ON THE INVARIANCE OF $q$-CONVEXITY AND HYPERCONVEXITY UNDER FINITE HOLOMORPHIC SURJECTIONS

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ABSTRACT. In this note we have proved that 0-convexity and hyperconvexity are invariant under finite holomorphic surjections. Invariance of cohomological $q$-convexity for the case of finite dimension also has been established.

It is known [7] that Steinness is invariant under finite holomorphic surjections. In this note we investigate the invariance property for 0-convexity generally, for cohomological completeness, for cohomological $q$-convexity of finite dimension, and for hyperconvexity.

1. The invariance of 0-convexity. We recall that a complex space $X$ is called $q$-convex if there exists an exhaustion function $\varphi$ on $X$ which is strictly $q$-pseudoconvex outside some compact $K \subset X$. It is known [2] that $X$ is 0-convex if and only if there exists a proper holomorphic map $\theta$ from $X$ onto a Stein space $S$ such that $\theta_* \mathcal{O}_X = \mathcal{O}_S$ and $\theta$ is biholomorphic outside some set of the form $\theta^{-1}A$, where $A$ is a finite subset of $S$. The proper surjection $\theta: X \to S$ with $S$ is Stein, such that $\theta_* \mathcal{O}_X = \mathcal{O}_S$ is said to be the Remmert reduction of $X$. By [2] if $X$ is holomorphically convex then there exists a Remmert reduction. In this section we prove the following theorem.

1.1 THEOREM. Let $\varphi: X \to Y$ be a finite holomorphic surjective map. Then $Y$ is 0-convex if and only if $X$ is.

The proof of Theorem 1.1 is based on the following assertion, essentially as in [9].

1.2 ASSERTION. A complex space $X$ is 0-convex if and only if $\dim H^1(X, \mathcal{S}) < \infty$ for every coherent ideal subsheaf $\mathcal{S} \subset \mathcal{O}_X$.

PROOF. The necessity follows from a theorem of Andreotti-Grauert [1].

Conversely, assume that $\dim H^1(X, \mathcal{S}) < \infty$ for every coherent ideal subsheaf $\mathcal{S}$ of $\mathcal{O}_X$. We have to prove that $X$ is 0-convex.

First we show that $X$ is holomorphically convex. Let $V = \{x_n\}_{n=1}^\infty$ be a discrete sequence in $X$ and let $J_V$ denote the ideal subsheaf of $\mathcal{O}_X$ associated to $V$. Consider the exact sequence:

\[
0 \to J_V \to \mathcal{O}_X \xrightarrow{\eta} \mathcal{O}_V \to 0.
\]

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By hypothesis and by exactness of the cohomology sequence associated to (1) we get
\[(2) \dim \mathcal{O}(V)/\text{Im} \eta = \dim C^{\infty}/\text{Im} \eta < \infty.\]

Let \( l_{\infty}(V) \) denote the subspace of \( \mathcal{O}(V) \) consisting of bounded functions on \( V \). Then \( \dim C^{\infty}/l_{\infty}(V) = \infty \). Thus by (2) it follows that \( \text{Im} \eta \setminus l_{\infty}(V) \neq \emptyset \). This implies that \( \sup|f(x_n)| = \infty \) for some \( f \in \mathcal{O}(X) \). Hence \( X \) is holomorphically convex.

Let \( \theta: X \to S \) be the Remmert reduction of \( X \). To prove that \( X \) is 0-convex it suffices to show that \( \theta: X \setminus K \to S \setminus \theta(K) \) is injective for some compact set \( K \subset X \).

For a contradiction suppose there is a discrete set \( V = \{x_1, y_1, \ldots\} \) such that \( \theta(x_n) = \theta(y_n) \) for every \( n \geq 1 \). For each \( n \) let \( \sigma_n \in \mathcal{O}(V) \) be given by the formula
\[\sigma_n(x_n) = 1, \quad \sigma_n|V - x_n = 0.\]

Since \( \theta_* \mathcal{O}_X = \mathcal{O}_S \) it is easy to see that \( \{\sigma_n \mod \eta\} (n = 1, 2, \ldots) \) is linearly independent in \( \mathcal{O}(V)/\text{Im} \eta \). This contradicts (2). Hence Assertion 1.2 is proved.

1.3 Assertion. Let \( \theta: X \to Y \) be an \( n \)-analytic covering and let \( Y \) be normal.

Then for every coherent ideal subsheaf \( \mathcal{I} \) of \( \mathcal{O}_Y \) there exists a morphism \( Q: \theta_* \mathcal{O}_Y \to \mathcal{I} \) such that \( Q \circ \theta_* = \text{id} \), where \( e: \mathcal{O} \to \theta_* \mathcal{O}_Y \) is the canonical injection.

Proof. Let \( V \subset Y \) be the branch locus of \( \theta \) and let \( U \subset Y \) be a Stein open subset of \( Y \) on which there exists an exact sequence
\[(4) \quad \mathcal{O}_V \to \mathcal{O}_V \to \mathcal{I} \to 0.\]

Then the sequence
\[(5) \quad \theta_* \mathcal{O}_V \to \theta_* \mathcal{O}_V \to \mathcal{I} \to 0 \]

is also exact. Note that \( \theta_* \mathcal{O}_V^{\eta} = \mathcal{O}_\eta \) for every \( m \geq 1 \).

Consider \( \sigma \in H^0(U, \theta_* \mathcal{O}_Y) = H^0(\theta^{-1}(U), \mathcal{I}) \). Since \( \theta^{-1}(U) \) is Stein \([7]\) we can find \( \beta \in H^0(\theta^{-1}(U), \mathcal{I}) \) such that \( \tilde{\eta} \beta = \sigma \). Since \( Y \) is normal, the formula
\[P_U(\beta)(z) = (1/n) \sum_{j=1}^n \beta(x_j) \]

for \( z \in U \setminus V \) and \( \theta^{-1}(z) = \{x_1, x_2, \ldots, x_n\} \), defines an element \( P_U(\beta) \in \mathcal{O}_{\eta}(U) \). Put
\[(6) \quad Q_U(\sigma) = \eta P_U(\beta).\]

It is easy to see that \( Q_U(\sigma) \) is independent of choice of \( \beta \in H^0(\theta^{-1}(U), \mathcal{O}_\eta) \), \( \tilde{\eta} \beta = \sigma \), and \( Q_U(\sigma) = \sigma \) for all \( \sigma \in H^0(U, \mathcal{I}) \).

Assume now that
\[(7) \quad \mathcal{O}_V \to \mathcal{O}_V \to \mathcal{I} \to 0 \]

in another exact sequence on \( U \). Then there exists a commutative diagram:
\[
\begin{array}{cccccc}
\mathcal{O}_V & \to & \mathcal{O}_V & \to & \mathcal{I} & \to & 0 \\
\downarrow & & \downarrow & & \| & & \\
\mathcal{O}_V & \to & \mathcal{O}_V & \to & \mathcal{I} & \to & 0
\end{array}
\]

By the commutativity of (8) it follows that \( Q_U \) is independent of choice of presentation. Hence \( Q = \{Q_U\} \) defines a morphism \( Q: \theta_* \mathcal{O} \to \mathcal{I} \) such that \( Q \circ \theta_* = \text{id} \). Assertion 1.3 is proved.
Now we are able to prove Theorem 1.1. Assume that $Y$ is $0$-convex. Since $\varphi$ is proper it follows that $X$ is holomorphically convex. Considering the commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\varphi} & Y \\
\theta_X & \downarrow & \theta_Y \\
S_X & \xrightarrow{\tilde{\varphi}} & S_Y
\end{array}
$$

(9)

where $\theta_X$ and $\theta_Y$ are Remmert reductions of $X$ and $Y$ respectively, it is easy to see that $\theta_X$ is biholomorphic outside some compact set $K$ in $X$. Hence $X$ is $0$-convex.

Conversely, assume that $X$ is $0$-convex. We prove that $Y$ is also $0$-convex.

(a) First we consider the case, where $\dim Y < \infty$.

We assume that the theorem has been proved for all complex spaces $Y$ of dimension $< m$. Now assume that $\dim Y = m$. Consider the commutative diagram:

$$
\begin{array}{ccc}
(X \times Y \tilde{Y})_{\text{red}} & \xrightarrow{\tilde{\varphi}} & \tilde{Y} \\
\downarrow \eta & & \downarrow \nu \\
X & \xrightarrow{\varphi} & Y
\end{array}
$$

(10)

of finite holomorphic surjective maps, where $\tilde{Y}$ is the normalization of $Y$. By the necessary condition already proved, $(X \times Y \tilde{Y})_{\text{red}}$ is $0$-convex. On the other hand, since $\tilde{\varphi}$ is finite and $\tilde{Y}$ normal it follows that $\tilde{\varphi}$ is a finite analytic $n$-covering for some $n$ [4]. Thus by 1.2 and 1.3 we infer that $\tilde{Y}$ is $0$-convex. To prove that $Y$ is $0$-convex by 1.2 it suffices to show that $\dim H^1(Y, \mathcal{S}) < \infty$ for every coherent ideal subsheaf $\mathcal{S}$ of $\mathcal{O}_Y$. Let $\mathcal{O}_Y$ denote the coherent analytic sheaf of germs of weakly holomorphic functions on $Y$ [4]. Put $\mathcal{D} = \mathcal{O}_Y : \mathcal{O}_Y$. Note that $\nu_*\mathcal{O}_Y = \mathcal{O}_Y$ and $\text{supp} \theta_Y/\mathcal{D} = N(Y)$ where $N(Y)$ denotes the nonnormal locus of $Y$. Let $\mathcal{V}$ be the coherent ideal subsheaf of $\mathcal{O}_Y$ which is the image of $\nu_*\mathcal{O}_Y = \nu^{-1}(\mathcal{D}\mathcal{O}_Y) \otimes (\mathcal{D}\mathcal{O}_Y) \otimes \mathcal{O}_Y$ under multiplication. By using the definition of $\mathcal{D}$ it follows that $\nu_*\mathcal{V} \subset \mathcal{S}$ and since $\tilde{Y}$ is $0$-convex and $\nu$ is finite we have [4]

$$
\dim H^1(Y, \nu_*\mathcal{V}) = \dim H^1(\tilde{Y}, \mathcal{V}) < \infty.
$$

(11)

Since $\nu$ is biholomorphic outside $\nu^{-1}(N(Y))$ it follows that

$$
\text{supp} \mathcal{S}/\nu_*\mathcal{V} \subset N(Y).
$$

(12)

Thus, using the induction hypothesis we get

$$
\dim H^1(Y, \mathcal{S}/\nu_*\mathcal{V}) = \dim H^1(N(Y), \mathcal{S}/\nu_*\mathcal{V}) < \infty.
$$

(13)

By (11) and (13) and by the exactness of the cohomology sequence associated to the exact sequence

$$
0 \rightarrow \nu_*\mathcal{V} \rightarrow \mathcal{S} \rightarrow \mathcal{S}/\nu_*\mathcal{V} \rightarrow 0,
$$

we infer that $\dim H^1(Y, \mathcal{S}) < \infty$.

(b) In the general case, let $Y = \bigcup_{j=1}^{\infty} V_j$, where $V$ is an irreducible branch of $Y$ for any $j > 1$. Since $\tilde{Y} = \bigcup_{j=1}^{\infty} \tilde{V}_j$, by the $0$-convexity of $\tilde{Y}$ it is easy to see that there exists $j_0$ such that $\tilde{V}_j$ is Stein for every $j > j_0$. Hence $V_j$ is also Stein for every $j > j_0$.
Put

\[ Y_0 = \bigcup_{j=1}^{j_0} V_j, \quad Y_k = Y_0 \cup \bigcup_{j=1}^{k} V_{j_0+j}. \]

By (a) \( Y_k \) is \( 0 \)-convex for every \( k \geq 0 \).

If \( Y_0 \) is Stein then \( Y \) is Stein by [7]. Now we assume that \( Y_0 \) is \( 0 \)-convex non-Stein. Thus \( Y_k \) is \( 0 \)-convex non-Stein for every \( k \geq 0 \). Let \( \theta_k: Y_k \to S_k \) be the Remmert reduction of \( Y_k \). Then we have the following diagram:

\[
\begin{array}{cccccc}
Y_0 & \to & Y_1 & \to & Y_2 & \to & \cdots \\
\theta_0 & \downarrow & \theta_1 & \downarrow & \theta_2 & \downarrow & \cdots \\
S_0 & \to & S_1 & \to & S_2 & \to & \cdots \\
\end{array}
\]

Let \( A_k \) be a finite subset of \( S_k \) such that \( \theta_k: Y_k - \theta_k^{-1}(A_k) \to S_k - A_k \) is biholomorphic. Since \( Y_k \) non-Stein, \( \theta_k^{-1}(y) \) is connected of positive dimension for every \( y \in A_k \) [2]. Then, since \( \bigcup_{j>j_0} V_j \) is Stein, and \( \theta_k^{-1}(A_k) \) is compact connected of positive dimension, it follows that

\[ \theta_k^{-1}(A_k) = \theta_0^{-1}(A_0) \text{ for every } k \geq 0. \]  

(14)

From (14) it is easy to see that there exists \( k_0 \) such that

\[ \tilde{\theta}_k \text{ is proper injective and } \theta_0^{-1}(A_0) \subseteq \text{Int}Y_k \text{ for every } k = k_0. \]  

(15)

Put \( S = \lim \to S_k \) and \( \theta = \lim \to \theta_k: Y \to S \). By (14)(15) we infer that \( \theta \) is proper, \( \theta | Y \setminus \theta_0^{-1}(A_0) \) is biholomorphic, and \( \theta_* \mathcal{O}_Y = \mathcal{O}_S \). Moreover, since \( S = \lim \to \tilde{\theta}_k(S_k) \) where \( \tilde{\theta}_k(S_k) \) are Stein closed subspaces of \( S \), it follows that \( S \) is holomorphically separated and holomorphically convex and thereby \( S \) is Stein. Hence \( Y \) is \( 0 \)-convex. This completes the proof of Theorem 1.1.

The following is an immediate consequence of Theorem 1.1.

1.4 Corollary. A complex space \( X \) is \( 0 \)-convex if and only if all its irreducible branches, except for finitely many which are \( 0 \)-convex, are Stein.

1.5 Corollary. Let \( \theta: X \to Y \) be a proper holomorphic surjective map which is finite outside a compact set. Then \( X \) is \( 0 \)-convex if and only if \( Y \) is.

Proof. Assume that \( Y \) is \( 0 \)-convex. Considering the commutative diagram (9) it is easy to see that \( \theta_X \) is finite outside a compact set. Hence by the Steinness of \( S_X \) we infer that \( X \) is \( 0 \)-convex. Now assume that \( X \) is \( 0 \)-convex. Consider the commutative diagram (16)

\[
\begin{array}{ccc}
X & \xrightarrow{\theta} & Y \\
\downarrow \theta_X & \nearrow \eta & \uparrow \theta' \\
S_X & \xleftarrow{\beta} & X' \\
\end{array}
\]

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in which $X'$ is the Stein factorization of $X$ for $\theta$, and $\eta, \beta$ are canonical maps and $\theta'$ is induced by $\theta$. It is easy to check that $\theta'$ is finite, $\beta$ is finite outside a compact set. This implies that $X'$ is 0-convex. By Theorem 1.1 we infer that $Y$ is 0-convex.

1.6 Corollary. Let $\theta: X \to Y$ be a proper holomorphic surjective map and $X$ be 0-convex. Then $Y$ is also 0-convex.

Proof. Considering the commutative diagrams (10) and (16) and by Theorem (1.1), without loss of generality we may assume that $Y$ is normal. Hence $Y$ is holomorphically convex. Consider the commutative diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{\theta} & Y \\
\downarrow{\theta_X} & & \downarrow{\theta_Y} \\
S_X & \xrightarrow{\tilde{\theta}} & S_Y
\end{array}
$$

Since $\theta_X$ is finite outside a compact set, and $\tilde{\theta}$ is finite, it follows that $\theta_Y$ is finite outside a compact set. Hence $Y$ is 0-convex.

1.7 Remark. Corollary (1.6) is not true for the holomorphically convex property [7].

2. The invariance of cohomological $q$-completness. A complex space $X$ is called cohomologically $q$-complete (resp. cohomologically $q$-convex) if and only if $H^p(X, \mathcal{F}) = 0$ (resp. dim $H^p(X, \mathcal{F}) < \infty$) for every coherent ideal subsheaf $\mathcal{F}$ of $\mathcal{O}_X$ and for every $p > q$.

In this section we prove the following theorem.

2.1 Theorem. Let $\varphi: X \to Y$ be a finite holomorphic surjective map. Then $X$ is cohomologically $q$-complete if and only if $Y$ is. If, moreover, dim $X < \infty$ then $X$ is cohomologically $q$-convex if and only if $Y$ is.

Proof. Since $\varphi$ is finite it follows that if $Y$ is cohomologically $q$-complete (resp. cohomologically $q$-convex) then $X$ is too.

As in the proof of Theorem 1.1(a) it follows that if dim $X < \infty$ and $X$ is cohomologically $q$-complete (resp. cohomologically $q$-convex), then so is $Y$.

Thus to find the proof of the theorem it suffices to prove the following

2.2 Assertion. Let $X = \bigcup_{k=1}^{\infty} X_k$, $X_k$ is the union of all irreducible branches of $X$ of dimension $< k$. If $X_k$ is cohomologically $q$-complete for every $k \geq 1$, then $X$ is also cohomologically $q$-complete.

Proof. Let $\mathcal{F}$ be a coherent ideal subsheaf of $\mathcal{O}_X$ and $J_k = \mathcal{F}_{X_k}$ — the ideal subsheaf of $\mathcal{O}_X$ associated to $X_k$. By $\eta_k: \mathcal{O}_X \to \tilde{\mathcal{O}}_{X_k}$ denotes the canonical map. Put $\mathcal{F}_k = \eta_k(\mathcal{F})$. Since any open set in $X$ is contained in some $X_k$ it follows that

$$
\mathcal{F} = \lim_{\leftarrow} \{ \mathcal{F}_k, \omega_k^j \},
$$

where $\omega_k^j: \mathcal{F}_k \to \mathcal{F}_j$ is a canonical map.

Let $\mathcal{U}$ be a Stein open covering of $X$. By hypothesis we have

$$
H^p(\mathcal{U}, \mathcal{F}_k) = H^p(X, \mathcal{F}_k) = H^p(X_k, \mathcal{F}_k) = 0
$$

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for every $p > q$ and so

\[(18) \quad \text{Im}\{H^{p-1}(\mathcal{U}, \mathcal{S}_{k+1}) \to H^{p-1}(\mathcal{U}, \mathcal{S}_k)\} = H^{p-1}(\mathcal{U}, \mathcal{S}_k)\]

for every $p > q$ and $k \geq 1$.

Consider $\sigma \in Z^p(\mathcal{U}, \mathcal{S}_k)$, $p > q$. By (17) for each $k \geq 1$ we find $\beta_k \in C^{p-1}(\mathcal{U}, \mathcal{S}_k)$ such that $\delta^{p-1}\beta_k = \eta\sigma$. Put $\beta_1 = \beta'_1$ and consider $\omega^1 \beta'_2 - \beta_1$. Since $\delta^{p-1}(\omega^1 \beta'_2 - \beta_1) = 0$, by (18) with $k = 1$ we find $\beta''_2 \in Z^{p-1}(\mathcal{U}, \mathcal{S}_2)$ such that $\omega^1 (\beta''_2 - \beta'_2) + \beta_1 = \delta^{p-2} \gamma$ for some $\gamma \in C^{p-2}(\mathcal{U}, \mathcal{S}_2)$.

Since $\mathcal{U}$ is a Stein open covering, there exists $\tilde{\gamma} \in C^{p-2}(\mathcal{U}, \mathcal{S}_2)$ such that $\omega^1 \tilde{\gamma} = \gamma$. Put

$$\beta_2 = -\beta''_2 + \beta'_2 + \delta^{p-2} \tilde{\gamma}.$$ 

Then $\delta^{p-1}\beta_2 = \eta_2 \sigma$ and $\omega^1 \beta_2 = \omega^1 (\beta'_2 - \beta''_2) + \omega^1 \delta^{p-1} \tilde{\gamma} = \omega^1 (\beta'_2 - \beta''_2) + \omega^1 \delta^{p-1} \omega^1 \tilde{\gamma} = \omega^1 (\beta'_2 - \beta''_2) + \beta_1 + \omega^1 (\beta''_2 - \beta'_2) = \beta_1$. Continuing this process we get a sequence $\{\beta_n\}$ such that for every $n \geq 1$:

$$\beta_n \in C^{p-1}(\mathcal{U}, \mathcal{S}_n), \quad \delta^{p-1}(\beta_n) = \eta_n(\sigma) \quad \text{and} \quad \omega^1_{n+1} \beta_{n+1} = \beta_n.$$ 

Thus $\beta = \{\beta_n\} \in C^{p-1}(\mathcal{U}, \mathcal{S})$ and $\delta^{p-1}\beta = \sigma$. Hence $H^{p}(X, \mathcal{S}) = 0$ and 2.2 is proved.

The following is an immediate consequence of Theorem 2.1.

2.3 Corollary. $X$ is cohomologically $q$-complete if and only if every irreducible branch of $X$ is.

3. The invariance of the hyperconvexity. We recall that a Stein space $X$ is called hyperconvex (resp. strongly hyperconvex) if there exists a plurisubharmonic (resp. strictly plurisubharmonic) negative exhaustion function on $X$ [8]. In this section the following theorem is proved.

3.1 Theorem. Let $\theta : X \to Y$ be a finite holomorphic surjective map of finite-dimensional complex spaces. Then:

(i) If $Y$ is strongly hyperconvex having a strictly plurisubharmonic negative exhaustion $C^2$-function, then $X$ is strongly hyperconvex.

(ii) If $Y$ is irreducible and $X$ is strongly hyperconvex having a strictly plurisubharmonic negative exhaustion $C^2$-function, then $Y$ is strongly hyperconvex.

We need the following.

3.2 Lemma. If $X$ is strongly hyperconvex and $Y$ is normal, then so is $Y$.

Proof. Let $\psi$ be a strictly plurisubharmonic negative exhaustion function of $X$. By the integer lemma [4] we infer that $\theta : X \to Y$ is an analytic covering. Thus we can define a function $\varphi$ on $Y$ by the formula

\[(19) \quad \varphi(y) = \text{Tr}_{\theta}(\psi)(y) = \sum_{\theta(x) = y} \psi(x)\]

(the points of $\theta^{-1}(y)$ being counted with the right multiplicity).
Since \( \psi < 0 \) it follows that \( \varphi \) is an exhaustion function. First we prove that \( \varphi \) is plurisubharmonic. By a theorem of Fornaess and Narasimham [5] it suffices to show that \( \varphi \sigma \) is subharmonic for any holomorphic map \( \sigma \) of unit disc \( D \subset C \) into \( Y \).

Given such a map \( \sigma: D \to Y \), consider the commutative diagram:

\[
\begin{array}{ccc}
(D \times Y \times X)_{\text{red}} & \xrightarrow{\hat{\theta}} & X \\
\downarrow \hat{\theta} & & \downarrow \theta \\
D & \to & Y
\end{array}
\]

in which \( \theta \) and \( \hat{\theta} \) are analytic coverings. It is easy to see that the branching order \( O_\theta(x) = O_\hat{\theta}(\sigma x) \) for any \( x \in (D \times X \times X)_{\text{red}} \). Thus \( (\text{Tr}_{\hat{\theta}}(\psi)) \sigma = \text{Tr}_{\theta}(\psi \sigma) \). Hence it remains to show that \( \text{Tr}_{\theta}(\psi \sigma) \) is subharmonic. The problem is local on \( D \), whence, without loss of generality, we can assume that there exists an embedding \( e: (D \times X \times X)_{\text{red}} \to \mathbb{C}^n \) for some \( n \). Then we have the commutative diagram:

\[
\begin{array}{ccc}
(D \times Y \times X)_{\text{red}} & \xrightarrow{\hat{\theta} = (\bar{\theta}, e)} & D \times \mathbb{C}^n \\
\downarrow \hat{\theta} & & \downarrow \bar{\pi} \\
D & \to & D
\end{array}
\]

in which \( \bar{\pi} | A: A \to D, A = \hat{e}(D \times Y \times X)_{\text{red}} \), is an analytic covering. Since

\[
\text{Tr}_{\hat{\theta}}(\psi \sigma) \circ \hat{e}^{-1} | A = \text{Tr}_{\bar{\theta}}(\psi \circ \bar{\sigma} \circ \bar{e}^{-1} | A),
\]

the subharmonicity of \( \text{Tr}_{\hat{\theta}}(\psi \sigma) \) follows from a lemma of [5].

If \( \sigma \) is a \( C^2 \)-function on a neighborhood \( V \) of a point \( y_0 \in Y \) such that partial derivatives of order \( \leq 2 \) have sufficiently small absolute values, then \( \psi + \sigma \theta \) is plurisubharmonic. Since \( \text{Tr}_{\theta}(\psi) + \sigma = \text{Tr}_{\hat{\theta}}(\psi + \sigma \theta) \) we infer that \( \text{Tr}_{\hat{\theta}}(\psi) + \sigma \) is plurisubharmonic. Thus \( \text{Tr}_{\hat{\theta}}(\psi) \) is strictly plurisubharmonic by definition. The lemma is proved.

3.3 Lemma. If \( Y \) is irreducible and \( \tilde{Y} \) is strongly hyperconvex, then so is \( Y \).

Proof. Since \( Y \) is irreducible, the normalization map \( \nu: \tilde{Y} \to Y \) is homeomorphic. Thus \( \psi \circ \nu^{-1} \) is a continuous negative exhaustion function on \( Y \), where \( \psi \) is that function on \( Y \). Since for every holomorphic map \( \sigma: D \to Y \) the map \( \nu^{-1} \sigma \) is holomorphic, as in the proof of the Lemma 3.2 we infer that \( \psi \nu^{-1} \) is strictly plurisubharmonic. Hence \( Y \) is strongly hyperconvex.

Proof of Theorem 3.1. (i) Let \( \varphi \) be a strictly plurisubharmonic negative exhaustion \( C^2 \)-function on \( Y \). We can assume that \( Y \) is embedded in \( \mathbb{C}^n \) for some \( n \). It is known [6] that there exists a relatively compact Stein open covering \( \{U_j\} \) of \( \mathbb{C}^n \) of finite order and a \( C^\infty \)-partition \( \{\rho_j\} \) of unity subordinate to \( \{U_j\} \) such that \( |D^\alpha \rho_j(x)| \leq C_\alpha \) for all \( \alpha \) and all \( j \). Since \( \theta^{-1}(U_j) \) is a relatively compact Stein open set, we may find a strictly plurisubharmonic nonnegative \( \psi_j \) \( C^\infty \)-function on \( \theta^{-1}(U_j) \). We set

\[
\psi(x) = \sum_j \rho_j(\theta x) \psi_j(x) + \varphi(\theta x).
\]
By calculating $\partial^2 \psi / \partial z \partial \bar{z}$ (in the local coordinate of $X$) we conclude that in choosing $\psi_j$ such that the absolute values of their partial derivatives of order $\leq 2$ is sufficiently small, $\psi_j(x)$ is a strictly plurisubharmonic negative exhaustion function of $X$. Hence $X$ is strongly hyperconvex.

(ii) Considering the commutative diagram:

$$(X \times Y \check{Y})_{\text{red}} \xrightarrow{\partial} \check{Y}$$

$$\check{v} \downarrow \quad \downarrow v$$

$$X \quad \check{\theta} \quad Y$$

of the finite surjective maps, by (i) and by Lemma 3.2 and 3.3 we get strong hyperconvexity of $Y$. The theorem is proved.

3.4 Remark. In [3] Diederich and Fornaess have proved that every Stein bounded domain in $\mathbb{C}^n$ with $C^2$-boundary has a strictly plurisubharmonic negative exhaustion $C^2$-function.

References


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