CHARACTER TABLE AND BLOCKS OF FINITE SIMPLE TRIALITY GROUPS $3D_4(q)$

D. I. DERIZIOTIS AND G. O. MICHLER

Abstract. Based on recent work of Spaltenstein [14] and the Deligne-Lusztig theory of irreducible characters of finite groups of Lie type, in this paper the character table of the finite simple groups $3D_4(q)$ is given. As an application we obtain a classification of the irreducible characters of $3D_4(q)$ into $r$-blocks for all primes $r > 0$. This enables us to verify Brauer’s height zero conjecture, his conjecture on the bound of irreducible characters belonging to a given block, and the Alperin-McKay conjecture for the simple triality groups $3D_4(q)$. It also follows that for every prime $r$ there are blocks of defect zero in $3D_4(q)$.

Introduction. Let $G_o = 3D_4(q)$ be a simple triality group defined over a finite field GF$(q)$ with $q = p^n$ elements, where $p > 0$ is a prime number and $n$ is a positive integer.

In [14] N. Spaltenstein computed the values of the eight unipotent irreducible characters of $G_o$. Using his results we determine the character table of $G_o$ in §4. In Theorem 4.3 the nonunipotent irreducible characters of $G_o$ are presented in the form of precise linear combinations of the virtual Deligne-Lusztig characters $R_{T,\Theta}$, where $\Theta$ is a linear character of the $\sigma$-fixed points of a $\sigma$-stable maximal torus $T$ of the corresponding algebraic group $G$. The values of the Deligne-Lusztig characters are given in Table 3.6.

By Lusztig’s Jordan form of the irreducible characters of a finite group of Lie type [11] each irreducible character $\chi$ of $G_o$ is of the form $\chi = \chi_{t,u}$, where $t$ is a semisimple element of $G_o$ and $\chi_u$ is a unipotent irreducible character of the centralizer $C_{G_o}(t)$ of $t$. The group theoretical structure of the centralizers $C_{G_o}(t)$ of the semisimple elements $t$ of $G_o$ is given in Proposition 2.2, and of the 7 (up to $G_o$-conjugacy) maximal tori $T_i$, $0 \leq i \leq 6$, in Proposition 1.2. It follows that $C_{G_o}(t)$ has at most three unipotent irreducible characters, namely the trivial 1, the Steinberg character $S_t$ or a unipotent character of degree either $qs = q(q + 1)$ or $qs' = q(q - 1)$. If $t \neq 1$ is regular, we write $\chi_t$ instead of $\chi_{t,1}$, in all other cases $\chi_{t,1}$, $\chi_{t,S_t}$, $\chi_{t,q}$, $\chi_{t,q'}$, or $\chi_{t,S_t,q}$, $\chi_{t,S_t,q'}$. A complete classification of the irreducible characters of $G_o$ with their degrees is given in Table 4.4.

On the set of conjugacy classes of semisimple elements $t$ of $G_o$ one can define an equivalence relation as follows. Two such conjugacy classes $t_{1}^\sigma$ and $t_{2}^\sigma$ are equivalent if and only if their centralizers $C_{G_o}(t_1)$ and $C_{G_o}(t_2)$ are $G_o$-conjugate. If $q$ is odd, there are 15 equivalence classes with representatives $s_i$, $1 \leq i \leq 15$, where $s_1 = 1$.

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and $s_2 \neq 1$ is the unique conjugacy class of involutions of $G_o$. If $q$ is even, the equivalence class of $s_2$ does not exist, and we have only 14 equivalence classes. Using the first author’s work on the Brauer complex [5] of $G_o$ and the computer, we obtain in Table 4.4 the numbers of semisimple conjugacy classes of $G_o$ belonging to a given equivalence class $[s_i]$, $1 \leq i \leq 15$. Applying then Proposition 2.2 and Spaltenstein’s characterization of the unipotent conjugacy classes of $G_o$ [14], we can give in Proposition 2.3 a complete classification of all conjugacy classes of $G_o$. In particular, we show that the number $k(G_o)$ of all conjugacy classes of $G_o$ is

$$k(G_o) = q^4 + q^3 + q^2 + q + 5,$$

if $2 | q$, and

$$k(G_o) = q^4 + q^3 + q^2 + q + 6,$$

if $2 \nmid q$.

By means of these results we determine in §5 the distribution of the irreducible characters of $G_o$ into $r$-blocks, where $r$ is a prime number dividing the group order $|G_o|$. If $r = p$, then by Humphrey’s theorem [10] $G_o$ has only the principal $p$-block $B_0$ and a block $B$ of defect zero consisting of the irreducible Steinberg character. For $r \neq p$ Theorem 5.9 asserts that each $r$-block $B$ with defect group $D$ determines, up to $G_o$-conjugacy, a unique semisimple $r'$-element $s$ of $G_o$ such that an irreducible character $\chi_{r,u}$ of $G_o$ belongs to $B$ if and only if $t$ is $G_o$-conjugate to $sy$ for some $y \in D$, and $\chi_u$ is an irreducible unipotent character of $C_{G_o}(sy)$ such that $sy\chi_u$ belongs to an $r$-block $B$ of $C_{G_o}(sy)$ with defect group $D$ satisfying $B = B^G$. This result can be considered to be an analogue of the Fong-Srinivasan characterization [8] of the $r$-blocks of the general linear and unitary groups.

In Corollary 5.11 we show that for all primes $r > 0$ and all $r$-blocks $B$ of $G_o$ with defect group $\delta(B) = G_o D$ the number of all irreducible characters of $G_o$ belonging to $B$ is bounded by $k(B) \leq |D|$. This verifies a well-known conjecture of R. Brauer, see [7], in the case of the simple triality groups. He also conjectured that an $r$-block $B$ of a finite group $G$ has only irreducible characters of height zero if and only if its defect group $\delta(B) = G D$ is abelian. In case $G = G_o$ this is shown for all primes $r$ in Corollary 5.10.

Let $k_0(B)$ be the number of irreducible characters of an $r$-block $B$ of $G$ with height zero. If $\delta(B) = G D$ denotes the defect group of $D$, $H = N_G(D)$, and $B_1$ is the Brauer correspondent of $B$ in $H$, then the Alperin-McKay conjecture asserts that $k_0(B) = k_0(B_1)$. In the case of $G = G_o$, we verify it for all primes $r$; see Corollary 5.12.

Another application of Table 4.4 yields that in $G_o$ there are $r$-blocks $B$ of defect zero for every prime $r > 0$; see Corollary 5.1.

Concerning the notation and terminology we refer to the books by Carter [2], Deriziotis [4], Feit [7], and Lusztig [11].

1. Notations and known results on $^3D_4(q)$. Let $G$ be a simple simply connected algebraic group of Dynkin diagram type $D_4$ over the algebraic closure $K$ of the prime field $GF(p) = F_p$, $p > 0$. Let $q = p^m$ for some positive integer $m$, and let $GF(q) = F_q$ be the field with $q$ elements. $F^*$ denotes the multiplicative group of every field $F$.

Let $T$ be a maximal torus of $G$, $\Phi$ the set of roots of $G$ relative to $T$, $Y = Hom(T, K^*)$ the group of rational characters of $T$, $Y = Hom(K^*, T)$ — the
group of one-parameter subgroups of $T$. On the real vector space $V = Y \otimes \mathbb{R}$ we have a Killing form $(\cdot, \cdot)$ which is transferred to an inner product $\langle \cdot, \cdot \rangle$ on the dual space $V^*$ of $V$ which can canonically be identified with the real vector space $X \otimes \mathbb{R}$. If $r$ is a root in $\Phi$, the coroot of $G$ associated to $r$ is defined to be the element $h_r$ of $Y$ such that $\langle h_r, h \rangle = 2r(h)/\langle r, r \rangle$, for all $h \in Y$. In $V$ there is an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ such that the coroots in $Y$ are the vectors $\pm e_i \pm e_j$, $1 \leq i, j \leq 4$. We fix the fundamental basis $\Delta = \{r_1, r_2, r_3, r_4\}$ in $\Phi$ for which the associated coroots are $h_1 = e_1 - e_2$, $h_2 = e_2 - e_3$, $h_3 = e_3 - e_4$, and $h_4 = e_3 + e_4$, respectively.

Let $\tau$ be the symmetry of the Dynkin diagram $D_4$ of $G$ with nodes $h_1$, $h_2$, $h_3$, and $h_4$ such that $\tau$: $h_1 \rightarrow h_3 \rightarrow h_4 \rightarrow h_1$ and $\tau(h_2) = h_2$. Then $\tau$ induces an isometry on $V$ which again is denoted by $\tau$. The triality automorphism $\sigma = \tau q$ of $G$ is induced by $\tau$ times the field automorphism $z \rightarrow z^q$ of $K$. The simple group $3D_4(q) = G_a = \{g \in G|\sigma(g) = g\}$ is called the Steinberg-Tits triality. Its order $|G_a| = q^{12}(q^8 + q^4 + 1)(q^6 - 1)(q^2 - 1)$.

The torus $T$ is $\sigma$-stable. The restriction of $\sigma = q\tau$ onto $T$ induces a linear transformation of $V$, again denoted by $\sigma$.

Let $h$: $\text{Hom}(X, K^*) \rightarrow T$ be defined as follows. For every $\chi \in \text{Hom}(X, K^*)$, $h(\chi) = t \in T$, where $\chi(\lambda) = \lambda(t)$ for all $\lambda \in X$. Then $h$ is an isomorphism.

Let $\lambda_1$, $\lambda_2$, $\lambda_3$, and $\lambda_4$ be the fundamental weights in $X$. Each element $h(\chi) \in T$ can uniquely be written as

$$h(\chi) = \prod_{i=1}^{4} h(x_{h_i}, z_i),$$

where $x_{h_i}(\lambda) = z^{\lambda(h_i)}$ for $r \in \Phi$, $z \in K^*$, and where $\chi(\lambda_i) = z_i$ for $1 \leq i \leq 4$.

Let $W$ be the Weyl group generated by all reflections $w_r$ at the hyperplanes of $V$ orthogonal to the coroots $h_r$, $r \in \Phi$. Then $\sigma$ acts on $W$ by $\sigma(w_r) = \sigma w_r \sigma^{-1} = \tau w_. \tau^{-1}$. In particular, $\sigma(w_r) = w_{\tau(r)}$. $W$ acts also on $T$ by $w \chi$, where $(w \chi)(\lambda) = \chi(w^{-1}(\lambda))$ for all $\lambda \in X$. Furthermore, $w_{\pm j}$ denotes the reflection at the hyperplane of $V$ orthogonal to the coroot $e_i \pm e_j$.

Let $r_0$ be the highest root of $\Phi$, and $\tilde{\Delta} = \Delta \cup \{-r_0\}$.

Let $J$ be an arbitrary $\tau$-invariant proper subset of $\tilde{\Delta}$, and $W$ the Weyl group of the torus $T$. The normalizer of $J$ in $W$ is denoted by $\Omega_J$. It is a $\sigma$-stable subgroup of $W$. Two elements $w_1$, $w_2 \in \Omega_J$ are called $\sigma$-equivalent if $w_1 = w w_2 \sigma(w^{-1})$ for some $w \in \Omega_J$. The $\sigma$-equivalence class of $w \in \Omega_J$ is denoted by $[w]$, and $H^1(\sigma, \Omega_J)$ is the set of all $\sigma$-equivalence classes $[w]$ of $\Omega_J$. The possibilities of $J$ and $\Omega_J$ are given in Table 1.0, up to $W$-conjugacy.

<table>
<thead>
<tr>
<th>$J$</th>
<th>$\Omega_J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_0 = {r_1, r_2, r_3, r_4}$</td>
<td>$\Omega_{J_0} = 1$</td>
</tr>
<tr>
<td>$J_1 = {r_1, r_3, r_4, -r_0}$</td>
<td>$\Omega_{J_1} = \langle w_{1+4} w_{2+3} \rangle \times \langle w_{1-4} w_{1+4} \rangle \simeq (Z_2)^2$</td>
</tr>
<tr>
<td>$J_2 = {r_1, r_3, r_4}$</td>
<td>$\Omega_{J_2} = \langle w_{1+2} \rangle \simeq Z_2$</td>
</tr>
<tr>
<td>$J_3 = {r_2, -r_0}$</td>
<td>$\Omega_{J_3} = \langle w_{1-3} w_{2+4} w_{2-4} \rangle \simeq Z_2$</td>
</tr>
<tr>
<td>$J_4 = {-r_0}$</td>
<td>$\Omega_{J_4} = \langle w_{1-2} \rangle \times \langle w_{3-4} \rangle \times \langle w_{3+4} \rangle \simeq (Z_2)^3$</td>
</tr>
<tr>
<td>$J_5 = \emptyset$</td>
<td>$\Omega_{J_5} = W$</td>
</tr>
</tbody>
</table>

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Let $\mathcal{C}_J$ be the collection of all $\sigma$-stable $G$-conjugates of $C_G(x)$ where $x$ is a semisimple element of $G$ with $r(x) = 1$ for all $r \in J$. Then the group $G_\sigma$ acts on $\mathcal{C}_J$ by conjugation. If $J = \emptyset$ is the empty set, then $G_\emptyset = W$, and $x$ is a regular element of $G$. There is a one-to-one correspondence between the $G_\sigma$-orbits of $\sigma$-stable maximal tori of $G$ and the classes of $H^1(\sigma, W)$, see [1, p. 186]. It is known for the triality $G_\sigma = D_4(q)$ that $|H^1(\sigma, W)| = 7$; cf. [14].

Let $T$ be a $\sigma$-stable maximal torus of $G$, with Weyl group $W = N_G(T)/T$. If $T'$ is a $\sigma$-stable maximal torus of $G$, then there is a unique class $[w_{J}] \in H^1(\sigma, W)$ with $j \in \{0, 1, \ldots, 6\}$ such that $T'_J$ is $G$-conjugate to $T_j = T_{w_{J}\sigma} = \{t \in T | w_{J}\sigma(t) = t\}$.

In particular, the element $h(x) = \prod_{i=1}^4 h(x_{i}, z_i) \in T$ belongs to $T_j$ if and only if

$$h(x) = w_J \sigma h(x) = \prod_{i=1}^4 h(x_{w_J\sigma}(z_i)),$$

For the sake of simplicity, each element $h(x) = \prod_{i=1}^4 h(x_{i}, z_i) \in T$ is denoted by

$$A(x) = \sum_{i=1}^4 A_i(x) = \prod_{i=1}^4 h(x_{i}, z_i).$$

For the sake of simplicity, each element $h(x) = \prod_{i=1}^4 h(x_{i}, z_i) \in T$ is denoted by

$$A(x) = (z_1, z_2, z_3, z_4).$$

With this notation we can parametrize all the elements of the tori $T_j$.

**Lemma 1.1.** Let $q \neq 2$. Let $T'$ be a maximal $\sigma$-stable torus of $G$ corresponding to the class $[w_{J}] \in H^1(\sigma, W)$, $j \in \{0, 1, \ldots, 6\}$, and let $T_j = T_{w_{J}\sigma}$. Then the Weyl group $W_j$ of $T_j$ is given by

$$W_j = C_{w_{J}\sigma}(w_{J}) = \{w \in W | w_{J\sigma}w = w_{J}\} \equiv N_G(T_j)/T_j.$$

**Proposition 1.2 (P. C. Gager).** The structure of the maximal tori $T_j$ of $G_\sigma$ and their Weyl groups $W_j$ is given in Table 1.1.

<table>
<thead>
<tr>
<th>$[w_{J}] \in H^1(\sigma, W)$</th>
<th>$T_j$</th>
<th>$W_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_0 = 1 \in W$</td>
<td>$T_0 = {(z_1, z_2, z_3, z_4)</td>
<td>z_1^{-1} = z_2^{-1} = 1}$</td>
</tr>
<tr>
<td>$w_1 = w_1^{-1}$</td>
<td>$T_1 = {(z_1, z_3^{-q}, z_4^{-q}, z_2^{-q})</td>
<td>z_1^{q^{-1}}q^{-1} = 1}$</td>
</tr>
<tr>
<td>$w_2 = -w_{20}$</td>
<td>$T_2 = {(z_1, z_2, z_3^{-q}, z_4^{-q})</td>
<td>z_1^{q^{-1}}q^{-1} = 1}$</td>
</tr>
<tr>
<td>$w_3 = w_1^{-1}z_{2^{-1}}$</td>
<td>$T_3 = {(z_1, z_2, z_3^{-q}, z_4^{-q})</td>
<td>z_1^{q^{-1}}q^{-1} = 1}$</td>
</tr>
<tr>
<td>$w_4 = -w_1^{-1}z_{2^{-1}}$</td>
<td>$T_4 = {(z_1, z_2, z_3^{-q}, z_4^{-q})</td>
<td>z_1^{q^{-1}}q^{-1} = 1}$</td>
</tr>
<tr>
<td>$w_5 = w_1^{-1}z_{2^{-1}}$</td>
<td>$T_5 = {(z_1, z_2, z_3^{-q}, z_4^{-q})</td>
<td>z_1^{q^{-1}}q^{-1} = 1}$</td>
</tr>
<tr>
<td>$w_6 = -1$</td>
<td>$T_6 = {(z_1, z_2, z_3^{-q}, z_4^{-q})</td>
<td>z_1^{q^{-1}}q^{-1} = 1}$</td>
</tr>
</tbody>
</table>

2. Structure of the centralizers of the semisimple elements and the determination of the conjugacy classes. For each $i \in \{0, 1, \ldots, 5\}$ let $E_i$ denote the Dynkin diagram type of the root system $\Phi_{J_i}$ generated by $J_i$. Let $\mathcal{C}_{J_i}$ be the collection of all $\sigma$-stable $G$-conjugates of $C_G(x)$, where $x$ is an element of the maximal torus $T$ of $G$. By
Corollary 3 of [3] there is a one-to-one correspondence between the $G_a$-orbits of $\mathcal{C}_J$ and the classes of $H^1(\sigma, \Omega_J)$). Therefore each $G_a$-orbit of $\mathcal{C}_J$ can be parametrized by a pair $(E_i, [w])$, where $[w] \in H^1(\sigma, \Omega_J)$.

**Proposition 2.1.** Let $s_i$ be a representative of a semisimple conjugacy class $s_i^{G_a}$ of $G_a$ whose centralizer $C_G(s_i)$ is in the orbit parametrized by the pair $(E_i, [w])$. Then the semisimple conjugacy classes are classified in Table 2.1.

<table>
<thead>
<tr>
<th>$(E_i, [w])$</th>
<th>$s_i$, $q$ even</th>
<th>$s_i$, $q$ odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(E_0, [1])$</td>
<td>$s_1 = (1,1,1,1)$</td>
<td>$s_1 = (1,1,1,1)$</td>
</tr>
<tr>
<td>$(E_1, [1])$</td>
<td>$s_2$</td>
<td>$s_2 = (t,1,1,1)$, $t^2 = 1, t \neq 1$</td>
</tr>
<tr>
<td>$(E_2, [1])$</td>
<td>$s_3 = (t,t^2,1,t)$, $t^{q-1} = 1, t \neq 1$</td>
<td>$s_3 = (t,t^2,1,t)$, $t^{q-1} = 1, t^2 \neq 1$</td>
</tr>
<tr>
<td>$(E_3, [1])$</td>
<td>$s_4 = (t,t^q,t^q)$, $t^{q^2+q+1} = 1, t \neq 1$</td>
<td>$s_4 = (t,t^q,t^q)$, $t^{q^2+q+1} = 1, t \neq 1$</td>
</tr>
<tr>
<td>$(E_4, [1])$</td>
<td>$s_5 = (t,t^q,t^q)$, $t^{q^2-1} = 1, t^{q^2+q-1} \neq 1$</td>
<td>$s_5 = (t,t^q,t^q)$, $t^{q^2-1} = 1, t^{q^2+q-1} \neq 1, t^2 \neq 1$</td>
</tr>
<tr>
<td>$(E_5, [1])$</td>
<td>$s_6 = (t_1,t_2,t_3,t_4)$, $t_1^{q^2-1} = 1, t_2 \neq 1, t_3^{q^2} \neq t_2$, $t_4^{q^2+q+1} \neq t_2$</td>
<td>$s_6 = (t_1,t_2,t_3,t_4)$, $t_1^{q^2-1} = 1, t_2 \neq 1, t_3^{q^2} \neq t_2$, $t_4^{q^2+q+1} \neq t_2$</td>
</tr>
<tr>
<td>$(E_6, [w_1+2])$</td>
<td>$s_7 = (t,t^q,t,t^q)$, $t^{q^2-1} = 1, t^2 \neq 1$</td>
<td>$s_7 = (t,t^2,t,t)$, $t^2 \neq 1, t^{q^2-1} = 1$</td>
</tr>
<tr>
<td>$(E_7, [w_1+2])$</td>
<td>$s_8 = (t,t^{q^2},t^{q^2})$, $t^{q^2(q^2-1)} = 1, t^{q^2-1} \neq 1 \neq t^{q^2}$</td>
<td>$s_8 = (t,t^{q^2},t^{q^2})$, $t^{q^2(q^2-1)} = 1, t^{q^2-1} \neq 1 \neq t^{q^2}$</td>
</tr>
<tr>
<td>$(E_8, [-w_1+2])$</td>
<td>$s_9 = (t,t^{q^2},t^{q^2})$, $t^{q^2-1} = 1, t^2 \neq 1$</td>
<td>$s_9 = (t,t^{q^2},t^{q^2})$, $t^{q^2-1} = 1, t^2 \neq 1$</td>
</tr>
<tr>
<td>$(E_9, [-w_1+2])$</td>
<td>$s_{10} = (t,t^{q^2},t^{q^2})$, $t^{q^2-1} = 1, t^2 \neq 1$</td>
<td>$s_{10} = (t,t^{q^2},t^{q^2})$, $t^{q^2-1} = 1, t^2 \neq 1$</td>
</tr>
<tr>
<td>$(E_{10}, [-w_1+2])$</td>
<td>$s_{11} = (t,t^{q^2+1},t^{q^2},t^{q^2})$, $t^{q^2(q^2-1)} = 1, t^{q^2-1} \neq 1 \neq t^{q^2}$</td>
<td>$s_{11} = (t,t^{q^2+1},t^{q^2},t^{q^2})$, $t^{q^2(q^2-1)} = 1, t^{q^2-1} \neq 1 \neq t^{q^2}$</td>
</tr>
<tr>
<td>$(E_{11}, [w_1+w_2-3])$</td>
<td>$s_{12} = (t_1,t_2,t_3,t_4,(t_1^{-1}t_2)^{q^2+1})$, $t_1^{q^2+q+1} = 1, t_1 \neq t_2$</td>
<td>$s_{12} = (t_1,t_2,t_3,t_4,(t_1^{-1}t_2)^{q^2+1})$, $t_1^{q^2+q+1} = 1, t_1 \neq t_2$</td>
</tr>
<tr>
<td>$(E_{12}, [-w_1+w_2-3])$</td>
<td>$s_{13} = (t_1,t_2,t_3,t_4,(t_1^{-1}t_2)^{q^2-1})$, $t_1^{q^2-1} = 1, t_1 \neq t_2$</td>
<td>$s_{13} = (t_1,t_2,t_3,t_4,(t_1^{-1}t_2)^{q^2-1})$, $t_1^{q^2-1} = 1, t_1 \neq t_2$</td>
</tr>
<tr>
<td>$(E_{13}, [w_1-w_2-3])$</td>
<td>$s_{14} = (t,t^{q^2+1},t^{q^2},t^{q^2})$, $t^{q^2+1} = 1, t \neq 1$</td>
<td>$s_{14} = (t,t^{q^2+1},t^{q^2},t^{q^2})$, $t^{q^2+1} = 1, t \neq 1$</td>
</tr>
<tr>
<td>$(E_{14}, [-1])$</td>
<td>$s_{15} = (t_1,t_2,t_3,t_4,(t_1^{-1}t_2)^{q^2})$, $t_1^{q^2+1} = 1, t_1 \neq t_2$, $t_2^{q^2+q+1} \neq 1$, $t_2 \neq t_3$, $t_3^{q^2+q+1} \neq 1$, $t_3 \neq t_4$, $t_4^{q^2+q+1} \neq 1$, $t_4 \neq t_1$</td>
<td>$s_{15} = (t_1,t_2,t_3,t_4,(t_1^{-1}t_2)^{q^2})$, $t_1^{q^2+1} = 1, t_1 \neq t_2$, $t_2^{q^2+q+1} \neq 1$, $t_2 \neq t_3$, $t_3^{q^2+q+1} \neq 1$, $t_3 \neq t_4$, $t_4^{q^2+q+1} \neq 1$, $t_4 \neq t_1$</td>
</tr>
</tbody>
</table>

Let $x \in G_a$ be semisimple contained in the maximal torus $T$ of $G$. By Proposition 2.3.2 of [4] there is a proper subset $J$ of $\Delta$ such that $C_G(x)$ is generated by $T$ and the root subgroups $X_r, r \in J$.
Let \( M = \{ X_r \mid r \in J \} \) and let \( S \) be the connected component of the center of \( C_G(x) \). Then \( M \) is semisimple, \( S \) is a torus, \( C_G(x) = MS \) and \( M \cap S \) is finite. Moreover, the order

\[
|C_G(x)| = |M_o| \cdot |S_o|.
\]

Furthermore, \( M_{au} \) denotes the subgroup of \( M_o \) generated by all its unipotent elements. Certainly \( M_{au} \) is a characteristic subgroup of \( C_G(x) \).

**Proposition 2.2.** Let \( s_i \neq 1 \) be a representative of a nonregular semisimple conjugacy class of \( G_o \). The structure of its centralizer \( C = C_G(s_i) \) is as given in Tables 2.2a and 2.2b.

In particular, \( M_o = M_{au} \) for every \( s_i \neq s_2 \).

**Table 2.2a. Even \( q \)**

| class  | \( M_{au} \)                  | \( S_o \)       | \( |C : M_{au} \cdot S_o| \) | \( C, C', \) or \( C/S_o \)                        |
|--------|------------------------------|----------------|----------------------------|-------------------------------------------------|
| \( s_3 \) | \( SL_2(q^3) \)            | \( Z_{q-1} \)   | 1                           | \( C = SL_2(q^3) \times Z_{q-1} \)             |
| \( s_4 \) if \( 3 \mid q-1 \) | \( SL_3(q) \)            | \( Z_{q^2+q+1} \) | 1                           | \( C = SL_3(q) \times Z_{q^2+q+1} \)         |
| \( s_5 \) if \( 3 \mid q-1 \) | \( SL_3(q) \)            | \( Z_{q^2+q+1} \) | 3                           | \( C/S_o = PGL_3(q) \)                        |
| \( s_5 \) | \( SL_2(q) \)             | \( Z_{q^3-1} \)  | 1                           | \( C = SL_2(q) \times Z_{q^3-1} \)             |
| \( s_7 \) | \( SL_2(q^3) \)           | \( Z_{q+1} \)    | 1                           | \( C = SL_2(q^3) \times Z_{q+1} \)            |
| \( s_9 \) if \( 3 \mid q+1 \) | \( SU_3(q) \)           | \( Z_{q^2-q+1} \)  | 1                           | \( C/S_o = PU_3(q) \)                        |
| \( s_9 \) if \( 3 \mid q+1 \) | \( SU_3(q) \)           | \( Z_{q^2-q+1} \)  | 3                           | \( C/S_o = PU_3(q) \)                        |
| \( s_{10} \) | \( SL_2(q) \)           | \( Z_{q^3-1} \)  | 1                           | \( C = SL_2(q) \times Z_{q^3-1} \)            |

**Table 2.2b. Odd \( q \)**

| class  | \( M_{au} \)          | \( S_o \)       | \( |C : M_{au} \cdot S_o| \) | \( C, C', \) or \( C/S_o \)                        |
|--------|------------------------|----------------|----------------------------|-------------------------------------------------|
| \( s_2 \) | \( SL_2(q^3) \times SL_2(q) \) | \( 1 \)       | 2                           | \( C' = SL_2(q^3) \times SL_2(q) \)             |
| \( s_3 \) | \( SL_2(q^3) \)       | \( Z_{q-1} \)   | 2                           | \( C/S_o = PGL_2(q^3) \)                        |
| \( s_4 \) if \( 3 \mid q-1 \) | \( SL_3(q) \)       | \( Z_{q^2+q+1} \) | 1                           | \( C = SL_3(q) \times Z_{q^2+q+1} \)         |
| \( s_4 \) if \( 3 \mid q-1 \) | \( SL_3(q) \)       | \( Z_{q^2+q+1} \) | 3                           | \( C/S_o = PGL_3(q) \)                        |
| \( s_5 \) | \( SL_2(q) \)        | \( Z_{q^3-1} \)  | 2                           | \( C/S_o = PGL_2(q) \)                        |
| \( s_7 \) | \( SL_2(q^3) \)      | \( Z_{q+1} \)    | 2                           | \( C/S_o = PGL_2(q^3) \)                      |
| \( s_9 \) if \( 3 \mid q+1 \) | \( SU_3(q) \)       | \( Z_{q^2-q+1} \)  | 1                           | \( C = SU_3(q) \times Z_{q^2-q+1} \)         |
| \( s_9 \) if \( 3 \mid q+1 \) | \( SU_3(q) \)       | \( Z_{q^2-q+1} \)  | 3                           | \( C/S_o = PU_3(q) \)                        |
| \( s_{10} \) | \( SL_2(q) \)        | \( Z_{q^3-1} \)  | 2                           | \( C/S_o = PGL_2(q) \)                        |
Proof. In Table 7 of Deriziotis [4, p. 140], for each centralizer $C_{G}(s_{i})$ the isogeny class of the groups $M_{a}$ and the orders of the cyclic groups $S_{a}$ are given. Using similar methods as in Iwahori’s paper [1, p. 281], the precise group structure of $C_{G}(s_{i})$ can be determined.

In order to find the mixed conjugacy classes, we use Spaltenstein’s [14] results on the orders of the centralizers $C_{G}(u_{j})$ of the unipotent elements $u_{j}$ of $G_{o}$, with the following notation. (See Table A.)

<table>
<thead>
<tr>
<th>unipotent class</th>
<th>notation of [14] for even $q$</th>
<th>for odd $q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_{1}$</td>
<td>$A_{1}$</td>
<td>$A'_{1}$</td>
</tr>
<tr>
<td>$u_{2}$</td>
<td>$A_{2}$</td>
<td>$A''_{2}$</td>
</tr>
<tr>
<td>$u_{3}$</td>
<td>$D_{4}(a_{1})$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$u_{4}$</td>
<td>$D_{4}(a_{1})$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$u_{5}$</td>
<td>$D_{4}(a_{1})$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$u_{6}$</td>
<td>$D_{4}(a_{1})$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$u_{7}$</td>
<td>$D_{4}(a_{1})$</td>
<td>$D_{4}$</td>
</tr>
</tbody>
</table>

Proposition 2.3. $G_{o}$ has $q^{3} + q^{2} + q$ and $q^{3} + q^{2} + q - 2$ mixed conjugacy classes with representatives $s_{j} \cdot u_{j} = u_{j} \cdot s_{j}$ for odd and even $q$, respectively, where $s_{j} \neq 1$ is a representative of a nonregular semisimple and $u_{j} \neq 1$ is a representative of a unipotent conjugacy class of $G_{o}$. These mixed conjugacy classes are given in Table 2.4.

Furthermore, if $k(G_{o})$ denotes the number of all conjugacy classes of $G_{o}$, then

$$k(G_{o}) = \begin{cases} 
q^{4} + q^{3} + q^{2} + q + 6, & \text{if } q \text{ is odd}, \\
q^{4} + q^{3} + q^{2} + q + 5, & \text{if } q \text{ is even}.
\end{cases}$$

<table>
<thead>
<tr>
<th>ss</th>
<th>unipotent classes of $C_{G}(s_{i})$</th>
<th>Number of mixed classes $(s_{j}u_{j})G_{o}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_{2}$</td>
<td>$u_{1}, u_{2}, u_{3}, u_{4}$</td>
<td>4</td>
</tr>
<tr>
<td>$s_{3}$</td>
<td>$u_{2}$</td>
<td>$\frac{1}{2}(q - 3)$</td>
</tr>
<tr>
<td>$s_{4}$</td>
<td>$u_{1}, u_{3}$</td>
<td>$q^{2} + q$</td>
</tr>
<tr>
<td>$s_{5}$</td>
<td>$u_{1}$</td>
<td>$\frac{1}{2}(q^{3} - q^{2} - q - 3)$</td>
</tr>
<tr>
<td>$s_{7}$</td>
<td>$u_{2}$</td>
<td>$\frac{1}{2}(q - 1)$</td>
</tr>
<tr>
<td>$s_{9}$</td>
<td>$u_{1}, u_{4}$</td>
<td>$q^{2} - q$</td>
</tr>
<tr>
<td>$s_{10}$</td>
<td>$u_{1}$</td>
<td>$\frac{1}{2}(q^{3} - q^{2} + q - 1)$</td>
</tr>
</tbody>
</table>

3. Deligne-Lusztig characters. In this section we determine the values of the Deligne-Lusztig characters of $G_{o} = D_{4}(q)$. Concerning the definition and the main properties of these class functions we refer to Carter [2] and Lusztig [11].

Let $T_{0}$ be a maximally split torus of the connected reductive group $G$, $X$ its character group, and $V = X \otimes \mathbb{R}$. Then $\sigma = q\tau$ acts on $V$. The relative rank $\text{rel rank } G$ of $G$ is the number of eigenvalues of $\sigma$ on $V$ which are equal to $q$; see Carter [2].

Definition. $\varepsilon_{G} = (-1)^{\text{rel rank } G}$.

By Corollary 6.5.7 of [2], $\varepsilon_{G} = \varepsilon_{T_{0}} = 1$ in our case $G_{o} = D_{4}(q)$.

Lemma 3.1. Let $s \neq 1$ be a semisimple element of $G_{o} = D_{4}(q)$. Then its centralizer $C_{G}(s)$ has sign $\varepsilon_{c_{G}(s)}$ which is given by Table B.
D. I. DERIZIOTIS AND G. O. MICHLER

Proof. This follows easily from Proposition 2.2 and Corollary 6.5.7 of [2].

Let $s$ be a semisimple element and $T$ a maximal torus of $G_0$. Then as in group theory we write $s \in G_0 T$, if $s^g \in T$ for some $g \in G_0$. In the following, $C(s)$ denotes the centralizer $C_{G_0}(s)$. If $\Theta$ is a linear character of $T$, then $R_{T,\Theta}$ is the corresponding Deligne-Lusztig character of $G_0$. For any unipotent element $u \in G_0$ the Green function $Q_T$ has value $Q_T(u) = R_{T,1}(u)$. If it is necessary to indicate the ambient group we also write $Q_T^G(u)$ and $R_{T,\Theta}^G$.

For the sake of completeness the following subsidiary result is given.

**Lemma 3.2.** Let $T$ be a maximal torus of $G_0 = 3D_4(q)$ with Weyl group $W_T$. Then for every linear character $\Theta$ of $T$ and every $x = su \in G_0$ in Jordan form the Deligne-Lusztig character $R_{T,\Theta}^G$ has value

$$R_{T,\Theta}^G(x) = \begin{cases} \frac{\epsilon_{C(s)} \epsilon_{T}|C(s)|}{|T|} \hat{\Theta}(s) & \text{if } u = 1 \text{ and } s \in G_0 T, \\ \hat{\Theta}(s) Q_T^C(s)(u) & \text{if } u \neq 1 \text{ and } s \in G_0 T, \\ 0 & \text{if } s \notin G_0 T, \end{cases}$$

where

$$\hat{\Theta}(s) = \frac{1}{|C_{W_T}(s)|} \sum_{w \in W_T} \Theta(wsw^{-1}).$$

With the notation of Propositions 2.2 and 2.3 we state

**Lemma 3.3.** Let $s \neq 1$ be a semisimple element of $G_0 = 3D_4(q)$ and $u$ a unipotent element of $C(s)$. Let $T$ be a maximal torus of $G_0$ contained in $C(s)$. Then the values $Q_T(u)$ of the Green functions of $C(s)$ are given by

(a) $Q_T(u) = 1$, if $s \in (s_i)_{G_0}$ and $i \in \{3, 5, 7, 10\}$.

(b)

<table>
<thead>
<tr>
<th>$C(s_2)$</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_{T_0}$</td>
<td>$q + 1$</td>
<td>$q^3 + 1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$Q_{T_1}$</td>
<td>$1 - q$</td>
<td>$q^3 + 1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$Q_{T_5}$</td>
<td>$q + 1$</td>
<td>$1 - q^3$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$Q_{T_6}$</td>
<td>$1 - q$</td>
<td>$1 - q^3$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

(c)

<table>
<thead>
<tr>
<th>$C(s_4)$</th>
<th>$u_1$</th>
<th>$u_3$</th>
<th>$C(s_9)$</th>
<th>$u_1$</th>
<th>$u_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_{T_0}$</td>
<td>$1 + 2q$</td>
<td>$1$</td>
<td>$Q_{T_2}$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$Q_{T_1}$</td>
<td>$1$</td>
<td>$1$</td>
<td>$Q_{T_6}$</td>
<td>$q + 1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$Q_{T_2}$</td>
<td>$1 - q$</td>
<td>$1$</td>
<td>$Q_{T}$</td>
<td>$1 - 2q$</td>
<td>$1$</td>
</tr>
</tbody>
</table>
Proof. Every Green function $Q_T$ is a linear combination of the unipotent characters of $C(s)$. Using then the character tables of $SL_2(q)$, $SL_3(q)$, $SU_3(q)$ of [6 and 13] it is easy to compute the given values of $Q_T(u)$, because Proposition 2.2 gives the group structure of $C(s)$.

With the notation of Propositions 1.2 and 2.1 we state the following result.

Lemma 3.4. Let $q \neq 2$ and $s$ be a semisimple element. Then for $0 \leq j \leq 6$, the centralizer $C_{W_j}(s)$ of $s$ in the Weyl group $W_j$ is as given in Table 3.4.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$C_{W(T_2)}(s)$</th>
<th>$C_{W(T_3)}(s)$</th>
<th>$C_{W(T_4)}(s)$</th>
<th>$C_{W(T_5)}(s)$</th>
<th>$C_{W(T_6)}(s)$</th>
<th>$C_{W(T_7)}(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$W(G_2) = D_{12}$</td>
<td>$Z_2 \times Z_2$</td>
<td>$Z_2 \times Z_2$</td>
<td>$Q_8 \cdot Z_3$</td>
<td>$Q_8 \cdot Z_3$</td>
<td>$Z_4$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>$Z_2 \times Z_2$</td>
<td>$Z_2 \times Z_2$</td>
<td>$Z_2 \times Z_2$</td>
<td>$Z_2 \times Z_2$</td>
<td>$Z_2 \times Z_2$</td>
<td>$Z_2 \times Z_2$</td>
</tr>
<tr>
<td>$s_3$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
</tr>
<tr>
<td>$s_4$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
</tr>
<tr>
<td>$s_5$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
</tr>
<tr>
<td>$s_6$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
</tr>
<tr>
<td>$s_7$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
</tr>
<tr>
<td>$s_8$</td>
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<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
</tr>
<tr>
<td>$s_9$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
</tr>
<tr>
<td>$s_{10}$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
</tr>
<tr>
<td>$s_{11}$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
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<td>$Z_2$</td>
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</tr>
<tr>
<td>$s_{12}$</td>
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</tr>
<tr>
<td>$s_{13}$</td>
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<td>$Z_2$</td>
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</tr>
<tr>
<td>$s_{14}$</td>
<td>$Z_2$</td>
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</tr>
<tr>
<td>$s_{15}$</td>
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<td>$Z_2$</td>
<td>$Z_2$</td>
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<td>$Z_2$</td>
</tr>
</tbody>
</table>

In the following subsidiary result $H \rtimes U$ denotes the semidirect product of the normal subgroup $H$ with the subgroup $U$ of the finite group $X$.

Lemma 3.5. Let $q = 2$. Let $s$ be a semisimple element and $W_j = N_{G_2}(T_j)/T_j$, $0 \leq j \leq 6$. Then the centralizer $C_{W_j}(s)$ of $s$ in $W_j$ is as given in Table 3.5.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$C_{W(T_2)}(s)$</th>
<th>$C_{W(T_3)}(s)$</th>
<th>$C_{W(T_4)}(s)$</th>
<th>$C_{W(T_5)}(s)$</th>
<th>$C_{W(T_6)}(s)$</th>
<th>$C_{W(T_7)}(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$SL_3(2) \rtimes Z_2$</td>
<td>$Z_2 \times Z_2$</td>
<td>$Q_8 \cdot Z_3$</td>
<td>$Q_8 \cdot Z_3$</td>
<td>$Z_4$</td>
<td>$W(G_2) = D_{12}$</td>
</tr>
<tr>
<td>$s_4$</td>
<td>SL_3(2)</td>
<td>$Z_2 \times Z_2$</td>
<td>$Z_2 \times Z_2$</td>
<td>$Z_2 \times Z_2$</td>
<td>$Z_2 \times Z_2$</td>
<td></td>
</tr>
<tr>
<td>$s_6$</td>
<td>SL_3(2)</td>
<td>$Z_2 \times Z_2$</td>
<td>$Z_2 \times Z_2$</td>
<td>$Z_2 \times Z_2$</td>
<td>$Z_2 \times Z_2$</td>
<td></td>
</tr>
<tr>
<td>$s_7$</td>
<td>$Z_2$</td>
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<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
</tr>
<tr>
<td>$s_8$</td>
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<tr>
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<tr>
<td>$s_{10}$</td>
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<td>$Z_2$</td>
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</tr>
<tr>
<td>$s_{12}$</td>
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<td>$Z_2$</td>
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<tr>
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<td>$Z_2$</td>
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<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
</tr>
</tbody>
</table>

Proof. Since no root vanishes on the tori $T_1$, $T_3$, $T_4$, $T_5$, and $T_6$ the proof of Lemma 3.4 remains valid for these cases by Veldkamp’s theorem [15].

By Proposition 2.2 $H = C_{G_2}(T_0) = SL_3(2) \rtimes Z_7$. Propositions 1.2 and 2.1 imply that $N_{G_2}(T_0)/H = Z_2$. Thus $N_{G_2}(T_0) = SL_3(2) \rtimes Z_2$. The remaining cases are proved similarly.
In order to give the values of the Deligne-Lusztig characters $R_{T, \Theta}$ we introduce the following

**Notation.** For a $\sigma$-stable maximal torus $T$ of $G$ we fix an isomorphism $T_\sigma \cong \hat{T}_\sigma = \text{Hom}(T_\sigma, \mathbb{C}^*)$. The linear character of $T_\sigma$ corresponding to $s \in T_\sigma$ under this isomorphism will be denoted by $\hat{s}$.

Let $T_j$ be a maximal torus of $G_\sigma$, $0 \leq j \leq 6$, and $s_i \in T_j$ be a representative of a semisimple conjugacy class of $G_\sigma$, where $i \in \{2, 3, \ldots, 15\}$.

### Table 3.6. Deligne-Lusztig characters

<table>
<thead>
<tr>
<th>$s_2$</th>
<th>$s_2 u_1$</th>
<th>$s_2 u_2$</th>
<th>$s_2 u_3$</th>
<th>$s_2 u_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{0,i}$</td>
<td>$(q^3 + 1)(q + 1)\eta_0(s_2)$</td>
<td>$(q + 1)\eta_0(s_2)$</td>
<td>$(q^3 + 1)\eta_0(s_2)$</td>
<td>$\eta_0(s_2)$</td>
</tr>
<tr>
<td>$R_{1,i}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$R_{2,i}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$R_{3,i}$</td>
<td>$0$</td>
<td>$0$</td>
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</tr>
<tr>
<td>$R_{4,i}$</td>
<td>$0$</td>
<td>$0$</td>
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</tr>
<tr>
<td>$R_{5,i}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$R_{6,i}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$s_3$</th>
<th>$s_3 u_2$</th>
<th>$s_4$</th>
<th>$s_4 u_1$</th>
<th>$s_4 u_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{0,i}$</td>
<td>$(q^3 + 1)\eta_0(s_3)$</td>
<td>$\eta_0(s_3)$</td>
<td>$(q + 1)(q^2 + q + 1)\eta_0(s_4)$</td>
<td>$(q + 1)\eta_0(s_4)$</td>
</tr>
<tr>
<td>$R_{1,i}$</td>
<td>$0$</td>
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<td>$0$</td>
</tr>
<tr>
<td>$R_{2,i}$</td>
<td>$0$</td>
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</tr>
<tr>
<td>$R_{3,i}$</td>
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<tr>
<td>$R_{4,i}$</td>
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<td>$R_{5,i}$</td>
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<tr>
<td>$R_{6,i}$</td>
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</tr>
</tbody>
</table>

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$G^*$ is isomorphic to the fixed points $G_{^*}$ of the triality endomorphism $\sigma^*$ of the adjoint group $G^*$ of type $D_4$, which is dual to $G$; see [2, p. 112]. As $G^*$ has a connected center, Lusztig has shown in [11] that there is a bijective map $\chi \mapsto (\chi_3, \chi_u)$ between the irreducible characters $\chi$ of $G^*$ and pairs $(\chi_3, \chi_u)$, where $\chi_3$ is a semisimple character of $G^*$ and $\chi_u$ is a unipotent character of the centralizer $C(s)$ of $s \in W$.

We set $R_{j,i} = R_{T_j,3}$, and

$$N_{j,i}(s) = \frac{1}{|C_{W_j}(s)|} \sum_{w \in W_j} \hat{s}_i(s^w),$$

where $W_j$ is the Weyl group of $T_j$. 


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a semisimple element \( s \in G \). Furthermore, this bijection satisfies the following conditions:

\[
(4.1) \quad \chi(1) = \chi_s(1)\chi_u(1),
\]

\[
(4.2) \quad (\chi, \varepsilon_T R_{T, \theta})_{C_n} = (\chi_u, \varepsilon_{C(s)} T R_{T,s})_{C(s)},
\]

because \( \varepsilon_G = 1 \) by Corollary 6.5.7 of [2].

The unipotent irreducible characters of \( SL_2(q') \) are the trivial character 1 and the Steinberg character \( St \). Following the notation of Simpson’s and Frame’s [13] character tables, besides 1 and \( St \) the groups \( SL_3(q) \) and \( SU_3(q) \) each have another unipotent irreducible character denoted by \( \chi_{qs} \) and \( \chi_{qs'} \), respectively, where \( s = q + 1 \) and \( s' = q - 1 \).

Using now the notation of Proposition 2.1 for the semisimple conjugacy classes \( s_i \neq 1, 2 \leq i \leq 15 \), and the structure of the centralizer \( C(s_i) \) given in Proposition 2.2, every irreducible character \( \chi = (\chi_s, \chi_u) \) of \( G_a = D_4(q) \) which is not unipotent can (up to conjugation) be uniquely denoted by

\[
\chi = \begin{cases} 
\chi_i, & \text{if } \chi = (\chi_i, \emptyset), \text{ and } s_i \text{ is regular} \\
\chi_{i,1}, & \text{if } \chi = (\chi_{s_i}, 1), \text{ and } s_i \neq 1 \text{ is not regular} \\
\chi_{i,St}, & \text{if } \chi = (\chi_{s_i}, St) \\
\chi_{i,qs}, & \text{if } \chi = (\chi_{s_i}, \chi_{qs}) \\
\chi_{i,qs'}, & \text{if } \chi = (\chi_{s_i}, \chi_{qs'}) \\
\chi_{i,StSt'}, & \text{if } i = 2 \text{ and } St, St' \text{ denote the Steinberg characters of } SL_2(q^3), SL_2(q), \text{ respectively.}
\end{cases}
\]

We keep Spaltenstein’s [14] notation of the unipotent irreducible characters of \( G_a \). Their values are given in [14].

Therefore the following result and the table of the Deligne-Lusztig characters complete the character table of \( G_a \).

**Theorem 4.3.** With the values of the Deligne-Lusztig characters \( R_{i,j} \) given in Table 3.6, the values of the nonunipotent irreducible characters \( \chi \) of \( G_a \) are determined as follows.

(a) \[
\begin{align*}
\chi_{2,1} &= \frac{1}{4} (R_{0,2} + R_{2,2} + R_{1,2} + R_{6,2}) \\
\chi_{2,St} &= \frac{1}{4} (R_{0,2} - R_{2,2} + R_{1,2} - R_{6,2}) \\
\chi_{2,St'} &= \frac{1}{4} (R_{0,2} + R_{2,2} - R_{1,2} - R_{6,2}) \\
\chi_{2,StSt'} &= \frac{1}{4} (R_{0,2} - R_{2,2} - R_{1,2} + R_{6,2})
\end{align*}
\]

(b) \[
\begin{align*}
\chi_{3,1} &= \frac{1}{2} (R_{0,3} + R_{2,3}) \\
\chi_{3,St} &= \frac{1}{2} (R_{0,3} - R_{2,3})
\end{align*}
\]
(c) \[\begin{align*}
\chi_{4,1} &= \frac{1}{6} (R_{0,4} + 3R_{1,4} + 2R_{3,4}) \\
\chi_{4,St} &= \frac{1}{6} (R_{0,4} - 3R_{1,4} + 2R_{3,4}) \\
\chi_{4,qS} &= \frac{1}{3} (R_{0,4} - R_{3,4})
\end{align*}\]

(d) \[\begin{align*}
\chi_{5,1} &= \frac{1}{2} (R_{0,5} + R_{1,5}) \\
\chi_{5,St} &= \frac{1}{2} (R_{0,5} - R_{1,5})
\end{align*}\]

(e) \[\chi_{6} = R_{0,6}\]

(f) \[\begin{align*}
\chi_{7,1} &= -\frac{1}{2} (R_{1,7} + R_{6,7}) \\
\chi_{7,St} &= -\frac{1}{2} (R_{1,7} - R_{6,7})
\end{align*}\]

(g) \[\chi_{8} = -R_{1,8}\]

(h) \[\begin{align*}
\chi_{9,1} &= -\frac{1}{6} (R_{6,9} + 3R_{2,9} + 2R_{4,9}) \\
\chi_{9,St} &= \frac{1}{6} (R_{6,9} - 3R_{2,9} + 2R_{4,9}) \\
\chi_{9,qS} &= -\frac{1}{3} (R_{6,9} - R_{4,9})
\end{align*}\]

(i) \[\begin{align*}
\chi_{10,1} &= -\frac{1}{2} (R_{2,10} + R_{6,10}) \\
\chi_{10,St} &= -\frac{1}{2} (R_{2,10} - R_{6,10})
\end{align*}\]

(j) \[\chi_{11} = -R_{2,11}\]

(k) \[\chi_{12} = R_{3,12}\]

(l) \[\chi_{13} = R_{4,13}\]

(m) \[\chi_{14} = R_{5,14}\]

(n) \[\chi_{15} = R_{6,15}\]

**Proof.** Let \(C(s_2)\) be the centralizer of the unique involution \(s_2 \neq 1\) in case \(q\) is odd. Its commutator subgroup \(C(s_2)' = \text{SL}_2(q^3) \rtimes \text{SL}_2(q)\) by Proposition 2.2. Since the central involution \(s_2\) of \(C(s_2)\) is in the kernel of each unipotent irreducible character of \(C(s_2)\)' the Green functions of \(C(s_2)\) are given by

\[R_{T_0,1}^{C(s_2)} = (1 + St) \otimes (1 + St') = 1 \otimes 1 + St \otimes 1 + 1 \otimes St' + St \otimes St',\]

\[R_{T_1,1}^{C(s_2)} = (1 + St) \otimes (1 - St') = 1 \otimes 1 + St \otimes 1 - 1 \otimes St' - St \otimes St',\]
\[ R_{T_2,1}^{C(s_2)} = (1 - St) \otimes (1 + St') \]
\[ = 1 \otimes 1 - St \otimes 1 + 1 \otimes St' - St \otimes St'. \]
\[ R_{T_3,1}^{C(s_3)} = (1 - St) \otimes (1 - St') \]
\[ = 1 \otimes 1 - St \otimes 1 - 1 \otimes St' + St \otimes St'. \]

By Lemma 3.1 \( \epsilon_{C(s_2)} = 1 \). Therefore we obtain from (4.2) the equations
\[ R_{0,2} = X_{2,1} + X_{2,St} + X_{2,St'} \]
\[ R_{1,2} = X_{2,1} - X_{2,St} - X_{2,St'} \]
\[ R_{2,2} = X_{2,1} - X_{2,St} + X_{2,St'} \]
\[ R_{6,2} = X_{2,1} + X_{2,St} + X_{2,St'} - X_{2,StSt'}. \]

This system of linear equations has the unique solution given in assertion (a).

The unipotent irreducible characters of \( \text{SL}_3(q) \) are 1, \( St \), and \( x_{q^2} \). By Simpson and Frame [13, p. 492], and Proposition 2.2 they extend uniquely to unipotent irreducible characters of \( C(s_4) \). Therefore the Green functions of \( C(s_4) \) are given by
\[ R_{0,1} = 1 + 2x_{q^2} + St, \]
\[ R_{1,1} = 1 - St, \]
\[ R_{3,1} = 1 - x_{q^2} + St. \]

By Lemma 3.1 \( \epsilon_{C(s_4)} = 1 \). Therefore we obtain from (4.2) the equations
\[ R_{0,4} = X_{4,1} + 2x_{4,q^2} + X_{4,St'}, \]
\[ R_{1,4} = X_{4,1} - X_{4,St'}, \]
\[ R_{3,4} = X_{4,1} - X_{4,q^2} + X_{4,St'}. \]

This system of linear equations has the unique solution given in (c). Using the same methods we obtain the assertions (h), (b), (d), (f) and (i).

By Corollary 7.3.5 of [2] \( \epsilon_{C(s_i)} R_{T_i,i} \) is irreducible, if \( s_i \) is a regular element of \( G_0 \). This completes the proof.

Proof of Table 4.4. The classification of the irreducible characters \( \chi_{s_i,u} \) follows from (4.1) and Proposition 2.2. Their degrees are computed by means of (4.1) and Theorem 8.4.8 of [2].

The numbers of irreducible characters in each family \( \chi_{s_i,u} \) equals the number \( N(E_i, [w]) \) of conjugacy classes of the semisimple elements \( s_i \) defined in Proposition 2.1. Let \( w \in \Omega_{J_i} \) and
\[ T(w, J_i) = \{ t \in T \mid w \sigma(t) = t, C_w(t) = W_i \}. \]
Define \( Z_{J_i}(w) = \{ y \in \Omega_{J_i} \mid y^{-1}w \sigma(y) = w \} \). Then Lemma 3.1 of [5] asserts that \( N(E_i, [w]) = |T(w, J_i)|/|Z_{J_i}(w)| \). As \( G \) is simply connected, Theorem 3.3 of [5] applies. Therefore \( |T(w, J_i)| = F_{w,J_i}(q) \), where \( F_{w,J_i}(X) \) is a polynomial with integral coefficients and degree \( \deg(F_{w,J_i}(X)) = 4 - |J_i| \). In particular, \( \deg(F_{1,0}(X)) = 4 \).
Using properties of the Brauer complex of $G$ in [5] the first author introduced a method for finding the polynomial $F_{w_0,j}(X)$ for fixed $w_0 \in \Omega_j$ and $J_j$. As an example we employ this method for the computation of $F_{x,0}(X)$, i.e., $w_0 = 1 \in \Omega_j = W$.

The $\sigma$-conjugacy class [1] of 1 in $\Omega_0 = W$ consists of 16 elements $w_0 \in W$ by Proposition 2.1. Let $n = n(w)$ be the order of $w \tau^{-1}$. Since $0 \not\in J_5 = \emptyset$ case (A) of [5] applies. Thus we have to find the number $k(1, \emptyset, q)$ of all $y \in \Omega \subseteq \text{Hom}(K^*, T)$ satisfying the following 16 systems of inequalities:

\[
\sum_{i=0}^{n-1} q^i(w \tau^{-1})^{i+1}(r_j)(\tau^{-1}(y)) > 0 \quad \text{for} \quad j = 1, 2, 3, 4, \quad \sum_{i=0}^{n-1} q^i(w \tau^{-1})^{i+1}(r_0)(\tau^{-1}(y)) < q^n - 1.
\]

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By [5] \( k(1, \varnothing, q) = F_{1, \varnothing}(q) \). Since \( F_{1, \varnothing}(X) \) is an integral polynomial of degree 4, its coefficients are easily found by interpolation, if these numbers \( k(1, \varnothing, q) \) can be determined for five different choices of \( q \). In case \( q \) is odd, we get the following numbers:

<table>
<thead>
<tr>
<th>( q )</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_{1, \varnothing}(q) )</td>
<td>0</td>
<td>15</td>
<td>94</td>
<td>317</td>
<td>796</td>
</tr>
</tbody>
</table>

Thus interpolation yields that \( F_{1, \varnothing}(X) = q^4 - 4q^3 + 2q^2 - 2q + 15 \).

As \( |Z_\varnothing(1)| = |Z_{J_5}(1)| = |W_6| = 12 \), it follows that

\[
N(E_5, 1) = \frac{1}{12}(q^4 - 4q^3 + 2q^2 + 15),
\]

which is the number of regular conjugacy classes of \( G_a \) intersecting \( T_0 \) nontrivially.

For even \( q \), we interpolate at \( q = 2, 4, 8, 16 \) and 32. Then the same method applies here as in all other remaining cases.

5. The blocks of irreducible characters. Let \( r > 0 \) be a prime number. In this section we determine the distribution of the irreducible characters \( \chi_{s,u} \) of \( G_a = 3D_4(q) \) into \( r \)-blocks. As an application we then obtain the validity of R. Brauer’s height zero conjecture, his conjecture on the number of irreducible characters in a block, and the Alperin-McKay conjecture for this class of simple groups.

Let \( R \) be a complete discrete rank one valuation ring with maximal ideal \( \text{max}(R) = \pi R \), residue class field \( F = R/\pi R \) of characteristic \( r > 0 \), and quotient field \( S = \text{quot}(R) \) of characteristic 0 such that \( S \) and \( F \) are splitting fields for the finite group \( G \). The block ideals of the \( r \)-block \( B \) of \( G \) in the group algebras \( FG, RG \) and \( SG \) are denoted by \( B, \hat{B} \) and \( B_S = \hat{B} \otimes_R S \) respectively. In particular, \( B = \hat{B} \otimes_R F \). The number of simple \( SG \)-modules of \( B_S \) is denoted by \( k(B) \), and \( k_0(B) \) is the number of irreducible characters \( \chi \) of \( G_a \) belonging to \( B \) with height \( h \chi = 0 \).

The number of irreducible modular characters of \( B \) is \( 1(B) \).

Let \( B \) be an \( r \)-block of a finite group \( G \) with defect group \( \delta(B) = GD \). Let \( H = N_G(D) \) and \( C = DC_G(D) \). By Brauer’s first main theorem there is a unique block \( B_1 \) of \( H \) with defect group \( \delta(B_1) = D \) such that \( B = B_1^G \); it is called the Brauer correspondent of \( B \) in \( H \).

The Alperin-McKay conjecture claims that \( k_0(B) = k_0(B_1) \). Brauer conjectured that in general \( k(B) \leq |D| \), and his height zero conjecture says that \( k_0(B) = k(B) \) if and only if \( \delta(B) \) is abelian.

Let \( B \) be an \( r \)-block of a finite group \( G \) with defect group \( \delta(B) = GD \). Then by Brauer’s extended first main theorem there is a block \( b \) of \( C = DC_G(D) \) with defect group \( D \) such that \( B = b^G \). Any such block \( b \) of \( C \) is called a root of \( B \). By Corollary 4.6 of [7, p. 204], \( b \) contains exactly one irreducible character \( \chi_s \) which has \( D \) in its kernel. This character \( \chi_s \) is called the canonical character of the block \( B \). If \( H = N_G(D) \), then \( \chi_s \) is uniquely determined by \( B \) up to \( H \)-conjugacy. The inertial subgroup \( T_H(b) = \{ x \in H \mid b^x = b \} = T_H(\chi_s) = \{ x \in H \mid \chi_s^x = \chi_s \} \).

By Theorem 4.3 every irreducible character \( \chi \) of \( G_a = 3D_4(q) \) is of the form \( \chi_{t,u} \), where \( t \) is a representative of a semisimple conjugacy class of \( G_a \), and where \( \chi_u \) is an...
irreducible unipotent character of $C_{G_a}(t)$. We now study the distribution of the irreducible characters of $G_a$ into $r$-blocks $B$ of $G_a$. Such a block $B$ is called unipotent, if $B$ contains a unipotent character of $G_a$.

**Corollary 5.1.** (a) For every prime number $r > 0$, $G_a = ^3D_4(q)$ contains $r$-blocks of defect zero.

(b) If $r \neq 2$, then $G_a$ contains unipotent $r$-blocks of defect zero.

**Proof.** Let $r^a$ be the order of a Sylow $r$-subgroup of $G_a$. Then by Lemma 4.19 of [7, p. 159], an irreducible character $\chi$ of $G_a$ belongs to an $r$-block $B$ with defect $d(B) = 0$ if and only if $r^a | \chi(1)$. Hence (b) follows immediately from Table 4.4.

If $r | q$, then (a) holds by Steinberg’s tensor product theorem. Let $r \neq q$. From (b) it follows that we may assume that $r = 2$. Then by Table 4.4 the $\frac{1}{4}(q^4 - q^2)$ irreducible characters $\chi_{14}$ yield 2-blocks of defect zero.

**Lemma 5.2.** Let $r$ be a prime number, $r \not\in \{2, 3, p\}$. If $D \cong I$ is a defect group of an $r$-block $B$ of $G_a$, then either $D$ is cyclic or a Sylow $r$-subgroup which is abelian and generated by 2 elements.

**Proof.** As $r \not\in \{2, 3, p\}$, $G_a$ has an abelian Sylow $r$-subgroup $S$ by Corollary 5.19 of [1, p. 212]. Proposition 1.2 asserts that $S$ has at most two generators.

If $D$ is not cyclic, then by Theorem 9.2 of [7, p. 231], there is a central element $1 \neq x \in D$ and an $r$-block $b$ of $C_{G_a}(x)$ such that $B = b^G$ and both blocks $B$ and $b$ have defect group $D$.

Using the character tables of $SL_2(q)$, $SL_3(q)$, and $SU_3(q)$ given in [6 and 13] it is easy to see that each $r$-block $b_1$ of any of these groups has a Sylow $r$-subgroup as defect group $\delta(b_1)$, if $\delta(b_1)$ is not cyclic. Therefore Proposition 2.2 implies that $\delta(b) = g_a D$ is a Sylow $r$-subgroup of $G_a$. This completes the proof.

**Proposition 5.3.** Let $q$ be odd, and let $2^a$ be the highest power of 2 dividing $q - 1$ or $q + 1$ if $q \equiv 1(4)$ or $q \equiv 3(4)$, respectively. Let $Q$ be a Sylow 2-subgroup of $SL_2(q)$ contained in a Sylow 2-subgroup $P$ of $G_a$, and let $Z$ be a cyclic 2-subgroup of $Q$ of order $|Z| = 2^a$. Then $P$ contains an involution $x$ such that $S = \{Q, x\}$ is a semidihedral subgroup of $P$ of order $|S| = 2^{a+2}$.

If $B$ is a 2-block of $G_a$ with defect group $D \neq 1$, then one of the following holds:

(a) $D = g_a P$ if and only if $B = B_0$ the principal 2-block of $G_a$.

(b) $D = S \ast Z$.

(c) $D = S$.

(d) $D = Z \times Z$.

(e) $D$ is isomorphic to a Klein four subgroup of $P$.

(f) $D$ is a cyclic Sylow 2-subgroup of a cyclic maximal torus of $G_a$.

**Proof.** $G_a$ has only one class of involutions $s_2 \neq 1$ by Proposition 2.1. Let $C = C_{G_a}(s_2)$. Then by Proposition 2.2 $C'$ is the central product $SL_2(q^2) \ast SL_2(q)$. Hence a Sylow 2-subgroup $P$ of $G_a$ contains a central product of two isomorphic generalized quaternion groups $Q$, and $|P : Q \ast Q| = 2$. 

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By Proposition 2.2 the defect group \( D \) of \( B \) may be chosen such that \( s_2 \in Z(D) \). Thus \( K = DC_{G_o}(D) \leq G \cdot C \). Let \( b \) be a root of \( B \) in \( K \). Then \( \delta(b) = K \cdot D \) and \( B = b^{G_o} \). Furthermore, \( B_1 = b^e \) exists by Lemma 6.1 of [7, p. 209], and \( \delta(B_1) = D \), because \( B = B^{G_o} = (b^e)^G_o = b^{G_o} \).

By the character table of \( U = SL_2(q) \) [6, p. 228], we know that the principal 2-block \( B_0(U) \) of \( U \) is the only 2-block of \( U \) with defect group \( Q \), and that all other 2-blocks of \( U \) have either the center \( Z(Q) \) of order \( |Z(Q)| = 2 \) or a cyclic group \( Z \) of order \( 2^a \geq 4 \) as a defect group. Up to isomorphism \( Q \) is also a Sylow 2-subgroup of \( SL_2(q) \). As \( C' = SL_2(q) \cdot SL_2(q^2) \), each block \( A \) of \( C' \) with defect group \( \delta(A) = E \) is mapped onto a block \( \tau(A) \) of \( C' \) such that \( \bar{C} = C'/\{s_2\} = PSL_2(q) \times PSL_2(q^2) \) with defect group \( \delta(\tau(A)) = E/\{s_2\} \). Hence \( E \) is isomorphic to one of the 2-subgroups \( Q \cdot Q, Q \cdot Z, Q, Z \cdot Z, Z \), or \( Z(Q) \). Since \( |C: C'| = 2 \) Green’s theorem [7, p. 107] implies that every 2-block \( A \) of \( C' \) induces up to a 2-block \( B' = A^C \) of \( C \). By Theorem 3.14 of [7, p. 201] \( \delta(B') \cap C' = \delta(A) = E \). Using now the group structure of \( P \) and \( C \) the assertions (b)-(f) follow.

Furthermore, by Brauer’s third main theorem the principal 2-block \( B_0 \) of \( G \) is the only 2-block of highest defect.

**Proposition 5.4.** Let \( 3 \nmid q \). If \( B \) is a 3-block of \( G_o \) with defect group \( D \), then one of the following statements holds:

(a) \( D \) is nonabelian if and only if \( D \) is a Sylow 3-subgroup of \( G_o \).

(b) \( D \) is a noncyclic Sylow 3-subgroup of a maximal torus of \( G_o \).

(c) \( D \) is a cyclic Sylow 3-subgroup of a cyclic maximal torus of \( G_o \).

**Proof.** (a) Let \( B \) be a 3-block of \( G_o \) with a nonabelian defect group \( D \). Either \( 3 \mid q - 1 \) or \( 3 \mid q + 1 \). Suppose that \( 3 \mid q - 1 \). By Theorem 9.2 of [7, p. 231], there is a central element \( 1 \neq x \in D \) of order 3 and an \( r \)-block \( b \) of \( C = G_o(x) \) such that \( B = b^{G_o} \), and \( D \) is a defect group of \( b \). By Proposition 2.2, \( C' = SL_3(q) \cdot Z \), where \( Z \) is a cyclic group with \( |Z| = q^2 + q + 1 \), and where the cyclic group \( C/C' \) of order 3 acts trivially on \( Z \). By the character table of [13, p. 487], only the principal 3-block of \( U = SL_3(q) \) has a Sylow 3-subgroup \( D_1 \) of \( U \) as a defect group. Since \( 3 \) divides \( |Z| \) only to the first power, \( C' \) has \( \frac{1}{2}(q^2 + q + 1) \) blocks \( b_1 \) with defect group \( D_1 \), and all other blocks of \( C' \) have abelian or cyclic defect groups. As \( C/C' \) acts trivially on \( Z \), it follows from Theorem 3.14 of [7, p. 201], that each 3-block \( (b_1)^C \) of \( C \) has a Sylow 3-subgroup \( P \) of \( C \) and thus of \( G_o \) as a defect group. By Proposition 2.2 all other blocks \( b_2 \) of \( C \) have abelian or cyclic defect groups. Hence \( D = c_nP \) and \( b \in \{(b_1)^C\} \) by Lemma 9.1 of [7, p. 230].

Because of Proposition 2.2 the same argument can also be applied in the case of \( 3 \mid q + 1 \). The converse implication is trivial.

So we may assume that \( D \) is abelian. The assertions (b) and (c) follow from Propositions 1.2 and 2.2 and the definition of a defect group, see [7, pp. 126 and 231].

**Lemma 5.5.** Let \( T_o \) be a maximal torus and \( D \) a Sylow \( r \)-subgroup of \( G_o \) contained in \( T \) where \( r \nmid p \). Let \( s \in T_o \) be an \( r \)-element of \( T_o \) and let \( \bar{s} \) denote the linear character
of $T$ corresponding to $s$. Then for every $y \in D$ the Deligne-Lusztig characters $R_{T,s}$ agree on all $r'$-elements $x \in G_o$.

**Proof.** Let the $r'$-element $x = tu \in G_o$ be in Jordan form, where $t$ is semisimple and $u$ is unipotent. By Lemma 3.2

$$R^G_{T,s}(x) = \begin{cases} \varepsilon_{C(t)} \varepsilon_T \frac{C(t)}{|T|} \overline{\tilde{sy}(t)}, & \text{if } u = 1, \text{ and } t \in G_o T, \\ \overline{\tilde{sy}(t)} Q_f(t)(u), & \text{if } u \neq 1, \text{ and } t \in G_o T, \\ 0, & \text{if } t \notin G_o T, \end{cases}$$

where

$$\tilde{sy}(t) = \frac{1}{|C_{W(T)}(t)|} \sum_{w \in W(T)} \tilde{sy}(w t w^{-1}),$$

and $W(T) = N_{G_o}(T) / T$. As $t' = w t w^{-1} \in T$ is an $r'$-element for each $w \in W(T)$, $\tilde{y}(t') = 1$ for each $y \in D$, because $D$ is a Sylow $r$-subgroup of $T$. Therefore

$$\tilde{sy}(t) = \frac{1}{|C_{W(T)}(t)|} \sum_{w \in W(T)} \tilde{s}(w t w^{-1}) \tilde{y}(w t w^{-1})$$

$$= \frac{1}{|C_{W(T)}(t)|} \sum_{w \in W(T)} \tilde{s}(w t w^{-1}) = \tilde{s}(t).$$

Hence $R^G_{T,s}(x) = R^G_{T,i}(x)$.

**Proposition 5.6.** Let $B$ be an $r$-block of $G_o$ with a cyclic defect group $\delta(B) = G_o D \neq 1$. Then the following assertions hold:

(a) $D$ is a Sylow $r$-subgroup of a cyclic maximal torus $T$ of $G_o$ such that $D \leq T \leq C = C_{G_o}(D)$.

(b) $B$ is either the principal $r$-block of $G_o$ or $B$ determines (up to $G_o$-conjugacy) uniquely a regular $r'$-element $s$ of $G_o$ contained in $T$ such that $\theta = \varepsilon_c \varepsilon_T R^C_{T,s}$ is the canonical character of a root $b$ of $B$ in $C$.

(c) $B$ is the principal $r$-block of $G_o$ if and only if $D$ is the Sylow $r$-subgroup of the Coxeter torus $T_5$ of $G_o$, and $B$ has $\frac{1}{2}(|D| - 1)$ exceptional irreducible characters $\chi_y$ with $1 \neq y \in D$, and 4 nonexceptional irreducible characters which are the 4 unipotent characters $1, St, \rho_1$ and $\delta D_4[-1]$.

(d) If $B$ is a nonprincipal $r$-block of $G_o$, then an irreducible character $\chi_{t,u}$ of $G_o$ belongs to $B$ if and only if $t \sim_{G_o} sy$ for some $y \in D$, and $\chi_u$ is a unipotent irreducible character of $C_{G_o}(sy)$ such that $\overline{sy} \chi_u$ belongs to an $r$-block $\beta$ of $C_{G_o}(sy)$ with $B = \beta G$.

If $y \neq 1$, then $sy$ is regular in $G_o$ and $\chi_{t,u} = \chi_{xy}$ is an exceptional character of $B$. If $s$ is regular in $G_o$, then $\chi_s$ is the only nonexceptional character of $B$. Otherwise $\chi_{s,1}$ and $\chi_{s,St}$ are the nonexceptional characters of $B$.

**Proof.** By Humphreys’ theorem [10] $r \equiv q$· So, if $q = 2$, then $r \in \{3, 7, 13\}$. In each case $|D| = r$, and all assertions are easily verified by means of Propositions 1.2, 2.2 and Table 4.4. Thus we may assume also that $q \neq 2$. Let $e$ be the smallest integer such that $r | q^e - 1$. Then $e \in \{1, 2, 3, 6, 12\}$.
(a) By Propositions 5.3 and 5.4 we may assume that \( r \not\in \{2, 3\} \). Let \( C = C_{G_\alpha}(D) \).

By Dade’s theorem [7, p. 270], there is an \( r \)-block \( b \) of \( C \) with defect group \( D \) such that \( B = b^C \). Then by Lemma 5.2 and Corollary 5.19 of [1, p. 212], \( D \) is contained in a maximal torus \( T \) of \( G_\alpha \) such that \( T \leq C = C_{G_\alpha}(D) \). Let \( x \neq 1 \) be a generator of \( D \).

If \( x \) is regular in \( G_\alpha \), then \( C_{G_\alpha}(x) = T \), and \( b \) is an \( r \)-block of \( T \) with defect group \( D \). Hence \( D \) is a Sylow \( r \)-subgroup of \( T \). Thus \( T \) is a cyclic Coxeter torus \( T_5 \) of \( G_\alpha \) by Proposition 1.2.

Suppose that \( x \) is not regular. Let \( e = 1 \). Then by Proposition 2.2 there is an \( r \)-block \( b_1 \) of \( C_1 = C_{G_\alpha}(s_3) \) or of \( C_2 = C_{G_\alpha}(s_4) \) with defect group \( \delta(b_1) = D \), because \( r \neq 2 \), and \( x \in S_\alpha \cong \mathbb{Z}_{q-1} \) or \( x \in S_\alpha \cong \mathbb{Z}_{q^3-1} \). Since all \( r \)-blocks of \( SL_2(q^k) \), \( k \in \{3, 1\} \), either have a Sylow \( r \)-subgroup as a defect group or have defect zero, it follows that \( D \) is a Sylow \( r \)-subgroup of \( S_\alpha = \mathbb{Z}_{q-1} \) or of \( S_\alpha = \mathbb{Z}_{q^3-1} \). Therefore \( D \) is contained in the maximal torus \( T_2 = \mathbb{Z}_{(q^3+1)(q-1)} \) or \( T_1 = \mathbb{Z}_{(q^3-1)(q+1)} \) of \( C = C_{G_\alpha}(D) \). Hence \( D \) is a Sylow \( r \)-subgroup of the cyclic maximal torus \( T_2 \) or \( T_1 \).

Similarly one can show that \( D \) is the Sylow \( r \)-subgroup of \( T_1 \) or \( T_2 \) for \( e \in \{2, 3, 6\} \). Hence (a) holds.

Let \( e = 1 \) and \( T = T_2 \). Suppose that \( r \neq 2 \). Then \( C = C_{G_\alpha}(s_3) \), where \( D \) is a Sylow \( r \)-subgroup of the central torus \( S_\alpha \) of \( C \) with order \( |S_\alpha| = q - 1 \). The canonical character \( \Theta \) of the root \( b \) of \( B \) in \( C \) has \( D \) in its kernel \( ker \Theta \) by [7, p. 205]. Furthermore, \( \Theta \) is projective as an irreducible character of \( C/D \) by Brauer’s extended first main theorem. Since the centralizer of all noncentral semisimple elements of \( SL_2(q^3) \) are cyclic, it follows from the character table [6, pp. 228 and 235], and Proposition 2.2 that there is a regular \( r \)-element \( s \) of \( C \) in \( T \) with order \( o(s)|q^3 + 1 \) such that \( \Theta = \epsilon_{C(D)} \epsilon_TR^{C}_{T,s} \), because \( e = 1 \). As \( D \leq ker \Theta \), Theorem 7.2.8 of [2] implies that \( D \leq ker \hat{s} \). Furthermore, \( s \in \{s_{10}, s_9, s_{11}\} \) up to \( G_\alpha \)-conjugacy.

If \( e = 1 \), and \( T = T_1 \), then the same argument shows that \( \Theta = \epsilon_{C(D)} \epsilon_TR^{C}_{T_3,s} \), where \( s = s_8 \) is a regular \( r \)-element of \( G_\alpha \) with order \( o(s)|q + 1 \).

Suppose that \( e = 2 \), and \( T = T_1 \). Then there exists a regular \( r \)-element \( s \) of \( C \) such that \( \Theta = \epsilon_{C(D)} \epsilon_TR^{C}_{T_3,s} \) is the canonical character of \( b \), where \( o(s)|q^3 - 1 \). Hence \( s \in \{s_5, s_8, s_4\} \). If \( e = 2 \), and \( T = T_2 \), then by the same argument \( s = s_{11} \).

Let \( e = 3 \). Then \( r|q^2 + q + 1 \), and \( C = C_{G_\alpha}(s_4) \) by Proposition 2.2. Furthermore, \( T = T_1 \). By the character table [13] of \( SL_2(q) \) there is a regular \( r \)-element \( s \) of \( C \) contained in \( T \) such that \( \Theta = \epsilon_{C(D)} \epsilon_TR^{C}_{T_3,s} \) is the canonical character of \( b \), where \( o(s)|q^2 - 1 \). Hence \( s \in \{s_8, s_7\} \).

If \( e = 6 \), then the same argument shows that \( s \in \{s_3, s_{11}\} \), \( T = T_2 \).

Let \( e = 12 \). Then \( D \) is a Sylow \( r \)-subgroup of the cyclic Coxeter torus \( T = T_5 \). Its elements \( t \neq 1 \) are regular by Table 4.4. In particular, \( C = C_{G_\alpha}(D) = T \). If \( B \) is not the principal \( r \)-block, then the canonical character \( \Theta \) of the root \( b \) of \( B \) in \( C \) is of the form \( \Theta = \hat{s} \), where \( s \) is a regular \( r \)-element of \( T \). So \( B \) is the principal \( r \)-block if and only if \( D \) is a Sylow \( r \)-subgroup of \( T_5 \) and \( s = 1 \).

If \( r = 2 \), then Proposition 5.3 asserts that the generator \( x \) of \( D \) has order \( 2^{e+1} \), and that \( D \) is a Sylow \( 2 \)-subgroup of a cyclic maximal torus \( T \). Hence \( x \) is regular by
Propositions 2.1 and 2.2, and \( C = C_{G_\alpha}(D) = T \). Therefore the canonical character \( \Theta \) of the root \( b \) of \( B \) is of the form \( \Theta = \hat{s} \), where \( s \in T \) is a regular \( r' \)-element of \( G \), because \( B \) has inertial index one by Dade’s theorem. Thus (b) holds.

Suppose that \( B \) is a nonprincipal \( r \)-block. Then in any case for \( e \in \{1, 2, 3, 5, 12\} \) we have shown that the canonical character \( \Theta \) of a root \( b \) of \( B \) in \( C = C_{G_\alpha}(D) \) is of the form \( \Theta = e_T R_{T,3}^C \), where \( s \) is a regular \( r' \)-element of \( C \) contained in a cyclic torus \( T \). Furthermore, \( D \leq \ker \hat{s} \), where \( \hat{s} \) is the linear character of \( T \) corresponding to \( s \in T \). By Propositions 2.1 and 2.2 \( sy \) is a regular element of \( G_\alpha \) for each \( 1 \neq y \in D \). Hence each \( \chi_{sy} = e_T R_{T,sy}^C \) with \( 1 \neq y \in D \leq T \) is an irreducible character of \( G_\alpha \) by Lemma 3.1 and Corollary 7.3.5 of [2].

Two irreducible characters of \( G_\alpha \) belong to the same \( r \)-block of \( G_\alpha \) if they agree on all \( r' \)-elements, see [7, p. 150 and p. 179]. Thus all irreducible characters \( \chi_{sy} \), \( 1 \neq y \in D \), belong to one \( r \)-block \( B_1 \) of \( G_\alpha \) by Lemma 5.5. Let \( D_1 \) be its cyclic defect group. For \( x \in D \chi_{sy}(x) = e_T R_{T,sy}^C \) by Lemmas 3.2 and 5.5, where

\[
\hat{s}(x) = \frac{1}{|C_{W(T)}(x)|} \sum_{w \in W(T)} \hat{s}(wxw^{-1}).
\]

As \( D \leq \ker \hat{s} \), \( \hat{s}(x) = |W(T): C_{W(T)}(x)| \neq 0 \) by Lemmas 3.4 and 3.5. Thus \( \chi_{sy}(x) \neq 0 \) for every \( x \in D \), and \( D = G_\alpha \) by Lemma 59.5 of [6] and (a). Let \( b_1 \) be a root of \( B_1 \) in \( C = C_{G_\alpha}(D) \). As shown above there is a regular element \( s_1 \) of \( T \) such that the canonical character of \( b_1 \) is of the form \( \Theta_1 = e_C e_T R_{T,3}^C \). Let \( \mathcal{H} \) be the Brauer homomorphism from the center \( ZF_G \) into \( ZFC \) with respect to \( D \). Let \( \omega_1 \) be the central character of \( \chi_{sy} \), and \( \omega_{s_1} \) the one of \( \Theta_1 \). As \( B_1 = b_1^{G_\alpha} \) it follows from Brauer’s extended first main theorem that \( \omega_1 = \omega_{s_1} \mathcal{H} \) on \( ZF_G \).

From [7, p. 144], we obtain for every \( r' \)-element \( x \in T \) that

\[
\frac{|x^{G_\alpha}| x_{sy}(x)}{\chi_{sy}(1)} \equiv \frac{|x^C| \Theta_1(x)}{\Theta_1(1)} \mod \pi R.
\]

Applying Lemma 3.2 and Theorem 7.5.1 of [2] we get

\[
\frac{|x^{G_\alpha}| e_{C_{G_\alpha}(x)} e_T C_G(x)|_p, s(x)}{|T| |G_\alpha : T|_p'} \equiv \frac{|x^C| e_{C(x)} e_T C_C(x)|_p, s_1(x)}{|T| |C : T|_p'}.
\]

Hence \( e_{C_{G_\alpha}(x)} |G_\alpha : C_G(x)|_p s(x) \equiv e_{C(x)} |C : C(x)|_p s_1(x) \). Corollary 6.5.7 of [2] and Proposition 2.2 assert that \( e_{C_{G_\alpha}(x)} = e_{C(x)} \) for all \( r' \)-elements \( x \in T \). Let \( C_1 = C_{W(T)}(x) \), and \( W_1 = W(T) \). Then

\[
\hat{s}(x) = \frac{1}{|C_1|} \sum_{w \in W_1} \hat{s}(wxw^{-1}) \equiv s_1(x) = \frac{1}{|C|} \sum_{w \in W} \hat{s}_1(wxw^{-1}).
\]

Since \(|T/D|, r \) = 1, and \( D \) is in the kernel of \( s \) and \( s_1 \), it follows that \( \hat{s} \) and \( \hat{s}_1 \) are \( W(T) \)-conjugate. Hence \( \Theta \) and \( \Theta_1 \) are \( N_{G_\alpha}(D) \)-conjugate, because \( W(T) = N_{G_\alpha}(D)/C_{G_\alpha}(D) \) by Proposition 1.2. Therefore \( B = B_1 \). Let \( t = |C_{W}(s)| \). Then \( t \) is
the inertial index of \( B \), and it follows from Dade's theorem [7, p. 177] that the 
\((|D| - 1)t^{-1}\) irreducible characters \( \chi_{sy}, 1 \neq y \in D \), are the exceptional characters of \( B \).

If \( s \) is regular in \( G_o \), then \( \chi_s = \epsilon_r R_{T, s} \) is an irreducible character of \( G_o \), which by the previous argument belongs to \( B \). By Lemma 3.4 \( t = 1 \). Hence by Dade's theorem \( \chi_s \) is the only nonexceptional character of \( B \).

Suppose that \( s \) is not regular. Let \( e = 1 \). Then \( T = T_2 \) and \( s \in \{ s_{10}, s_9 \} \) by the proof of (b). \( B \) has inertial index \( t = 2 \) by Lemma 3.4. Let \( s = s_{10} \). Then by Theorem 4.3

\[
\chi_{s,1} = -\frac{1}{2}(R_{2,10} + R_{6,10}), \quad \chi_{s,St} = -\frac{1}{2}(R_{2,10} - R_{6,10}).
\]

Using Proposition 2.2 and Tables 3.6 and 4.4 it follows that \( |x^{G_o}|R_{6,10}(x)/\chi_{s,u}(1) \equiv 0 \mod \pi R \) for every \( r' \)-element \( x \) of \( G_o \). Let \( 1 \neq y \in D \). Then \( \chi_{sy} \) belongs to \( B \). Since \( e = 1 \), \( v = \chi_{sy}(1)/2\chi_{s,u}(1) \equiv 1 \mod \pi R \) by Table 4.4. \( \chi_{sy}(x) = -R_{2,s}(x) \) for every \( r' \)-element \( x \in G_o \) by Theorem 4.3 and Lemma 5.5. Hence

\[
\frac{|x^{G_o}|\chi_{s,u}(x)}{\chi_{s,u}(1)} = u \frac{|x^{G_o}|\chi_{sy}(x)}{\chi_{sy}(1)} \equiv \frac{|x^{G_o}|\chi_{sy}(x)}{\chi_{sy}(1)}.
\]

Therefore \( \chi_{10,1} \) and \( \chi_{10,St} \) are the two nonexceptional characters of \( B \). Now let \( s = s_9 \). Then by Theorem 4.3

\[
\chi_{s,1} = -\frac{1}{8}(R_{6,9} + 3R_{2,9} + 2R_{4,9}), \quad \chi_{s,St} = \frac{1}{8}(R_{6,9} - 3R_{2,9} + 2R_{4,9}).
\]

Using Proposition 2.2 and Tables 3.6 and 4.4 it follows that

\[
|\frac{x^{G_o}|R_{6,9}(x)}{\chi_{s,u}(1)} \equiv 0 \equiv \frac{|x^{G_o}|R_{6,9}(x)}{\chi_{s,u}(1)} \mod \pi R
\]

for every \( r' \)-element \( x \) of \( G_o \). Applying the above argument again we see that \( \chi_{9,1} \) and \( \chi_{9,St} \) are the nonexceptional characters of \( B \).

Let \( e = 3 \). Then \( T = T_1 \) and \( s = s_7 \) by the proof of b). Now

\[
\chi_{s,1} = -\frac{1}{6}(R_{1,7} + R_{6,7}), \quad \chi_{s,St} = -\frac{1}{3}(R_{1,7} - R_{6,7})
\]

by Theorem 4.3. Using Proposition 2.2 and Tables 3.6 and 4.4 it follows that for every \( r' \)-element \( x \in G_o \)

\[
|\frac{x^{G_o}|R_{6,7}(x)}{\chi_{s,u}(1)}\equiv 0 \mod \pi R.
\]

Let \( 1 \neq y \in D \). Then \( \chi_{sy} \) belongs to \( B \). Since \( e = 3 \),

\[
v = \frac{\chi_{sy}(1)}{2\chi_{s,u}(1)} = \frac{q^3 + 1}{2q^2} \equiv 1 \mod \pi R
\]

by Table 4.4, where \( n \in \{0, 3\} \). From Theorem 4.3 and Lemma 5.5 follows that \( \chi_{sy}(x) = -R_{1,s}(x) \) for every \( r' \)-element \( x \) of \( G_o \). Hence

\[
\frac{|x^{G_o}|\chi_{s,u}(x)}{\chi_{s,u}(1)} = u \frac{|x^{G_o}|\chi_{sy}(x)}{\chi_{sy}(1)} \equiv \frac{|x^{G_o}|\chi_{sy}(x)}{\chi_{sy}(1)}.
\]
Therefore $\chi_7$ and $\chi_{7, St}$ are the two nonexceptional characters of $B$.

Replacing $q$ by $-q$ the cases $e = 2, 6$ follow from the cases $e = 1, 3$, respectively.

Let $\beta$ be the $r$-block of $C_{G_4}(s)$ containing the unipotent characters $s\chi_u$ corresponding to $\chi_{s,u}$ of $B$. Then $B = \beta^{C_{G_4}(s)}$ by [7, p. 136], because the linear character $s$ of $T = C_{G_4}(s) \cap C_{G_4}(D)$ is the canonical character of $\beta$. Hence (d) holds.

Finally let $B$ be the principal $r$-block. Then $D$ is the Sylow $r$-subgroup of the Coxeter torus $T_s$. Therefore every element $1 \neq y \in D$ is regular by Propositions 2.1 and 2.2, and $B$ has inertial index $i = 4$ by Proposition 1.2. Hence $B$ has $\frac{1}{4}(|D| - 1)$ irreducible nonexceptional characters $\chi_y$ with $1 \neq y \in D$ by Lemma 5.5 and the proof of (b). Furthermore, the following 4 unipotent irreducible characters $1, St, \rho_1$ and $3D_4[-1]$ belong to $B$ by Table 4.4. Hence by Dade's theorem we have found all the characters of $B$. This completes the proof.

**Lemma 5.7.** Let $r \not\in \{ p, 2, 3\}$, and let $B$ be an $r$-block of $G_4 = 3D_4(q)$ with a noncyclic defect group $D$. Let $H = N_{G_4}(D)$. Then:

(a) $C_{G_4}(D) = T$ is a maximal torus of $G_4$.

(b) Up to $G_4$-conjugacy there exists a unique $r'$-element $s \in T$ and a root $b$ of $B$ in $C_{G_4}(D) = T$ such that the linear character $s$ of $T$ is the canonical character of $B$.

(c) Let $W(T)$ be the Weyl group of $T$. Then the inertial subgroup $T_H(b) = T(D \cdot C_{W(T)}(s))$, where $D \cdot C_{W(T)}(s)$ denotes the split extension of $D$ by $C_{W(T)}(s)$ induced by the action of $W(T)$ on $T$.

(d) If $B_1$ is the Brauer correspondent of $B$ in $H$, then its $r$-adic block ideal $\hat{B}_1$ is Morita equivalent to the group algebra $R[D \cdot C_{W(T)}(s)]$, and $k(B_1) = k_0(B_1) = k(S[D \cdot C_{W(T)}(s)]) = |D|$.

**Proof.** (a) As $D$ is not cyclic, Lemma 5.2 asserts that $D$ is an abelian Sylow $r$-subgroup of $G_4$. By Corollary 5.19 of [1, p. 212] and Proposition 2.2, $C_{G_4}(D) = T$ is a maximal torus of $G_4$.

(b) Let the $r$-block $b$ of $C_{G_4}(D) = T$ be a root of $B$, and let $\Theta \in \text{Irr}_D(b)$ be the canonical character of $B$. Certainly $\Theta$ is a linear character of $T$. As $D \subseteq \ker\Theta$ there is up to $H$-conjugacy a unique element $s$ of $T$ such that $\Theta = s \in \text{Irr}_s(T)$. As $r \not\in \{2, 3, p\}$, it follows from Proposition 1.2 and the lemma of Schur and Zassenhaus that $D \cdot C_{W(T)}(s)$ is the split extension of $D$ by $C_{W(T)}(s)$ induced by the action of $W(T)$ on $T$.

(d) Let $B' = b^{T_H(b)}$ be the block of $T_H(b)$ with the same block idempotent as $b$, and let $\hat{B}$ be its $r$-adic block ideal. If $\hat{B}_1$ denotes the $r$-adic block ideal of the Brauer correspondent $B_1$ of $B$ in $H$, then by Theorem 2.5 of [7, p. 197], the algebras $\hat{B}$ and $\hat{B}_1$ are Morita equivalent, and $k(B') = k_0(B_1) = k_0(B_1)$.

It is easy to see that $\hat{B}' = R[D \cdot C_{W(T)}(s)]$. By Lemmas 3.4 and 3.5 $|C_{W(T)}(s)|$ divides 24. As $D$ is abelian and $r \not\in \{2, 3\}$, it follows that $k(B') = k_0(B') = k(S[D \cdot C_{W(T)}(s)])$. 

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Hence \( k(B_1) = k_0(B_1) \leq |D| \), the latter inequality follows by application of Lemmas 3.4 and 3.5 and the structure of the group algebra \( S[D \cdot C_{W(T)}(s)] \).

**Proposition 5.8.** Let \( B \) be an \( r \)-block of \( G_\sigma \) with a noncyclic abelian defect group \( D \). Then the following assertions hold:

(a) \( C_{G_\sigma}(D) = T \) is a maximal torus of \( G_\sigma \), and \( D \) is a Sylow \( r \)-subgroup of \( T \).

(b) Up to \( G_\sigma \)-conjugacy there exists a unique \( r' \)-element \( s \in T \) and a root \( b \) of \( B \) in \( C_{G_\sigma}(D) = T \) such that the linear character \( \hat{s} \) of \( T \) is the canonical character of \( B \).

(c) If \( H = N_{G_\sigma}(D) \), and \( T_H(b) \) denotes the inertial subgroup of \( b \) in \( H \), then \( T_H(b)/T = C_{W(T)}(s) \), where \( W(T) = N_{G_\sigma}(T)/T \).

(d) An irreducible character \( \chi_{t,u} \) of \( G_\sigma \) belongs to \( B \) if and only if \( t \sim G_\sigma s y \) for some \( y \in D \), and \( \chi_u \) is a unipotent irreducible character of \( C_{G_\sigma}(s y) \) such that \( s y \chi_u \) belongs to an \( r \)-block \( \beta \) of \( C_{G_\sigma}(s y) \) with \( B = \beta G_\sigma^* \).

(e) \( B \) is the principal \( r \)-block of \( G_\sigma \) if and only if \( s = 1 \) and \( r \geq 5 \).

(f) The number \( l(B) \) of modular irreducible characters of \( B \) equals the number of unipotent irreducible characters of \( C_{G_\sigma}(s) \), provided \( s \neq 1 \).

**Proof.** By Humphreys’ theorem [10] \( r \neq q \). If \( r \neq 3 \), then assertions (a), (b), and (c) hold by Proposition 5.3 and Lemmas 5.7 and 5.2.

Let \( r = 3 \). Then by Lemma 5.4 \( B \) is a Sylow 3-subgroup of a maximal torus \( T \) of \( G_\sigma \). As \( D \) is not cyclic, \( T \) is isomorphic to \( T_0, T_3, T_4, \) or \( T_6 \) by Proposition 1.2. From Corollary 5.19 of [1, p. 212], and Proposition 2.2 it follows that \( C_{G_\sigma}(D) = T \). Thus (a) holds, and (b) can now be shown as in Lemma 5.7. If \( \hat{s} \in \mathcal{T} \) denotes the canonical character of \( B \), then \( T_H(b) = T_H(\hat{s}) = \{ h \in H | s^h = s \} \). Hence (c) holds also for \( p = 3 \).

(e) is a consequence of (b) and Brauer’s third main theorem, because by Propositions 5.3 and 5.4 we may assume that \( r \geq 5 \).

(d) Fix \( s \in T = C_{G_\sigma}(D) \) such that its corresponding linear character \( \hat{s} \) of \( T \) is the canonical character of a block \( b \) of \( C_{G_\sigma}(D) \) satisfying \( B = b G_\sigma^* \). Then \( s \) is uniquely determined by \( B \) up to \( G_\sigma \)-conjugation and \( \text{Irr}_s(b) = \{ s y | y \in D \} \). Furthermore, \( D \leq \text{ker} \hat{s} \).

Let \( \mathcal{H} \) be the Brauer homomorphism with respect to \( D \) from \( ZFG_\sigma \) into \( ZFC_{G_\sigma}(D) \). As \( T = C_{G_\sigma}(D) \) and \( B = b G_\sigma^* \), Brauer’s extended first main theorem implies that on each \( r \)-regular conjugacy class \( x^{G_\sigma} \) of \( G_\sigma \) with defect group \( D \) the central linear character \( \lambda \) of \( B \) agrees with \( \tau(x) = 1/|C_{\mathcal{H}(x)}(D)| \sum_{w \in \mathcal{H}} \mathcal{H}(w x w^{-1}) \), where \( \mathcal{H} = H/T \). Since by (a) \( D \) is a Sylow \( r \)-subgroup of \( T \) it follows from Proposition 1.2 that \( W(T) = H/T \). Using now the notation introduced before Table 3.6 we obtain \( \tau(x) = N_{T, \hat{s}}(x) \). Therefore by [7, p. 144], an irreducible character \( \chi_{t,u} \) of \( G_\sigma \) belongs to \( B \) if and only if

\[
(*) \quad \frac{|x^{G_\sigma}|_{\chi_{t,u}(x)}}{\chi_{t,u}(1)} = N_{T, \hat{s}}(x) \mod R \text{ for every } r'-element } x \text{ of } T
\]

with defect group \( D \).

Suppose that the irreducible character \( \chi_{t,u} \) of \( G_\sigma \) belongs to \( B \). If \( f = f^2 \neq 0 \) denotes the block idempotent of \( B \), then \( \lambda(f) = 1 \in F \). By Lemma 7.2 of [7, p. 179] \( f \) is a linear combination of \( r \)-regular class sums of \( G_\sigma \). Therefore \( \chi_{t,u} \neq 0 \) for some
$r'$-element $x$. Hence $t \in T$ (up to $G_\sigma$-conjugacy) by Lemma 3.2 and Theorem 4.3. Let $t = zy = yz$, where $z \in T$ is $r$-regular and $y \in D$. By Lemma 5.5, $R_{T, \hat{z}}$ and $R_{T, z}$ agree on all $r'$-elements $x \in G_\sigma$, and $D$ is in the kernel of $\hat{z} \in \hat{T}$.

As $q \neq 2$, we obtain from Theorem 4.3, Table 3.6, and Lemma 3.4 that for all unipotent irreducible characters $\chi_u$ in $C_{G_\sigma}(t)$ and all $r'$-elements $x \in T$ the following consequences hold mod $\pi R$.

$$x_{t,u}(1) \equiv \frac{1}{|C_{W(T)}(t)|} \epsilon_{T_i} \epsilon_{G_\sigma} \frac{|G_\sigma : T|_{p'}}{|C_{W(T)}(t)|}.$$ 

$$x_{t,u}(x) = \frac{\epsilon_{C(t)} \epsilon_T}{|C_{W(T)}(t)|} R_{T,i}(x).$$

Here we denote $C_{G_\sigma}(t)$ and $C_{G_\sigma}(x)$ by $C(t)$ and $C(x)$, respectively. Hence, using Lemma 3.2, one obtains

$$\frac{|x^{G_\sigma}| x_{t,u}(x)}{x_{t,u}(1)} = \frac{|G_\sigma : C(x)| \epsilon_{C(t)} \epsilon_T R_{T,i}(x) |C_{W(T)}(t)|}{|C_{W(T)}(t)| |G_\sigma : T|_{p'}}$$

$$= \frac{|G_\sigma : C(x)| \epsilon_{C(t)} \epsilon_T R_{T,i}(x)}{|C(x) : T|_{p'}}$$

$$= \frac{|G_\sigma : C(x)| \epsilon_{C(t)} \epsilon_T R_{T,i}(x)}{|C(x) : T|_{p'}}$$

$$= \frac{|G_\sigma : C(x)| \epsilon_{C(t)} \epsilon_T R_{T,i}(x)}{|T|_p}$$

$$= \epsilon_{C(t)} \epsilon_T R_{T,i}(x)$$

because $T \in \{ T_0, T_3, T_4, T_6 \}$, and therefore $\epsilon_{C(t)} \epsilon_{C(t)} = 1$, by Proposition 2.2. Since the right-hand side is independent of the unipotent irreducible character $\chi_u$ of $C(t)$, it follows that $x_{t,u_1}$ and $x_{t,u_2}$ belong to the same $r$-block $B$ of $G_\sigma$, whenever $\chi_u$ and $\chi_u'$ are two unipotent irreducible characters of $C(t)$. Furthermore, $\mathcal{N}_{T,i}(x) \equiv \mathcal{N}_{T,\hat{z}}(x)$, because $R_{T,i} = R_{T,\hat{z}}$ and $R_{T,\hat{z}}$ agree on all $r'$-elements $x \in T$. Now $x_{t,u} \in B$ implies that $\mathcal{N}_{T,\hat{z}}(x) \equiv \mathcal{N}_{T,\hat{z}}(x) \mod \pi R$. Since $D$ is in the kernels of $\hat{s}$ and $\hat{z}$, and since $D$ is the Sylow $r$-subgroup of $T$, it follows that $z$ and $s$ are $W(t)$-conjugate.

For every fixed $G_\sigma$-conjugacy class $y^{G_\sigma}$ of $G_\sigma$ meeting $D$ let $s_{yG_\sigma} = \{ x_{sy,v} \}$, where $x_u$ runs through all the unipotent irreducible characters of $C = C_{G_\sigma}(sy)$, and where $\hat{s}$ denotes the canonical character of the root $b$ of $B$ in $T = C_{G_\sigma}(D)$. Observe that by Proposition 2.2 $C_{G_\sigma}(sy)$ does not contain any unipotent irreducible characters $\chi_u$ if and only if $sy$ is a regular element of $T$; in this case $(sy)_{G_\sigma}$ consists only of the irreducible character $\chi_{sy} = \pm R_{T,\hat{s}}$ of $G_\sigma$. The above argument with $zy$ replaced by $sy$ shows that for a fixed $sy$ all $x_{sy,v} \in \text{Irr}_S(B)$, where $\chi_u$ runs through all unipotent irreducible characters of $C = C_{G_\sigma}(sy)$. Thus we have shown that $\text{Irr}_S(B) = \bigcup_{y \in G_\sigma \Delta} (sy)$.
Let \( y \) be a representation of a conjugacy class \( yG° \) with \( yG° \cap D \neq \emptyset \). If \( sy \) is not regular, then Proposition 2.2 and the character tables of [6 and 13] imply that all unipotent irreducible characters \( \chi_u \) of \( C = C_{G_a}(sy) \) belong to the principal \( r \)-block of \( C \). Hence the irreducible characters \( \chi_u sy \) of \( C \) belong to an \( r \)-block \( \hat{B} \) of \( C \) with root \( b \) in \( T = C_G(D) \), because \( \hat{s} \) is a canonical character of \( \beta \). In particular, \( B = \beta G° \) by Brauer’s extended first main theorem. If \( sy \) is regular, then \( b = \{ sy \mid y \in D \} = \beta \) and \( C_{G_a}(sy) = T \). Hence \( B = \beta G° \). This completes the proof of (d).

(f) By (d) we know that \( B \) determines up to \( G_a \)-conjugacy a unique \( r' \)-element \( s \neq 1 \) of \( G_a \) representing the canonical character \( \hat{s} \) of \( B \) in \( T = C_G(D) \). The number of \( G_a \)-conjugacy classes of maximal tori \( T \) containing \( s \) equals by Proposition 2.2 the number \( |sG_a| \) of unipotent irreducible characters \( \chi_u \) of \( G_a(s) \). By Lemma 5.5 and Theorem 4.3 each irreducible character \( \chi_{t,u} \) of \( B \) restricted to the \( r' \)-elements is a linear combination of the \( R_T s \). Hence \( l(B) = |sG_a| \).

**Theorem 5.9.** Let \( B \) be an \( r \)-block of \( G_a = 3D_4(g) \) with defect group \( \delta(B) = G_a D \neq 1 \), where the prime \( r \) does not divide \( q \). Then the following assertions hold:

(a) \( C = DC_{G_a}(D) \) contains a maximal torus \( T \) such that \( H = N_{G_a}(D) \leq N_G(T) \).

(b) Up to \( G_a \)-conjugacy there exists a unique \( r' \)-element \( s \in T \) and a root \( b \) of \( B \) in \( C = DC_{G_a}(D) \) such that the linear character \( \hat{s} \) of \( T \) is the canonical character of \( B \).

(c) If \( T_H(b) \) denotes the inertial subgroup of \( b \) in \( H \), then \( T_H(b)/C = C_{W(T)}(s) \), where \( W(T) = N_G(T)/T \).

(d) \( B \) is the principal \( r \)-block of \( G_a \) if and only if \( s = 1 \) and \( D \) is a Sylow \( r \)-subgroup of \( G_a \).

(e) An irreducible character \( \chi_{t,u} \) of \( G_a \) belongs to \( B \) if and only if \( t \simeq G_a sy \) for some \( y \in D \), and \( \chi_u \) is a unipotent irreducible character of \( C_{G_a}(sy) \) such that \( sy \chi_u \) belongs to an \( r \)-block \( \beta \) of \( C_{G_a}(sy) \) with \( B = \beta G° \).

(f) The number \( l(B) \) of modular irreducible characters of \( B \) equals the number of unipotent irreducible characters of \( C_{G_a}(S) \), provided \( s \neq 1 \).

**Proof.** If the defect group \( D \) of \( B \) is cyclic or abelian, then all assertions hold by Propositions 5.6 and 5.8. Hence we may assume that \( D \) is not abelian. Thus \( r \in \{2, 3\} \) by Lemma 5.2. The proof of (f) is the same as in Proposition 5.8.

Suppose that \( r = 3 \), and that \( 3 \mid q - 1 \). Then \( q \neq 2 \). By Proposition 5.4 \( D \) is a Sylow 3-subgroup of \( G_a \). Theorem 9.2 of [7, p. 231] asserts that there is a central element \( 1 \neq x \in D \) of order 3 and a 3-block \( b_1 \) of \( C_1 = C_{G_a}(x) \) such that \( B = b_1 G° \), and \( b_1 \) has defect group \( \delta(b) = D \). From Proposition 2.2 and the proof of Proposition 5.4 it follows that \( C_1 = U \times Z \), where \( Z \) is a cyclic group of order \( k = \frac{1}{3}(q^2 + q + 1) \), and where \( U \) is a nonsplit extension of \( SL_3(q) \) by a cyclic group \( C_1/C_1' \) of order 3. In particular, each block \( b_1 \) is of the form \( b_1 = b_0 \otimes z \), where \( z \) denotes a 3-block of defect zero of \( Z \), and \( b_0 \) denotes the principal 3-block of \( U \). Because of the structure of \( C_1 \) we have \( C = DC_{G_a}(D) \leq C_1 \), and \( C \) contains a maximal torus \( T \supseteq Z \) such that \( T = G_a T_0 \), as \( 3 \mid q - 1 \). Now \( D \) normalizes \( T \) by Corollary 5.19 of [1, p. 212]. Since \( D \cap T \) is a Sylow 3-subgroup of \( T \) and also the largest abelian normal subgroup of \( D \), it follows from Propositions 1.2 and 2.2 that \( H = N_{G_a}(D) \leq N_T(T) \).
The 3-block $z$ of $Z$ consists of one linear character $\hat{s}$ of $\hat{T}$, because $Z \leq T$. As $Z \leq C = DC_{G_0}(D_1) \leq C_1$ and $b_1 = b_0 \otimes z$, it follows from Brauer's extended first main theorem that $\hat{s}$ is the canonical character of a common root block $b$ of $B$ and $b_1$ in $C$. Certainly

$$T_H(b) = \left\{ h \in N_{G_0}(D) \mid s^h = s \right\} = \left\{ h \in N_{G_0}(T) \mid s^h = s \right\}$$

by Proposition 1.2. Therefore $T_H(b)/C \cong C_{W(T)}(s)$.

Since $b_1 = b_0 \otimes z$, and $z = \{\hat{s}\}$, it follows from Brauer’s third main theorem that $B = B_0$ is the principal 3-block of $G_0$ if and only if $s = 1$ and $D$ is a Sylow 3-subgroup of $G_0$.

As shown in the proof of Proposition 5.4, the cyclic group $C_1/C_1'$ of order 3 acts trivially on $Z$. Thus $y \in C_{G_0}(s)$ for every $y \in D$.

Let $\chi_u$ be a fixed unipotent irreducible character of $C_y = C_{G_0}(sy)$. Now $D \cap T'$ is a Sylow 3-subgroup of $T'$ for every maximal torus $T' \subseteq C_y$ containing $sy$. Therefore the irreducible characters $\chi_{sy,u}$ of $G_0$ agree on all 3’-conjugacy classes of $G_0$ by the proof of Lemma 5.5 and Theorem 4.3, because $D \cap T' \subseteq \ker s$ and $s \in T'$. Hence Osima’s theorem and Lemma 4.2 of [7, p. 150], imply that all $\chi_{sy,u}$ with $y \in D$ belong to the same 3-block $B'$ of $G_0$. By Proposition 2.2 the unipotent irreducible characters $\chi_u$ of $C_y$ belong to the principal 3-block $b_0^*$ of $C_y$. Hence by the structure of $C_y$ the irreducible characters $sy\chi_u$ belong to one 3-block $b_y$ of $C_y = C_{G_0}(sy)$ with defect group $D_2 = D \cap C_y$ and canonical character $\hat{s}$. As $C_{C_y}(D_2) \subseteq C_y$, Lemma 6.1 of [7, p. 209] asserts the existence of $(b_y)^{G_0}$, and $B = (b_y)^{G_0}$ by Brauer's extended first main theorem, because both blocks have the same canonical character. Applying now the proof of Proposition 5.8(d) we see that $B' = B$. Hence all irreducible characters $\chi_{sy,u}$ of $G_0$ such that $sy\chi_u$ belongs to a 3-block $b_y$ of $C_{G_0}(sy)$ with $(b_y)^{G_0} = B$ are contained in $B$.

As $3 \mid q - 1$, it follows from Theorem 4.3 and Table 4.4 that $B_0$ contains the unipotent irreducible characters

$$U(B_0) = \left\{ 1, [\varepsilon_1], [\varepsilon_2], St, \rho_1, \rho_2, 3D_4[1] \right\}.$$

Since the order of the Sylow 3-subgroup $P$ divides the degree of $3D_4[-1]$, this unipotent irreducible character belongs to a 3-block of $G_0$ with defect zero. Hence all other 3-blocks of $G_0$ with positive defect are not unipotent.

For every fixed $G_0$-conjugacy class $y_i^{G_0}$ of $G_0$ meeting $D$ let $sy_{G_0} = \{\chi_{sy,u}\}$, where $\chi_u$ runs through all the unipotent irreducible characters of $C_y = C_{G_0}(sy)$. Let $y_1 = 1, y_2, \ldots, y_t$ be representatives of these conjugacy classes of 3-elements. As no $y_i$ is conjugate to the involution $s_2$ it follows from Proposition 2.2 that $C_{y_i} = C_{G_0}(y_i) = C_{G_0}(y_i')$ for $i = 2, 3, \ldots, t$ provided $sy_i$ is of type $s_4$ or $s_5$ and $s \neq 1$ or $s = 1$ and $y_i$ is of type $s_3, s_4$, or $s_5$. In particular, the irreducible characters $sy_i\chi_u$ of $C_{y_i}$ belong to one 3-block $b_{y_i}$ of $C_{y_i} = C_{G_0}(y_i) = C_G(y_i)$ with $B = (b_{y_i})^{G_0}$, and the number of 3-modular characters of $b_{y_i}$ is $l(b_{y_i}) = l(sy_i)G_0$ for $i = 2, 3, \ldots, t$, because no $sy_i$ is regular by Propositions 2.1 and 2.2 and Table 4.4. An application of Theorem 68.4
of [6] now yields that the number of ordinary irreducible characters of $B$ is
\[ k(B) = \sum_{i=1}^{t} l(b_{\gamma_i}) = \sum_{i=1}^{t} |(s_{\gamma_i})_{G_a}|. \]
This completes the proof of (e) in the case $r = 3$ and $3 \mid q - 1$.

If $3 \mid q + 1$, then $s_3$, $s_4$, and $s_5$ are replaced by the representatives $s_7$, $s_9$, and $s_{10}$, respectively. Furthermore, it follows from Theorem 4.3 and Table 4.4 that the principal 3-block $B_0$ contains the unipotent irreducible characters $U(B_0) = \{1, [\varepsilon_1], [\varepsilon_2], St, p_2, 3D_4[-1], 3D_4[1]\}$, and in this case $\rho_1$ is of defect zero. With these changes the above argument applies in this case. Hence Theorem 5.9 holds for $r = 3$.

So we may assume that $r = 2$, and $q$ is odd. With the notation of Proposition 5.3 it follows that $D$ is one of the 2-groups $P$, $S \ast Z$, or $S$, where $P$ is a Sylow 2-subgroup of $G_a$, $S$ is a semidihedral group of order $|S| = 2^{a+2}$ and $Z$ is a cyclic group of order $|Z| = 2^a$. Furthermore, by Propositions 1.2, 2.1, and 5.3 we may assume that $q = 1 \mod 4$, because the case $q = 3 \mod 4$ follows similarly.

Suppose that $D$ is a semidihedral group of order $|D| = 2^{a+2}$, where $2^a$ is the highest power of 2 dividing $q - 1$. As $D$ is a defect group of the 2-block $B$, Theorem 3.15 of [12] and the proof of Proposition 5.3 imply that $k(B) = 2^a + 4$, $k_0(B) = 4$, $k_1(B) = 2^a - 1$ and $k_v(B) = 1$, where $k_i(B)$ denotes the number of irreducible characters of $B$ with highest $i$, and where $v = 2^a$. By Theorem 9.2 of [7, p. 231], there is a central element $1 \neq x \in D$ and a 2-block $b_i$ of $C_i = C_{G_a}(x)$ such that $B = b_i^{G_a}$ and $D$ is a defect group of $b_1$. Again by the proof of Proposition 5.3 we may assume that $x$ is either of type $s_7$ or $s_{10}$. Let $x$ be of type $s_7$. Then by Propositions 1.2 and 2.1 $C_1$ contains a maximal torus $T_1 \asymp Z((q^2 - 1)/(q + 1))$ such that $C = DC_{G_a}(D) \supseteq T_1$. Furthermore, there exists up to $G_a$-conjugacy a unique element $s \in T_1$ of odd order dividing $q + 1$ such that the linear character $\delta$ of $T_1$ is the canonical character of a common root block $b$ of $C$ of the blocks $B$ and $b_1$. Each irreducible character $\chi_{sy,u}$ of $G_a$ with $y \in D$ belongs to $B$ by the proof of Proposition 5.8(d), Lemma 5.5, and Theorem 4.3. The center $Z(D)$ of $D$ has order 4. Applying Propositions 1.2 and 2.2 and Lemma 3.4 we see that there are two conjugacy classes of $G_a$ of the form $sy$ with $y \in Z(D)$. As $D$ is a Sylow 2-subgroup of $C_{G_a}(sy)$ for $y \in Z(D)$ it follows from Proposition 2.2 and Table 4.4 that each of the four irreducible characters $\chi_{sy,u}$ with $y \in Z(D)$ has height zero. Since $k_0(B) = 4$, all other irreducible characters of $B$ have positive height. By Proposition 1.2 $T_1$ has a cyclic Sylow 2-subgroup $\langle y \rangle$ of order $2^{a+1}$. Therefore $y^i \not\in Z(D)$ for $1 \leq i \leq 2^a - 1$. Proposition 2.1 and Table 4.4 assert that each element $sy^i$ of $T_1$ is regular. Thus each irreducible character $\chi_{sy}^i, 1 \leq i \leq 2^a - 1$, of $B$ has height 1 by Table 4.4. As $q \equiv 1 \mod 4$ the Sylow 2-subgroup of the maximal torus $T_6$ of $G_a$ is a Klein four group by Proposition 1.2. Applying again Table 4.4 and Proposition 2.1, we see that there is a $y \in D$ such that $sy$ is a regular element of $T_6$. Hence $\chi_{sy}$ is an irreducible character of $B$ with height $v = 2^a$. Therefore we have found all irreducible characters of $B$. Replacing $T_1$ and $s_7$ by $T_2$ and $s_{10}$, respectively, the remaining case is proved similarly. Hence all assertions (a)-(e) hold for blocks $B$ of $G_a$ with a semidihedral defect group $D$, because the same arguments hold in the case $q \equiv 3 \mod 4$. 

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Suppose that \( D = S \star Z \), where \( S \) is a semidihedral group of order \(|S| = 2^a+2\) and \( Z \) is a cyclic group of order \(|Z| = 2^a\). Let \( q \equiv 1 \mod 4 \). By Theorem 9.2 of \([7, p. 231]\), there is a central element \( 1 \neq x \in D \) and a 2-block \( b_1 \) of \( C = C_{G_a}(x) \) such that \( B = b_1^{G_a} \) and \( D \) is a defect group of \( b_1 \). Again by proof of Proposition 5.3 we may assume that \( x \) is either of type \( s_3 \) or \( s_5 \). In both cases it follows that \( C_1 \) contains a maximal torus \( T_0 = \mathbb{Z}_q x^{-1} \times \mathbb{Z}_q x^{-1} \) such that \( C = DC_{G_a}(D) \supseteq T_0 \). Let \( x \) be of type \( s_3 \), and let \( b \) be a root of \( B \) and therefore of \( b_1 \) in \( C \). Then there exists up to \( G_a \)-conjugacy a unique element \( s \in T_0 \) of odd order dividing \( q - 1 \) and of type \( s_3 \) such that the linear character \( \delta \) of \( T_0 \) is the canonical character of \( B \). Applying again Lemma 5.5, Theorem 4.3, and the proof of Proposition 5.8(d) it follows that each irreducible character \( \chi_{s,v,u} \) of \( G_a \) with \( v \in D \) belongs to \( B \). Using now Proposition 2.2 and Theorem 68.4 of \([6]\) as in the case \( r = 3 \) it follows that we have found all irreducible characters of \( B \). The remaining cases \( x \sim G_a^s 5 \) and \( q \equiv 3 \mod 4 \) are dealt with similarly.

By Proposition 5.3 only the principal 2-block \( B_0 \) of \( G_a \) has a Sylow 2-subgroup \( P \) as a defect group. Furthermore, the above argument shows that each irreducible character \( \chi_{y,v,u} \) with \( y \in D \) belongs to \( B_0 \). Therefore all unipotent irreducible characters of \( G_a \) are contained in \( B_0 \). This completes the proof of Theorem 5.9.

As a first application of Theorem 5.9 we verify Brauer’s height zero conjecture in the case of the simple triality groups \( G_a = 3D_4(q) \).

**Corollary 5.10.** Let \( B \) be an \( r \)-block of \( G \) with defect group \( D \). Then every irreducible character \( \chi \) of \( G_a \) belonging to \( B \) has height zero if and only if \( D \) is abelian.

**Proof.** If \( r \nmid q \), then by Humphreys \([10]\) we may assume that \( B = B_0 \), the principal \( r \)-block of \( G_a \). The Sylow \( r \)-subgroup of \( G_a \) has order \( q^{12} \) and is not abelian. By Table 4.4, \( B_0 \) has unipotent irreducible characters of positive height. So we may suppose that \( r \nmid q \).

By Lemma 5.2 and Propositions 5.3 and 5.4 the \( r \)-block \( B \) has an abelian defect group \( D \) if and only if \( D \) is a Sylow \( r \)-subgroup of a maximal torus \( T \) of \( G_a \). Hence, if \( D \) is abelian, then Theorem 5.9 and Table 4.4 imply that all irreducible characters of \( B \) have height zero. Suppose that \( D \) is not abelian. Then \( r \in \{2, 3\} \) by Lemma 5.2. By Theorem 5.9(a) and (b) \( C = DC_{G_a}(D) \) contains a maximal torus \( T \) of \( G_a \) such that there is up to \( G_a \)-conjugacy a unique \( r' \)-element \( s \in T \) which corresponds to the canonical character of \( B \). In the proof of Theorem 5.9(e) we have shown that for every \( y \in D \) which is not \( G_a \)-conjugate to a central element of \( D \) the irreducible characters \( \chi_{s,v,u} \) of \( B \) have positive height. This completes the proof.

Brauer’s conjecture on the number \( k(B) \) of irreducible characters of an \( r \)-block \( B \) of \( G_a \) follows also.

**Corollary 5.11.** Let \( B \) be an \( r \)-block of \( G_a \) with defect group \( D \). Then \( k(B) \leq |D| \).

**Proof.** Since \( k(B) = 1 \) for every \( r \)-block of defect zero, we may assume that \(|D| \neq 1 \).

If \( r \nmid q \), then \( B \) is the principal block of \( G_a \) by \([10]\), and

\[
k(B) = \begin{cases} 
q^4 + q^3 + q^2 + q + 4, & \text{if } 2 \mid q, \\
q^4 + q^3 + q^2 + q + 5, & \text{if } 2 \nmid q,
\end{cases}
\]
by Proposition 2.3. As $|D| = q^{12}$, we get $k(B) \leq |D|$.

Suppose that $r \nmid q$. If $D$ is abelian, then $D$ is a Sylow $r$-subgroup of a maximal torus $T$ of $G_\alpha$ by Lemma 5.2, Proposition 5.3, and Proposition 5.4. Therefore Proposition 1.2 asserts that $D$ can be generated by one or two elements. Thus $k(B) \leq |D|$ by Theorem 10.13 of [7, p. 316].

If $D$ is nonabelian, then $r \in \{2, 3\}$ by Lemma 5.2. Let $r = 3$. Then $3 \nmid q$, and $D$ is a Sylow 3-subgroup of $G_\alpha$ by Proposition 5.4. Suppose that $3^a$ is the highest power of 3 dividing $q - 1$. By Theorem 5.9 there is a semisimple 3'-element $s$ of $G_\alpha$ such that each irreducible character $\chi$ of $B$ is of the form $\chi = \chi_{s\gamma, u}$, where $y$ is $G_\alpha$-conjugate to an element of $D$, and where $\chi_u$ is a unipotent irreducible character of $C_{G_\alpha}(sy)$. Furthermore, $B = B_0$ if and only if $s = 1$. Since by the proof of Theorem 5.9 the principal 3-block $B_0$ has 7 unipotent irreducible characters, it follows from Proposition 2.2 and Theorem 5.9(e) that

$$k(B) = \begin{cases} 6 + 4 \cdot 3^a, & \text{if } s \neq 1, \\ 10 + 4 \cdot 3^a, & \text{if } s = 1. \end{cases}$$

In any case $k(B) \leq 3^{2+2a} = |D|$. The same argument applies, if $3 | q + 1$.

Let $r = 2$. Then $2 \nmid q$, and $D \in \{P, S \ast Z, S\}$ by Proposition 5.3, where $P$ is a Sylow 2-subgroup of $G_\alpha$ of order $|P| = 2^{2+2a}$, $S$ is a semidihedral group of order $|S| = 2^{a+2}$, and $Z$ is a cyclic group of order $|Z| = 2^a$, and where $2^a$ is the highest power of 2 dividing $q - 1$ or $q + 1$. Furthermore, the principal 2-block $B_0$ is the only 2-block of highest defect, and it has 8 unipotent irreducible characters by the proof of Theorem 5.9(e). Another application of Proposition 2.2, Table 4.4, and Theorem 5.9(e) yields that for $2^a | q - 1$

$$k(B) = \begin{cases} 14 + 2^a & \text{if } D = G_\alpha P \text{ and } q = 3, \\ 14 + 2^{a+1} & \text{if } D = G_\alpha P \text{ and } q \neq 3, \\ 2 + 2^{a+1} & \text{if } D = G_\alpha S \ast Z, \\ 4 + 2^a & \text{if } D = G_\alpha S. \end{cases}$$

Hence $k(B) \leq |D|$ in any of these cases. If $2^a | q + 1$, then the assertion follows by a similar count for $k(B)$. This completes the proof.

We also can verify the Alperin-McKay conjecture for the simple triality groups.

**Corollary 5.12.** Let $B$ be an $r$-block with defect group $D$ of $G_\alpha$. Let $B_1$ be the Brauer correspondent of $B$ in $H = N_{G_\alpha}(D)$. Then $k_0(B) = k_0(B_1)$.

**Proof.** If $r = p | q$, then by [9] $k_0(B) = k_0(B_1)$, as was pointed out by Feit [7, p. 171].

Let $r \nmid q$. If $D$ is abelian, then $k(B) = k_0(B)$ by Corollary 5.10. Furthermore, $k(B) = k(B_1)$ by Propositions 5.6 and 5.8 and Lemmas 3.4 it and 3.5. Using the proof of Lemma 5.7 it follows that $k_0(B_1) = k(B)$. Hence $k_0(B) = k_0(B_1)$.

Let $D$ be nonabelian. If $r = 2$, then by the proof of Theorem 5.9 and Corollary 5.10 it follows (with the notation of Proposition 5.3) that

$$k_0(B) = k_0(B_1) = \begin{cases} 8 & \text{if } D = G_\alpha P, \\ |Z| & \text{if } D = G_\alpha S \ast Z, \\ 4 & \text{if } D = G_\alpha S. \end{cases}$$

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So we may assume that \( r = 3 \). As \( D \) is not abelian, it is a Sylow 3-subgroup of \( G_a \) by Proposition 5.4.

If \( q = 2 \), then \( G_a \) has only the principal 3-block \( B_0 \) as a 3-block of highest defect. By Table 4.4 and the proof of Theorem 5.9 the set of irreducible characters of \( B_0 \) with height zero is

\[
\text{Ir}r^0(B_0) = \{1, [e_1], [e_2], \text{St}, 3D_4[-1], 3D_4[1], x_{9,1}, x_{9, St}, x_{9, qs'}\},
\]

where \( s_a \) denotes a representative of order 3. As \( q = 2 \) it follows from Proposition 1.2, Lemma 3.5, and Theorem 5.9 that \( H = N_{G_a}(D) = N_{G_a}(T_6) \). Therefore \( H = U_3(2) \) by Proposition 2.2. Let \( b_0 \) be the principal 3-block of \( H \). Using the character table of Simpson and Frame \[1\] it follows that \( \text{Ir}r^0(b_0) = \{1, 1_1, 1_2, 2, 2_2, 8, 8_1, 8_2\} \), where the irreducible characters of \( H \) with height zero are denoted by their degrees. As \( B_1 = b_0 \) by Brauer's third main theorem, we obtain that \( k_0(B) = 9 = k_0(B_1) \).

Thus we may assume that \( q > 2 \). Hence either \( 3|q - 1 \) or \( 3|q + 1 \). Let \( 3\mid q - 1 \). Since the number of 3-blocks with highest defect equals the number of 3-regular conjugacy classes with highest defect, it follows from Table 4.4 and Proposition 2.2 that \( G_a \) has the principal 3-block \( B_0 \) and \( \frac{1}{3}(q^2 + q - 2) \) many nonunipotent 3-blocks \( B \) with defect group \( \delta(B) = G_a \). Let \( B = B_0 \). By the proof of Theorem 5.9

\[
\text{Ir}r^0(B_0) = \{1, [e_1], [e_2], \text{St}, \rho_1, \rho_1, x_{4,1}, x_{4, St}, x_{4, qs'}\},
\]

where \( s_a \) denotes a representative of order 3. From Theorem 5.9, Proposition 1.2 and Lemma 3.4, it follows that \( H = N_{G_a}(D) = N_{G_a}(T_6) \) and \( H/T_6 = D_{12} \). Using the action of \( D_{12} \) on \( T_6 \) it is easy to see that the Brauer correspondent \( B_1 \) of \( B_0 \) has \( k_0(B_1) = k_0(b_0) = 9 \). Thus \( k_0(B_0) = 9 = k_0(B_1) \).

Now let \( B \) be a nonprincipal 3-block, and \( b \) its Brauer correspondent in \( H = N_{G_a}(D) = N_{G_a}(T_6) \). Then \( \text{Ir}r^0(B) = \{x_{4,1}, x_{4, St}, x_{4, qs}\} \), where \( s_a = yc \) with \( y \) an element of order 3 in the center of \( S L_3(q) \) and \( c \neq 1 \) a fixed representative of 3'-conjugacy class of the cyclic group \( S_a = Z_{q^2 + q + 1} \) described in Proposition 2.2. Also \( b \) is determined by the conjugacy class \( e_{H}^H \), as follows from Brauer's first main theorem. From Theorem 5.9(c) and Lemma 3.4 it follows that \( k_0(b) = k_0(B) \).

The remaining case \( 3|q + 1 \) follows similarly, with \( q \) replaced by \((-q)\), which means replacing \( s_a \) by \( s_0 \) and \( \rho_1, \rho_2 \) by \( 3D_4[-1], 3D_4[1] \), respectively. This completes the proof.

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