CONJUGACY CLASSES IN ALGEBRAIC MONOIDS

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ABSTRACT. Let $M$ be a connected linear algebraic monoid with zero and a reductive group of units $G$. The following theorem is established.

THEOREM. There exist affine subsets $M_1, \ldots, M_k$ of $M$, reductive groups $G_1, \ldots, G_k$ with antiautomorphisms $\theta^*$, surjective morphisms $\theta_i : M_i \rightarrow G_i$, such that: (1) Every element of $M$ is conjugate to an element of some $M_i$, and (2) Two elements $a, b$ in $M_i$ are conjugate in $M$ if and only if there exists $x \in G_i$ such that $x\theta_i(a)x^* = \theta_i(b)$. As a consequence, it is shown that $M$ is a union of its inverse submonoids.

Introduction. The objects of study in this paper are connected linear algebraic monoids $M$ with zero. This means by definition that the underlying set of $M$ is an irreducible affine variety and that the product map is a morphism (i.e. a polynomial map). We will further assume that the group of units $G$ of $M$ is reductive. This means [1, 3] that the unipotent radical of $G$ is trivial. Then by [6, 10], $M$ is unit regular, i.e. $M = E(M)G$ where $E = E(M) = \{ e \in M | e^2 = e \}$. In this paper we study the conjugacy classes of $M$. An initial study was made by the author [8], where the general problem was reduced to nilpotent elements. The approach here is quite different, yielding a more complete answer. To be precise, we show that there exist affine subsets $M_1, \ldots, M_k$ of $M$, reductive groups $G_1, \ldots, G_k$ with antiautomorphisms $\theta^*$, surjective morphisms $\theta_i : M_i \rightarrow G_i$, $i = 1, \ldots, k$, such that: (1) Every element of $M$ is conjugate to an element of some $M_i$, and (2) If $a, b \in M_i$, then $a$ is conjugate to $b$ in $M$ if and only if there exists $x \in G_i$ such that $x\theta_i(a)x^* = \theta_i(b)$. As an application of this result, we show that $M$ is a union of its inverse submonoids. An inverse semigroup is a semigroup $S$ with the property that for each $a \in S$, there exists a unique $\bar{a} \in S$ such that $a\bar{a}a = a$ and $\bar{a}a\bar{a} = \bar{a}$. See [2]. Finally in §3, we use our main results to briefly analyze the conjugacy classes of nilpotent elements.

1. Preliminaries. Throughout this paper $\mathbb{Z}^+$ will denote the set of all positive integers and $K$ an algebraically closed field. Let $G$ be a connected linear algebraic group defined over $K$. The radical $R(G)$ is the maximal closed connected normal solvable subgroup of $G$ and the unipotent radical $R_u(G)$ is the group of unipotent elements of $R(G)$. We will assume throughout that $G$ is a reductive group, i.e., $R_u(G) = 1$. Then $R(G) \subseteq C(G)$, the center of $G$. Moreover $G = R(G)G_0$ where $G_0 = (G, G)$ is a semisimple group, i.e. $R(G_0) = 1$. Also [3, Theorem 27.5] $G_0$ is a product of the simple closed normal subgroups of $G$. In particular we have the following.

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FACT 1.1. If $H$ is a closed normal subgroup $G$, then $G = HCG(H)$. If $H_1, H_2, H'_1, H'_2$ are closed normal subgroups of $G$ with $G = H_1 H_2 = H'_1 H'_2$ then

$$G = (H_1 \cap H'_1)(H_1 \cap H'_2)(H_2 \cap H'_1)(H_2 \cap H'_2)R(G).$$

A connected diagonalizable subgroup of $G$ is called a torus. Let $T$ be a maximal torus of $G$. Then

$$R(G) \subseteq C(G) \subseteq C_G(T) = T.$$  

$W = N_G(T)/T$ is called the Weyl group of $G$ and is finite. A maximal closed connected solvable subgroup of $G$ is called a Borel subgroup. Let $B_1, B_2$ be Borel subgroups of $G$ with $T \subseteq B_1 \cap B_2$. Then [3, Theorem 28.3] $G$ is expressible as the following disjoint union:

$$G = \bigcup_{\sigma \in W} B_1 \sigma B_2.$$  

This is called the Bruhat decomposition of $G$. A subgroup of $G$ containing a Borel subgroup is called parabolic. Let $P$ be a parabolic subgroup of $G$ with $T \subseteq P$. Then there exists a parabolic subgroup $P^-$ of $G$ such that $T \subseteq P^-$ and $L = P \cap P^-$ is a reductive group. $P^-$ is called the opposite parabolic subgroup of $P$ relative to $T$ and $L$ is called a Levi factor of $P$. If $U = R_u(P)$, then [1, 3], $P = LU$ is a semidirect product. This is called the Levi decomposition of $P$. By Fact 1.1, we have

FACT 1.2. Let $G = G_1 G_2$ where $G_1, G_2$ are closed connected normal subgroups of $G$. Let $P$ be a parabolic subgroup of $G$. Then $P_i = P \cap G_i$ is a parabolic subgroup of $G_i$ ($i = 1, 2$) and $P = P_1 P_2$. If $P = LU$ is a Levi decomposition of $P$, then $P_i = L_i U_i, L = L_1 L_2, U = U_1 U_2$, where $L_i = L \cap G_i, U_i = U \cap G_i, i = 1, 2$.

The following result follows from [1, Theorem 28.7].

FACT 1.3. Let $P, Q$ be parabolic subgroups of $G$ with Levi decompositions, $P = L_1 U_1, Q = L_2 U_2$ such that $T \subseteq L_1 \cap L_2$. Then

$$P \cap Q = (U_1 \cap U_2)(L_1 \cap L_2)(L_1 \cap U_2)(L_1 \cap U_2).$$

By a (linear) algebraic monoid, we mean a monoid $M$ such that the underlying set is an affine variety and the product map is a morphism. The identity component of $M$ is denoted by $M^\circ$. We will assume that $M$ is connected (i.e. $M = M^\circ$) and that $M$ has a zero. We will further assume that the group of units $G$ of $M$ is reductive. Then by [6, 10], $M$ is unit regular, i.e. $M = E(M)G$. Here $E(M)$ is the idempotent set of $M$. We fix a maximal torus $T$ of $G$. We fix a maximal torus $T$ of $G$. If $\Gamma \subseteq E(T)$, then we let

$$C^\Gamma_G(\Gamma) = \{a \in G|ae = eae \text{ for all } e \in \Gamma\},$$

$$C^\Gamma_L(\Gamma) = \{a \in G|ea = eae \text{ for all } e \in \Gamma\}.$$  

Then $C_G(\Gamma) = \overline{C^\Gamma_G(\Gamma)} \cap \overline{C^\Gamma_L(\Gamma)}$ is a reductive group. If $e \in E(T)$, then by [5, 7], $C^\Gamma_G(e), C^\Gamma_L(e)$ are opposite parabolic subgroups of $G$. We let

$$G^\Gamma_e = \{a \in G|ae = e\}^\circ, \quad C^\Gamma_e = \{a \in G|ea = e\}^\circ,$$

$$G_e = \{a \in G|ae = ea = e\}^\circ = G^\Gamma_e \cap C_G(e),$$

$$G^\Gamma_e = \{a \in G|ae = ea = e\}.$$ 

Since $G_e \triangleleft C_G(e)$, we have by Fact 1.1,

$$C_G(e) = G_e C_G(e).$$
In particular, $\hat{G}_e = G_e T_e$. Now $eC_G(e)$ is the group of units of $eMe$ by [4]. In particular, $eC_G(e) = eC_G(e)$. Hence we have the surjective homomorphism: $a \to ea$ from $C_G(e)$ onto the reductive group $eC_G(e)$. Thus

$$R_u(C_G(e)) \subseteq G_e \triangleleft C_G(e).$$

Similarly

$$R_u(C_G(e)) \subseteq G_e \triangleleft C_G(e).$$

Since $C_G(e) = R_u(C_G(e))C_G(e)$, we get

$$G_e = R_u(C_G(e))G_e.$$

**LEMMA 1.4.** Let $e, f \in E(T)$. Then

$$C_G(e, f) = (G_e \cap G_f)((G_f \cap C_G(T_e))(G_e \cap C_G(T_f)))C_G(T_e, T_f).$$

**PROOF.** Now $C_G(f) \cap C_G(e)$ is a parabolic subgroup of $C_G(e)$ with Levi factor $C_G(e, f)$. Since $C_G(e) = G_e C_G(G_e)$, we have by Fact 1.2,

$$C_G(e, f) = [C_G(f) \cap G_e][C_G(f) \cap C_G(G_e)].$$

Similarly

$$C_G(e, f) = [C_G(e) \cap G_f][C_G(e) \cap C_G(G_f)].$$

Since $C_G(G_e) \subseteq C_G(T_e)$, $C_G(G_f) \subseteq C_G(T_f)$, we are done by Fact 1.1.

**LEMMA 1.5.** Let $e, f \in E(T)$. Then

(i) $G_e \cap C_G(f) = [G_e \cap C_G(T_f)][G_e \cap G_f],$

(ii) $G_e \cap C_G(f) = [G_e \cap C_G(T_f)][G_e \cap G_f].$

**PROOF.** We prove (i), as the proof of (ii) is similar. By Fact 1.3,

$$C_G(e, f) \cap C_G(f) = [R_u(C_G(e)) \cap C_G(f)][C_G(e) \cap C_G(f)].$$

Since $R_u(C_G(e)) \subseteq G_e$, we obtain

$$G_e \cap C_G(f) = [R_u(C_G(e)) \cap C_G(f)][G_e \cap C_G(f)].$$

By Facts 1.2, 1.3,

$$R_u(C_G(e)) \cap C_G(f) = [R_u(C_G(e)) \cap C_G(f)][R_u(C_G(e)) \cap C_G(f)]$$

$$\subseteq [G_e \cap G_f][R_u(C_G(e)) \cap G_f][R_u(C_G(e)) \cap C_G(G_f)]$$

$$\subseteq [G_e \cap G_f][G_e \cap C_G(T_f)].$$

Now $C_G(e) \cap C_G(f)$ is a parabolic subgroup of $C_G(e)$ with Levi decomposition

$$[C_G(e, f)][R_u(C_G(f)) \cap C_G(e)].$$

So by Fact 1.2,

$$G_e \cap C_G(f) = [G_e \cap C_G(f)][G_e \cap R_u(C_G(f))]$$

$$\subseteq [G_e \cap C_G(f)][G_e \cap G_f].$$

By Lemma 1.4,

$$C_G(e, f) = (C_G(e) \cap C_G(T_f)).$$

Since $G_e \cap C_G(T_f) \triangleleft C_G(e, f)$ and since the radical of $G_e \cap C_G(f)$ is contained in $T_e \subseteq G_e \cap C_G(T_f)$, we obtain

$$G_e \cap C_G(f) = (G_e \cap G_f)(G_e \cap C_G(T_f)).$$

Since $G_e \cap G_f \triangleleft G_e \cap C_G(f)$, the result follows.
LEMMA 1.6. Let $e \in E(\tilde{T})$. Then $C_G(T_e) = T_eC_G(G_e)$.

PROOF. Since $G_e \triangleleft C_G(G_e)$,

$$C_G(G_e) \subseteq C_G(T_e) \subseteq C_G(e) = G_eC_G(G_e).$$

So

$$C_G(T_e) = C_G(G_e)[G_e \cap C_G(T_e)] = C_G(G_e)T_e.$$

LEMMA 1.7. Let $e, f \in E(\tilde{T})$. Then

$$C^*_G(e) \cap C^*_G(f) = [G^*_e \cap C_G(T_f)][C_G(T_e, T_f)][G^*_f \cap C_G(T_e)][G^*_e \cap G^*_f].$$

PROOF. By Fact 1.3,

$$C^*_G(e) \cap C^*_G(f) = [G^*_e \cap C^*_G(e) \cap C^*_G(f)] = C^*_G(e)T_f.$$

Now $G^*_e \cap G^*_f \triangleleft C^*_G(e) \cap C^*_G(f)$. Also if $a \in G^*_f \cap C_G(T_e)$, $b \in G^*_e \cap C_G(T_f)$, then $a^{-1}b^{-1}ab \in G^*_e \cap G^*_f$. Moreover $C_G(T_e, T_f)$ normalizes $G^*_e \cap C_G(T_f)$ and $G^*_f \cap C_G(T_e)$. So we are done by Lemmas 1.4, 1.5.

LEMMA 1.8. Let $e, f \in E(\tilde{T})$, $a \in G^*_e$, $b \in C_G(T_e)$. If $ab \in C^*_G(f)$, then $a, b \in C^*_G(f)$. If $ab \in C^*_G(f)$, then $a, b \in C^*_G(f)$.

PROOF. Suppose $ab \in C^*_G(f)$. Now $a = a_1a_2$ for some $a_1 \in R_u(C^*_G(e))$, $a_2 \in G_e$. Then $a_2b \in C_G(e)$. So by Fact 1.3, $a_1, a_2 \in C^*_G(f)$. Then $a_2b \in C^*_G(e) \cap C^*_G(f)$. So by Fact 1.2, $a_2b = uv$ for some $u \in G_e \cap C^*_G(f)$, $v \in C_G(G_e) \cap C^*_G(f)$. So $u^{-1}a_2 = vb^{-1} \in G_e \cap C_G(T_e) = T_e \subseteq T \subseteq C^*_G(f)$. So $b \in C^*_G(f)$. Hence $a \in C^*_G(f)$.

PROPOSITION 1.9. Let $\Gamma \subseteq E(\tilde{T})$, $e_1, \ldots, e_{k+1} = f \in \Gamma$. Let $V = C_G(\Gamma)$, $Y_0 = G^*_f$, $Y_1 = G^*_e$, $Y_i = C_G(e_1, \ldots, e_{i-1}) \cap G^*_e$, for $i = 2, \ldots, k + 1$. Then

$$Y_0 \cdots Y_{k+1} \cap V = \prod_{i=1}^{k+1} V_{e_i}.$$
Moreover \( y_0 y_2 \cdots y_{k+1} \in C_G(T_{e_i}) \). By Lemma 1.8, \( y_0', y_0 y_2 \cdots y_{k+1} \in V \). So \( y_1' \in V_{e_1} \). By the induction hypothesis \( y_0 y_2 \cdots y_{k+1} \in V_{e_2} \cdots V_{e_{k+1}} \). This completes the proof.

2. **Main section.** We fix a connected linear algebraic monoid \( M \) with zero 0 and a reductive group of units \( G \). As usual two elements \( a, b \in M \) are conjugate \((a \sim b)\) if \( x^{-1} a x = b \) for some \( x \in G \). Note that for \( a \in M \), \( g \in G \), \( a g \sim g a \). We fix a maximal torus \( T \) of \( G \). Let \( W = N_G(T)/T \) denote the Weyl group of \( G \). We let \( \mathcal{R}, \mathcal{L}, \mathcal{K} \) denote the usual Green’s relations on \( M \) [2]. If \( a, b \in M \), then \( a \mathcal{K} b \) means \( aM = bM \), \( a \mathcal{L} b \) means \( Ma = Mb \), \( \mathcal{K} = \mathcal{R} \cap \mathcal{L} \). Let \( e \in E(T), \sigma = nT \in W \). Then we let

\[
\sigma = \sigma^{-1} e \sigma = n^{-1} e n \in E(T).
\]

We let

\[
M_{e, \sigma} = e C_G(e^\theta | \theta \in \langle \sigma \rangle ) \sigma.
\]

Our first result is that every element of \( M \) is conjugate to an element of some \( M_{e, \sigma} \).

**Lemma 2.1.** Let \( e \in E(T), \sigma = nT \in W, k \in \mathbb{Z}^+, x, y \in C_G(e^{\sigma^j} | 0 \leq j \leq k-1), x \in G^{t_{e^k}} \). Then \( exyn \sim eyn \).

**Proof.** We prove by induction on \( k \). First let \( k = 1 \). Then

\[
exyn = xynx = yne^x = yne^\sigma = yn = eyn.
\]

In general let \( k > 1 \). Then

\[
exyn = xynx = ynxn^{-1} y^{-1} yn = ex'yn
\]

where \( x' = ynxn^{-1} y^{-1} \in C_G(e^{\sigma^j} | 0 \leq j \leq k-2) \cap G_{t_{e^{k-1}}} \). So by the induction hypothesis \( ex'yn \sim eyn \).

**Theorem 2.2.** Every element of \( M \) is conjugate to an element of some \( M_{e, \sigma} \).

**Proof.** Let \( a \in M \). By [8, Corollary 2.3], there exists a maximal torus \( T_1 \) of \( G, e, f \in E(T_1) \) such that \( e \mathcal{K} f \). Since all maximal tori of \( G \) are conjugate, we can assume that \( T = T_1 \). There exists \( \theta = mT \in W \) such that \( e^\theta = f \). Thus \( e \mathcal{R} em \mathcal{L} f \). So \( em \mathcal{K} a \). Since \( eC_G(e) \) is the \( \mathcal{K} \)-class of \( e \), we see that \( a \in eC_G(e)m = eC_G(e) \theta \). Suppose inductively that \( a \in eC_G(e^\theta | j = 0, \ldots, k) \). Let \( H = C_G(e^\theta | j = 0, \ldots, k) \). So there exists \( x \in H \) such that \( a = exm \). By [5], \( C_H(e^{\theta^{k+1}}), C_H(\theta e^{\theta^{-1}}) \) are parabolic subgroups of \( H \) containing \( T \). By the Bruhat decomposition there exists \( \pi = n_1 T \in W(H), x_1 \in C_H(e^{\theta^{k+1}}), x_2 \in C_H(\theta e^{\theta^{-1}}) \) such that \( x = x_1 n_1 x_2 \). So

\[
exm = ex_1 n_1 x_2 m \sim (m^{-1} x_2 m)ex_1 n_1 m.
\]

Now \( m^{-1} x_2 m \in C_G(e^\theta | j = 1, \ldots, k+1) \cap C_G(e) \). So

\[
m^{-1} x_2 me = z e \quad \text{for some} \ z \in C_G(e^\theta | j = 0, \ldots, k+1).
\]

Thus

\[
a \sim ezx_1 n_1 m, \quad zx_1 \in C_H^j(e^{\theta^{k+1}}).
\]
Let $\lambda = \pi \theta = n_1 m T \in W$. We claim that $e^{\lambda_j} = e^{\theta_j}$ for $j = 0, \ldots, k+1$. For $j = 0$, this is obvious. So assume $e^{\theta_j} = e^{\lambda_j}$, $j \leq k$. Then $\pi \in C_W(e^{\theta_j})$. So
\[
e^{\lambda_{j+1}} = \left( e^{\theta_j} \right)^\lambda = \left( e^{\theta_j} \right)^\pi = e^{\theta_{j+1}}.
\]
Thus $y = xz_1 \in C_H^{e^{\lambda_{k+1}}}$. Hence $y = y_1 y_2$ for some $y_1 \in H^{e^{\lambda_{k+1}}}$, $y_2 \in C_H^{e^{\lambda_{k+1}}}$. By Lemma 2.1,
\[
a \sim e_{y_1} n y_1 m \sim e_{y_2} n_1 m, \quad y_2 \in C_G \left( e^{\lambda_j} \mid j = 0, \ldots, k+1 \right).
\]
Continuing this process, we see that there exist $\sigma \in nT \in W$ and $u \in C_G(e^{\sigma_j} \mid 0 \leq j \leq |W|) = C_G(e^\gamma \mid \gamma \in \langle \sigma \rangle)$ such that $a \sim e_{u^n}$. Then clearly $e_{u^n} \in M_{e,\sigma}$. This completes the proof of the theorem.

Schein [13] has shown that the full transformation semigroup on any set is a union of its inverse subsemigroups. The corresponding result, for the full matrix semigroup over a field, follows from the Fitting decomposition.

**Theorem 2.3.** (i) If $F$ is a commutative, idempotent submonoid of $M$, then $F_{NG}(F)$ is the maximal unit regular inverse submonoid of $M$ with idempotent set $F$.
(ii) If $F$ is a subsemilattice of $E(T)$ with $1 \in F$, then
\[F_{NG}(F) = F C_G(F) N_W(F).
\]
(iii) If $e \in E(T)$, $\sigma \in W$, $F = \langle 1, e^\theta \mid \theta \in \langle \sigma \rangle \rangle$, then $M_{e,\sigma} \subseteq F_{NG}(F)$.
(iv) $M$ is a union of its unit regular inverse submonoids.

**Proof.** (i) That $F_{NG}(F)$ is a submonoid of $M$ is obvious. Let $a \in F_{NG}(F)$, $a^2 = a$. So $a = fu$ for some $f \in F$, $u \in N_G(F)$. Then $fu = f$. Since $M$ is a matrix semigroup and $f$, $ufu^{-1}$ commute, we see that $f = uf^{-1}$. Thus $a = fu = fu f = f \in F$. So $F$ is the idempotent set of $F_{NG}(F)$. It follows that $F_{NG}(F)$ is the maximal unit regular submonoid of $M$ with idempotent set $F$. Since $F$ is commutative, it follows [2] that $F_{NG}(F)$ is an inverse semigroup.
(ii) Let $a \in N_G(F)$. Clearly $T \subseteq C_G(F)$. So $aTa^{-1} \subseteq C_G(aFa^{-1}) = C_G(F)$. So $T$, $aTa^{-1}$ are maximal tori of $C_G(F)$. Hence $b^{-1}aTa^{-1}b = T$ for some $b \in C_G(F)$.
Hence $b^{-1}a \in N_G(T) \cap N_G(F)$. So $a = b(b^{-1}a) \in C_G(F) N_W(F)$.
(iii), (iv) follow from (ii) and Theorem 2.2.

Now fix $e \in E(T)$, $\sigma = nT \in W$. Let $f = e^\sigma$, $\alpha + 1$ the order of $\sigma$. Let
\[V = C_G(e^\theta \mid \theta \in \langle \sigma \rangle)\]
So $V$ is a reductive group, $T \subseteq V$, $V^\sigma = V$, $M_{e,\sigma} = eV\sigma$. Now $\hat{V}_e = \{a \in V \mid ae = ea = e\} = \hat{T_e}V_e$ is a closed normal subgroup of $V$. Let
\[\Omega = \prod_{\theta \in \langle \sigma \rangle} \hat{V}_e^\theta = \prod_{\theta \in \langle \sigma \rangle} \hat{V}_e^\theta.
\]
Then $\Omega$ is a closed normal subgroup of $V$. If $x \in V$, let $x^* = nx^{-1}n^{-1} \in V$. Then $\Omega^* = \Omega$. So * induces an antiautomorphism * on the reductive group $G_{e,\sigma} = V/\Omega$. Define $\xi: M_{e,\sigma} \rightarrow G_{e,\sigma}$ as follows: If $a = evn \in M_{e,\sigma}$, $v \in V$, then, $\xi(a) = v\Omega \in G_{e,\sigma}$. Since $\hat{V}_e \subseteq \Omega$, $\xi$ is well defined. Note further that if $G_{e,\sigma}$ is replaced by $eV/e\Omega$ (which is isomorphic to $G_{e,\sigma}$ as an abstract group), then $\xi$ would also be a morphism of varieties.
THEOREM 2.4. Let $a, b \in M_{e, \sigma}$. Then $a$ is conjugate to $b$ in $M$ if and only if there exists $x \in G_{e, \sigma}$ such that $x \xi(a) x^* = \xi(b)$.

PROOF. For $a, b \in M_{e, \sigma}$, define $a \equiv b$ if $x \xi(a) x^* = \xi(b)$ for some $x \in G_{e, \sigma}$. We are to show that $\equiv = \sim$. Let

$$A = \{a \in V | e_u_n \sim e_a u_n \text{ for all } u \in V\}.$$ 

Clearly $\hat{V}_e \subseteq A$. Let $a, b \in A$. Then for $u \in V$, $e_a b u n \sim e_b u n \sim e u n$. So $A^2 \subseteq A$. Now let $a \in A$, $u \in V$. Then

$$e(na^{-1})u n \sim a^{-1} u n = e(a^{-1} u n)n$$

$$\sim e(n^{-1} u n)n = n^{-1} u n \sim e u n.$$ 

Thus $nA^{-1}n \subseteq A$. It follows that $\Omega \subseteq A$. Now let $m_1, m_2 \in M_{e, \sigma}$ such that $m_1 \equiv m_2$. Let $m_1 = e_u n$, $m_2 = e_v n$ where $u, v \in V$. Then there exists $x \in V$ such that $v \in \Omega x u n x^{-1} n^{-1}$. Since $\Omega \subseteq A$,

$$m_1 = e v n \sim e u n x^{-1} n^{-1} n = e u n x^{-1}$$

$$= x e u n x^{-1} \sim e u n = m_2.$$ 

This shows that $\equiv \subseteq \sim$.

Conversely let $m_1, m_2 \in M_{e, \sigma}$ such that $m_1 \sim m_2$. Then there exists $X_1 \in G$ such that

(1) $$X_1 m_1 = m_2 X_1.$$ 

Let $m_1 = e_u n$, $m_2 = e_v n$ where $u, v \in V$. Then by (1),

$$X_1 e \sim X_1 e u n = e v n X_1 \sim e u n.$$ 

So $X_1 e = e X_1 e$ and $X_1 \in C_G^e(e)$. Also by (1),

$$f X_1 = n^{-1} e u n \sim m_2 X_1 = X_1 m_1 \sim m_1 \sim f.$$ 

Thus $X_1 \in C_G^e(e) \cap C_G^f(f)$. By Lemma 1.7, $X_1 \in X[G_e^e \cap G_f^f]$ for some

$$X \in [C_G(T_e) \cap C_G^e][C_G(T_f) \cap C_G^f].$$ 

Since $m_1 = e m_1$, $m_2 = m_2 f$, we see by (1) that

(2) $$X e u n = e v n X.$$ 

Now $X = axb$ for some

(3) $$a \in C_G(T_e) \cap G_f^f, \quad x \in C_G(T_e, T_f), \quad b \in C_G(T_f) \cap G_e^e.$$ 

So by (2), $e a x u n = e v n x b$. Then $e a x u = e v n x b n^{-1}$ and $e a u, v n x b n^{-1} \in C_G(e)$. So

(4) $$a x u = v n x b n^{-1} z \quad \text{for some } z \in \hat{G}_e.$$ 

Now $n x b n^{-1} \in C_G(T_e)$. So by Lemma 1.6, $n x b n^{-1} = \eta t$ for some $t \in T$, $\eta \in C_G(G_e)$. So $v t \in V \subseteq C_G(e)$. So $v t = v' v''$ for some $v' \in C_V(V_e) \subseteq C_G(T_e)$, $v'' \in V_e \subseteq G_e$. Also $u = u' u''$ for some $u' \in C_V(V_e) \subseteq C_G(T_e)$, $u'' \in V_e \subseteq G_e$. Then

$$a x u = v n x b n^{-1} z = v t \eta z = v' v'' \eta z = v' \eta v'' z.$$ 

So

(5) $$a x u' = v' \eta (v'' z (u'')^{-1}).$$
Let \( z' = v''z(u'')^{-1} \). Then \( z' \in \hat{G}_e \). So \( z'h = hz' = h \) for all \( h \in E(M) \) with \( h \leq e \). Now \( axu', v', \eta \in C_G(T_e) \). So by (5), \( z' \in C_G(T_e) \). Thus \( z'h = hz' \) for all \( h \in E(T) \) with \( h \geq e \). So for any maximal chain \( \Gamma \) of \( E(T) \) with \( e \in \Gamma \), \( z' \in C_G(\Gamma) = T \subseteq V \).

Let \( u_1 = u'(z')^{-1}v'' \in V \). Then by (5),

\[
axu_1 = v'v'' = v'v'' \eta = vtn = vnxbn^{-1}.
\]

Also \( z = (v'')^{-1}z' u'' \in V \cap \hat{G}_e = \hat{V}_e \). So

(6) \[
axu_1 = vnxbn^{-1}, \quad u_1, v \in V, \quad z \in \hat{V}_e.
\]

Now \( xb \in C^r_{G}(e) \). So \( nxbn^{-1} \in C^r_{G}(\sigma \sigma^{-1}) \). Thus \( ax \in C^r_{G}(\sigma \sigma^{-1}) \). By (3), Lemma 1.8, \( a, x \in C^r_{G}(\sigma \sigma^{-1}) \). So \( x \in C^r_{G}(\sigma \sigma^{-1}) \cap C_G(T_e, T_f) \). Hence we can factor

(7) \[
x = y_1x_1 \quad \text{for some} \quad y_1 \in C^r_{\sigma i\sigma^{-1}} \cap C_G(T_e, T_f), \quad x_1 \in C_G(T_{\sigma i\sigma^{-1}}, T_e, T_f).
\]

Also \( a \in C^r_G(\sigma \sigma^{-1}) \cap C_G(T_e) \cap G^l_f \). So working within \( C_G(T_e) \) and applying Lemma 1.5, we can factor

(8) \[
a = c_1a_1 \quad \text{for some} \quad c_1 \in C^r_{\sigma i\sigma^{-1}} \cap G^l_f \cap C_G(T_e), \quad a_1 \in C_G(T_e, T_{\sigma i\sigma^{-1}}) \cap G^l_f.
\]

Now by (6),

\[
c_1a_1y_1x_1u_1 = vnyn_1x_1bn^{-1}.
\]

So

(9) \[
wa_1x_1u_1 = vnyn_1x_1n^{-1}
\]

where

\[
w = c_1a_1y_1a_1^{-1}[((a_1x_1u_1)(nb^{-1}n^{-1})(a_1x_1u_1)^{-1}] \in C^r_G(\sigma \sigma^{-1}).
\]

Suppose now inductively that

(10) \[
x = y_1 \cdots y_kx_k,
\]

where

(11) \[
y_i \in C_G(T_f, T_{i+1\sigma^{-1}}|j = 0, \ldots, i - 1) \cap G^r_{\sigma i\sigma^{-1}}, \quad i = 1, \ldots, k;
\]
\[
x_k \in C_G(T_f, T_{i\sigma^{-1}}|j = 0, \ldots, k).
\]

Further assume that there exist

\[
w_i \in C_G(T_{i\sigma^{-1}}|i + 1 \leq j \leq k) \cap G^r_{\sigma i\sigma^{-1}}, \quad i = 1, \ldots, k;
\]
\[
a_k \in C_G(T_{i\sigma^{-1}}|i = 0, \ldots, k) \cap G^l_f
\]

such that

(12) \[
w_k \cdots w_1a_kx_ku_1 = vnyn_kx_kn^{-1}.
\]

Note that (7)--(9) show (10)--(12) to be valid for \( k = 1 \). Now

\[
vnyn_kx_kn^{-1} \in C^r_G(\sigma^{k+1}e \sigma^{-k-1}).
\]

So by (12), \( w_k \cdots w_1a_kx_k \in C^r_G(\sigma^{k+1}e \sigma^{-k-1}) \). Repeated use of Lemma 1.8 shows that \( u_1, \ldots, w_k, a_k, x_k \in C^r_G(\sigma^{k+1}e \sigma^{-k-1}) \). So by Lemma 1.5, we can factor for
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\[ w_i = q_i w_i', \quad w_i' \in C_G(T_{\sigma^i e \sigma^{-j}} | i + 1 \leq j \leq k + 1) \cap G_{\sigma^i e \sigma^{-j}-i}, \]
\[ q_i \in G_{\sigma^{i+1} e \sigma^{-i}} \cap C_G(T_{\sigma^i e \sigma^{-j}} | i + 1 \leq j \leq k), \]
\[ a_k = c_k a_{k+1}, \quad a_{k+1} \in C_G(T_{\sigma^i e \sigma^{-j}} | 0 \leq j \leq k + 1) \cap G_{\sigma^i}, \]
\[ c_{k+1} \in C_G(T_{\sigma^i e \sigma^{-j}} | 0 \leq j \leq k) \cap G_{\sigma^{k+1} e \sigma^{-k-1}} \cap G_{\sigma^i}, \]
\[ x_k = y_k x_{k+1}, \quad x_{k+1} \in C_G(T_{\sigma^i}, T_{\sigma^i e \sigma^{-j}} | 0 \leq j \leq k + 1), \]
\[ y_{k+1} \in C_G(T_{\sigma^i}, T_{\sigma^i e \sigma^{-j}} | 0 \leq j \leq k) \cap G_{\sigma^{k+1} e \sigma^{-k-1}}. \]

Let
\[ q_i' = w_k \cdots w_{i+1} q_i w_k \cdots w_{i+1}^{-1} \in G_{\sigma^i e \sigma^{-j}}^{-1}, \quad i = 1, \ldots, k - 1, \]
\[ c_{k+1}' = w_k' \cdots w_i' c_{k+1} w_k \cdots w_i^{-1} \in G_{\sigma^i e \sigma^{-j}}^{-1}, \]
\[ y_{k+1}' = w_k' \cdots w_1' a_{k+1} y_{k+1} (w_k' \cdots w_1' a_{k+1})^{-1} \in G_{\sigma^i e \sigma^{-j}}^{-1}, \]
\[ p = q_k q_{k-1}' \cdots q_1 q_{k+1}' y_{k+1}' \in G_{\sigma^i e \sigma^{-j}}^{-1}. \]

Then
\[ w_k \cdots w_1 a_k x_k = p w_k' \cdots w_i' a_{k+1} x_{k+1}. \]

So by (12),
\[ w_{k+1}' \cdots w_1' a_{k+1} x_{k+1} u_1 = v y_{k+1} x_{k+1} n^{-1}, \]

where
\[ w_{k+1}' = v n y_k^{-1} n^{-1} v^{-1} p \in G_{\sigma^i e \sigma^{-j}}^{-1}. \]

This completes the induction step in (10)–(12). So (10) is valid for all \( k \in \mathbb{Z}^+ \). In particular it is valid for \( k = \alpha \), where \( \sigma^{\alpha+1} = 1 \). Then
\[ x = y_1 \cdots y_\alpha x_\alpha, \quad x_\alpha \in C_G(e^\theta | \theta \in \langle \sigma \rangle) = V. \]

Now by (4), (6)
\[ axu = vnxbn^{-1} z, \quad z \in \hat{V}_e. \]

Let \( Y_0 = G_{\sigma^i}^l, \quad Y_1 = G_{\sigma^i e \sigma^{-j}}^l, \)
\[ Y_i = C_G(\sigma^j e \sigma^{-j} | j = 1, \ldots, i - 1) \cap G_{\sigma^i e \sigma^{-j}}^l, \quad i \geq 2. \]

Then \( Y_j \) normalizes \( Y_i \) for \( j \geq i \geq 1 \). Also \( V \) normalizes \( Y_i \) for all \( i \). By (3), (11) we see that
\[ a \in Y_0, \quad nbn^{-1} \in Y_1, \quad y_i \in Y_i, \quad 1 = 1, \ldots, \alpha. \]

Also, since \( \sigma^{\alpha+1} = 1 \), we see by (11) that
\[ ny_\alpha n^{-1} \in Y_{\alpha+1}, \quad 1 = 1, \ldots, \alpha - 1, \quad ny_\alpha n^{-1} \in V_e. \]

Since \( x_\alpha, u, v \in V \) and \( V^\sigma = V \), we see by (13)–(16),
\[ v(x_\alpha u x_\alpha^*)^{-1} = v n x_\alpha n^{-1} u^{-1} x_\alpha^{-1}, \]
\[ = axuz^{-1} n b^{-1} x^{-1} x_\alpha n^{-1} u^{-1} x_\alpha^{-1}, \]
\[ = [a y_1 \cdots y_\alpha x_\alpha u z^{-1} u^{-1} x_\alpha^{-1}] (x_\alpha u), \]
\[ \times [n b^{-1} x^{-1} (y_1 \cdots y_\alpha^{-1}) x_\alpha n^{-1}] (x_\alpha u)^{-1} \]
\[ \in \hat{V}_e Y_0 Y_1 \cdots Y_\alpha. \]

Since \( \sigma^{\alpha e \sigma^{-\alpha}} = f \), we see by Proposition 1.9 that \( v(x_\alpha u x_\alpha^*)^{-1} \in \Omega \). Thus \( m_1 = eun = evn = m_2 \). This completes the proof of the theorem.
The proof of the above theorem shows

**COROLLARY 2.5.** Let \( a, b \in M_{e, \sigma} \). Then \( a \sim b \) if and only if there exists \( x \in V = C_G(e^\theta | \theta \in \{\sigma\}) \) such that \( x^{-1}ax = b \).

**COROLLARY 2.6.** Let \( D = eC_G(e) \) denote the group of units of \( eMe \), \( h \in E(eT), \theta = mt \in C_W(e) \). Then \( M_{h, \theta} = (eMe)_{h, e\theta} \) and \( G_{h, \theta} \cong D_{h, e\theta} \). If \( a, b \in M_{h, \theta} \), then \( a \) is conjugate to \( b \) in \( M \) if and only if \( a \) is conjugate to \( b \) in \( eMe \).

**PROOF.** Let 

\[ V = C_G(h^\gamma | \gamma \in \{\theta\}), \quad Y = C_D(h^\gamma | \gamma \in \{\theta\}) \]

Let \( a \in Y \). Then \( a = ex \) for some \( x \in C_G(e) \). For \( \gamma \in \{\theta\} \),

\[ xh^\gamma = xeh^\gamma = ah^\gamma = h^\gamma a = h^\gamma ex = h^\gamma x. \]

So \( x \in C_V(e) \) and \( Y = eC_V(e) \). Now \( V = V_hC_V(V_h) = V_hC_V(e) \). Hence

\[ M_{h, \theta} = hV\theta = hC_V(e)\theta = heC_V(e)\theta = hY\theta = (eMe)_{h, e\theta}. \]

Let \( \Omega = \prod_{\gamma \in \{\theta\}} V_\gamma \). Since \( V = V_hC_V(e), h \leq e \),

\[ G_{h, \theta} = V/\Omega \cong C_V(e)/C_\Omega(e) \cong eV/eC_\Omega(e). \]

By Proposition 1.9,

\[ C_\Omega(e) = \prod_{\gamma \in \{\theta\}} [V_\gamma \cap C_G(e)]. \]

It follows that \( eV/eC_\Omega(e) = D_{h, e\theta} \). We are now done by Theorem 2.4.

**CONJECTURE 2.7.** Let \( a, b \in eMe \). Then \( a \) is conjugate to \( b \) in \( M \) if and only if \( a \) is conjugate to \( b \) in \( eMe \).

**CONJECTURE 2.8.** Let \( Y = \{M_{e, \sigma} | e \in E(T), \sigma \in W\} \), \( Y_0 \) the set of maximal elements (with respect to inclusion) of \( Y \). Then if \( Y_1, Y_2 \in Y_0, a \in Y_1, b \in Y_2, a \sim b, \) then \( Y_1^\theta = Y_2 \) for some \( \theta \in W \).

Let \( g \in G \). Then the map: \( x \rightarrow gx^{-1}g^{-1} \) is an antiautomorphism of \( G \). We will call such an antiautomorphism an inner antiautomorphism.

**EXAMPLE 2.9.** Let \( n \in Z^+, M = M_n(K) \). Let \( h = \prod_{\theta \in \{\sigma\}} e^\theta, r \) the rank of \( h \).

Then \( G_{e, \sigma} \cong GL(r, K) \) and \( * \) is an inner antiautomorphism.

**EXAMPLE 2.10.** Let \( M = \{A \otimes B | A, B \in M_2(K)\} \). Then the possibilities for \( G_{e, \sigma} \) are \( G, SL(2, K), PGL(2, K), G_m, \{1\} \). In all cases, \( * \) is inner.

**CONJECTURE 2.11.** If the simple components of \( G \) are all of type \( A_t \), then \( * \) is necessarily inner.

By [3, Theorem 27.4], an antiautomorphism of a semisimple group is the composition of an inner antiautomorphism and an automorphism determined by an automorphism of the Dynkin diagram of the group.

**CONJECTURE 2.12.** For all \( t \in R(G_{e, \sigma}), t^* = t^{-1} \) and hence \( * \) is completely determined by its action on the semisimple group \( G_{e, \sigma}' = (G_{e, \sigma}, G_{e, \sigma}) \).

3. Nilpotent elements. We continue from [8] the analysis of conjugacy classes of nilpotent elements of \( M \). It was shown in [8] that the conjugacy classes of minimal nilpotent elements (in the \( J \)-class ordering) is always finite. Renner [12] has introduced the finite fundamental inverse monoid \( Ren(M) = N_G(T)/T \) and
used it to generalize the Bruhat decomposition to $M$. We easily have

**Proposition 3.1.** Let $e \in E(\bar{T})$, $\sigma = nT \in W$, $k \in \mathbb{Z}^+$. Then the following conditions are equivalent:

(i) $a^k = 0$ for some $a \in M_{e,\sigma}$,
(ii) $M_{e,\sigma}^k = 0$,
(iii) $(e\sigma)^k = 0$ in $\text{Ren}(M)$,
(iv) $e^\sigma \cdots e^\sigma = 0$.

Since $V = C_G(e^\theta | \theta \in (\sigma))$ is a reductive group, we see that any closed normal subgroup of $V$ containing $T$, must equal $V$. Thus

**Proposition 3.2.** Let $e \in E(\bar{T})$, $\sigma \in W$. Then $G_{e,\sigma}$ is trivial if and only if $T = \prod_{\theta \in \sigma} T_e^\theta$.

In particular, we see that $G_{e,\sigma}$ trivial implies that $e\sigma$ is nilpotent. If the groups $G_{e,\sigma}$ are trivial for all nilpotent $e\sigma$, then by Theorems 2.2, 2.4, the number of conjugacy classes of nilpotent elements in $M$ is finite.

**Conjecture 3.3.** The number of conjugacy classes of nilpotent elements of $M$ is finite if and only if the groups $G_{e,\sigma}$ are trivial for all nilpotent $e\sigma$.

**Example 3.4.** If $M = M_n(K)$, then we see by Example 2.9 that the groups $G_{e,\sigma}$ are trivial for nilpotent $e\sigma$.

**Example 3.5.** Let $G_0 = \{A \otimes (A^{-1})^t | A \in \text{SL}(3, K)\}$, $G = K^*G_0$, $M = \overline{KG_0}$. Let $S = M \setminus G$. Then

$$E(S) = \{e \otimes f | e^2 = e, f^2 = f \in M_3(K), ef^t = f^t e = 0\}.$$  

In particular

$$e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in E(M).$$

Also if

$$\sigma = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in W(G),$$

then $e^\sigma = f$ and $(e\sigma)^2 = 0$. The group $G_{e,\sigma}$ can be seen to be the one dimensional torus with $*$ being given by $x \mapsto x^{-1}$. Thus by Theorem 2.4, the number of conjugacy classes of nilpotent elements of $M$ is infinite. However if $C$ denotes the center of $G$, then the number of conjugacy classes of nilpotent elements in $M/C$ is finite.

**Example 3.6.** Suppose $\text{char } K \neq 2$, $n \in \mathbb{Z}^+$, $n \geq 2$. For $r \in \mathbb{Z}^+$, let $J_r$ denote the $r \times r$ matrix

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

Let $G_0$ consist of all $A \in \text{SL}(2n + 1, K)$ such that

$$A^t \begin{bmatrix} 1 & 0 \\ 0 & J_{2n} \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & J_{2n} \end{bmatrix}.$$
Thus [3, §7.2], $G_0$ is the special orthogonal group of type $B_n$. Let $G = K^*G_0$, $M = K^*G_0$. Then

$$e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_n \end{bmatrix} \in E(M).$$

If

$$\sigma = \begin{bmatrix} \pm 1 & 0 \\ 0 & J_{2n} \end{bmatrix} \in W(G),$$

then $e^\sigma = f$ and $(ee)^2 = 0$. It can be seen that $G_{e,\sigma} \cong \text{PGL}(n, K)$ with the antiautomorphism $^*$ on $G_{e,\sigma}$ given by $A \rightarrow J_nA^tJ_n$. Thus by Theorem 2.4, the number of conjugacy classes of nilpotent elements of $M$ is infinite. This gives a counterexample to [8, Conjectures 4.5, 4.6]. Note also that for $n \geq 3$, $^*$ is not inner.

The above examples suggest

**Conjecture 3.7.** Suppose that the center of $G$ is one dimensional. Then the number of conjugacy classes of nilpotent elements of $M$ is finite if and only if $\text{Ren}(M)$ is isomorphic to the symmetric inverse semigroup of some finite set.

**References**


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