde{Y}, \mathcal{V}) < \infty.$$

Since ν is biholomorphic outside $\nu^{-1}(N(Y))$ it follows that

$$(12) \quad \text{supp } \mathcal{S} / \nu_* \mathcal{V} \subset N(Y).$$

Thus, using the induction hypothesis we get

$$(13) \quad \dim H^1(Y, \mathcal{S} / \nu_* \mathcal{V}) = \dim H^1(N(Y), \mathcal{S} / \nu_* \mathcal{V}) < \infty.$$

By (11) and (13) and by the exactness of the cohomology sequence associated to the exact sequence

$$0 \rightarrow \nu_* \mathcal{V} \rightarrow \mathcal{S} \rightarrow \mathcal{S} / \nu_* \mathcal{V} \rightarrow 0,$$

we infer that $\dim H^1(Y, \mathcal{S}) < \infty$.

(b) In the general case, let $Y = \bigcup_{j=1}^{\infty} V_j$, where V is an irreducible branch of Y for any $j \geq 1$. Since $\tilde{Y} = \prod_1^{\infty} \tilde{V}_j$, by the 0-convexity of \tilde{Y} it is easy to see that there exists j_0 such that \tilde{V}_j is Stein for every $j > j_0$. Hence V_j is also Stein for every $j > j_0$.

Put

$$Y_0 = \bigcup_{j=1}^{j_0} V_j, \quad Y_k = Y_0 \cup \bigcup_{j=1}^k V_{j_0+j}.$$

By (a) Y_k is 0-convex for every $k \geq 0$.

If Y_0 is Stein then Y is Stein by [7]. Now we assume that Y_0 is 0-convex non-Stein. Thus Y_k is 0-convex non-Stein for every $k \geq 0$. Let $\theta_k: Y_k \rightarrow S_k$ be the Remmert reduction of Y_k . Then we have the following diagram:

$$\begin{array}{ccccccc} Y_0 & \xrightarrow{i_0} & Y_1 & \xrightarrow{i_1} & Y_2 & \xrightarrow{i_2} & \dots \\ \theta_0 \downarrow & & \downarrow \theta_1 & & \downarrow \theta_2 & & \\ S_0 & \xrightarrow{\tilde{i}_0} & S_1 & \xrightarrow{\tilde{i}_1} & S_2 & \xrightarrow{\tilde{i}_2} & \dots \end{array}$$

Let A_k be a finite subset of S_k such that $\theta_k: Y_k - \theta_k^{-1}(A_k) \rightarrow S_k - A_k$ is biholomorphic. Since Y_k non-Stein, $\theta_k^{-1}(y)$ is connected of positive dimension for every $y \in A_k$ [2]. Then, since $\bigcup_{j>j_0} V_j$ is Stein, and $\theta_k^{-1}(A_k)$ is compact connected of positive dimension, it follows that

$$(14) \quad \theta_k^{-1}(A_k) = \theta_0^{-1}(A_0) \quad \text{for every } k \geq 0.$$

From (14) it is easy to see that there exists k_0 such that

$$(15) \quad \tilde{i}_k \text{ is proper injective and } \theta_0^{-1}(A_0) \subset \text{Int } Y_k \quad \text{for every } k = k_0.$$

Put $S = \varinjlim S_k$ and $\theta = \varinjlim \theta_k: Y \rightarrow S$. By (14)(15) we infer that θ is proper, $\theta|_{Y \setminus \theta_0^{-1}(A_0)}$ is biholomorphic, and $\theta_* \mathcal{O}_Y = \mathcal{O}_S$. Moreover, since $S = \varinjlim \tilde{i}_k(S_k)$ where $\tilde{i}_k(S_k)$ are Stein closed subspaces of S , it follows that S is holomorphically separated and holomorphically convex and thereby S is Stein. Hence Y is 0-convex. This completes the proof of Theorem 1.1.

The following is an immediate consequence of Theorem 1.1.

1.4 COROLLARY. *A complex space X is 0-convex if and only if all its irreducible branches, except for finitely many which are 0-convex, are Stein.*

1.5 COROLLARY. *Let $\theta: X \rightarrow Y$ be a proper holomorphic surjective map which is finite outside a compact set. Then X is 0-convex if and only if Y is.*

PROOF. Assume that Y is 0-convex. Considering the commutative diagram (9) it is easy to see that θ_X is finite outside a compact set. Hence by the Steinness of S_X we infer that X is 0-convex. Now assume that X is 0-convex. Consider the commutative diagram (16)

$$(16) \quad \begin{array}{ccc} X & \xrightarrow{\theta} & Y \\ \theta_X \downarrow & \searrow \eta & \uparrow \theta' \\ S_X & \xleftarrow{\beta} & X' \end{array}$$

in which X' is the Stein factorization of X for θ , and η, β are canonical maps and θ' is induced by θ . It is easy to check that θ' is finite, β is finite outside a compact set. This implies that X' is 0-convex. By Theorem 1.1 we infer that Y is 0-convex.

1.6 COROLLARY. *Let $\theta: X \rightarrow Y$ be a proper holomorphic surjective map and X be 0-convex. Then Y is also 0-convex.*

PROOF. Considering the commutative diagrams (10) and (16) and by Theorem (1.1), without loss of generality we may assume that Y is normal. Hence Y is holomorphically convex. Consider the commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\theta} & Y \\ \theta_X \downarrow & & \downarrow \theta_Y \\ S_X & \xrightarrow{\tilde{\theta}} & S_Y \end{array}$$

Since θ_X is finite outside a compact set, and $\tilde{\theta}$ is finite, it follows that θ_Y is finite outside a compact set. Hence Y is 0-convex.

1.7 REMARK. Corollary (1.6) is not true for the holomorphically convex property [7].

2. The invariance of cohomological q -completeness. A complex space X is called cohomologically q -complete (resp. cohomologically q -convex) if and only if $H^p(X, \mathcal{S}) = 0$ (resp. $\dim H^p(X, \mathcal{S}) < \infty$) for every coherent ideal subsheaf \mathcal{S} of \mathcal{O}_X and for every $p > q$.

In this section we prove the following theorem.

2.1 THEOREM. *Let $\varphi: X \rightarrow Y$ be a finite holomorphic surjective map. Then X is cohomologically q -complete if and only if Y is. If, moreover, $\dim X < \infty$ then X is cohomologically q -convex if and only if Y is.*

PROOF. Since φ is finite it follows that if Y is cohomologically q -complete (resp. cohomologically q -convex) then X is too.

As in the proof of Theorem 1.1(a) it follows that if $\dim X < \infty$ and X is cohomologically q -complete (resp. cohomologically q -convex), then so is Y .

Thus to find the proof of the theorem it suffices to prove the following

2.2 ASSERTION. Let $X = \bigcup_{k=1}^{\infty} X_k$, X_k is the union of all irreducible branches of X of dimension $< k$. If X_k is cohomologically q -complete for every $k \geq 1$, then X is also cohomologically q -complete.

PROOF. Let \mathcal{S} be a coherent ideal subsheaf of \mathcal{O}_X and $J_k = \mathcal{F}_{X_k}$ —the ideal subsheaf of \mathcal{O}_X associated to X_k . By $\eta_k: \mathcal{O}_X \rightarrow \tilde{\mathcal{O}}_{X_k}$ denotes the canonical map. Put $\mathcal{S}_k = \eta_k(\mathcal{S})$. Since any open set in X is contained in some X_k it follows that

$$\mathcal{S} = \varprojlim \{ \mathcal{S}_k, \omega_k^j \},$$

where $\omega_k^j: \mathcal{S}_k \rightarrow \mathcal{S}_j$ is a canonical map.

Let \mathcal{U} be a Stein open covering of X . By hypothesis we have

$$(17) \quad H^p(\mathcal{U}, \mathcal{S}_k) = H^p(X, \mathcal{S}_k) = H^p(X_k, \mathcal{S}_k) = 0$$

for every $p > q$ and so

$$(18) \quad \text{Im}\{H^{p-1}(\mathcal{U}, \mathcal{S}_{k+1}) \rightarrow H^{p-1}(\mathcal{U}, \mathcal{S}_k)\} = H^{p-1}(\mathcal{U}, \mathcal{S}_k)$$

for every $p > q$ and $k \geq 1$.

Consider $\sigma \in Z^p(\mathcal{U}, \mathcal{S})$, $p > q$. By (17) for each $k \geq 1$ we find $\beta'_k \in C^{p-1}(\mathcal{U}, \mathcal{S}_k)$ such that $\delta^{p-1}\beta'_k = \eta_k\sigma$. Put $\beta_1 = \beta'_1$ and consider $\omega_2^1\beta'_2 - \beta_1$. Since $\delta^{p-1}(\omega_2^1\beta'_2 - \beta_1) = 0$, by (18) with $k = 1$ we find $\beta''_2 \in Z^{p-1}(\mathcal{U}, \mathcal{S}_2)$ such that

$$\omega_2^1(\beta''_2 - \beta'_2) + \beta_1 = \delta^{p-2}\gamma \quad \text{for some } \gamma \in C^{p-2}(\mathcal{U}, \mathcal{S}_1).$$

Since \mathcal{U} is a Stein open covering, there exists $\tilde{\gamma} \in C^{p-2}(\mathcal{U}, \mathcal{S}_2)$ such that $\omega_2^1\tilde{\gamma} = \gamma$. Put

$$\beta_2 = -\beta''_2 + \beta'_2 + \delta^{p-2}\tilde{\gamma}.$$

Then $\delta^{p-1}\beta_2 = \eta_2\sigma$ and $\omega_2^1\beta_2 = \omega_2^1(\beta'_2 - \beta''_2) + \omega_2^1\delta^{p-1}\tilde{\gamma} = \omega_2^1(\beta'_2 - \beta''_2) + \delta^{p-1}\omega_2^1\tilde{\gamma} = \omega_2^1(\beta'_2 - \beta''_2) + \beta_1 + \omega_2^1(\beta''_2 - \beta'_2) = \beta_1$. Continuing this process we get a sequence $\{\beta_n\}$ such that for every $n \geq 1$:

$$\beta_n \in C^{p-1}(\mathcal{U}, \mathcal{S}_n), \quad \delta^{p-1}(\beta_n) = \eta_n(\sigma) \quad \text{and} \quad \omega_{n+1}^n\beta_{n+1} = \beta_n.$$

Thus $\beta = \{\beta_n\} \in C^{p-1}(\mathcal{U}, \mathcal{S})$ and $\delta^{p-1}\beta = \sigma$. Hence $H^p(X, \mathcal{S}) = 0$ and 2.2 is proved.

The following is an immediate consequence of Theorem 2.1.

2.3 COROLLARY. *X is cohomologically q -complete if and only if every irreducible branch of X is.*

3. The invariance of the hyperconvexity. We recall that a Stein space X is called hyperconvex (resp. strongly hyperconvex) if there exists a plurisubharmonic (resp. strictly plurisubharmonic) negative exhaustion function on X [8]. In this section the following theorem is proved.

3.1 THEOREM. *Let $\theta: X \rightarrow Y$ be a finite holomorphic surjective map of finite-dimensional complex spaces. Then:*

(i) *If Y is strongly hyperconvex having a strictly plurisubharmonic negative exhaustion C^2 -function, then X is strongly hyperconvex.*

(ii) *If Y is irreducible and X is strongly hyperconvex having a strictly plurisubharmonic negative exhaustion C^2 -function, then Y is strongly hyperconvex.*

We need the following.

3.2 LEMMA. *If X is strongly hyperconvex and Y is normal, then so is Y .*

PROOF. Let ψ be a strictly plurisubharmonic negative exhaustion function of X . By the integer lemma [4] we infer that $\theta: X \rightarrow Y$ is an analytic covering. Thus we can define a function φ on Y by the formula

$$(19) \quad \varphi(y) = \text{Tr}_\theta(\psi)(y) = \sum_{\theta x=y} \psi(x)$$

(the points of $\theta^{-1}(y)$ being counted with the right multiplicity).

Since $\psi < 0$ it follows that φ is an exhaustion function. First we prove that φ is plurisubharmonic. By a theorem of Fornaess and Narasimham [5] it suffices to show that $\varphi\sigma$ is subharmonic for any holomorphic map σ of unit disc $D \subset C$ into Y .

Given such a map $\sigma: D \rightarrow Y$, consider the commutative diagram:

$$\begin{array}{ccc} (D \times_Y X)_{\text{red}} & \xrightarrow{\tilde{\sigma}} & X \\ \tilde{\theta} \downarrow & & \downarrow \theta \\ D & \rightarrow & Y \end{array}$$

in which θ and $\tilde{\theta}$ are analytic coverings. It is easy to see that the branching order $O_{\tilde{\theta}}(x) = O_{\tilde{\theta}}(\sigma x)$ for any $x \in (D \times_Y X)_{\text{red}}$. Thus $(\text{Tr}_{\tilde{\theta}}\psi)\sigma = \text{Tr}_{\tilde{\theta}}(\psi\tilde{\sigma})$. Hence it remains to show that $\text{Tr}_{\tilde{\theta}}(\psi\tilde{\sigma})$ is subharmonic. The problem is local on D , whence, without loss of generality, we can assume that there exists an embedding $e: (D \times_Y X)_{\text{red}} \rightarrow C^n$ for some n . Then we have the commutative diagram:

$$\begin{array}{ccc} (D \times_Y X)_{\text{red}} & \xrightarrow{\tilde{e}=(\tilde{\theta}, e)} & D \times C^n \\ \tilde{\theta} \searrow & & \downarrow \tilde{\pi} \\ & & D \end{array}$$

in which $\tilde{\pi}|_A: A \rightarrow D$, $A = \tilde{e}(D \times_Y X)_{\text{red}}$, is an analytic covering. Since

$$\text{Tr}_{\tilde{\theta}}(\psi\tilde{\sigma}) \circ \tilde{e}^{-1}|_A = \text{Tr}_{\tilde{\pi}}(\psi \circ \tilde{\sigma} \circ \tilde{e}^{-1}|_A),$$

the subharmonicity of $\text{Tr}_{\tilde{\theta}}(\psi\tilde{\sigma})$ follows from a lemma of [5].

If σ is a C^2 -function on a neighborhood V of a point $y_0 \in Y$ such that partial derivatives of order ≤ 2 have sufficiently small absolute values, then $\psi + \sigma\theta$ is plurisubharmonic. Since $\text{Tr}_{\theta}(\psi) + \sigma = \text{Tr}_{\theta}(\psi + \sigma\theta)$ we infer that $\text{Tr}_{\theta}(\psi) + \sigma$ is plurisubharmonic. Thus $\text{Tr}_{\theta}\psi$ is strictly plurisubharmonic by definition. The lemma is proved.

3.3 LEMMA. *If Y is irreducible and \tilde{Y} is strongly hyperconvex, then so is Y .*

PROOF. Since Y is irreducible, the normalization map $\nu: \tilde{Y} \rightarrow Y$ is homeomorphic. Thus $\psi \circ \nu^{-1}$ is a continuous negative exhaustion function on Y , where ψ is that function on Y . Since for every holomorphic map $\sigma: D \rightarrow Y$ the map $\nu^{-1}\sigma$ is holomorphic, as in the proof of the Lemma 3.2 we infer that $\psi\nu^{-1}$ is strictly plurisubharmonic. Hence Y is strongly hyperconvex.

PROOF OF THEOREM 3.1. (i) Let φ be a strictly plurisubharmonic negative exhaustion C^2 -function on Y . We can assume that Y is embedded in C^n for some n . It is known [6] that there exists a relatively compact Stein open covering $\{U_j\}$ of C^n of finite order and a C^∞ -partition $\{\rho_j\}$ of unity subordinate to $\{U_j\}$ such that $|D^\alpha \rho_j(x)| \leq C_\alpha$ for all α and for all j . Since $\theta^{-1}(U_j)$ is a relatively compact Stein open set, we may find a strictly plurisubharmonic nonnegative ψ_j C^∞ -function on $\theta^{-1}(U_j)$. We set

$$\psi(x) = \sum_j \rho_j(\theta x) \psi_j(x) + \varphi(\theta x).$$

By calculating $\partial^2\psi/\partial z\partial\bar{z}$ (in the local coordinate of X) we conclude that in choosing ψ_j such that the absolute values of their partial derivatives of order ≤ 2 is sufficiently small, $\psi(x)$ is a strictly plurisubharmonic negative exhaustion function of X . Hence X is strongly hyperconvex.

(ii) Considering the commutative diagram:

$$\begin{array}{ccc} (X \times_Y \tilde{Y})_{\text{red}} & \xrightarrow{\tilde{\theta}} & \tilde{Y} \\ \bar{\nu} \downarrow & & \downarrow \nu \\ X & \xrightarrow{\theta} & Y \end{array}$$

of the finite surjective maps, by (i) and by Lemma 3.2 and 3.3 we get strong hyperconvexity of Y . The theorem is proved.

3.4 REMARK. In [3] Diederich and Fornaess have proved that every Stein bounded domain in C^n with C^2 -boundary has a strictly plurisubharmonic negative exhaustion C^2 -function.

REFERENCES

1. A. Andreotti and H. Grauert, *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France **90** (1962), 193–259.
2. H. Cartan, *Quotients of complex spaces*, Contributions to Function Theory, Tata Inst. of Fund. Res., Bombay, 1960, pp. 1–15.
3. K. Diederich and J. E. Fornaess, *Pseudoconvex domain: bounded strictly plurisubharmonic exhaustion functions*, Invent. Math. **39** (1977), 129–141.
4. G. Fischer, *Complex analytic geometry*, Springer-Verlag, Berlin and New York, 1976.
5. J. E. Fornaess and R. Narasimhan, *The Levi problem on complex spaces with singularities*, Math. Ann. **243** (1980), 47–72.
6. B. Malgrange, *Ideal of differentiable functions*, Oxford Univ. Press, 1966.
7. A. Markoe, *Invariant of holomorphic convexity*, Math. Ann. **217** (1975), 267–270.
8. R. Narasimhan, *A note on Stein spaces and their normalizations*, Ann. Scuola Norm. Sup. Pisa **16** (1962), 327–333.
9. _____, *The Levi problem for complex spaces. II*, Math. Ann. **146** (1962), 195–216.

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