A NEW PROOF THAT TEICHMÜLLER SPACE IS A CELL

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ABSTRACT. A new proof is given, using the energy of a harmonic map, that Teichmüller space is a cell.

In [2] the authors developed a new approach to Teichmüller’s famous theorem on the dimension of the unramified moduli space for compact Riemann surfaces. Teichmüller’s theorem states (roughly) that the space $\mathcal{T}$ of conformally inequivalent Riemann surfaces of genus $p$, $p > 1$ (with some topological restrictions) is homeomorphic to Euclidean $\mathbb{R}^{6p-6}$. In proving homeomorphism Teichmüller had put a complete Finsler metric on this space. In [2] we showed that $\mathcal{T}$ naturally carried the structure of a $C^\infty$ connected and simply connected differentiable manifold of dimension $6p - 6$. The proof of this was straightforward and used only splitting results for symmetric tensors and a standard existence theorem in elliptic partial differential equations. Using somewhat deeper results from the theory of harmonic functions between Riemannian manifolds and a result of Earle and Eells, we were then able to show that our moduli space $\mathcal{T}$ was a contractible manifold.

The purpose of this note is to show that there is a straightforward proof that our Teichmüller space is diffeomorphic to $\mathbb{R}^{6p-6}$. This completes the program of giving the main classical results of Teichmüller strictly in terms of concepts from Riemannian geometry as was formulated in [2, 3, 4].

1. A quick review of the Fischer-Tromba approach to Teichmüller theory. Let $M$ be a compact oriented surface without boundary. Let $\mathcal{C}$ denote the space of complex structures compatible with the given orientation, $\mathcal{D}$ the space of $C^\infty$ diffeomorphisms, $\mathcal{D}_0$ those homotopic (and hence isotopic) to the identity, and $\mathcal{M}_{-1}$ those Riemannian metrics on $M$ with Riemann scalar curvature negative one. If $c = \{\phi_i, U_i\}, \bigcup U_i = M$, is a complex coordinate atlas for $M$ and $f \in \mathcal{D}$, then $\{\phi_i \circ f, f^{-1}(U_i)\}$ is a complex coordinate atlas for $M$ which we designate as $f^*c$. If $g \in \mathcal{M}_{-1}$, then for each $x \in M$, $g(x): T_xM \times T_xM \to \mathbb{R}$ is a positive definite symmetric quadratic form on $M$. By $f^*g$ we mean the form $g(f(x))(df(x)^-, df(x)^-)$.

One can then form the quotient spaces $\mathcal{M}_{-1}/\mathcal{D}, \mathcal{M}_{-1}/\mathcal{D}_0, \mathcal{C}/\mathcal{D}, \mathcal{C}/\mathcal{D}_0$. The main result of [2] is

**Theorem 1.1.** The spaces $\mathcal{T} = \mathcal{M}_{-1}/\mathcal{D}_0$ and $\mathcal{C}/\mathcal{D}_0$ naturally have the structure of a $C^\infty$ connected and simply connected finite-dimensional manifold of dimension

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6p − 6. Moreover there is a naturally defined equivariant diffeomorphism from $M_{-1}$ to $C$ which passes to a diffeomorphism of $M_{-1}/D_0$ with $C/D_0$. The space $C/D_0 \cong M_{-1}/D_0$ is the Teichmüller space of $M$.

We should remark that the true Riemann space of moduli $R = M_{-1}/D = C/D$ is not a smooth manifold but does have the structure of an algebraic variety.

For purposes of exposition we wish to describe how one puts a differentiable structure on $M_{-1}/D_0$ and to see what the natural tangent space is to this manifold. To see how the diffeomorphism between $M_{-1}/D_0$ and $C/D_0$ is constructed the reader is referred to [3]. Let us think of $M_{-1}$ as an infinite dimensional submanifold of the space of $C^\infty$ symmetric two tensors $S_2$ on $M$. For $g \in M_{-1}$, the tangent space $T_gO_g$ to the orbit $O_g$ of $D_0$ at $g$ consists of all symmetric tensors of the form $L_xg$, the Lie derivative of $g$ with respect to some vector field $X$ on $M$. In this case $X$ will be uniquely determined by $h \in T_gM_{-1}$. The next splitting result of symmetric tensors is basic to our theory.

**Theorem 1.2.** Every $h \in T_gM_{-1}$ can be expressed uniquely as a direct sum $h = h^{TT} + L_xg$ where $h^{TT}$ is a symmetric two tensor on $M$ which is trace free and divergence free. This implies that in a conformal coordinate system (with respect to the metric $g$), $h^{TT}$ has a local representation as

$$h^{TT} = u dx^2 - u dy^2 - 2v dx dy = \text{Re}\{(u + iv)(dx + idy)^2\}$$

where $u + iv$ is a holomorphic function of the local coordinates $z = x + iy$, and Re designates the real part.

Thus every $h \in T_gM_{-1}$ can be expressed uniquely as a direct sum

$$h = \text{Re}(\xi(z) dz^2) + L_xg$$

where $\xi(z) dz^2$ is a holomorphic quadratic differential on $M$ with respect to the complex structure induced by $g$. Moreover every such holomorphic quadratic differential occurs in decomposition (1).

Now the $C^\infty$ manifold structure on $M_{-1}/D_0$ follows as a consequence of fact that $D_0$ acts freely and that as a consequence of the theorem of Riemann-Roch the space of holomorphic quadratic differentials on $M$ has finite dimension $6p − 6$.

We summarize these facts as

**Theorem 1.3.** The tangent space to the manifold $M_{-1}/D_0$ at an element $[g] \in M_{-1}/D_0$ can be naturally identified with those symmetric two tensors which are trace free and divergence free and also (by taking real parts) with the holomorphic quadratic differentials on $M$, holomorphic with respect to the complex structure induced by $g$.

As we already stated, we are viewing $M_{-1}$ as a differentiable submanifold of the space of all symmetric tensors $S_2$. There is a natural "weak" $L_2$ Riemannian structure $\langle \langle \ , \ \rangle \rangle$ on $M_{-1}$, $\langle \langle \ , \ \rangle \rangle_g : T_gM_{-1} \times T_gM_{-1} \to R$ defined by

$$\langle \langle h^1, h^2 \rangle \rangle_g = \int_M h^1 \cdot h^2 \ d\mu(g)$$

where, $m$ local coordinates,

$$h^1 \cdot h^2 = g^{ab} g^{cd} h^1_{ac} h^2_{bd}$$

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TEICHMÜLLER SPACE IS A CELL

and where \{g^{ab}\} denotes the local representation of the inverse to the matrix \{g_{ij}\} of \(g\), \(d\mu(g)\) is the volume element of \(g\), and where the Einstein summation convention is used. One can also give an intrinsic formulation of (2) avoiding local coordinates, as follows.

Using the metric \(g\) we can transform \(h^1, h^2\) into \(1:1\) tensors \(H^1, H^2\) satisfying

\[
g(x)(H^i_x X_x, Y_x) = h^i(x)(X_x, Y_x), \quad i = 1, 2,
\]

for all \(X_x, Y_x \in T_x M\). Then each \(H^i\) is symmetric with respect to \(g\) and for \(x \in M\) the trace \(\text{tr}(H^1_x H^2_x)\) is a well-defined function (of \(x\)) on \(M\). Then (2) is equivalent to

\[
\langle h^1, h^2 \rangle_g = \int_M \text{tr}(H^1 H^2) \, d\mu(g).
\]

This \(L^2\)-Riemannian metric is \(\mathcal{D}\) invariant, a fact which follows immediately from the change of variables formula. Thus \(\mathcal{D}\) acts on \(M_{-1}\) as a group of isometries.

The important remark is that (1) is an \(L^2\)-orthogonal decomposition.

2. Dirichlet’s functional on Teichmüller space. Let \(g_0 \in M_{-1}\) and \([g_0]\) denote its class in \(M_{-1}/\mathcal{D}_0\). This fixed \(g_0\) will act as our base point. Let \(g \in M_{-1}\) be any other metric and let \(s: M \to M\) be viewed as a map from \((M, g)\) to \((M, g_0)\).

Using the metrics \(g\) and \(g_0\) one defines Dirichlet's energy functional

\[
E_g(s) = \frac{1}{2} \int_M |ds|^2 \, d\mu(g)
\]

where \(|ds|^2 = \text{trace}_g ds \otimes ds\) depends on both metrics \(g\) and \(g_0\), and again \(d\mu(g)\) is the volume element induced by \(g\).

We may assume that \((M, g_0)\) is isometrically embedded in some Euclidean \(\mathbb{R}^k\), which is possible by the Nash-Moser embedding theorem. Thus we can think of \(s: (M, g) \to (M, g_0)\) as a map into \(\mathbb{R}^k\) with Dirichlet’s integral having the equivalent form

\[
E_g(s) = \frac{1}{2} \sum_{i=1}^k \int_M g(x) (\nabla g s^i(x), \nabla g s^i(x)) \, d\mu(g).
\]

For fixed \(g\), the critical points of \(E\) are then said to be harmonic maps. From [1, 5 and 8] we have the following result.

THEOREM 2.1. Given metrics \(g\) and \(g_0\) there exists a unique harmonic map \(s(g): (M, g) \to (M, g_0)\). Moreover \(s(g)\) depends differentiably on \(g\) in any \(H^r\) topology, \(r > 2\), and \(s(g)\) is a \(C^\infty\) diffeomorphism.

Consider the function \(g \to E_g(s(g))\). This function on \(M_{-1}\) is \(\mathcal{D}\)-invariant and thus can be viewed as a function on Teichmüller space. To see this one must show that \(E_{f^*g}(s(f^*(g))) = E_g(s(g))\). Let \(c(g)\) be the complex structure associated to \(g\) given by Theorem 1.1. For \(f \in \mathcal{D}_0, f: (M, f^*c(g)) \to (M, c(g))\) is a holomorphic map, and consequently since the composition of harmonic maps and holomorphic maps is still harmonic, we may conclude, by uniqueness, that \(S(f^*g) = S(g) \circ f\). Since Dirichlet’s functional is invariant under complex holomorphic changes of coordinates, it follows immediately that

\[
E_{f^*(g)}(s(g) \circ f) = E_g(s(g)).
\]

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Consequently for \( [g] \in \mathcal{M}_{-1}/\mathcal{D}_0 \), define the \( C^\infty \) smooth function \( \hat{E}: \mathcal{M}_{-1}/\mathcal{D}_0 \to \mathbb{R} \) by \( \hat{E}([g]) = E_g(s(g)) \). We wish now to prove the main theorem of this note, namely

**Theorem 2.2.** Teichmüller space \( \mathcal{M}_{-1}/\mathcal{D}_0 \) is \( C^\infty \) diffeomorphic to \( \mathbb{R}^{6p-6} \).

To prove this result it suffices to show that \( \hat{E} \) has the following properties:

(i) \( \hat{E} \) is a proper map, i.e. the inverse image of bounded sets in \( \mathbb{R} \) under \( \hat{E} \) is compact in \( \mathcal{M}_{-1}/\mathcal{D}_0 \).

(ii) \( [g_0] \) is the only critical point of \( \hat{E} \).

(iii) \( [g_0] \) is a nondegenerate minimum.

Once (i) through (iii) are established the result follows immediately from the application of the well-known gradient deformations of Morse theory.

The proof of (i) follows from ideas due to Mumford, Schoen and Yau [7], and a result on equicontinuity of harmonic maps (Jost [6, p. 20]). Using a result of Mumford, Schoen and Yau show that \( E: \mathcal{M}_{-1}/\mathcal{D}_0 \to \mathbb{R} \) is proper; that is, \( E \) is proper on the true space \( \mathcal{R} \) of Riemann moduli.

Now suppose that \( E[g_n] \) is a bounded sequence. It then follows from [7] that \( \{g_n\} \) represents a sequence of a class of metrics in \( \mathcal{M}_{-1}/\mathcal{D}_0 \) all of whose injectivity radii are strictly bounded below. By a version of Mumford’s theorem due to Tromba [10], it follows that there is a subsequence of \( g_n \), call it again \( g_n \) and a sequence of diffeomorphisms \( f_n \in \mathcal{D} \) such that \( f_n^*g_n \) converges.

Let \( \gamma_n = f_n^*g_n; r_n = s_n \circ f_n \). Then \( E(\gamma_n, r_n) \) is a bounded sequence of real numbers, the \( \gamma_n \) all have injectivity radii strictly bounded below, and \( r_n: (M, \gamma_n) \to (M, g_0) \) is harmonic. We claim that one can find a subsequence \( f_n \) all of which are in the same homotopy class.

Suppose not. Then there is a subsequence of the \( f_n \) all in distinct homotopy classes. Again call the subsequence \( f_n \). From Jost’s result, the \( r_n = s_n \circ f_n \) are equicontinuous. Since the \( s_n \) are all homotopic to the identity, this gives a contradiction.

Thus we may assume the \( f_n \) are in one homotopy class, \( f_n = h_n \circ f, f \in \mathcal{D} \) fixed and \( h_n \in \mathcal{D}_0 \). Then necessarily \( h_n^*g_n \) (or more simply \([g_n]\)) converges. This proves properness on \( \mathcal{M}_{-1}/\mathcal{D}_0 \).

To show (ii), again let \( s = s(g): (M, g_0) \to (M, g_0) \) be the unique harmonic map determined by \( g \) and \( g_0 \). Let \( \mathcal{N}_g(z) dz^2 \) be the quadratic differential defined by

\[
\mathcal{N}_g(z) dz^2 = \sum_{i=1}^{k} \frac{\partial s^i}{\partial z} \cdot \frac{\partial s^i}{\partial \overline{z}} dz^2
\]

where \( s^i \) is the \( i \)th component function of \( s: (M, g) \to (M, g_0) \subseteq \mathbb{R}^k \), and \( z = x + iy \) are local conformal coordinates on \( (M, g) \). We next prove

**Theorem 2.3.** \( \mathcal{N}_g(z) dz^2 \) is a holomorphic quadratic differential on \( (M, c(g)) \).

**Proof.** Let \( \Omega \) denote the second fundamental form of \( (M, g_0) \subseteq \mathbb{R}^k \). Thus for each \( p \in M \), \( \Omega(p): T_p M \times T_p M \to T_p M^\perp \). Let \( \Delta \) denote the Laplacian maps from \( (M, g) \) to \( (M, g_0) \), and \( \Delta_\beta \) denote the Laplace-Beltrami operator on functions. Then if \( s \) is harmonic we have

\[
0 = \Delta s = \Delta_\beta s + \sum_{j=1}^{2} \Omega(s)(ds(e_j), ds(e_j))
\]
with $e_1(p), e_2(p)$ an orthonormal basis for $T_pM$ (w.r.t. the matrix $g$). $\mathcal{N}_g$ will be holomorphic if
\[ \frac{\partial}{\partial z} \left( \sum_{i=1}^{k} \frac{\partial s^i}{\partial z} \cdot \frac{\partial s^i}{\partial z} \right) = 0. \]
But this equals
\[ 2 \sum_{i=1}^{k} \Delta g s^i \cdot \frac{\partial s^i}{\partial z} \]
and by (4) we see that this in turn equals
\[ -2 \sum_{i=1}^{k} \sum_{j=1}^{2} \Omega^i(s) (ds(e_j), ds(e_j)) \cdot \frac{\partial s^i}{\partial z} \]
\[ = -2 \sum_{i=1}^{k} \sum_{j=1}^{2} \left\{ \sum_{i=1}^{k} \Omega^i(s) \left( ds(e_j), ds(e_j) \frac{\partial s^i}{\partial x} \right) + i\Omega(s) \left( ds(e_j), ds(e_j) \frac{\partial s^i}{\partial y} \right) \right\}. \]
Since $\Omega(p)$ takes value in $T_pM^\perp$ it follows that both the real and the imaginary parts of this expression vanish. □

From 1.2 we saw that $\zeta = \text{Re} (\mathcal{N}_g(z) dz^2)$ is a trace free divergence free symmetric two tensor on $(M, g)$. Let $\rho \in T[g]_M_{-1}/D_0$. We know from 1.3 that we may think of $\rho$ as a trace free divergence free symmetric two tensor. From [10] we have the following result:

**THEOREM 2.4.** $D \tilde{E}(\{g\}) = -\langle (\xi, \rho) \rangle_g$. Thus $[g]$ is a critical point of $\tilde{E}$ if $\rho = 0 \equiv \text{Re}(\mathcal{N}_g(z) dz^2)$, or if $\mathcal{N}_g(z) dz^2 \equiv 0$.

**THEOREM 2.5.** $\mathcal{N}_g(z) dz^2 = 0$ implies that $[g] = [g_0]$.

**PROOF.** $\mathcal{N}_g(z) dz^2 = \{ |s_x|^2 - |s_y|^2 + 2i(s_x, s_y) \} dz^2$. Thus $\mathcal{N}_g(z) dz^2$ implies that $s$ is weakly conformal. Since $s$ is a diffeomorphism it is conformal. Thus $s: (M, c(g)) \to (M, c(g_0))$ is holomorphic and hence $[g] = [g_0]$.

It remains to show (iii). It is clear that since $\mathcal{N}_g(z) dz^2 \equiv 0 (s(g_0) = \text{id})$ that $[g_0]$ is a critical point.

Let $\rho, \nu \in T[g_0]_M_{-1}/D_0$ be trace free, and divergence free symmetric two tensors. Then a straightforward computation yields

**THEOREM 2.6.** The second derivative or Hessian of $\tilde{E}$ at $[g_0]$ is given by the formula
\[ D^2 \tilde{E}(\{g_0\})(\rho, \nu) = 2 \int_M \rho \cdot \nu \, d\mu(g_0) = 2 \langle (\rho, \nu) \rangle_{g_0}. \]
Thus the Hessian of $\tilde{E}$ at $[g_0]$ is essentially the natural inner product on $T[g_0]_M_{-1}/D_0$ and hence a positive definite quadratic form. This concludes the proof of our main result 2.2.

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