VMO, ESV, AND TOEPLITZ OPERATORS
ON THE BERGMAN SPACE

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ABSTRACT. This paper studies the largest $C^*$-subalgebra $Q$ of $L^\infty(D)$ such that the Toeplitz operators $T_f$ on the Bergman space $L^2_a(D)$ with symbols $f$ in $Q$ have a symbol calculus modulo the compact operators. $Q$ is characterized by a condition of vanishing mean oscillation near the boundary. I also give several other necessary and sufficient conditions for a bounded function to be in $Q$. After decomposing $Q$ in a “nice” way, I study the Fredholm theory of Toeplitz operators with symbols in $Q$. The essential spectrum of $T_f(f \in Q)$ is shown to be connected and computable in terms of the Stone-Cech compactification of $D$. The results in this article partially answer a question posed in [3] and give several new necessary and sufficient conditions for a bounded analytic function on the open unit disc to be in the little Bloch space $B_0$.

1. Introduction. Let $D$ be the open unit disc in the complex plane $C$. Consider the Bergman space $L^2_a(D)$ of analytic functions in $L^2(D,dA)$, where $dA = \frac{1}{\pi} r \, dr \, d\theta$ is the normalized area measure on $D$. For any function $f$ in $L^\infty(D, dA)$, the Toeplitz operator $T_f: L^2_a(D) \rightarrow L^2_a(D)$ and the Hankel operator $H_f: L^2_a(D) \rightarrow L^2(D)$ are defined by

$$T_f g = P(fg), \quad H_f g = (I - P)(fg), \quad g \in L^2_a(D),$$

where $P: L^2(D) \rightarrow L^2(D)$ is the orthogonal projection. It is well known that Toeplitz operators and Hankel operators are related by

$$T_{|f|^2} - T_f T_f = H_f H_f.$$

In [3], Sheldon Axler raised the question of characterizing the functions $f \in L^\infty(D)$ such that $H_f$ is compact. This is equivalent to characterizing functions $f \in L^\infty(D)$ such that the semi-self-commutator $T_{|f|^2} - T_f T_f$ is compact. Axler answered a special case of this problem in [1]. He proved that for any analytic function $f$ on $D$, $H_f$ is compact if and only if $f \in B_0$, the “little Bloch” space.

Recall that for Toeplitz operators $T_f$ and Hankel operators $H_f$ ($f \in L^\infty(S^1)$) on the Hardy space $H^2$ of the unit circle $S^1$, it is well known [15] that $H_f$ is compact if and only if $f \in C(S^1) + H^\infty$; $H_f$ and $H_f$ are compact if and only if $f \in \text{VMO}(S^1)$ [22, 23]. For Toeplitz operators $T_f$ and Hankel operators $H_f$ ($f \in L^\infty(C^n)$) on the Bergman space $L^2_a(C^n, d\mu)$ of $C^n$ with the Gaussian measure $d\mu$, L. A. Coburn and C. A. Berger in [7] proved that $H_f$ is compact if and only if $H_f$ and $H_f$ are compact if and only if $f = f_1 + f_2$ with $f_1 \in \text{ESV}(C^n)$ and $T_{|f_2|^2}$ compact.
In this paper, we introduce a new space \( VMO_0(D) \) of integrable functions on \( D \) and use it to characterize the functions \( f \in L^\infty(D) \) such that \( H_f \) and \( H_f^* \) are compact. \( VMO_0(D) \), roughly speaking, is the space of integrable functions on \( D \) with vanishing mean oscillation near the boundary of \( D \). The usual area \( VMO \) [10] fails to work in this situation because a Toeplitz operator \( T_f \) on \( L^2_a(D) \), up to a compact perturbation, only depends on the behavior of \( f \) near the boundary of \( D \) [2, 6, 9, 16]. The (mean) oscillation of \( f \) inside \( D \) does not affect \( T_f \) in the Calkin algebra.

In §§3 and 4, we study the space \( VMO_0(D) \) and one of its important subspaces, \( ESV(D) \). Several equivalent conditions for a function to be in \( VMO_0(D) \) or \( ESV(D) \) are proved. We also prove that for \( f \in H^\infty(D) \), \( f \in VMO_0(D) \) if and only if \( f \in ESV(D) \) if and only if \( f \in B_0 \). The so-called Berezin symbol [7, 5] serves as a basic tool to study \( VMO_0(D) \) and Toeplitz operators. Some basic properties of Berezin symbol are first established in §2. §5 is devoted to the proof of the main theorem: For \( f \in L^\infty(D) \), \( H_f \) and \( H_f^* \) are compact if and only if \( f \) is in \( VMO_0(D) \). Notice that this theorem also solves the “symbol calculus” problem of finding the largest \( C^* \)-subalgebra \( Q \) of \( L^\infty(D) \) such that the map \( \xi : Q \to B(L^2_a(D))/K \) defined by \( \xi(f) = T_f + K \) is a \( C^* \)-algebra homomorphism, where \( K \) is the compact ideal of the full algebra \( B(L^2_a(D)) \) of bounded linear operators on \( L^2_a(D) \). \( B(L^2_a(D))/K \) is the Calkin algebra. The theorem simply says that \( Q = L^\infty(D) \cap VMO_0(D) \). In §6 we discuss the Fredholm theory of Toeplitz operators with symbols in \( Q \). The conformal invariance of \( VMO_0 \) is discussed in §7. §8 concludes the paper with some open problems and possible generalizations.

2. The Berezin symbol of Toeplitz operators. Recall that \( L^2_a(D) \) has reproducing kernel
\[
K(z, \bar{w}) = 1/(1 - z\bar{w})^2.
\]
For any \( w \in D \), we can define a unit vector \( k_w \) in \( L^2_a(D) \) by
\[
k_w(z) = \frac{K(z, \bar{w})}{\sqrt{K(w, \bar{w})}} = \frac{1 - |w|^2}{(1 - z\bar{w})^2}, \quad z \in D.
\]
The \( k_w \)'s are called the normalized reproducing kernels.

Now given any bounded linear operator \( S \) on \( L^2_a(D) \), we define a bounded continuous function \( \tilde{S} \) on \( D \) [5] by
\[
\tilde{S}(z) = \langle Sk_z, k_z \rangle, \quad z \in D.
\]
\( \tilde{S} \) is called the Berezin symbol of \( S \). For any function \( f \in L^\infty(D, dA) \), we define \( \tilde{f} = \tilde{T}_f \), so that
\[
\tilde{f}(z) = \langle Tf k_z, k_z \rangle = \langle fk_z, k_z \rangle = \int_D f(w)|k_z(w)|^2 dA(w).
\]
We also call \( \tilde{f} \) the Berezin symbol of \( f \). Notice that
\[
|\tilde{f}(z)| = |\langle Tf k_z, k_z \rangle| \leq \|T_f\| \|k_z\| \leq \|T_f\| \leq \|f\|_\infty,
\]
so \( \|\tilde{f}\|_\infty \leq \|f\|_\infty \). The map \( f \mapsto \tilde{f} : L^\infty(D, dA) \to L^\infty(D, dA) \) is linear, contractive (hence continuous), and order-preserving. It is easy to see that \( \tilde{f} = \tilde{\tilde{f}} \) for any
Let $H^\infty(D)$ denote the Banach algebra of bounded holomorphic functions on $D$. We have

**Proposition 1.** For any $f \in H^\infty(D)$ and $z \in D$, $T_f k_z = \tilde{f}(z)k_z$.

**Proof.** First recall the reproducing property of $K(z, w)$:

$$f(z) = \int_D K(z, w)f(w)\,dA(w)$$

for any $f \in L^2_a(D)$ and $z \in D$. The Toeplitz operator $T_f$ is an integral operator:

$$(T_f g)(z) = \int_D K(z, w)f(w)g(w)\,dA(w)$$

for any $f \in L^\infty(D)$ and $g \in L^2_a(D)$. Now if $f \in H^\infty(D)$, we have

$$(T_f K(\cdot, z))(w) = \int_D K(w, u)K(u, z)f(u)\,dA(u)$$

so

$$T_f K(\cdot, z) = \tilde{f}(z)K(\cdot, z).$$

Dividing both sides by $\sqrt{K(z, z)}$, we get $T_f k_z = \tilde{f}(z)k_z$.

**Proposition 2.** For any $f \in H^\infty(D) + \overline{H^\infty(D)}$, $\tilde{f} = f$.

**Proof.** Since the map $f \mapsto \tilde{f}$ is linear and conjugation-preserving, it suffices to prove the result for $f \in H^\infty(D)$. But in this case, we have $T_f k_z = f(z)k_z$ by Proposition 1. Thus

$$\tilde{f}(z) = \langle T_f k_z, k_z \rangle = \langle f(z)k_z, k_z \rangle = f(z)\langle k_z, k_z \rangle = f(z).$$

**Proposition 3.** For any $f \in L^\infty(D)$, the following are equivalent:

1. $\lim_{|z| \to 1^-} (\tilde{f}(z) - \tilde{f}(z)\tilde{g}(z)) = 0$ for all $g \in L^\infty(D)$;
2. $\lim_{|z| \to 1^-} (|\tilde{f}(z)|^2 - |\tilde{f}(z)|^2) = 0$.

**Proof.** First it is easy to establish the following two identities:

$$|\tilde{f}(z)|^2 - |	ilde{f}(z)|^2 = \frac{1}{2} \int_D \int_D |f(w) - f(u)|^2|k_z(w)|^2|k_z(u)|^2\,dA(w)dA(u);$$

$$\tilde{f}g(z) - \tilde{f}(z)\tilde{g}(z) = \frac{1}{2} \int_D \int_D (f(u) - f(w))(g(u) - g(w))|k_z(u)|^2|k_z(w)|^2\,dA(w)dA(u).$$

Then the Cauchy-Schwarz inequality gives

$$|\tilde{f}(z) - \tilde{f}(z)\tilde{g}(z)| \leq (|\tilde{f}(z)|^2 - |\tilde{f}(z)|^2)(|\tilde{g}(z)|^2 - |\tilde{g}(z)|^2).$$

Now the desired result follows easily from this inequality.
COROLLARY.

\[ \tilde{Q} = \left\{ f \in L^\infty(D) \mid \lim_{|z| \to 1^-} (|f|^2(z) - |\tilde{f}(z)|^2) = 0 \right\} \]

is a \( C^* \)-subalgebra of \( L^\infty(D) \).

PROOF. Let \( f_1, f_2 \in \tilde{Q} \). Then for any \( g \in L^\infty(D) \), we have

\[
\tilde{f}_i g(z) - \tilde{f}_i(z) \tilde{g}(z) \to 0 \quad \text{as} \quad |z| \to 1^-, \quad i = 1, 2.
\]

So

\[
(f_1 + f_2) g(z) - (f_1 + f_2)(z) \tilde{g}(z) \to 0
\]

as \( |z| \to 1^- \), and

\[
(f_1 f_2) g(z) - f_1 f_2(z) \tilde{g}(z) \to 0
\]

as \( |z| \to 1^- \), thus \( f_1 + f_2 \) and \( f_1 f_2 \) are in \( \tilde{Q} \) by the previous proposition. \( \tilde{Q} \) is obviously selfadjoint and closed under scalar multiplication. \( \tilde{Q} \) is norm-closed since \( f \mapsto \tilde{f} \) is continuous. Therefore, \( \tilde{Q} \) is a \( C^* \)-subalgebra of \( L^\infty(D) \).

Before going on, we have some remarks:

(1) \( k_z \to 0 \) weakly in \( L^2_0(D) \) as \( |z| \to 1^- \), so if \( S \) is a compact operator on \( L^2_0(D) \), then

\[
\tilde{S}(z) = \langle Sk_z, k_z \rangle \to 0 \quad \text{as} \quad |z| \to 1^-.
\]

(2) If \( f \) is a polynomial in \( z \), then it is well known that \( T_{|f|^2} - T_f T_{\tilde{f}} \) is compact \cite{9}, so

\[
|f|^2(z) - |\tilde{f}(z)|^2 = |\tilde{f}|^2(z) - |f(z)|^2 = \langle (T_{|f|^2} - T_f T_{\tilde{f}})k_z, k_z \rangle \to 0 \quad \text{as} \quad |z| \to 1^-.
\]

Propositions 1 and 2 are used here. Thus \( \tilde{Q} \) contains all the polynomials. By the Stone-Weierstrass theorem, \( \tilde{Q} \) contains \( C(D) \), the algebra of all continuous complex-valued functions on the closed disc \( \tilde{D} \).

(3) Propositions 1–3 extend to general domains in \( C^n \) without change of proofs. However, for an arbitrary domain \( \Omega \) in \( C^n \), one does not have \( k_z \to 0 \) weakly as \( z \) goes to the boundary of \( \Omega \).

Let \( \tilde{B} = \{ f \in L^\infty(D) \mid |f(z)| \to 0 \quad \text{as} \quad |z| \to 1^- \} \).

PROPOSITION 4. \( \tilde{Q} \cap \tilde{B} \) is a closed selfadjoint ideal of \( \tilde{Q} \), and the following conditions are all equivalent:

(1) \( f \in \tilde{Q} \cap \tilde{B} \);
(2) \( |f| \in \tilde{B} \);
(3) \( |f|^2 \in \tilde{B} \).

PROOF. If \( f \in \tilde{Q} \cap \tilde{B} \) and \( g \in \tilde{Q} \), then \( \tilde{f} g(z) = (\tilde{f} g(z) - \tilde{f} z \tilde{g}(z)) + \tilde{f} z \tilde{g}(z) \to 0 \) as \( |z| \to 1 \), so \( f g \in \tilde{Q} \cap \tilde{B} \). Thus \( \tilde{Q} \cap \tilde{B} \) is an ideal in \( \tilde{Q} \). The selfadjointness and closedness (in the sup-norm topology) of \( \tilde{Q} \cap \tilde{B} \) are trivial.
Next we prove that (1)–(3) are all equivalent.

(2) \Leftrightarrow \ (3) follows from the following inequalities:

\[
|f|^2(z) = \int_D |f(w)|^2 |k_z(w)|^2 dA(w) \\
\leq \|f\|_\infty \int_D |f(w)||k_z(w)|^2 dA(w) = \|f\|_\infty |f|(z); \\
|f|(z) = \int_D |f(w)||k_z(w)|^2 dA(w) \\
\leq \sqrt{\int_D |f(w)|^2 |k_z(w)|^2 dA(w)} = \sqrt{|f|^2(z)}.
\]

(3) \Rightarrow (1). Suppose \(|f|^2 \in \hat{B}\), i.e. \(|f|^2(z) \to 0\) as \(|z| \to 1^-\). Then

\[
0 \leq |f|^2(z) - |\hat{f}(z)|^2 \leq |f|^2(z) \to 0 \quad (|z| \to 1^-),
\]

so \(f \in \hat{Q}\). But \(|\hat{f}(z)| \leq \widehat{|f|^2}(z) \to 0\) \((|z| \to 1^-)\), so we have \(f \in \hat{Q} \cap \hat{B}\).

(1) \Rightarrow (3). If \(f \in \hat{Q} \cap \hat{B}\), then \(\hat{f}(z) \to 0\) and \(|f|^2(z) - |\hat{f}(z)|^2 \to 0\) as \(|z| \to 1^-\),

so \(|f|^2(z) \to 0\) \((|z| \to 1^-)\), i.e. \(|f|^2 \in \hat{B}\).

In [7], Coburn and Berger pointed out that for Toeplitz operators on \(L^2_q(C^n, d\mu)\), where \(d\mu\) is the so-called Gaussian measure on \(C^n\), the Berezin symbol \(\tilde{f}\) is just the solution of the heat equation on \(C^n = \mathbb{R}^{2n}\) at time \(t = \frac{1}{2}\) with initial values \(f\). We expect that the same thing happens on the unit disc, but no such equation has been found yet.

3. VMO\(\_\delta\)(D). For any \(z \in D\), let

\[
S_z = \{w \in D \mid |w| \geq |z|, |\arg w - \arg z| \leq 1 - |z|\}.
\]

Now we can give the definition of VMO\(\_\delta\)(D).

DEFINITION. A function \(f \in L^1(D, dA)\) is in VMO\(\_\delta\)(D) if

\[
\lim_{|z| \to 1^-} \frac{1}{|S_z|} \int_{S_z} |f(w) - \frac{1}{|S_z|} \int_{S_z} f(u) dA(u)| A(w) = 0,
\]

where \(|S_z| = (1 + |z|)(1 - |z|)^2\) is the measure of \(S_z\) and VMO\(\_\delta\) stands for “vanishing mean oscillation near the boundary”.

The main theorem of this section is the equality

\[
\tilde{Q} = L^\infty(D) \cap \text{VMO}\_\delta(D).
\]

LEMMA 1. If \(\delta \in (0, 1)\) is close enough to \(1\), then \(|1 - z| \leq |1 - \delta e^{i(1 - \delta)}|\) for all \(z\) in \(S_\delta\).

PROOF. Given \(z \in S_\delta\), write \(z = re^{i\theta}\). Then

\[
|1 - \delta e^{i(1 - \delta)}|^2 - |1 - z|^2 = \delta^2 - 2\delta \cos(1 - \delta) - r^2 + 2r \cos \theta \\
\geq \delta^2 - 2\delta \cos(1 - \delta) - r^2 + 2r \cos(1 - \delta) \\
= (\delta - r)(\delta + r) - 2(\delta - r) \cos(1 - \delta) \\
= (r - \delta)(2\cos(1 - \delta) - \delta - r) \\
\geq (r - \delta)(2\cos(1 - \delta) - 1 - \delta).
\]
For $\delta$ close enough to 1, we have 
\[
\cos(1 - \delta) \geq 1 - (1 - \delta)^2/2,
\]
thus
\[
2\cos(1 - \delta) - 1 - \delta \geq 2 - (1 - \delta)^2 - 1 - \delta = 2\delta - \delta^2 - \delta = \delta - \delta^2 > 0.
\]
This completes the proof of the lemma.

**Lemma 2.** If $\delta \in (0, 1)$ is very close to 1, then $|1 - \delta e^{i(1 - \delta)}| \leq 2(1 - \delta)$.

**Proof.** The equality
\[
|1 - \delta e^{i(1 - \delta)}|^2 = 1 + \delta^2 - 2\delta \cos(1 - \delta)
\]
and L'Hôpital's rule give us the limit
\[
\lim_{\delta \to 1^-} \frac{|1 - \delta e^{i(1 - \delta)}|^2}{(1 - \delta)^2} = 2.
\]
So for $\delta$ close enough to 1, we must have
\[
|1 - \delta e^{i(1 - \delta)}|^2 \leq 4(1 - \delta)^2.
\]

**Lemma 3.** For any $\epsilon > 0$, there are $\sigma$ and $\delta_0$ in $(0, 1)$ such that
\[
\int_{\mathbb{D} - S_{\delta_0}} \frac{(1 - |z|^2)^2}{|1 - \bar{z}w|^4} \, dA(w) < \epsilon
\]
whenever $z = |z|e^{i\theta} \in \mathbb{D}$, $0 < 1 - |z| < \delta_0$, and $1 - |z| = \sigma(1 - \delta)$.

**Proof.** Let $r = |z|$. A change of variable gives
\[
\int_{\mathbb{D} - S_{\delta_0}} \frac{(1 - |z|^2)^2}{|1 - \bar{z}w|^4} \, dA(w) = \int_{\mathbb{D} - S_\delta} \frac{(1 - r^2)^2}{|1 - rw|^4} \, dA(w) = \frac{\pi}{\pi} \text{Area}(\mathbb{D} - S_\delta),
\]
where $F: \mathbb{D} \to \mathbb{D}$ is the map defined by $F(w) = (r - w)/(1 - rw)$.

Notice that we have used the fact that $(1 - |z|^2)^2/|1 - \bar{z}w|^4$ is the Jacobian of the map $w \mapsto (z - w)/(1 - \bar{z}w)$.

Now suppose that $\sigma$ is any number in $(0, 1)$ and $1 - r = \sigma(1 - \delta)$. We want to estimate $|1 - F(w)|$ for all $w$ in $\mathbb{D} - S_\delta$. If $w \in \mathbb{D} - S_\delta$, then either $|w| < \delta$ or $|w| \geq \delta$ but $|\arg w| > 1 - \delta$.

**Case 1.** $|w| < \delta$. In this case, we have
\[
|1 - F(w)| = \left|\frac{1 - r - w}{1 - rw}\right| = \frac{(1 - r)|1 + w|}{|1 - rw|} \leq \frac{2(1 - r)}{1 - \delta} = \frac{2\sigma(1 - \delta)}{1 - \delta(1 - \sigma(1 - \delta))} = \frac{2\sigma(1 - \delta)}{(1 - \delta)(1 + \sigma\delta)} = \frac{2\sigma}{1 + \sigma\delta} \leq 2\sigma.
\]

**Case 2.** $|w| \geq \delta$, $|\arg w| > 1 - \delta$. In this case, we have
\[
|1 - rw|^2 \geq 1 + r^2|w|^2 - 2r|w|\cos(1 - \delta)
\]
\[
\geq 1 + r^2|w|^2 - 2r|w|(1 - (1 - \delta)^2/2 + (1 - \delta)^4/24)
\]
\[
= (1 - r|w|)^2 + r|w|(1 - \delta)^2 - r|w|(1 - \delta)^4/12
\]
\[
\geq (1 - \delta)^2 + \delta r(1 - \delta)^2 - \frac{1}{12}(1 - \delta)^4
\]
\[
= (1 - \delta)^2(\sigma^2 + \delta r - \frac{1}{12}(1 - \delta)^2),
\]
thus
\[ |1 - F(w)| = \frac{(1 - r)|1 + w|}{|1 - rw|} \leq \frac{2(1 - \delta)\sigma}{|1 - rw|} \leq \frac{2\sigma(1 - \delta)}{(1 - \delta)\sqrt{\sigma^2 + \delta r - \frac{1}{12}(1 - \delta)^2}} = \frac{2\sigma}{\sqrt{\sigma^2 + \delta r - \frac{1}{12}(1 - \delta)^2}}.\]

Since we are only concerned with \( \delta \) close to 1, we may assume \( \delta > \frac{1}{2} \) (\( \Rightarrow r > \delta > \frac{1}{2} \)). Thus
\[ \sigma^2 + \delta r - \frac{1}{12}(1 - \delta)^2 > \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{12}(1 - \frac{1}{2})^2 > \frac{1}{6}, \]
which gives \( |1 - F(w)| \leq 6\sigma \).

Combining Cases 1 and 2, we have proved that \( |1 - F(w)| \leq 6\sigma \) whenever \( w \in \mathbf{D} - S_\delta, 1 - r = \sigma(1 - \delta) \) (\( \delta > \frac{1}{2} \)). Therefore, if \( \sigma \) is small enough, \( F(\mathbf{D} - S_\delta) \) is concentrated around 1, so \( \text{Area } F(\mathbf{D} - S_\delta) \) is small. This completes the proof of Lemma 3. [Note: \( \delta > \frac{1}{2} \Rightarrow 1 - r = \sigma(1 - \delta) < \frac{1}{2}\sigma \), so \( \delta_0 \) can be chosen to be \( \frac{1}{2}\sigma \).]

**Lemma 4.** For \( f \in L^\infty(\mathbf{D}) \), we have
\[
\frac{1}{|S_z|} \int_{S_z} \left| f(w) - \frac{1}{|S_z|} \int_{S_z} f(u) dA(u) \right| dA(w) \leq \sqrt{\frac{1}{|S_z|^2} \int_{S_z} \int_{S_z} |f(w) - f(u)|^2 dA(w) dA(u)};
\]

(1)
\[
\frac{1}{|S_z|^2} \int_{S_z} \int_{S_z} |f(w) - f(u)|^2 dA(w) dA(u) \leq 4\|f\|_\infty \left| f(w) - \frac{1}{|S_z|} \int_{S_z} f(u) dA(u) \right| dA(w).
\]

(2)

**Proof.** (1) follows from the Cauchy-Schwarz inequality, while (2) follows from the following identity:
\[
\frac{1}{|S_z|^2} \int_{S_z} \int_{S_z} |f(w) - f(u)|^2 dA(w) dA(u) = \frac{2}{|S_z|^2} \int_{S_z} \left( f(w) - \frac{1}{|S_z|} \int_{S_z} f(u) dA(u) \right) dA(w) + \frac{2}{|S_z|^2} \int_{S_z} \left( f(w) - \frac{1}{|S_z|} \int_{S_z} f(u) dA(u) \right) dA(u).
\]

**Corollary.** For \( f \in L^\infty(\mathbf{D}) \), \( f \in \text{VMO}\sigma(\mathbf{D}) \) if and only if
\[
\lim_{|z| \to 1^-} \frac{1}{|S_z|^2} \int_{S_z} \int_{S_z} |f(w) - f(u)|^2 dA(w) dA(u) = 0.
\]

**Theorem 1.** \( \tilde{Q} = \text{VMO}\sigma(\mathbf{D}) \cap L^\infty(\mathbf{D}) \).

**Proof.** First we prove the inclusion \( \tilde{Q} \subset \text{VMO}\sigma(\mathbf{D}) \). Given \( z = |z|e^{i\theta} \in \mathbf{D} \) and \( f \in \tilde{Q} \),
\[
|f|^2(z) - |\tilde{f}(z)|^2 = \frac{1}{2}(1 - |z|^2)^4 \int_{\mathbf{D}} \int_{\mathbf{D}} \frac{|f(w) - f(u)|^2}{|1 - \bar{z}w|^4|1 - \bar{z}u|^4} dA(w) dA(u) \geq \frac{1}{2}(1 - |z|^2)^4 \int_{S_z} \int_{S_z} \frac{|f(w) - f(u)|^2}{|1 - \bar{z}w|^4|1 - \bar{z}u|^4} dA(w) dA(u).
\]
For \(w, u \in S_z\), we have \(\tilde{z}w, \tilde{z}u \in S_{|z|^2}\). Thus if \(|z|\) is close enough to 1, 
\[
|1 - \tilde{z}w| \leq |1 - |z|^2| e^{i(1 - |z|^2)} \leq 2(1 - |z|^2)
\]
by Lemmas 1 and 2. Similarly, \(|1 - \tilde{z}u| \leq 2(1 - |z|^2)\). So 
\[
|f|^2 \! (z) - |\tilde{f}(z)|^2 \geq \frac{1}{2^n} \int_{S_z} \left( \int_{S_u} |f(w) - f(u)|^2 dA(w) dA(u) \right)
\]

Notice that 
\[
|S_z|^2 = (1 + |z|)^2 (1 - |z|)^4 \sim (1 - |z|^2)^4.
\]
So we have 
\[
|f|^2 \! (z) - |\tilde{f}(z)|^2 \to 0 \Rightarrow \lim_{|z| \to 1} \frac{1}{|S_z|^2} \int_{S_z} \left( \int_{S_z} |f(w) - f(u)|^2 dA(w) dA(u) \right)
\]
which means \(f \in \tilde{Q} \Rightarrow f \in \text{VMO}_D(D)\) by Lemma 4.

Next we prove the other inclusion:

\(\text{VMO}_D(D) \cap L^\infty(D) \subset \tilde{Q}\).

Given \(z = |z|e^{i\theta} \in D, \delta \in (0, 1), \) and \(f \in \text{VMO}_D(D) \cap L^\infty(D), \)
\[
|f|^2 \! (z) - |\tilde{f}(z)|^2 = \frac{1}{2} \int_D \left( \int_D \frac{1}{1 - \tilde{z}w} |f(w) - f(u)|^2 dA(u) \right)
\]

\[
= \frac{1}{2} \int_D \left( \int_D \frac{1}{1 - \tilde{z}w} |f(w) - f(u)|^2 dA(u) \right)
\]

\[
+ \frac{1}{2} \left( \int_D - S_{\delta e^i\theta} \int_D + \int_D \int_D - S_{\delta e^i\theta} - \int_D - S_{\delta e^i\theta} \int_D - S_{\delta e^i\theta} \right)
\]

\[
\cdot \frac{|f(w) - f(u)|^2}{1 - \tilde{z}w} dA(w) dA(u)
\]

\[
\leq \frac{1}{2} \int_D \left( \int_D \frac{1}{1 - \tilde{z}w} |f(w) - f(u)|^2 dA(u) \right)
\]

\[
+ \frac{1}{2} \left( \int_D - S_{\delta e^i\theta} \int_D + \int_D - S_{\delta e^i\theta} \int_D - S_{\delta e^i\theta} \right)
\]

\[
\cdot \frac{|f(w) - f(u)|^2}{1 - \tilde{z}w} dA(w) dA(u)
\]

\[
= \frac{1}{2} \left( \int_D \frac{1}{1 - \tilde{z}w} |f(w) - f(u)|^2 dA(u) \right)
\]

\[
+ \frac{1}{2} \int_D \left( \int_D \frac{1}{1 - \tilde{z}w} |f(w) - f(u)|^2 dA(u) \right)
\]

\[
= \frac{(1 + |z|)^4 (1 + \delta)^2}{2} \int_D \frac{1}{1 - \tilde{z}w} dA(w)
\]

\[
\cdot \int_{S_{\delta e^i\theta}} \frac{1}{|S_{\delta e^i\theta}|^2} \int_{S_{\delta e^i\theta}} |f(w) - f(u)|^2 dA(w) dA(u)
\]

\[
+ 4||f||^2 \infty \int_D - S_{\delta e^i\theta} \frac{1}{1 - \tilde{z}w} dA(w) dA(u).
\]
Now given any \( \varepsilon > 0 \), by Lemma 3, there exist \( \sigma \in (0,1) \) and \( \delta_0 \in (0,1) \) such that
\[
\int_{D-S_{\delta_0 \varepsilon \theta}} \frac{(1 - |z|^2)^2}{|1 - \bar{z}w|^4} \, dA(w) < \varepsilon
\]
whenever \( 0 < 1 - |z| < \delta_0 \), \( 1 - |z| = (1 - \delta)\sigma \). Thus
\[
\left| \hat{f}(z) - |\hat{f}(z)|^2 \right|
\leq 4 \|f\|_{\infty}^2 \varepsilon + \frac{(1 + |z|)^4(1 + \delta)^2}{2\sigma^4} \int_{S_{\delta_0 \varepsilon \theta}} 1 \int_{S_{\delta_0 \varepsilon \theta}} |f(w) - f(u)|^2 \, dA(w) \, dA(u)
\]
whenever \( 0 < 1 - |z| < \delta_0 \) and \( 1 - |z| = (1 - \delta)\sigma \).

Now using Lemma 4 we get
\[
\lim_{|z| \to 1^-} (\left| \hat{f}(z) - |\hat{f}(z)|^2 \right|) = 0 ;
\]
(Note: \( |z| \to 1 \Rightarrow \delta \to 1 \), \( \sigma \) is fixed.) Since \( \varepsilon \) is arbitrary, we have
\[
\lim_{|z| \to 1^-} (\left| \hat{f}(z) - |\hat{f}(z)|^2 \right|) = 0 ;
\]
and so \( f \in \hat{Q} \). This completes the proof of Theorem 1.

**Theorem 2.** For \( f \in L^\infty(D, dA) \), \( f \in \hat{Q} \cap \hat{B} \) iff
\[
(1) \quad \lim_{|z| \to 1^-} \frac{1}{|S_z|} \int_{S_z} |f(w)| \, dA(w) = 0.
\]

**Proof.** Suppose that (1) holds. Consider
\[
\left| \hat{f}(z) \right| = (1 - |z|^2)^2 \int_D \frac{|f(w)|}{|1 - \bar{z}w|^4} \, dA(w), \quad z = |z|e^{i\theta}.
\]
Given \( \varepsilon > 0 \), by Lemma 3 there are \( \sigma \in (0,1) \) and \( \delta_0 \in (0,1) \) such that
\[
\int_{D-S_{\delta_0 \varepsilon \theta}} \frac{(1 - |z|^2)^2}{|1 - \bar{z}w|^4} \, dA(w) < \varepsilon
\]
whenever \( 0 < 1 - |z| < \delta_0 \) and \( 1 - |z| = (1 - \delta)\sigma \), so
\[
\left| \hat{f}(z) \right| \leq \|f\|_{\infty} \varepsilon + (1 - |z|^2)^2 \int_{S_{\delta_0 \varepsilon \theta}} \frac{|f(w)|}{|1 - \bar{z}w|^4} \, dA(w)
\]
\[
\leq \|f\|_{\infty} \varepsilon + \frac{(1 - |z|^2)^2}{(1 - |z|)^4} \int_{S_{\delta_0 \varepsilon \theta}} |f(w)| \, dA(w)
\]
whenever \( 0 < 1 - |z| < \delta_0 \) and \( 1 - |z| = (1 - \delta)\sigma \). Since
\[
\frac{(1 - |z|^2)^2}{(1 - |z|)^4} = \frac{(1 + |z|)^2}{\sigma^2(1 - \delta)^2} \leq \frac{4}{\sigma^2(1 - \delta)^2}
\]
and \( |S_{\delta_0 \varepsilon \theta}| = (1 + \delta)(1 - \delta)^2 \), thus
\[
(2) \quad \left| \hat{f}(z) \right| \leq \|f\|_{\infty} \varepsilon + \frac{4(1 + \delta)}{\sigma^2|S_{\delta_0 \varepsilon \theta}|} \int_{S_{\delta_0 \varepsilon \theta}} |f(w)| \, dA(w)
\]
whenever $0 < 1 - |z| < \delta_0$ and $1 - |z| = \sigma(1 - \delta)$. Let $|z| \to 1^{-}$ in (2). Then 
\[
\lim_{|z| \to 1^{-}} \overline{f}(z) \leq \|f\|_{\infty} \varepsilon.
\]
Since $\varepsilon$ is arbitrary, we have $|f| \in \widetilde{B}$, thus $f \in \widetilde{Q} \cap \widetilde{B}$.

Conversely, if $f \in \widetilde{Q} \cap \widetilde{B}$, then $|f| \in \widetilde{B}$. But 
\[
\overline{f}(z) = (1 - |z|^2)^2 \int_{D} \frac{|f(w)|}{|1 - \bar{z}w|^4} dA(w)
\geq (1 - |z|^2)^2 \int_{S_z} \frac{|f(w)|}{|1 - \bar{z}w|^4} dA(w),
\]
and $|1 - \bar{z}w| \leq 2(1 - |z|^2)$ for $|z|$ close enough to 1 and $w \in S_z$ by Lemmas 1 and 2. So there is $\delta \in (0, 1)$ such that 
\[
(3) \quad \overline{f}(z) \geq \frac{1}{2^{4}(1 - |z|^2)^2} \int_{S_z} |f(w)| dA(w)
\]
for all $\delta < |z| < 1$. Notice that $(1 - |z|^2)^2 = (1 + |z|)|S_z|$. So (3) says that $|\overline{f}(z) \to 0 (|z| \to 1^{-})$ implies 
\[
\frac{1}{|S_z|} \int_{S_z} |f(w)| dA(w) \to 0 \quad (|z| \to 1^{-}).
\]
This completes the proof of Theorem 2.

4. ESV(D). Let $f$ be in $L^\infty(D, dA)$. We say $f$ is in ESV(D) if for any $\varepsilon > 0$ and $\sigma \in (0, 1)$, there is $\delta_0 > 0$ such that $|f(z) - f(w)| < \varepsilon$ whenever $|z|, |w| \in [1 - \delta, 1 - \sigma \delta], \delta < \delta_0$ and $|\arg z - \arg w| \leq \max(1 - |z|, 1 - |w|)$.

Recall that for $z \in D$,
\[
S_z = \{w \in D \mid |w| \geq |z|, \ |\arg z - \arg w| \leq 1 - |z|\}.
\]
Then it is clear that $f \in \text{ESV}(D)$ if and only if for any $\varepsilon > 0$, $\sigma \in (0, 1)$, there exists $\delta_0 > 0$ such that $|f(z) - f(w)| < \varepsilon$ whenever $w \in S_z$ and $|z|, |w| \in [1 - \delta, 1 - \sigma \delta], \delta < \delta_0$.

It is also easy to see that $f \in \text{ESV}(D)$ if and only if for any $\varepsilon > 0$, $\sigma \in (0, 1)$ and for any positive number $k$, there exists a positive number $\delta_0$ such that $|f(z) - f(w)| < \varepsilon$ whenever $|z|, |w| \in [1 - \delta, 1 - \sigma \delta], \delta < \delta_0$, and $|\arg z - \arg w| \leq k \max(1 - |z|, 1 - |w|)$. In particular, if 
\[
S'_z = \{w \in D \mid |w| \geq |z|, \ |\arg w - \arg z| \leq (1 - |z|)/2\},
\]
then $f \in \text{ESV}(D)$ if and only if for any $\varepsilon > 0$, and $\sigma \in (0, 1)$, there exists $\delta_0 > 0$ such that $|f(z) - f(w)| < \varepsilon$ whenever $w \in S'_z$ and $|z|, |w| \in [1 - \delta, 1 - \sigma \delta], \delta < \delta_0$.

The notation ESV is borrowed from [6] and [7], where it stands for “eventually slowly varying”. In [22] and [23], Sarason also introduced the concept of ESV in a special case, but used a different notation, namely, SO, standing for “slowly oscillating”. ESV is indeed an oscillation condition. It is stronger than the mean-oscillation condition as shown in Theorem 5.

ESV(D) is a relatively large class of functions in $L^\infty(D, dA)$. It is easy to see that $C(\overline{D})$ is strictly contained in it.
Let $f \in L^\infty(D)$ and $z \in D$, and define
\[
\hat{f}(z) = \frac{1}{|S'_z|} \int_{S'_z} f(w) \, dA(w).
\]

**Remark.** By the corollary to Lemma 4, it is easy to see that $f \in \hat{Q}$ if and only if $|\hat{f}|^2(\cdot) - |\hat{f}(\cdot)|^2 \to 0$ as $|z| \to 1^{-}$.

**Theorem 3.** If $f \in \hat{Q} = \text{VMO}_D(D) \cap L^\infty(D)$, then $\hat{f} \in \text{ESV}(D)$.

**Proof.** Given $\varepsilon > 0$, $\sigma \in (0, 1)$, choose $\delta_0 > 0$ such that
\[
\frac{1}{|S'_z|^2} \int_{S'_z} \int_{S_z} |f(w) - f(u)| \, dA(u) \, dA(w) < \frac{\varepsilon \sigma^2}{8}
\]
whenever $0 < 1 - |z| < \delta_0$.

Now if $1 - \delta \leq |z_1| \leq |z_2| \leq 1 - \sigma \delta$, $\delta < \delta_0$, and
\[
|\arg z_1 - \arg z_2| \leq \frac{1}{2} \max(1 - |z_1|, 1 - |z_2|) = \frac{1}{2}(1 - |z_1|),
\]
then $S_{z_2} \subset S_{z_1}$, so we have
\[
|\hat{f}(z_1) - \hat{f}(z_2)| = \left| \frac{1}{|S'_{z_1}| |S'_{z_2}|} \int_{S'_{z_1}} \int_{S'_{z_2}} (f(u) - f(w)) \, dA(u) \, dA(w) \right|
\leq \frac{1}{|S'_{z_1}| |S'_{z_2}|} \int_{S'_{z_1}} \int_{S'_{z_2}} |f(u) - f(w)| \, dA(u) \, dA(w)
\leq \frac{1}{|S'_{z_1}| |S'_{z_2}|} \int_{S_{z_1}} \int_{S_{z_2}} |f(u) - f(w)| \, dA(u) \, dA(w)
= \frac{4}{|S_{z_1}| |S_{z_2}|} \int_{S_{z_1}} \int_{S_{z_2}} |f(u) - f(w)| \, dA(u) \, dA(w).
\]

But
\[
\frac{|S_{z_1}|}{|S_{z_2}|} = \frac{(1 + |z_1|)(1 - |z_1|)^2}{(1 + |z_2|)(1 - |z_2|)^2} \leq 2 \left( \frac{1 - |z_1|}{1 - |z_2|} \right)^2 \leq \frac{2}{\sigma^2}.
\]
(Note: $1 - \delta \leq |z_1| \leq |z_2| \leq 1 - \sigma \delta \Rightarrow (1 - |z_1|)/(1 - |z_2|) \leq 1/\sigma$.) So we have
\[
|\hat{f}(z_1) - \hat{f}(z_2)| \leq \frac{8 \sigma^2}{\sigma^2} \cdot \frac{1}{|S_{z_1}|^2} \int_{S_{z_1}} \int_{S_{z_2}} |f(u) - f(w)| \, dA(u) \, dA(w) \leq \frac{8 \sigma^2 \varepsilon}{8} = \varepsilon.
\]
Thus $\hat{f}$ is in $\text{ESV}(D)$.

**Lemma 5.** If $f \in \text{ESV}(D)$, then
\[
\lim_{|z| \to 1^{-}} \frac{1}{|S_z|} \int_{S_z} |f(z) - f(u)| \, dA(u) = 0.
\]

**Proof.** For $0 < |z| < \delta < 1$, let
\[
A_1 = \{ w \in S_z \mid |w| \leq \delta \}, \\
A_2 = \{ w \in S_z \mid |w| \geq \delta \}.
\]
Then \( S_z = A_1 \cup A_2 \) and
\[
\frac{1}{|S_z|} \int_{S_z} |f(z) - f(u)| \, dA(u)
\leq \frac{1}{|A_1|} \int_{A_1} |f(z) - f(u)| \, dA(u) + \frac{1}{|S_z|} \int_{A_2} |f(z) - f(u)| \, dA(u).
\]
Given any \( \varepsilon \in (0, 1) \), choose \( \delta_0 \in (0, 1) \) such that

\[
(*) \quad |f(r_1 e^{i\theta_1}) - f(r_2 e^{i\theta_2})| < \varepsilon
\]
whenever \( r_1, r_2 \in [1 - \varepsilon, 1 - \varepsilon r] \), \( r \leq \delta_0 \), and \( |\theta_1 - \theta_2| \leq \max(1 - r_1, 1 - r_2) \).

Now let \( |z| > 1 - \delta_0 \) and \( \delta = 1 - \varepsilon(1 - |z|) \). Then \( |z| < \delta < 1 \). For any \( u \in A_1 \), \( |f(z) - f(u)| < \varepsilon \) by \((*)\). So for each \( |z| > 1 - \delta_0 \), we have
\[
\frac{1}{|S_z|} \int_{S_z} |f(z) - f(u)| \, dA(u) \leq \varepsilon + \frac{1}{|S_z|} \int_{A_2} |f(z) - f(u)| \, dA(u)
\leq \varepsilon + 2|A_2|/|S_z||f||\infty.
\]
Since
\[
|S_z| = (1 + |z|)(1 - |z|)^2,
\]
\[
|A_2| = (1 - |z|)(1 - \delta^2) = \varepsilon(1 - |z|)^2(2 - \varepsilon(1 - |z|)),
\]
\[
\frac{|A_2|}{|S_z|} \leq \frac{2\varepsilon(1 - |z|)^2}{(1 + |z|)(1 - |z|)^2} \leq 2\varepsilon,
\]
\( |z| > 1 - \delta_0 \) implies
\[
\frac{1}{|S_z|} \int_{S_z} |f(z) - f(u)| \, dA(u) \leq (1 + 4|f||\infty)\varepsilon.
\]
This completes the proof of Lemma 5.

**THEOREM 4.** If \( f \in \tilde{Q} = \text{VMO}_\theta(D) \cap L^\infty(D) \), then \( f - \hat{f} \in \tilde{Q} \cap \tilde{B} \).

**PROOF.** Let \( g = f - \hat{f} \). Then
\[
\frac{1}{|S_z|} \int_{S_z} |g(w)| \, dA(w) \leq \frac{1}{|S_z|} \int_{S_z} |f(w) - \hat{f}(z)| \, dA(w)
\]
\[
+ \frac{1}{|S_z|} \int_{S_z} |f(z) - \hat{f}(w)| \, dA(w).
\]
The second term goes to 0 as \( |z| \to 1^- \) by Lemma 5 and Theorem 3. Next we estimate the first term.
\[
\frac{1}{|S_z|} \int_{S_z} |f(w) - \hat{f}(z)| \, dA(w)
\]
\[
= \frac{1}{|S_z|} \int_{S_z} \left| \frac{1}{|S'_z|} \int_{S'_z} (f(w) - f(u)) \, dA(u) \right| \, dA(w)
\]
\[
\leq \frac{1}{|S_z||S'_z|} \int_{S_z} \int_{S'_z} |f(w) - f(u)| \, dA(u)
\]
\[
\leq \frac{2}{|S_z|^2} \int_{S_z} \int_{S_z} |f(w) - f(u)| \, dA(w) \, dA(u),
\]
this goes to 0 as \( |z| \to 1 \) since \( f \) is in \( \text{VMO}_\theta(D) \).

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**Lemma 6.** \( ESV(D) \subset \tilde{Q} = VMO_0(D) \cap L^\infty(D). \)

**Proof.** Let \( f \in ESV(D) \). Then \( |f|^2 \in ESV(D) \) since \( ESV(D) \) is a \( C^* \)-subalgebra of \( L^\infty(D) \). By the corollary to Lemma 5, we have \( f(z) - \hat{f}(z) \to 0 \) as \( |z| \to 1^- \), \( |\hat{f}(z)|^2 - |f(z)|^2 \to 0 \) as \( |z| \to 1^- \). Moreover,

\[
|\hat{f}(z)|^2 - |f(z)|^2 = (|\hat{f}(z)| + |f(z)|) |\hat{f}(z)| - |f(z)| \\
\leq 2\|f\|_\infty |\hat{f}(z) - f(z)| \to 0 \quad \text{as} \quad |z| \to 1^-.
\]

So

\[
|\hat{f}(z)|^2 - |f(z)|^2 = (|\hat{f}(z)|^2 - |f(z)|^2) + (|f(z)|^2 - |\hat{f}(z)|^2) \to 0 \quad \text{as} \quad |z| \to 1^-,
\]

and we have \( f \in VMO_0(D) \). Since \( ESV(D) \subset L^\infty(D) \), \( f \in VMO_0(D) \cap L^\infty(D) = \tilde{Q} \).

Summarizing the above results, we have proved the following theorem.

**Theorem 5.** \( \tilde{Q} = ESV(D) + \tilde{Q} \cap \tilde{B} \). A decomposition is given by \( f = \hat{f} + (f - \hat{f}) \).

**Corollary 1.**

\( ESV(D) \cap \tilde{B} = \{ f \in L^\infty(D, dA) \mid f(z) \to 0 \text{ as } |z| \to 1^- \} \).

**Proof.** If \( f(z) \to 0 \) as \( |z| \to 1^- \), then obviously \( f \in ESV(D) \) (just by definition). On the other hand,

\[
\hat{f}(z) = \int_D f \left( \frac{z-w}{1-\bar{z}w} \right) dA(w) \to 0 \quad (|z| \to 1^-)
\]

by the dominated convergence theorem. So \( f \in \tilde{B} \), and hence \( f \in ESV(D) \cap \tilde{B} \).

Conversely, if \( f \in ESV(D) \cap \tilde{B} \), then \( f(z) - \hat{f}(z) \to 0 \) as \( |z| \to 1^- \) and

\[
|\hat{f}(z)| \leq \frac{1}{|S_z^z|} \int_{S_z^z} |f(w)| dA(w) \\
\leq \frac{2}{|S_z^z|} \int_{S_z^z} |f(w)| dA(w) \to 0 \quad \text{as} \quad |z| \to 1^-.
\]

since \( f \in \tilde{Q} \cap \tilde{B} \). Therefore, \( f(z) = (f(z) - \hat{f}(z)) + \hat{f}(z) \to 0 \) as \( |z| \to 1^- \).

**Corollary 2.** For \( f \in \tilde{Q} = VMO_0(D) \cap L^\infty(D) \), \( f \in ESV(D) \) iff \( f(z) - \hat{f}(z) \to 0 \) (\( |z| \to 1^- \)).

**Proof.** The “only if” part follows from the corollary to Lemma 5. If \( f(z) - \hat{f}(z) \to 0 \) as \( |z| \to 1^- \), then \( f - \hat{f} \in ESV(D) \). So \( f = (f - \hat{f}) + \hat{f} \in ESV(D) \).

**Theorem 6.** If \( f \in ESV(D) \), then \( f(z) - \hat{f}(z) \to 0 \) as \( |z| \to 1^- \).

**Proof.**

\[
f(z) - \hat{f}(z) = (1 - |z|^2)^2 \int_D \frac{f(z) - f(w)}{|1 - \bar{z}w|^4} dA(w),
\]

\[
|f(z) - \hat{f}(z)| \leq (1 - |z|^2)^2 \int_D \frac{|f(z) - f(w)|}{|1 - \bar{z}w|^4} dA(w).
\]
Given $\varepsilon > 0$, by Lemma 3, there are $\sigma$ and $\delta_0$ in $(0,1)$ such that
\[
\int_{D - S_{\delta_0 \theta}} \frac{(1 - |z|^2)^2}{|1 - \bar{z}w|^4} dA(w) < \varepsilon \quad (z = |z|e^{i\theta})
\]
whenever $0 < 1 - |z| < \delta_0$ and $1 - |z| = (1 - \delta)\sigma$. But $f \in \text{ESV}(D)$, so there exists $\delta_1 \in (0,1)$ such that
\[
|f(r_1e^{i\theta_1}) - f(r_2e^{i\theta_2})| < \varepsilon \sigma^2
\]
whenever $r_1, r_2 \in [1 - \lambda, 1 - \sigma\lambda]$, $\lambda < \delta_1$, and $|\theta_1 - \theta_2| \leq \max(1 - r_1, 1 - r_2)$.

Let $\delta_2 = \min(\delta_0, \delta_1)$. Then for $0 < 1 - |z| < \delta_2$ and $1 - |z| = \sigma(1 - \delta)$, we have
\[
|f(z) - \tilde{f}(z)| \leq 2\|f\|_\infty \int_{D - S_{\delta_2 \theta}} \frac{(1 - |z|^2)^2}{|1 - \bar{z}w|^4} dA(w)
\]
\[
+ (1 - |z|^2)^2 \int_{S_{\delta_2 \theta}} \frac{|f(z) - f(w)|}{|1 - \bar{z}w|^4} dA(w)
\]
\[
\leq 2\|f\|_\infty \varepsilon + \frac{(1 - |z|^2)^2}{(1 - |z|)^4} \int_{S_{\delta_2 \theta}} |f(z) - f(w)| dA(w)
\]
\[
\leq 2\|f\|_\infty \varepsilon + \frac{4}{(1 - |z|)^2} \int_{S_{\delta_2 \theta}} |f(z) - f(\delta e^{i\theta})| dA(w)
\]
\[
+ \frac{4}{(1 - |z|)^2} \int_{S_{\delta_2 \theta}} |f(z) - f(\delta e^{i\theta})| dA(w).
\]
Since $|f(z) - f(\delta e^{i\theta})| < \varepsilon \sigma^2$ by (4), and
\[
\frac{|S_{\delta_2 \theta}|}{(1 - |z|)^2} = \frac{(1 + \delta)(1 - \delta)^2}{(1 - |z|)^2} = \frac{1 + \delta}{\sigma^2} \leq \frac{2}{\sigma^2},
\]
we must have
\[
|f(z) - \tilde{f}(z)| \leq 2\|f\|_\infty \varepsilon + \frac{\delta}{\sigma^2} \cdot \frac{1}{|S_{\delta_2 \theta}|} \int_{S_{\delta_2 \theta}} |f(w) - f(\delta e^{i\theta})| dA(w) + \delta \varepsilon
\]
\[
(0 < 1 - |z| < \delta_2).
\]
Because
\[
\lim_{|z| \to 1^-} \frac{1}{|S_{\delta_2 \theta}|} \int_{S_{\delta_2 \theta}} |f(w) - f(\delta e^{i\theta})| dA(w) = 0
\]
by Lemma 5, we have
\[
\lim_{|z| \to 1^-} |f(z) - \tilde{f}(z)| \leq 2\|f\|_\infty \varepsilon + \delta \varepsilon.
\]
This completes the proof of $\lim_{|z| \to 1^-} (f(z) - \tilde{f}(z)) = 0$ for any $f \in \text{ESV}(D)$.

**THEOREM 7.** For $f \in \tilde{Q} = \text{VMO}_0(D) \cap L^\infty(D, dA)$, we have

1. $\tilde{f} \in \text{ESV}(D)$;
2. $f - \tilde{f} \in \tilde{Q} \cap \tilde{B}$.

**PROOF.** (1) By Theorem 5, $f = f_1 + f_2$, where $f_1 \in \text{ESV}(D)$ and $f_2 \in \tilde{Q} \cap \tilde{B}$.

Taking the Berezin symbol of $f$, we get $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$. 

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Now \( f_2 \in \tilde{Q} \cap \tilde{B} \) implies \( \tilde{f}_2(z) \to 0 \) as \( |z| \to 1^- \), so \( \tilde{f}_2 \in \operatorname{ESV}(D) \). \( f_1 \in \operatorname{ESV}(D) \) implies \( f_1(z) - \tilde{f}_1(z) \to 0 \) as \( |z| \to 1^- \), so \( f_1 - \tilde{f}_1 \in \operatorname{ESV} \). Thus
\[
\tilde{f}_1 = (\tilde{f}_1 - f_1) + f_1 \in \operatorname{ESV}(D).
\]
Hence
\[
\tilde{f} = \tilde{f}_1 + \tilde{f}_2 \in \operatorname{ESV}(D).
\]

(2). \( f - \tilde{f} = f_1 + f_2 - \tilde{f}_1 - \tilde{f}_2 = (f_1 - \tilde{f}_1) + f_2 - \tilde{f}_2 \). \( f_1(z) - \tilde{f}_1(z) \to 0 \) (as \( |z| \to 1^- \)) implies \( f_1 - \tilde{f}_1 \in \tilde{Q} \cap \tilde{B} \). \( \tilde{f}_2(z) \to 0 \) (as \( |z| \to 1^- \)) implies \( \tilde{f}_2 \in \tilde{Q} \cap \tilde{B} \). So \( f - \tilde{f} \in \tilde{Q} \cap \tilde{B} \).

**COROLLARY 1.** If \( f \in \tilde{Q} \), then \( \tilde{f}(z) - \tilde{f}(z) \to 0 \) as \( |z| \to 1^- \).

**PROOF.** If \( f \in \tilde{Q} \), then \( \tilde{f} \) and \( \tilde{f} \) are \( \operatorname{ESV}(D) \) by Theorems 3 and 7, so \( \tilde{f} - \tilde{f} \) are \( \operatorname{ESV}(D) \). On the other hand,
\[
\tilde{f} - \tilde{f} = (\tilde{f} - f) + (f - \tilde{f}) \in \tilde{Q} \cap \tilde{B}
\]
by Theorems 4 and 7. Therefore,
\[
\tilde{f} - \tilde{f} \in \operatorname{ESV}(D) \cap \tilde{Q} \cap \tilde{B} = \operatorname{ESV}(D) \cap \tilde{B}
\]
Applying Corollary 1 to Theorem 5, we get \( \tilde{f}(z) - \tilde{f}(z) \to 0 \) as \( |z| \to 1^- \).

**COROLLARY 2.** For \( f \in \tilde{Q} \), we have \( f \in \operatorname{ESV}(D) \) iff \( f(z) - \tilde{f}(z) \to 0 \) as \( |z| \to 1^- \).

**PROOF.** If \( f \in \operatorname{ESV}(D) \), then \( f(z) - \tilde{f}(z) \to 0 \) (\( |z| \to 1^- \)) by Theorem 6. If \( f(z) - \tilde{f}(z) \to 0 \), then \( f - \tilde{f} \in \operatorname{ESV}(D) \), but \( \tilde{f} \in \operatorname{ESV}(D) \) by Theorem 7, so \( f = (f - \tilde{f}) + \tilde{f} \in \operatorname{ESV}(D) \).

**REMARK.** For the identity \( \tilde{Q} = \operatorname{ESV}(D) + \tilde{Q} \cap \tilde{B} \), we have found two canonical decompositions:
\[
f = \tilde{f} + (f - \tilde{f}) \quad \text{and} \quad f = \tilde{f} + (f - \tilde{f}).
\]

**THEOREM 8.** For \( f \in \operatorname{LOO}(D) \), we have \( f \in \operatorname{ESV}(D) \) iff \( \| f(z) - f \circ b_z \|_{L^2} \to 0 \) as \( |z| \to 1^- \), where \( b_z(w) = (z - w)/(1 - zw) \), and the norm is just the usual \( L^2 \)-norm.

**PROOF.** For \( f \in \operatorname{LOO}(D) \), it is easy to check the following identity:
\[
\| f(z) - f \circ b_z \|_{L^2}^2 = \| \tilde{f}(z) - |f(z)| \|^2 + |f(z) - f(z)|^2.
\]
If the left-hand side of (5) goes to 0 as \( |z| \to 1^- \), then \( \| \tilde{f}(z) - |f(z)| \|^2 \to 0 \) (\( |z| \to 1^- \)) and \( |f(z) - f(z)| \to 0 \). The first limit says that \( f \) is in \( \tilde{Q} \), the second limit and Corollary 2 to Theorem 7 imply that \( f \) in \( \operatorname{ESV}(D) \).

Conversely, if \( f \in \operatorname{ESV}(D) \subset \tilde{Q} \), the \( \| \tilde{f}(z) - |f(z)| \|^2 \to 0 \) and \( |f(z) - \tilde{f}(z)| \to 0 \) as \( |z| \to 1^- \), so the left-hand side of (5) goes to 0 as \( |z| \to 1^- \).

**LEMMA 7.** There is an absolute constant \( C \) such that
\[
\int_D |f(z) - f(0)|^2 \, dA(z) \leq C \int_D (1 - |z|^2)^2 |f'(z)|^2 \, dA(z)
\]
for all \( f \in \operatorname{Hoo}(D) \).
PROOF. Using Green's formula, we can easily prove (see p. 236 of [13])
\[ \int_{|z|<r} |f'(z)|^2 \log \frac{r}{|z|} \, dA(z) = \frac{1}{4\pi} \int_0^{2\pi} |f(re^{i\theta}) - f(0)|^2 \, d\theta. \]

It is also known (p. 237 of [13]) that
\[ \int_{|z|<r} |f'(z)|^2 \log \frac{r}{|z|} \, dA(z) \leq C \int_{|z|<r} |f'(z)|^2 \left( 1 - \frac{|z|^2}{r} \right) \, dA(z) \]
for all \( f \in H^\infty(D) \), where \( C \) is an absolute constant, i.e. \( C \) does not depend on \( f \).

Now integrating the above inequality with respect to \( rd\theta \), we get
\[ \frac{1}{4\pi} \int_0^1 r \, dr \int_0^{2\pi} |f(re^{i\theta}) - f(0)|^2 \, d\theta \leq C \int_0^1 r \, dr \int_{|z|<r} |f'(z)|^2 \left( 1 - \frac{|z|^2}{r} \right) \, dA(z), \]
or
\[ \frac{1}{4} \int_D |f(z) - f(0)|^2 \, dA(z) \leq \frac{C}{2} \int_D \left[ 1 - |z|^2 + |z|^2 \log |z|^2 \right] |f'(z)|^2 \, dA(z). \]

Power series expansion shows that
\[ 1 - |z|^2 + |z|^2 \log |z|^2 \leq (1 - |z|^2)^2, \]
so we have
\[ \int_D |f(z) - f(0)|^2 \, dA(z) \leq 2C \int_D |f'(z)|^2 (1 - |z|^2)^2 \, dA(z). \]

THEOREM 9. For \( f \in H^\infty(D) \), the following are all equivalent:
1. \( f \in \operatorname{ESV}(D) \);
2. \( f \in \operatorname{VMO}_0(D) \);
3. \( f \in \tilde{Q} \);
4. \( f \in \mathcal{B}_0 \), where \( \mathcal{B}_0 \) is the "little Bloch" space consisting of all the analytic functions \( g \) on \( D \) such that \( |g'(z)|(1 - |z|^2) \to 0 \) as \( |z| \to 1^- \).

PROOF. (2) and (3) are equivalent by Theorem 1. That (1) implies (3) follows from the fact that \( \operatorname{ESV}(D) \subset \tilde{Q} \). If \( f \in \tilde{Q} \), then
\[ \|f(z) - f \circ b_z\|_{L^2}^2 = |\tilde{f}(z)|^2 = |\tilde{f}(z)|^2 + |\tilde{f}(z) - f(z)|^2 \to 0 \]
as \( |z| \to 1^- \) since \( \tilde{f} = f \) for \( f \in H^\infty(D) \), so \( f \in \operatorname{ESV}(D) \) by Theorem 8. Thus we have proved that (3) implies (1).

Next we prove the equivalence of (3) and (4).

If we replace \( f \) by \( f \circ b_{z_0} \) in Lemma 7, then the inequality becomes
\[ |\tilde{f}^2(z_0) - |\tilde{f}(z_0)|^2 \leq C \int_D \left( 1 - \frac{|z_0 - z|^2}{1 - \bar{z}_0 z} \right) \left| f' \left( \frac{z_0 - z}{1 - \bar{z}_0 z} \right) \right|^2 \, dA(z). \]

Now if \( f \in \mathcal{B}_0 \), then
\[ \left( 1 - \frac{|z_0 - z|^2}{1 - \bar{z}_0 z} \right)^2 \left| f' \left( \frac{z_0 - z}{1 - \bar{z}_0 z} \right) \right|^2 \to 0 \]
as \(|z_0| \to 1^-\) for any fixed \(z \in \mathbb{D}\). Thus by the dominated convergence theorem, we have \(\|f(z_0) - f(z_0)^2\| \to 0\) as \(|z_0| \to 1^-\). This shows that (4) implies (3).

To prove (3) implies (4), we use the Bergman formula

\[
f(z) - f(0) = \int_{\mathbb{D}} \frac{f(w) - f(0)}{(1 - zw)^2} dA(w), \quad f \in H^\infty(\mathbb{D}).
\]

Taking derivative on both sides, we get

\[
f'(z) = \int_{\mathbb{D}} \frac{2 \bar{w}(f(w) - f(0))}{(1 - zw)^3} dA(w).
\]

Let \(z = 0\), then

\[
|f'(0)|^2 \leq 4 \int_{\mathbb{D}} |f(w) - f(0)|^2 dA(w), \quad f \in H^\infty(\mathbb{D}).
\]

Replacing \(f\) by \(f \circ b_z\), we get

\[
|f'(z)|^2 (1 - |z|^2)^2 \leq 4(|f|^2(z) - |f(z)|^2), \quad z \in \mathbb{D}.
\]

This completes the proof of Theorem 9.

5. Symbol calculus of Toeplitz operators. In this section, we are going to determine the largest \(C^*\)-subalgebra \(Q\) of \(L^\infty(\mathbb{D}, dA)\) such that the map \(\xi: Q \to \mathcal{B}(L^2_0(\mathbb{D}))/\mathcal{K}\), defined by \(\xi(f) = T_f + \mathcal{K}\), is a \(C^*\)-algebra homomorphism, where \(\mathcal{K}\) is the compact ideal of the full algebra \(\mathcal{B}(L^2_0(\mathbb{D}))\) of bounded linear operators on \(L^2_0(\mathbb{D})\). First we establish the existence of such an algebra.

Let

\[
\Gamma = \{f \in L^\infty(\mathbb{D}) | T_g T_f - T_f T_g \in \mathcal{K} \text{ for all } g \in L^\infty(\mathbb{D})\},
\]

\[
Q = \Gamma \cap \overline{\Gamma},
\]

\[
B = \{f \in L^\infty(\mathbb{D}) | T_f \in \mathcal{K}\}.
\]

**Proposition 5.** For \(f \in L^\infty(\mathbb{D})\), the following are all equivalent:

1. \(f \in \Gamma\);
2. \(H_f\) is compact;
3. \(T_{|f|^2} - T_f T_f\) is compact.

**Proof.** The proof is the same as in [6].

**Proposition 6.** For \(f \in L^\infty(\mathbb{D})\), the following are all equivalent:

1. \(f \in Q\);
2. \(H_f\) and \(H_{\bar{f}}\) are compact;
3. \(T_{|f|^2} - T_{f \bar{f}} T_f\) and \(T_{|f|^2} - T_{f \bar{f}} T_f\) are compact.

**Proof.** The proof follows from Proposition 5.

**Proposition 7.** \(Q\) is a \(C^*\)-subalgebra of \(L^\infty(\mathbb{D})\); \(Q \cap B\) is a closed selfadjoint ideal of \(Q\).

**Proof.** The proof is the same as in [6] and [7].

**Remark.** Propositions 6 and 7 imply that \(Q\) is the largest \(C^*\)-subalgebra of \(L^\infty(\mathbb{D})\) such that the map \(\xi: Q \to \mathcal{B}(L^2_0(\mathbb{D}))/\mathcal{K}\) is a \(C^*\)-algebra homomorphism.
The kernel of this homomorphism is \( Q \cap B \). Thus if we let \( \tau(Q) \) denote the \( C^* \)-subalgebra of \( \mathcal{B}(L^2_a(D)) \) generated by all the operators \( T_f \) with \( f \in Q \), then

\[
(6) \quad Q/\mathcal{Q} \cap B \cong \tau(Q)/\mathcal{K}
\]
as \( C^* \)-algebras. (6) is traditionally called the symbol calculus of Toeplitz operators. So far \( Q \) has only been defined abstractly. Next we want to determine \( Q \). The main theorem is that for a function \( f \in L^\infty(D) \), \( f \in Q \) if and only if \( f \in \text{VMO}_a(D) \); \( f \in Q \cap B \) if and only if \( f \in \overline{Q} \cap B \).

**Proposition 8** (C. A. Berger). The operator \( P: L^\infty(D) \to L^2_a(D) \) is compact.

**Proof.** Given a bounded sequence \( \{f_n\} \) in \( L^\infty(D) \), say, \( \|f_n\|_{\infty} \leq M \ (n = 1, 2, \ldots) \). We want to find a subsequence \( \{f_{n_k}\} \) such that \( \{Pf_{n_k}\} \) converges in \( L^2_a(D) \).

Recall that

\[
Pf_n(z) = \int_D \frac{f_n(w)}{(1 - zw)^2} \, dA(w), \quad z \in D.
\]

Now if \( |z| \leq \delta < 1 \), then

\[
|Pf_n(z)| \leq \int_D \frac{M \, dA(w)}{|1 - zw|^2} \leq \frac{M}{(1 - \delta)^2}, \quad n = 1, 2, \ldots.
\]

So \( \{Pf_n\} \) is uniformly bounded on every compact subset of \( D \). Since \( \{Pf_n\} \) is a sequence of analytic functions on \( D \), by Arzela’s theorem, there is a subsequence \( \{Pf_{n_k}\} \) which converges to \( h \in L^2_a(D) \) uniformly on every compact subset of \( D \). (Note: \( h \in L^2_a(D) \) by Fatou’s lemma.) It remains to prove that

\[
(7) \quad \|Pf_{n_k} - h\|_{L^2} \to 0 \quad (k \to +\infty).
\]

For any \( z \in D \), we have

\[
|Pf_n(z)| \leq M \int_D \frac{dA(w)}{|1 - zw|^2} = -\frac{M}{2|z|^2} \ln(1 - |z|^2).
\]

Since \( \int_D \frac{1}{2}|z|^{-2} \ln(1 - |z|^2))^2 \, dA(z) < +\infty \), (7) follows from the dominated convergence theorem.

**Lemma 8.** If \( \{f_n\} \) is a sequence of real-valued functions in \( L^2(D) \) such that \( \|f_n - h\|_{L^2} \to 0 \ (n \to +\infty) \) for some \( h \in L^2_a(D) \), then \( h \) is a constant.

**Proof.** Write \( h = u + iv \). Then

\[
|f_n(z) - h(z)|^2 = (f_n(z) - u(z))^2 + (v(z))^2,
\]

so

\[
\|f_n - h\|_{L^2}^2 = \|f_n - u\|_{L^2}^2 + \|v\|_{L^2}^2 \geq \|v\|_{L^2}^2.
\]

Let \( n \to +\infty \), we have \( v = 0 \). Thus \( h \) is real-valued. Since \( h \) is analytic, \( h \) must be a constant.
\textbf{Theorem 10.} \( Q \subset \tilde{Q} = \text{VMO}_0(D) \cap L^\infty(D) \).

\textbf{Proof.} Given \( f \in Q \), we want to prove \( |\hat{f}(z)| - |\tilde{f}(z)|^2 \to 0 \ (|z| \to 1^-) \). Since \( Q \) and \( \tilde{Q} \) are selfadjoint, we might as well assume that \( f \) is real-valued.

It is easy to check that
\[ |\hat{f}(z)| - |\tilde{f}(z)|^2 = \|\tilde{f}(z) - f \circ b_z\|_{L^2}^2 \geq 0, \]
where \( b_z(w) = (z - w)/(1 - \bar{z}w) \). We prove the theorem by contradiction.

Suppose \( \lim_{|z| \to 1^-} \|\tilde{f}(z) - f \circ b_z\| > 0 \).

Then there exists \( \rho > 0 \) and \( |z_n| \to 1^- \) such that
\[ \|\tilde{f}(z_n) - f \circ b_{z_n}\|^2 > \rho, \quad n = 1, 2, \ldots. \] (8)

Because \( f \in Q \), \( H_f = (I - P)M_f P \) is compact, so
\[ \|(I - P)f_k\|_{L^2} \to 0 \quad (|z| \to 1^-) \] (9)

since \( k_z \to 0 \) weakly as \( |z| \to 1^- \).

For each \( z \in D \), define a unitary operator \( U_z \) on \( L^2(D) \) as follows:
\[ U_z f(w) = \frac{1 - |z|^2}{(1 - \bar{z}w)^2} f \left( \frac{z - w}{1 - \bar{z}w} \right), \quad w \in D, \ f \in L^2(D). \]

It is easy to check that \( U_z^* = U_z \) and \( L^2(D) \) is a reducing subspace of \( U_z \), so \( U_z^* P = PU_z \).

Now using the equality \( f_k = U_z(f \circ b_z) \) and (9), we get
\[ \|(I - P)U_z(f \circ b_z)\|_{L^2} \to 0 \quad (|z| \to 1^-). \] But \( (I - P)U_z = U_z(I - P) \) and \( U_z \) is unitary, so we must have
\[ \|(I - P)f \circ b_z\|_{L^2} \to 0 \quad (|z| \to 1^-). \] (10)

Notice that \( \|f \circ b_z\| \to \|f\| \) for all \( z \in D \), so by Proposition 8, there is a subsequence \( \{z_{n_k}\} \) of \( \{z_n\} \) and \( h \in L^2(D) \) such that
\[ \|P(f \circ b_{z_{n_k}}) - h\|_{L^2} \to 0 \quad (k \to +\infty). \] (11)

Now (10) + (11) implies that
\[ \|f \circ b_{z_{n_k}} - h\|_{L^2} \to 0 \quad (k \to +\infty). \] (12)

By Lemma 8, \( h \) is a constant. Therefore,
\[ \tilde{f}(z_{n_k}) = (f \circ b_{z_{n_k}}, 1) \to (h, 1) = h \]
as \( k \to +\infty. \) Thus
\[ \|f \circ b_{z_{n_k}} - \tilde{f}(z_{n_k})\|_{L^2} \leq \|f \circ b_{z_{n_k}} - h\|_{L^2} + \|h - \tilde{f}(z_{n_k})\|_{L^2} \]
\[ = \|f \circ b_{z_{n_k}} - h\|_{L^2} + |\tilde{f}(z_{n_k}) - h| \to 0 \quad \text{as } k \to +\infty, \]
a contradiction to (8).

\textbf{Remark.} The proof of Theorem 10 is a modification of the corresponding result in an early version of [7].
PROPOSITION 9. Let

\[ M_p = \sup_{z \in D} \int_D \frac{dA(w)}{|1 - z \bar{w}|^p (1 - |w|^2)^{p/2}}. \]

Then \( M_p < +\infty \) for \( p < \frac{4}{3} \).

PROOF. See [1], or 1.4.10 of [21].

LEMMA 9. Let \( D_\delta = \{ z \in D \mid |z| < \delta \} \), \( \delta \in (0, 1) \). Then \( M_{XD_{\delta}} H_f \) is Hilbert-Schmidt as an operator from \( L^2_\alpha(D) \) to \( L^2(D) \) for all \( f \) in \( L^\infty(D) \).

PROOF. For \( |z| \geq \delta \), \( M_{XD_{\delta}} H_f g(z) = 0 \). For \( |z| < \delta \),

\[ M_{XD_{\delta}} H_f g(z) = H_f g(z) = \int_D \frac{f(z) - f(w)}{(1 - z \bar{w})^2} g(w) dA(w). \]

Thus for all \( z \in D \) and \( g \in L^2_\alpha(D) \),

\[ |M_{XD_{\delta}} H_f g(z)| \leq \int_D \frac{|f(z) - f(w)|}{(1 - \delta)^2} |g(w)| dA(w). \]

The operator \( A \) on \( L^2(D) \) defined by

\[ A g(z) = \int_D |f(z) - f(w)| g(w) dA(w) \]

is Hilbert-Schmidt since the integral kernel is in \( L^2(D \times D) \), so \( M_{XD_{\delta}} H_f \) is Hilbert-Schmidt by (13).

THEOREM 11. \( ESV(D) \subset Q \).

PROOF. Let \( f \in ESV(D) \). Then by Theorem 8, \( \|f(z) - f \circ b_z\|_{L^2} \to 0 \) as \( |z| \to 1^- \). Given \( \varepsilon > 0 \), choose \( \delta \in (0, 1) \) such that \( \|f(z) - f \circ b_z\|_{L^2} < \varepsilon^3 \) for all \( \delta \leq |z| < 1 \). Then for all \( \delta \leq |z| < 1 \), we have

\[
\int_D \frac{|f(z) - f(w)|}{|1 - z \bar{w}|^2 \sqrt{1 - |w|^2}} dA(w) = \frac{1}{\sqrt{1 - |z|^2}} \int_D \frac{|f(z) - f \circ b_z(w)|}{|1 - z \bar{w}| \sqrt{1 - |w|^2}} dA(w)
\]

\[
\leq \frac{M}{\sqrt{1 - |z|^2}} \left( \int_D |f(z) - f \circ b_z(w)|^6 dA(w) \right)^{1/6}
\]

\[
\leq \frac{M(2\|f\|_\infty)^{2/3}}{\sqrt{1 - |z|^2}} \left( \int_D |f(z) - f \circ b_z(w)|^2 dA(w) \right)^{1/6}
\]

\[
= \frac{M(2\|f\|_\infty)^{2/3}}{\sqrt{1 - |z|^2}} \|f(z) - f \circ b_z\|_{L^2}^{1/3} < \frac{M(2\|f\|_\infty)^{2/3}\varepsilon}{\sqrt{1 - |z|^2}},
\]

where \( M = M_{6/5} \) in Proposition 9.

It is easy to check that

\[ H_f g(z) = \int_D \frac{f(z) - f(w)}{(1 - z \bar{w})^2} g(w) dA(w), \quad g \in L^2_\alpha. \]

So

\[ |H_f g(z)| \leq \int_D \frac{|f(z) - f(w)|}{|1 - z \bar{w}|^2} |g(w)| dA(w). \]
The Cauchy-Schwarz inequality shows that
\[ |H_f g(z)|^2 \leq \int_{D} \frac{|f(z) - f(w)|}{|1 - z\bar{w}|^2 \sqrt{1 - |w|^2}} dA(w) \cdot \int_{D} \frac{|f(z) - f(w)|}{|1 - z\bar{w}|^2 \sqrt{1 - |w|^2}|g(w)|^2} dA(w). \]

Thus for all \( \delta \leq |z| < 1 \),
\[ |H_f g(z)|^2 \leq \frac{M(2\|f\|_\infty)^{2/3}}{\sqrt{1 - |z|^2}} \cdot \int_{D} \frac{|f(z) - f(w)|}{|1 - z\bar{w}|^2 \sqrt{1 - |w|^2}|g(w)|^2} dA(w). \]

Write \( \overline{M} = M(2\|f\|_\infty)^{5/3} \). Then
\[ \int_{|z| \geq \delta} |H_f g(z)|^2 dA(z) \leq \overline{M} \varepsilon \int_{D} \frac{dA(z)}{\sqrt{1 - |z|^2}} \cdot \int_{D} \frac{|g(w)|^2 dA(w)}{\sqrt{1 - |z|^2}} \leq \overline{M} M_1 \varepsilon \int_{D} |g(w)|^2 dA(w) = \overline{M} M_1 \varepsilon \|g\|^2. \]

This implies that \( \|M_{XD} H_f\| \leq \overline{M} M_1 \varepsilon \), namely, \( \|H_f - M_{XD} H_f\| \leq \overline{M} M_1 \varepsilon \). Since \( M_{XD} H_f \) is compact and \( \varepsilon \) is arbitrary, \( H_f \) is compact, and so \( f \in \Gamma \). Because \( f \) is arbitrary and ESV(D) is selfadjoint, we have proved ESV(D) \( \subset \Gamma \cap \overline{\Gamma} = Q \).

**Theorem 12.** \( \tilde{Q} \cap \tilde{B} \subset Q \cap B \).

**Proof.** Given \( f \in \tilde{Q} \cap \tilde{B} \), we have \( |f| \in \tilde{B} \). To prove \( f \in Q \cap B \), it suffices to prove \( |f| \in B \), that is, \( T_{|f|} \) is compact.

Recall that
\[ T_{|f|} g(z) = \int_{D} \frac{|f(w)|}{(1 - z\bar{w})^2} g(w)dA(w) \]
for \( g \in L_2^2(D) \) and \( z \in D \). So the Cauchy-Schwarz inequality gives
\[ |T_{|f|} g(z)|^2 \leq \int_{D} \frac{|f(w)|^2 dA(w)}{|1 - z\bar{w}|^2 \sqrt{1 - |w|^2}} \cdot \int_{D} \frac{\sqrt{1 - |w|^2}|g(w)|^2}{|1 - z\bar{w}|^2} dA(w). \]

But
\[ \int_{D} \frac{|f(w)|^2 dA(w)}{|1 - z\bar{w}|^2 \sqrt{1 - |w|^2}} = \int_{D} \frac{|f \circ b_z(w)|^2}{\sqrt{1 - |z|^2}} \cdot \int_{D} \frac{dA(w)}{|1 - z\bar{w}|^2 \sqrt{1 - |w|^2}} \leq \frac{M_{6/5}}{\sqrt{1 - |z|^2}} \left( \int_{D} |f \circ b_z(w)|^2 dA(w) \right)^{1/6} \]
\[ \leq \frac{M_{6/5}}{\sqrt{1 - |z|^2}} \left( \int_{D} |f \circ b_z(w)| dA(w) \right)^{1/6}, \]

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and we have
\[ |T_{|f|}g(z)|^2 \leq \frac{M}{1-|z|^2} (\int |f(z)|^2 )^{1/6} \int_D \frac{\sqrt{1-|w|^2} |g(w)|^2}{|1-z\bar{w}|^2} dA(w), \]
where \( M = M_6/5 \| f \|_{11/6}^{11/6} \).

Given \( \varepsilon > 0 \), choose \( \delta \in (0, 1) \) such that \( |f(z)| < \varepsilon^6 \) whenever \( \delta < |z| < 1 \). Then
\[
\int_{|z| \geq \delta} |T_{|f|}g(z)|^2 dA(z) \leq M\varepsilon \int_D |g(w)|^2 \sqrt{1-|w|^2} dA(w) \cdot \int_D \frac{dA(z)}{|1-z\bar{w}|^2 \sqrt{1-|z|^2}}
\[
= M\varepsilon \int_D |g(w)|^2 dA(w) \int_D \frac{dA(z)}{|1-z\bar{w}|\sqrt{1-|z|^2}}
\[
\leq M' M_1 \varepsilon \int_D |g(w)|^2 dA(w) = MM_1 \varepsilon \| g \|_{L^2}^2.
\]

So \( \| M_{XD-D\delta} T_{|f|} \| ^2 \leq MM_1 \varepsilon \), that is,
\[
\| T_{|f|} - M_{XD\delta} T_{|f|} \| ^2 \leq MM_1 \varepsilon.
\]

Since \( M_{XD\delta} T_{|f|} \) is compact as an operator from \( L^2(D) \) to \( L^2(D) \) and \( \varepsilon \) is arbitrary, \( T_{|f|} \) must be compact.

**REMARK.** Since \( k_z \to 0 \) weakly as \( |z| \to 1^- \), we have \( Q \cap B \subset \tilde{Q} \cap \tilde{B} \) trivially.

Theorems 11 and 12 and the decomposition \( \tilde{Q} = ESV(D) + Q \) show that \( \tilde{Q} \subset Q \). In summary, we have proved the following main theorem.

**THEOREM 13.**

1. \( Q = \tilde{Q} = \text{VMO}_0(D) \cap L^\infty(D) \).
2. \( Q \cap B = \tilde{Q} \cap \tilde{B} \).

**COROLLARY 1 (S. AXLER [1]).** Let \( f \in H^\infty(D) \). Then \( H_f \) is compact if and only if \( f \) is in the “little Bloch” space \( B_0 \).

**PROOF.** It follows from Theorems 9 and 13 and the fact that \( H_f = 0 \).

**COROLLARY 2.** \( Q \) and \( Q \cap B \) are invariant under Möbius transformations.

**PROOF.** This follows from the facts that \( Q = \tilde{Q} \) and \( Q \cap B = \tilde{Q} \cap \tilde{B} \) and \( \tilde{f}(b_\lambda(z)) = f \circ b_\lambda(z) \) (simply a change of variable formula), where the \( b_\lambda \)'s are Möbius transformations.

**6. Fredholm theory of Toeplitz operators with symbols in Q.** The isomorphism \( Q/Q \cap B \cong \tau(Q)/K \) and the decomposition \( Q = ESV + Q \cap B \) will serve as basic tools for our study of Fredholm theory of Toeplitz operators with symbols in \( Q \). Let \( BC(D) \) be the \( C^* \)-algebra of all bounded continuous functions on \( D \), and \( C_0(D) \) be the space of continuous functions \( f \) on \( D \) with the property that \( f(z) \to 0 \) as \( |z| \to 1^- \). Consider the algebra \( BCESV \) defined as \( BC(D) \cap ESV \). Since \( \tilde{f} \in BC(D) \) for any \( f \in L^\infty(D) \), the equality \( f = \tilde{f} + (f - \tilde{f}) \) gives a decomposition
\[ Q = BCESV + Q \cap B. \]
Notice that $\text{BCESV} \cap (Q \cap B) = C_0(D)$, so we have

$$\frac{Q}{Q \cap B} = (\text{BCESV} + Q \cap B)/Q \cap B \cong \text{BCESV}/C_0(D).$$

Also we should mention that

$$\frac{Q}{Q \cap B} \cong \text{ESV}/V_0(D),$$

where $V_0(D)$ consists of all functions $f$ in $L^\infty(D)$ with $f(z) \to 0$ as $|z| \to 1^-$.

Let $\beta D$ be the Stone-Čech compactification of $D$. Any bounded continuous functions $f$ on $D$ has a unique continuous extension to $\beta D$: we also denote this extension of $f$ to $\beta D$ by $f$, so there should be no confusion about this.

**Theorem 14.** If $f \in Q$, then $\sigma_e(T_f) = \tilde{f}(\beta D - D)$, where $\sigma_e(T_f)$ is the essential spectrum of $T_f$.

**Proof.** Since $f \in Q$, we know $T_{f - \tilde{f}}$ is compact. Thus $\sigma_e(T_f) = \sigma_e(T_{f - \tilde{f}})$. $\tilde{f}$ is in BCESV. Mimicking [7], we can prove that for any $g \in \text{BCESV}$, $g + C_0(D)$ is invertible in $\text{BCESV}/C_0(D)$ if and only if there are $\delta, \varepsilon$ in $(0,1)$ such that $|g(z)| \geq \varepsilon$ for all $\delta \leq |z| < 1$. By the symbol calculus

$$\text{BCESV}/C_0(D) \cong \tau(\text{BCESV})/K,$$ 

$T_g + K$ is invertible in $\tau(\text{BCESV})/K$ if and only if there are $\delta, \varepsilon$ in $(0,1)$ such that $|g(z)| \geq \varepsilon$ for all $\delta \leq |z| < 1$. Therefore

$$\sigma_e(T_f) = \bigcap_{\delta \in (0,1)} \overline{\tilde{f}(D - D_\delta)},$$

where $D_\delta = \{z \in D \mid |z| < \delta\}$. The compactness of $\beta D$ and the continuity of $\tilde{f}$ yield $\tilde{f}(\overline{D - D_\delta}) = \tilde{f}(\overline{D - D_\delta}) = \tilde{f}(\beta D - D_\delta)$. So we get

$$\sigma_e(T_f) = \bigcap_{\delta \in (0,1)} \tilde{f}(\beta D - D_\delta).$$

On the other hand, if $\lambda \in \bigcap_{\delta \in (0,1)} \tilde{f}(\beta D - D_\delta)$, then $\lambda = \tilde{f}(z_\delta)$, $z_\delta \in \beta D - D_\delta$, $\delta \in (0,1)$. Consider the sequence $\{z_{1-1/n}\}$. The compactness of $\beta D$ implies that there exists a subsequence $\{z_{1-1/n_k}\}$ and $z \in \beta D$ such that $z_{1-1/n_k} \to z$ as $k \to +\infty$. It is clear that $z \in \beta D - D$ since $D$ is open in $\beta D$. The continuity of $\tilde{f}$ and the equality $\lambda = f(z_\delta)$ give $\lambda = \tilde{f}(z) \in \tilde{f}(\beta D - D)$. Hence $\tilde{f}(\beta D - D) = \bigcap_{\delta \in (0,1)} \tilde{f}(\beta D - D_\delta)$, and the proof is complete.

**Corollary 1.** For $f \in Q$, $T_f$ is Fredholm if and only if $\tilde{f}$ is nonvanishing on $\{z \mid z \in D, \ |z| \geq \delta\}$ for some $\delta \in (0,1)$.

**Corollary 2.** If $f \in \text{BCESV}$, then $\sigma_e(T_f) = f(\beta D - D)$, hence $T_f$ is Fredholm if and only if $f$ is nonvanishing on $\{z \mid z \in D, \ |z| \geq \delta\}$ for some $\delta \in (0,1)$.

**Proof.** For $f \in \text{BCESV}$, $f - \tilde{f}$ is in $C_0(D)$, so $f(\beta D - D) = \tilde{f}(\beta D - D)$. 

**Corollary 3.** If $f \in Q$, then $\sigma_e(T_f)$ is connected.

**Proof.** $\sigma_e(T_f) = \bigcap_{\delta \in (0,1)} \overline{\tilde{f}(D - D_\delta)}$ is the intersection of a nested family of compact connected sets, so it is connected. See [7].
REMARK. As $C^*$-algebras, $C(\beta D)$ is isomorphic to $BC(D)$. Under the isomorphism, $C_0(D)$ is the closed ideal of $C(\beta D)$ consisting of functions $f$ on $\beta D$ such that $f$ is identically zero on $\beta D - D$.

If $f$ is in $BCESV$ and $T_f$ is Fredholm, then we know that there are $\delta, \varepsilon \in (0, 1)$ such that $|f(z)| \geq \varepsilon$ for all $\delta \leq |z| < 1$. For any $r \in (\delta, 1)$, we have a continuous map $f_r : \partial D \to C - \{0\}$ defined by $f_r(e^{i\theta}) = f(re^{i\theta})$. Given any two $r_1, r_2 \in [\delta, 1)$, $f_{r_1}$ and $f_{r_2}$ are homotopic in the obvious way. So the winding numbers of $f_{r_1}$ and $f_{r_2}$ are equal and independent of the choice of $\delta$. Denote the common winding number by $N_f$. Then by monodromy as used in [7], we can prove

**Theorem 15.** If $f \in Q$ and $T_f$ is Fredholm, then $\text{Ind}(T_f) = -N_f$, where $\text{Ind}(T_f)$ is the Fredholm index of $T_f$, i.e. $\text{Ind}(T_f) = \text{dimension of kernel } T_f - \text{dimension of kernel } T_f$.

REMARK. $BCESV$ has played a significant role in our analysis. It seems interesting to know the structure of $BCESV$ as a $C^*$-algebra. $BCESV$ contains $C(\bar{D})$ as a proper $C^*$-subalgebra. Let $M$ be the maximal ideal space of $BCESV$. $M$ is connected since for any $f \in BCESV$, $\sigma(f) = \overline{f(D)}$ is connected (so there is no idempotent in $BCESV$ with spectrum $\{0, 1\}$). For any $\lambda \in \mathbb{D}$, the evaluation functional on $BCESV$ at $\lambda$ is in $M$, denoted by $F_\lambda$. The map $\lambda \mapsto F_\lambda$ is a one-to-one map of $D$ into $M$. Let $\mathcal{D}$ be the image of this map. We put the induced topology on $\mathcal{D}$. Let $f_0 \in BCESV$ be the function $f_0(z) = z$ for all $z \in D$. Then we have the following

**Theorem 16.** Let $F \in M$ be a multiplicative linear functional on $BCESV$. Then $F \in \mathcal{D}$ if and only if $|F(f_0)| < 1$.

**Proof.** The "only if" part is obvious. We prove the "if" part.

Suppose $|F(f_0)| < 1$. Let $z_0 = F(f_0) \in \mathbb{D}$. We want to prove $F(f) = f(z_0)$ for all $f$ in $BCESV$. By the Stone-Weierstrass approximation theorem, it is easy to show that $F(f) = f(z_0)$ for all $f$ in $C(\overline{D})$. Choose a function $\varphi \in \overline{D}$ so that $\varphi \equiv 1$ on a neighborhood $U \subset D$ of $z_0$ and $\varphi \equiv 0$ on a neighborhood $V$ of $\partial D$. Now for any $f \in BCESV$, $f\varphi \in C(\overline{D})$. Thus $F(f\varphi) = (f\varphi)(z_0) = f(z_0)\varphi(z_0)$. On the other hand, the multiplicativity of $F$ given $F(f\varphi) = F(f)F(\varphi) = F(z_0)\varphi(z_0) = F(f)$. Hence $F(f) = f(z_0)$ for all $f \in BCESV$.

**Corollary.** $\mathcal{D}$ is open in $M$, and hence $BCESV/C_0(D) \cong C(M - \mathcal{D})$.

**Proof.** The map $F \mapsto F(f_0)$ from $M$ to $C$ is continuous. By the above theorem, $\mathcal{D}$ is the inverse image of $\mathbb{D}$ under this map, so $\mathcal{D}$ is open in $M$.

**Remark.** This corollary says that $M - \mathcal{D}$ is homeomorphic to the maximal ideal space of $BCESV/C_0(D) \cong \tau(Q)/K$.

**Remark.** $M - \mathcal{D}$ is connected since $BCESV/C_0(D)$ has no idempotent element with spectrum $\{0, 1\}$ by Corollary 3 to Theorem 14.

7. A conformal invariant description of $VMO_{\beta}$. In this section, we are going to give another characterization of $VMO_{\beta}$. Also we will describe the relationship between $VMO_{\beta}(D)$ and the usual $VMO(D)$.

For $z_0 \in D$ and $r \in (0, 1)$, let
\[ D(z_0, r) = \{ z \in D : |(z_0 - z)/(1 - \bar{z}_0 z)| < r \}. \]
$D(z_0, r)$ is called the pseudohyperbolic disc centered at $z_0$ with radius $r$. It is actually a Euclidean disc (see [13]) contained in $D$ with center

$$c = \frac{1 - r^2}{1 - r^2|z_0|^2} z_0$$

and radius

$$R = r \frac{1 - |z_0|^2}{1 - r^2|z_0|^2}.$$ 

Thus the normalized Lebesgue measure of $D(z_0, r)$ is

$$|D(z_0, r)| = r^2(1 - |z_0|^2)^2/(1 - |z_0|^2)^2.$$

**Theorem 17.** For $f \in L^\infty(D, dA)$, we have $f \in \text{VMO}_\Theta$ if and only if

$$\lim_{|z| \to 1} \frac{1}{|D(z, r)|} \int_{D(z, r)} |f(w) - f(w)|^2 dA(w) = 0$$

for each $r \in (0, 1)$.

**Proof.** Let

$$I(z, r) = \frac{1}{|D(z, r)|^2} \int_{D(z, r)} \int_{D(z, r)} |f(u) - f(w)|^2 dA(w) dA(u).$$

Since $f$ is bounded, it suffices to show that $f \in \text{VMO}_\Theta \Leftrightarrow I(z, r) \to 0$ as $|z| \to 1$ for each $r \in (0, 1)$.

A change of variable shows that

$$I(z, r) = \frac{1}{|D(z, r)|^2} \int_{|w| \leq r} \int_{|u| \leq r} \left| f \left( \frac{z - w}{1 - \bar{z}w} \right) - f \left( \frac{z - u}{1 - \bar{z}u} \right) \right|^2$$

$$\leq \frac{1}{|D(z, r)|^2} \frac{(1 - |z|^2)^4}{(1 - r)^8} \int_D \int_D \left| f \left( \frac{z - w}{1 - \bar{z}w} \right) - f \left( \frac{z - u}{1 - \bar{z}u} \right) \right|^2 dA(w) dA(u)$$

$$= \frac{2(1 - r^2|z|^2)^4}{r^4(1 - r)^8} \left( |f(z)|^2 - |f(z)|^2 \right)$$

$$\leq \frac{2}{r^4(1 - r)^8} \left( |f(z)|^2 - |f(z)|^2 \right).$$

Thus $f \in \text{VMO}_\Theta \Rightarrow f \in \hat{Q} \Rightarrow I(z, r) \to 0$ as $|z| \to 1$ for each $r \in (0, 1)$.

On the other hand,

$$I(z, r) \geq \frac{(1 - |z|^2)^4}{28|D(z, r)|^2} \int_{|w| \leq r} \int_{|u| \leq r} \left| f \left( \frac{z - w}{1 - \bar{z}w} \right) - f \left( \frac{z - u}{1 - \bar{z}u} \right) \right|^2 dA(w) dA(u)$$

$$= \frac{(1 - r^2|z|^2)^4}{28r^4} \left[ \int_D \int_D - \int_{D - Dr} \int_{D - Dr} - \int_{D - Dr} \int_{D - Dr} \right]$$

$$\geq \frac{(1 - r^2|z|^2)^4}{28r^4} \left[ \int_D \int_D - \|f\|_{\infty} |D - Dr| \right].$$
that is,
\[ 2(|\tilde{f}|^2(z) - |\check{f}(z)|^2) \leq \frac{2^8 r^4}{(1 - r^2|z|^2)^4} I(z, r) + 8\|f\|_\infty^2 |D - D_r|. \]

Now if \( I(z, r) \to 0 \) as \(|z| \to 1\) for each \( r \in (0, 1)\), then
\[ 2 \lim_{|z| \to 1} \left( |\tilde{f}|^2(z) - |\check{f}(z)|^2 \right) \leq 8\|f\|_\infty^2 |D - D_r| \]
for each \( r \in (0, 1)\). Letting \( r \to 1 \) yields
\[ \lim_{|z| \to 1} \left( |\tilde{f}|^2(z) - |\check{f}(z)|^2 \right) = 0, \]
namely, \( f \in \check{Q} = \text{VMO}_0 \cap L^\infty \).

**Corollary 1.** For \( f \in L^\infty(D, dA) \), we have \( f \in Q \cap B \) if and only if
\[ \lim_{|z| \to 1} \frac{1}{|D(z, r)|} \int_{D(z, r)} |f(w)| dA(w) = 0 \]
for each \( r \in (0, 1) \).

The proof of Corollary 1 is very similar to that of the theorem, so we omit it.

For any \( f \in L^\infty(D, dA) \), define a continuous function \( \hat{f}_r(z) \) on \( D \) as follows:
\[ \hat{f}_r(z) = \frac{1}{|D(z, r)|} \int_{D(z, r)} f(u) dA(u). \]

Then we have

**Corollary 2.** For \( f \in L^\infty(D, dA) \), we have
\[ f \in Q \Leftrightarrow |\tilde{f}|^2(z) - |\check{f}_r(z)|^2 \to 0 \ (|z| \to 1) \quad \text{for any} \ r \in (0, 1), \]
\[ f \in Q \cap B \Leftrightarrow |\tilde{f}_r(z)| \to 0 \ (|z| \to 1) \quad \text{for any} \ r \in (0, 1). \]

**Proof.** The second equivalence is just the above Corollary 1. The first equivalence follows from the identity
\[ I(z, r) = 2(|\tilde{f}|^2(z) - |\check{f}_r(z)|^2). \]

**Theorem 18.** For \( f \in L^\infty(D, dA) \), we have
\[ f \in Q \Leftrightarrow \lim_{|z| \to 1} \frac{1}{|D(z, r)|} \int_{D(z, r)} |f(w) - \tilde{f}(z)|^2 dA(w) = 0, \]
\[ f \in ESV \Leftrightarrow \lim_{|z| \to 1} \frac{1}{|D(z, r)|} \int_{D(z, r)} |f(w) - f(z)|^2 dA(w) = 0. \]

**Proof.** Recall that
\[ f \in Q \Leftrightarrow \|f \circ b_z - \tilde{f}(z)\|_{L^2} \to 0 \quad \text{as} \ |z| \to 1, \]
\[ f \in ESV \Leftrightarrow \|f \circ b_z - f(z)\|_{L^2} \to 0 \quad \text{as} \ |z| \to 1. \]

Now the theorem can be proved by using the same techniques as in the proof of Theorem 17.
COROLLARY 1. For \( f \in Q \), we have

\[ f \in \text{ESV} \iff \hat{f}_r(z) - f(z) \to 0 \quad \text{as} \quad |z| \to 1 \quad \text{for each} \quad r \in (0, 1). \]

PROOF. The proof follows from Corollary 2 to Theorem 17 and the identity

\[ \frac{1}{|D(z, r)|} \int_{D(z, r)} |f(w) - f(z)|^2 dA(w) = \int_{D(z, r)} |f_r(w)|^2 - |\hat{f}_r(z)|^2 + |\hat{f}_r(z) - f(z)|^2. \]

COROLLARY 2. For \( f \in Q \), we have

\[ \hat{f}_r(z) - \tilde{f}(z) \to 0 \quad \text{as} \quad |z| \to 1 \quad \text{for each} \quad r \in (0, 1). \]

PROOF.

\[ |\hat{f}_r(z) - \tilde{f}(z)| \leq \frac{1}{|D(z, r)|} \int_{D(z, r)} |f(w) - \tilde{f}(z)| dA(w). \]

Now the assertion follows from Theorem 18 and the Schwarz inequality.

COROLLARY 3. For \( f \in Q \), we have \( \hat{f}_r \in \text{ESV} \) and \( f - \hat{f}_r \in Q \cap B \).

PROOF. It follows from Corollary 2 and the fact that \( \tilde{f} \in \text{ESV} \) and \( f - \tilde{f} \in Q \cap B \).

COROLLARY 4. Given \( f \in L^\infty(D) \), we have

\[ f \in \text{ESV} \iff \lim_{|z| \to 1^-} \sup_{w \in D(z, r)} |f(z) - f(w)| = 0 \]

for all \( r \in (0, 1) \).

PROOF. “\( \Rightarrow \)” follows from the second statement of the theorem.

To prove “\( \Rightarrow \)”, given any \( r \in (0, \frac{1}{2}) \) and consider \( \hat{f}_r(z) \) on \( D \). Suppose \( w \in D(z, r) \). Then

\[ |\hat{f}_r(z) - \hat{f}_r(w)| \leq \frac{1}{|D(z, r)||D(w, r)|} \int_{D(z, r)} \int_{D(w, r)} |f(u) - f(v)| dA(u)dA(v) \]

\[ \leq \frac{|D(z, 2r)|^2}{|D(z, r)||D(w, r)| |D(z, 2r)|^2} \cdot \int_{D(z, 2r)} \int_{D(z, 2r)} |f(u) - f(v)| dA(u)dA(v). \]

Since \( f \in \text{ESV} \Rightarrow f \in \text{VMO}_a(D) \), we have

\[ \lim_{|z| \to 1^-} \sup_{w \in D(z, r)} |\hat{f}_r(z) - \hat{f}_r(w)| = 0 \quad (r \in (0, \frac{1}{2})), \]

but \( f(z) - \hat{f}_r(z) \to 0 \) as \( |z| \to 1^- \), hence

\[ \lim_{|z| \to 1^-} \sup_{w \in D(z, r)} |f(z) - f(w)| = 0 \quad (r \in (0, \frac{1}{2})). \]

By a finite covering argument, we get

\[ \lim_{|z| \to 1^-} \sup_{w \in D(z, r)} |f(z) - f(w)| = 0 \]

for all \( r \in (0, 1) \).

Finally, we discuss the relationship between \( \text{VMO}_a(D) \) and the usual area \( \text{VMO}(D) \). Recall that \( f \in \text{VMO}(D) \) if and only if given \( \varepsilon > 0 \), there is \( \delta \in (0, 1) \) such that

\[ \frac{1}{|D|} \int_{D} |f(w) - \frac{1}{|D|} \int_{D} f(u) dA(u)| dA(w) < \varepsilon \]
whenever $D$ is a disc contained in $D$ with radius $\leq \delta$. For any $r \in (0,1)$, the pseudohyperbolic disc $D(z,r)$ centered at $z$ is a Euclidean disc contained in $D$ with radius $R = r \frac{1 - |z|^2}{1 - r^2|z|^2}$ which goes to 0 as $|z| \to 1$. Thus if $f \in VMO(D)$, then for any $r \in (0,1)$ we have

$$\lim_{|z| \to 1} \frac{1}{|D(z,r)|} \int_{D(z,r)} f(w) - f(u) dA(u) dA(w) = 0.$$ 

**Theorem 19.** If $f \in L^\infty(D)$, then $f \in VMO(D) \Rightarrow f \in VMO_\partial(D)$.

**Remark.** The converse of Theorem 19 is obviously false. For example, if $f$ is the characteristic function of any closed square contained in $D$, then $f \in Q \cap B \subset VMO_\partial(D)$ while $f \notin VMO(D)$. Even for bounded continuous functions $f$ on $D$, the converse of Theorem 19 does not hold. However, if $f \in H^\infty(D)$, then $f \in VMO_\partial(D) \Leftrightarrow f \in VMO(D)$.

**8. Open questions and possible generalizations.** All the results in this paper are concerned with essentially bounded functions on $D$. It is clear that many concepts and techniques apply to unbounded functions. First we make some definitions.

**Definition 1.** A function $f \in L^1(D)$ is said to be in $BMO_\partial(D)$ if

$$\sup_{z \in D} \frac{1}{|S_z|} \int_{S_z} f(w) - f(u) dA(u) dA(w) < +\infty.$$ 

It is obvious that $VMO_\partial(D) \subset BMO_\partial(D)$.

For a function $f \in L^2(D,dA)$, the Toeplitz operator $T_f$ is an unbounded operator in general. However, we always have $k_z \in D(T_f)$. Thus, the Berezin symbol $\hat{f}$ is well defined in this case. Also $\hat{f}$ is well defined. Our first problem is to generalize Theorem 1:

**Problem 1.** For $f \in L^2(D,dA)$, prove that the following are all equivalent:

(a) $H_f$ and $H_f$ are compact;
(b) $T_{|f|^2} - T_f T_f$ and $T_{|f|^2} - T_f T_f$ are compact;
(c) $f \in VMO_\partial(D)$;
(d) $|\hat{f}|^2(z) - |\hat{f}(z)|^2 \to 0$ as $|z| \to 1$;
(e) $|\hat{f}|^2(z) - |\hat{f}(z)|^2 \to 0$ as $|z| \to 1$.

An analogous problem is

**Problem 2.** For $f \in L^2(D,dA)$, prove that the following are all equivalent:

(a) $H_f$ and $H_f$ are bounded;
(b) $T_{|f|^2} - T_f T_f$ and $T_{|f|^2} - T_f T_f$ are bounded;
(c) $f \in BMO_\partial(D)$;
(d) $|\hat{f}|^2(z) - |\hat{f}(z)|^2$ is bounded on $D$;
(e) $|\hat{f}|^2(z) - |\hat{f}(z)|^2$ is bounded on $D$. 

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For $f \in L^2(D, dA)$, let
\[
\|f\|_1 = \sup_{z \in D} \frac{1}{|S_z|} \int_{S_z} \left| f(w) - \frac{1}{|S_w|} \int_{S_w} f(u) dA(u) \right| dA(w),
\]
\[
\|f\|_2 = \sup_{z \in D} \frac{1}{|S_z|} \sqrt{\int_{S_z} \int_{S_z} |f(u) - f(w)|^2 dA(u) dA(w)},
\]
\[
\|f\|_3 = \sup_{z \in D} \sqrt{|f|^2(z) - |\hat{f}(z)|^2}.
\]

**Problem 3.** Show that $\| \cdot \|_i$ $(i = 1, 2, 3)$ are complete norms on $BMO_\partial(D)$ modulo the constant functions and show that they are equivalent.

In the theory of BMO and VMO [13], Fefferman’s duality theorem is one of the most important and deepest results, so it is very natural to propose:

**Problem 4.** Formulate and prove a duality theorem about $BMO_\partial(D)$.

New characterizations of $BMO_\partial(D)$ and $VMO_\partial(D)$ are also worth further investigation.

Finally, I am very curious about the possible generalizations of the above concepts and results to general strongly pseudo-convex domains $\Omega$ in $\mathbb{C}^n$. The definitions of Berezin symbol, $Q$, and $\hat{Q}$ can be carried over word by word. It seems to me that a reasonable definition of $BMO_\partial(\Omega)$ and $VMO_\partial(\Omega)$ as well as ESV($\Omega$) should involve the geometry of $\Omega$ and $\partial \Omega$. A connection between geometry and operator theory is expected in the further study of this direction.

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