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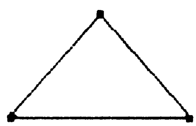
ON CERTAIN 3-GENERATOR ARTIN GROUPS

CRAIG C. SQUIER

ABSTRACT. We describe the three 3-generator Artin groups that correspond to the three sets $\{p, q, r\}$ of positive integer solutions of $p^{-1} + q^{-1} + r^{-1} = 1$. In each case, we show that the Artin group is a free product with amalgamation or HNN extension involving finitely generated free groups and subgroups of finite index.

We study the 3-generator Artin groups with the following Coxeter diagrams

S: 

T: 

U: 

(For presentations, see the relevant sections below.) In the notation of [**3**, p. 199], S has type $\tilde{B}_2$, T has type $\tilde{A}_2$ and U has type $\tilde{G}_2$. (In fact, the Coxeter diagrams for S, T and U are the smallest for which the corresponding Coxeter group is infinite; they correspond to positive integer solutions to $p^{-1} + q^{-1} + r^{-1} = 1$). In particular, the corresponding Coxeter groups are infinite so that the Brieskorn-Saito treatment [**4**] of Artin groups of "finite type" does not apply.

Our main result is to show that S, T and U are semidirect products of a free group of countably infinite rank with an appropriate 2-generator Artin group. (In the case of S, we actually obtain two such descriptions.) Following a suggestion of D. L. Johnson [private communciation], we show that S is an HNN extension of a free group of rank 2 defined by an automorphism of a subgroup of index 2 in the free group. We also show that T and U are both free products with amalgamation of a free group of rank 4 and a free group of rank 3 amalgamated along a free group of rank 7 which has index 2 in the first factor and index 3 in the second factor. (We also indicate how to obtain a more complicated free product with amalgamation description of S.)

An easy consequence of the results presented here is a solution of the word problem for S, T and U. (A solution of the word problem for T is known. For example, Appel's treatment of Artin groups of "large type" applies to T—see [**2**]. Also, [**1**] contains a description of T—similar to ours—which suffices to solve the

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word problem. Finally, it can be shown that T is isomorphic to a subgroup of the ordinary braid group on 4 strands and therefore has a solvable word problem.)

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1. Preliminaries. We shall need several well-known results from combinatorial group theory. Here they are:

(1.1) PROPOSITION. *Suppose that G is a group with a presentation on generators $X \cup Y$ (with X and Y disjoint) and relations $R \cup S$ where each relation in R involves elements of X only and the relations in S may be described as follows: for each $x \in X$ and $y \in Y$, there is a unique relation $xyx^{-1} = s_x(y)$ in S where $s_x(y)$ is a word involving elements of Y only and, furthermore, each relation in S arises in this manner. Suppose further that each function s_x extends to an automorphism of the free group on Y and that these automorphisms satisfy the relations in R. Then the subgroup of G generated by Y is a free group on Y, the subgroup of G generated by X has presentation with generators X and relations R and G is the corresponding semidirect product.*

PROOF. The hypotheses describe a special case of the presentation of a semidirect product.

(1.2) PROPOSITION. *Let G be a free product with amalgamation of two groups G_1 and G_2 amalgamated along a common subgroup H. Let $\tilde{G}$ be a group and let $f\colon \tilde{G} \to G$ be a surjective homomorphism. Then $\tilde{G}$ is a free product with amalgamation of $f^{-1}(G_1)$ and $f^{-1}(G_2)$ amalgamated along their common subgroup $f^{-1}(H)$.*

PROOF. The easiest proof of this uses trees [**6**]: since G is a free product with amalgamation, G acts on a tree (without inversion) with quotient an interval consisting of two vertices $v_1 \neq v_2$ and an edge e connecting them; moreover, there exists an edge $\tilde{e}$ (in the tree) covering e with endpoints $\tilde{v}_1 \neq \tilde{v}_2$ such that G_1, G_2 and H are the stabilizers of $\tilde{v}_1$, $\tilde{v}_2$ and $\tilde{e}$, respectively. The homomorphism f induces an action of $\tilde{G}$ on the tree; clearly, the corresponding quotient graph is an interval covered by $\tilde{e}$ and the stabilizers of $\tilde{v}_1$, $\tilde{v}_2$ and $\tilde{e}$ are $f^{-1}(G_1)$, $f^{-1}(G_2)$ and $f^{-1}(H)$, as required.

(1.3) PROPOSITION. *Let G be an HNN extension of a group H with stable letter a which realizes an isomorphism $\alpha\colon H_1 \to H_2$ between two subgroups of H. (If $g \in H_1$, then $aga^{-1} = \alpha(g)$ in G.) Let $\tilde{G}$ be a group, let $\tilde{f}\colon \tilde{G} \to G$ be a surjective homomorphism and choose $\tilde{a} \in \tilde{G}$ so that $f(\tilde{a}) = a$. Then $\tilde{G}$ is an HNN extension of $f^{-1}(H)$ with stable letter $\tilde{a}$ which realizes an isomorphism from $f^{-1}(H_1)$ to $f^{-1}(H_2)$.*

PROOF. Again, we use trees [**6**]. Here, G acts on a tree (without inversion) with quotient a loop consisting of a single vertex v and edge e and there exists an edge $\tilde{e}$ (in the tree) covering e with endpoints $\tilde{v}_1 \neq \tilde{v}_2$ such that H and H_1 are the stabilizers of $\tilde{v}_1$ and $\tilde{e}$, respectively, and $a(\tilde{v}_2) = \tilde{v}_1$. As above, f induces an action of $\tilde{G}$ on the tree, the quotient is a loop covered by e, the stabilizers of $\tilde{v}_1$ and $\tilde{e}$ are $f^{-1}(H)$ and $f^{-1}(H_1)$ and $\tilde{a}(\tilde{v}_2) = \tilde{v}_1$, as required.

(1.4) PROPOSITION. *Let F be a free group of rank r and let H be a subgroup of index k in F. Then H is a free group of rank $rk - k + 1$.*

PROOF. See, for example, [**5**, p. 16].

Our only application of (1.4) will be: if H has rank 7 and $k = 2$ or 3, then $r = 4$ or 3, respectively. See (3.3) and (4.3) below.

2. The group S. Let S denote the group defined by the following presentation: generators a, b, c and relations $abab = baba$, $bcbc = cbcb$, $ac = ca$. Let B denote the group defined by the following presentation: generators a, b and relation $abab = baba$. Clearly, the assignment $a \mapsto a$, $b \mapsto b$, $c \mapsto 1$ defines a homomorphism from S onto B; let K denote the kernel of this homomophism.

We begin by showing that K is a free group on a countably infinite set of generators. Define $x = c$. Clearly, K is the normal closure of x in S. Define $y = bxb^{-1}$. Clearly, the relation $ac = ca$ is equivalent to $axa^{-1} = x$ and the relation $bcbc = cbcb$ is equivalent to $byb^{-1} = y^{-1}xy$.

To determine aya^{-1}, note that $b^{-1}xb = xyx^{-1}$. Also, define $\Delta = abab = baba$ and note that $a\Delta = \Delta a$ and $b\Delta = \Delta b$. Thus

$$\begin{aligned} aya^{-1} &= abxb^{-1}a^{-1} = abaxa^{-1}b^{-1}a^{-1} \\ &= abab(b^{-1}xb)b^{-1}a^{-1}b^{-1}a^{-1} = \Delta xyx^{-1}\Delta^{-1}. \end{aligned}$$

For each integer n, define $x_n = \Delta^n x \Delta^{-n}$ and $y_n = \Delta^n y \Delta^{-n}$. This leads to the following presentation of S:

(2.1) PROPOSITION. *S has the following presentation: generators a, b together with, for each integer n, x_n and y_n and relations $abab = baba$ together with, for each integer n, the following relations*:

$$\begin{aligned} ax_na^{-1} &= x_n, & bx_nb^{-1} &= y_n, \\ ay_na^{-1} &= x_{n+1}y_{n+1}x_{n+1}^{-1}, & by_nb^{-1} &= y_n^{-1}x_ny_n. \end{aligned}$$

PROOF. Clearly, a, b and $x_0 = x$ generate S and $abab = baba$. As noted above, the other defining relations of S are equivalent to $ax_0a^{-1}x_0$ and (via the definition $y_0 = bx_0b^{-1}$) $by_0b^{-1} = y_0^{-1}x_0y_0$; moreover, the formula for ay_0a^{-1} is a consequence of these relations and the definitions $x_1 = \Delta x_0 \Delta^{-1}$ and $y_1 = \Delta y_0 \Delta^{-1}$. The relations in (2.1) for arbitrary n are an easy consequence of the relations for $n = 0$, using the definitions of x_n and y_n and the fact that a and b commute with Δ. Finally, using the formula $\Delta = abab$, it can easily be shown that $\Delta x_n \Delta^{-1} = x_{n+1}$ and $\Delta y_n \Delta^{-1} = y_{n+1}$ are consequences of the relations in (2.1). In turn, the relations $x_n = \Delta^n x_0 \Delta^{-n}$ and $y_n = \Delta^n y_0 \Delta^{-n}$ are consequences of (2.1). It follows that (2.1) is a presentation of S.

(2.2) COROLLARY. *K is a free group on the x_n's and y_n's. S is a semidirect product of K with B.*

PROOF. By (1.1), it suffices to show that the conjugations described in (2.1) define automorphisms of the free group on formal symbols x_n and y_n which satisfy the defining relations of B. This will be left to the reader.

We give a second description of S as a semidirect product based on the homomorphism from S onto B defined by $a \mapsto a$, $b \mapsto b$, $c \mapsto a$. Letting K denote the kernel, define $x = ca^{-1}$ and $y = bxb^{-1}$. Substituting $c = xa$ into $ac = ca$ easily gives $axa^{-1} = x$. Substituting $c = xa$ into $bcbc = cbcb$ gives $bxabxa = xabxab$ which, using $bx = yb$ and $xa = ax$ can be rewritten as $ybabax = xaybab$. In turn, letting $\Delta = abab = baba$ as above, we get $y\Delta x = xaya^{-1}\Delta$ so that $aya^{-1} = x^{-1}y\Delta x\Delta^{-1}$.

To determine byb^{-1}, note that $y = xaya^{-1}\Delta x^{-1}\Delta^{-1}$. Thus

$$\begin{aligned} byb^{-1} &= (bxb^{-1})(baya^{-1}b^{-1})(b\Delta x^{-1}\Delta^{-1}b^{-1}) = y(babxb^{-1}a^{-1}b^{-1})(\Delta bx^{-1}b^{-1}\Delta^{-1}) \\ &= y(babaxa^{-1}b^{-1}a^{-1}b^{-1})(\Delta y^{-1}\Delta^{-1}) = y(\Delta x\Delta^{-1})(\Delta y^{-1}\Delta^{-1}). \end{aligned}$$

Defining x_n and y_n in terms of x and y as above, we find:

(2.1′) Proposition. *S has the following presentation: generators a, b together with, for each integer n, x_n and y_n and relations $abab = baba$ together with, for each integer n, the following relations*:

$$ax_na^{-1} = x_n, \quad bx_nb^{-1} = y_n,$$
$$ay_na^{-1} = x_n^{-1}y_nx_{n+1}, \quad by_nb^{-1} = y_nx_{n+1}y_{n+1}^{-1}.$$

Proof. As above.

The analogue of (2.2) also holds here.

To describe S as an HNN extension, we use (1.3) and the following description of B as an HNN extension: define $\Pi = ba$. In terms of Π and a, the defining relation of B becomes $a\Pi^2a^{-1} = \Pi^2$, so that B is an HNN extension of an infinite cyclic group (generated by Π) with stable letter a which induces the identity automorphism on the subgroup of index 2 in the infinite cyclic group.

(2.3) Theorem. *S is an HNN extension of a free group of rank 2 defined by an automorphism of a subgroup of index 2 in the free group.*

Proof. Using (2.1) and (2.1′), we obtain two such descriptions. Defining $\Pi = ba$ as above, note that in both (2.1) and (2.1′), $\Pi x_n\Pi^{-1} = y_n$ and $\Pi y_n\Pi^{-1} = x_{n+1}$. In each case, it follows easily that the subgroup of S generated by Π and K is a free group of rank 2 (freely generated by Π and $x = x_0$). Thus (2.3) follows from (1.3).

For completeness, we explicitly describe the HNN structures of S just obtained. In both cases, S is generated by a, Π and x. (Recall that in (2.1), $x = c$ and that in (2.1′), $x = ca^{-1}$.) In both cases, the subgroup of index 2 in the free group on Π and x is freely generated by Π^2, x and $\Pi x\Pi^{-1}$. In (2.1), $a\Pi^2a^{-1} = \Pi^2$, $axa^{-1} = x$, and $a\Pi x\Pi^{-1}a^{-1} = \Pi^2x\Pi x\Pi^{-1}x^{-1}\Pi^{-2}$. In (2.1′), $a\Pi^2a^{-1} = \Pi^2$, $axa^{k-1} = x$, and $a\Pi x\Pi^{-1}a^{-1} = x^{-1}\Pi x\Pi x\Pi^{-2}$.

We conclude this section by indicating how (1.2) may be used to obtain a free product with amalgamation description of T. To do this we need a similar description of B. Note that B modulo Δ is a free product of an infinite cyclic group with a cyclic group of order 2. (There are essentially two ways to achieve this: for one, choose a as the generator of the infinite cyclic group for the other, choose b.) It follows that B is a free product with amalgamation of a free abelian group of rank 2 with an infinite cyclic group amalgamated along a direct factor of the first and a subgroup of index 2 in the second. (This follows from the fact that Δ belongs to the center of B.) Then (1.2) can be applied (using either homomorphism from S onto B above) to decompose S as a free product with amalgamation.

3. The group T. Let T denote the group defined by the following presentation: generators a, b, c and relations $aba = bab$, $bcb = cbc$, $aca = cac$. Let A denote the group defined by the following presentation: generators a, b and relation $aba = bab$. It is easily to verify that the assignment $a \mapsto a$, $b \mapsto b$, $c \mapsto b^{-1}ab$ defines a homomorphism from T onto A; let K denote the kernel of this homomorphism:

We begin by describing K as in §2. Define $x = bcb^{-1}a^{-1}$. Clearly, K is the normal closure of x in T. Note that $c = b^{-1}xab$. Substituting for c in $bcb = cbc$ gives $bxab = xabxa$ and in $aca = cac$ gives $baxb = axbax$.

Define $y = axa^{-1}$ and $z = bxb^{-1}$. Also, define $\Delta = aba = bab$ and note that $a\Delta = \Delta b$ and that $b\Delta = \Delta a$; it follows that Δ^2 commutes with a and b. To determine aza^{-1}, substitute zb for bx wherever possible in $bxab = xabxa$. This gives $zbab = xazba$, which is equivalent to $aza^{-1} = x^{-1}z$ (using $aba = bab$). To determine byb^{-1}, substitute ya for ax on the left and for the first occurrence on the right in $baxb = axbax$. This gives $byab = yabax$, which is equivalent to $byb^{-1} = y\Delta x\Delta^{-1}$ as above. To determine bzb^{-1}, note first that $z = xaza^{-1}$. Thus

$$bzb^{-1} = bxaza^{-1}b^{-1} = (bxb^{-1})(babxb^{-1}a^{-1}b^{-1}) = z\Delta x\Delta^{-1}.$$

Finally, to determine aya^{-1}, note that $y = byb^{-1}\Delta x^{-1}\Delta^{-1}$. Thus

$$\begin{aligned} aya^{-1} &= abyb^{-1}\Delta x^{-1}\Delta^{-1}a^{-1} = (abaxa^{-1}b^{-1}a^{-1})(a\Delta x^{-1}\Delta^{-1}a^{-1}) \\ &= (\Delta x\Delta^{-1})(\Delta bx^{-1}b^{-1}\Delta^{-1}) = \Delta xz^{-1}\Delta^{-1}. \end{aligned}$$

Since a and b commute with Δ^2, but not with Δ, it is convenient to introduce copies u, v, w of x, y, z: for each integer n, define $x_n = \Delta^{2n}x\Delta^{-2n}$, $y_n = \Delta^{2n}y\Delta^{-2n}$, $z_n = \Delta^{2n}z\Delta^{-2n}$, $u_n = \Delta^{2n+1}x\Delta^{-(2n+1)}$, $v_n = \Delta^{2n+1}y\Delta^{-(2n+1)}$, and $w_n = \Delta^{2n+1}z\Delta^{-(2n+1)}$.

(3.1) PROPOSITION. *T has the following presentation: generators a, b together with, for each integer n, x_n, y_n, z_n, u_n, v_n and w_n and relations $aba = bab$ together with, for each integer n, the following relations*:

$$\begin{aligned} &ax_na^{-1} = y_n, && bx_nb^{-1} = z_n, \\ &ay_na^{-1} = u_nw_n^{-1}, && by_nb^{-1} = y_nu_n, \\ &az_na^{-1} = x_n^{-1}z_n, && bz_nb^{-1} = z_nu_n, \\ &au_na^{-1} = w_n, && bu_nb^{-1} = v_n, \\ &av_na^{-1} = v_nx_{n+1}, && bv_nb^{-1} = x_{n+1}z_{n+1}^{-1}, \\ &aw_na^{-1} = w_nx_{n+1}, && bw_nb^{-1} = u_n^{-1}w_n. \end{aligned}$$

PROOF. In outline, the proof is similar to the proof of (2.1). As in (2.1), the relations here involving x_n, y_n and z_n follow from the relations for $n = 0$, which have already been treated, using the fact that Δ^2 commutes with a and b. The relations involving u_n, v_n and w_n follow similarly, using the fact that, in general, $a\Delta P\Delta^{-1}a^{-1} = \Delta bPb^{-1}\Delta^{-1}$ and $b\Delta P\Delta^{-1}b^{-1} = \Delta aPa^{-1}\Delta^{-1}$. As in (2.1), the definitions of x_n, y_n, z_n, u_n, v_n and w_n are consequences of the relations in (3.1).

(3.2) COROLLARY. *K is a free group on the x_n's, y_n's, z_n's, u_n's, v_n's, and w_n's. T is a semidirect product of K with A.*

PROOF. As in the proof of (2.2), this follows easily, after checking a few details, from (1.1). Again, details will be left to the reader.

The description of T as a free product with amalgamation follows easily, using (1.2), from the following well-known description of A: letting $\Pi = ab$ and, as above, $\Delta = aba$, A is a free product with amalgamation of an infinite cyclic group generated by Δ and an infinite cyclic group generated by Π amalgamated along a subgroup of index 2 in the first and index 3 in the second ($\Delta^2 = \Pi^3$).

(3.3) THEOREM. *T is a free product with amalgamation of a free group of rank 4 and a free group of rank 3 amalgamated along a subgroup of index 2 in the first and index 3 in the second.*

PROOF. By (1.2) and (3.2), T is a free product with amalgamation of the subgroup of T generated by Δ and K and the subgroup of T generated by Π and K amalgamated along the subgroup of T generated by Δ^2 and K. That the indexes are correct is automatic.

It remains to be seen that the various groups just described are free groups of the indicated ranks. Using the fact that conjugation by Δ^2 increases subscripts by 1, it follows easily from (3.2) that the subgroup of T generated by Δ^2 and K is a free group of rank 7 freely generated by Δ^2, x_0, y_0, z_0, u_0, v_0 and w_0. (A similar argument shows that the subgroup of T generated by Δ and K is a free group of rank 4 freely generated by Δ, x_0, y_0 and z_0.) Unfortunately, the same analysis applied to the subgroup of T generated by Π and K is somewhat more difficult. Fortunately, a theorem of Stallings [7] applies: a torsion-free group with a free subgroup of finite index is free. Since A is torsion-free, (3.2) shows that T is torsion-free. Thus Stallings' Theorem applies; (3.3) follows by a rank argument (1.4).

We conclude this section by giving an explicit description of the free product with amalgamation decomposition of T. The amalgamating subgroup is a free group of rank 7 freely generated by Q, x, y, z, u, v, w. The free factor of rank 4 is freely generated by Δ, x, y, z where $Q = \Delta^2$, $u = \Delta x\Delta^{-1}$, $v = \Delta y\Delta^{-1}$, and $w = \Delta z\Delta^{-1}$. The free factor of rank 3 is freely generated by Π, x, y where $Q = \Pi^3$, $z = x\Pi x\Pi^{-1}$, $u = \Pi y\Pi^{-1}$, $v = \Pi^2 y\Pi x^{-1}\Pi^{-3}$, and $w = \Pi^2 x\Pi^{-2}$.

4. The group U. Let U denote the group defined by the following presentation: generators a, b, c and relations $aba = bab$, $bcbcbc = cbcbcb$, $ac = ca$. Let A denote the same group as in §3. It is easy to verify that the assignment $a \mapsto a$, $b \mapsto b$, $c \mapsto a^3$ defines a homomorphism from U onto A; let K denote the kernel of this homomorphism.

We begin by describing K as in §2. Define $x = ca^{-3}$. Clearly, K is the normal closure of x in U. Note that $c = xa^3$. Substituting for c in $ac = ca$ gives $axa^3 = xa^4$, which simplifies to $axa^{-1} = x$. Substituting for c in $bcbcbc = cbcbcb$ gives

$$bxa^3bxa^3bxa^3 = xa^3bxa^3bxa^3b.$$

Using the relation $xa = ax$ from above (judiciously) and the definition $\Delta = aba = bab$ from §3, this equality becomes

$$bxa^2\Delta xa\Delta axa = xa^2\Delta xa\Delta axab.$$

Rewriting $a\Delta$ as Δb wherever possible (again from §3), we obtain

$$bx\Delta b^2x\Delta baxa = x\Delta b^2x\Delta baxab.$$

Define $y = bxb^{-1}$. Replacing bx with yb wherever it occurs above, we find

$$yb\Delta byb\Delta baxa = x\Delta byb\Delta baxab.$$

Rewriting $b\Delta$ as Δa and replacing aba with Δ gives

$$y\Delta aby\Delta^2xa = x\Delta by\Delta^2xab.$$

Define $z = byb^{-1}$. Replacing by with zb above and then using $b\Delta^2 = \Delta^2b$ and $bx = yb$ gives

$$y\Delta az\Delta^2yba = x\Delta z\Delta^2ybab.$$

We want to determine aza^{-1}. Note first that

$$aya^{-1} = abxb^{-1}a^{-1} = abaxa^{-1}b^{-1}a^{-1} = \Delta x\Delta^{-1}.$$

Thus

$$\begin{aligned} x\Delta z\Delta^2ybab &= y\Delta az\Delta^2yba = y\Delta(aza^{-1})a\Delta^2yba = y\Delta(aza^{-1})\Delta^2ayba \\ &= y\Delta(aza^{-1})\Delta^2(\Delta x\Delta^{-1})aba. \end{aligned}$$

Cancelling $bab = aba$ on the right and solving for aza^{-1} gives

$$aza^{-1} = \Delta^{-1}y^{-1}x\Delta z\Delta^2y\Delta x^{-1}\Delta^{-3}.$$

We use this equality to determine bzb^{-1}:

$$\begin{aligned} \Delta y\Delta^{-1} &= babyb^{-1}a^{-1}b^{-1} = baza^{-1}b^{-1} \\ &= b\Delta^{-1}y^{-1}x\Delta z\Delta^2y\Delta x^{-1}\Delta^{-3}b^{-1} \\ &= \Delta^{1}ay^{-1}x\Delta z\Delta^2y\Delta x^{-1}a^{-1}\Delta^{-3} \\ &= \Delta^{-1}(\Delta x^{-1}\Delta^{-1})ax\Delta z\Delta^2y\Delta a^{-1}x^{-1}\Delta^{-3} \\ &= x^{-1}\Delta^{-1}xa\Delta z\Delta^2yb^{-1}\Delta x^{-1}\Delta^{-3} \\ &= x^{-1}\Delta^{-1}x\Delta bzb^{-1}\Delta^2z\Delta x^{-1}\Delta^{-3} \end{aligned}$$

and solving for bzb^{-1} gives

$$bzb^{-1} = \Delta^{-1}x^{-1}\Delta x\Delta y\Delta^2x\Delta^{-1}z^{-1}\Delta^{-2}.$$

Define x_n, y_n, z_n, u_n, v_n and w_n as in §3.

(4.1) PROPOSITION. *U has the following presentation: generators a, b together with, for each integer n, x_n, y_n, z_n, u_n, v_n, and w_n and relations $aba = bab$ together with, for each integer n, the following relations*

$$\begin{aligned} ax_na^{-1} &= x_n, & bx_nb^{-1} &= y_n, \\ ay_na^{-1} &= u_n, & by_nb^{-1} &= z_n, \\ az_na^{-1} &= v_{n-1}^{-1}u_{n-1}z_ny_{n+1}u_{n+1}^{-1}, & bz_nb^{-1} &= u_{n-1}^{-1}x_nv_nu_{n+1}z_{n+1}^{-1}, \\ au_na^{-1} &= v_n, & bu_nb^{-1} &= u_n, \\ av_na^{-1} &= w_n, & bv_nb^{-1} &= x_{n+1}, \\ aw_na^{-1} &= x_n^{-1}u_ny_{n+1}x_{n+2}w_{n+1}^{-1}, & bw_nb^{-1} &= y_n^{-1}x_nw_nv_{n+1}x_{n+2}^{-1}. \end{aligned}$$

PROOF. See (3.1).

(4.2) COROLLARY. *K is a free group on the x_n's, y_n's, z_n's, u_n's, v_n's, and w_n's. U is a semidirect product of K with A.*

PROOF. See (3.2).

Finally, using the description of A given in §3, we have

(4.3) THEOREM. *U is a free product with amalgamation of a free group of rank 4 and a free group of rank 3 amalgamated along a subgroup of index 2 in the first and index 3 in the second.*

PROOF. See (3.3).

We conclude with an explicit description of the free product with amalgamation decomposition of U. The amalgamating subgroup and free factor of rank 4 are exactly as described in §3. Here, the free factor of rank 3 is freely generated by Π, x, y where $Q = \Pi^3$, $z = \Pi^{-1}x^{-1}\Pi x\Pi^2y\Pi^3x\Pi^{-1}y^{-1}\Pi^{-3}$, $u = \Pi x\Pi^{-1}$, $v = \Pi^2x\Pi^{-2}$, and $w = \Pi^2y\Pi^{-2}$.

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DEPARTMENT OF MATHEMATICAL SCIENCES, STATE UNIVERSITY OF NEW YORK AT BINGHAMTON, BINGHAMTON, NEW YORK 13901