CONJUGACY CLASSES IN ALGEBRAIC MONOIDS

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ABSTRACT. Let M be a connected linear algebraic monoid with zero and a reductive group of units G. The following theorem is established.

THEOREM. There exist affine subsets $M_1, \ldots, M_k$ of M, reductive groups $G_1, \ldots, G_k$ with anti-automorphisms $\theta_i$, surjective morphisms $\theta_i : M_i \rightarrow G_i$, such that: (1) Every element of M is conjugate to an element of some $M_i$, and (2) Two elements $a, b$ in $M_i$ are conjugate in M if and only if there exists $x \in G_i$ such that $x\theta_i(a)x^* = \theta_i(b)$. As a consequence, it is shown that M is a union of its inverse submonoids.

Introduction. The objects of study in this paper are connected linear algebraic monoids M with zero. This means by definition that the underlying set of M is an irreducible affine variety and that the product map is a morphism (i.e., a polynomial map). We will further assume that the group of units G of M is reductive. This means [1, 3] that the unipotent radical of G is trivial. Then by [6, 10], M is unit regular, i.e., $M = E(M)G$ where $E = E(M) = \{ e \in M | e^2 = e \}$. In this paper we study the conjugacy classes of M. An initial study was made by the author [8], where the general problem was reduced to nilpotent elements. The approach here is quite different, yielding a more complete answer. To be precise, we show that there exist affine subsets $M_1, \ldots, M_k$ of M, reductive groups $G_1, \ldots, G_k$ with anti-automorphisms $\theta_i$, surjective morphisms $\theta_i : M_i \rightarrow G_i$, $i = 1, \ldots, k$, such that: (1) Every element of M is conjugate to an element of some $M_i$, and (2) If $a, b \in M_i$, then a is conjugate to b in M if and only if there exists $x \in G_i$ such that $x\theta_i(a)x^* = \theta_i(b)$. As an application of this result, we show that M is a union of its inverse submonoids. An inverse semigroup is a semigroup S with the property that for each $a \in S$, there exists a unique $\bar{a} \in S$ such that $a\bar{a}a = a$ and $\bar{a}a\bar{a} = \bar{a}$. See [2]. Finally in §3, we use our main results to briefly analyze the conjugacy classes of nilpotent elements.

1. Preliminaries. Throughout this paper $Z^+$ will denote the set of all positive integers and K an algebraically closed field. Let G be a connected linear algebraic group defined over K. The radical $R(G)$ is the maximal closed connected normal solvable subgroup of G and the unipotent radical $R_u(G)$ is the group of unipotent elements of $R(G)$. We will assume throughout that G is a reductive group, i.e., $R_u(G) = 1$. Then $R(G) \subseteq C(G)$, the center of G. Moreover $G = R(G)G_0$ where $G_0 = (G, G)$ is a semisimple group, i.e. $R(G_0) = 1$. Also [3, Theorem 27.5] $G_0$ is a product of the simple closed normal subgroups of G. In particular we have the following.

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FACT 1.1. If $H$ is a closed normal subgroup $G$, then $G = H C_G(H)$. If $H_1, H_2, H'_1, H'_2$ are closed normal subgroups of $G$ with $G = H_1 H_2 = H'_1 H'_2$ then

$$G = (H_1 \cap H'_1)(H_1 \cap H'_2)(H_2 \cap H'_1)(H_2 \cap H'_2) R(G).$$

A connected diagonalizable subgroup of $G$ is called a torus. Let $T$ be a maximal torus of $G$. Then

$$R(G) \subseteq C(G) \subseteq C_G(T) = T.$$

$W = N_G(T)/T$ is called the Weyl group of $G$ and is finite. A maximal closed connected solvable subgroup of $G$ is called a Borel subgroup. Let $B_1, B_2$ be Borel subgroups of $G$ with $T \subseteq B_1 \cap B_2$. Then [3, Theorem 28.3] $G$ is expressible as the following disjoint union:

$$G = \bigcup_{\sigma \in W} B_1 \sigma B_2.$$

This is called the Bruhat decomposition of $G$. A subgroup of $G$ containing a Borel subgroup is called parabolic. Let $P$ be a parabolic subgroup of $G$ with $T \subseteq P$. Then there exists a parabolic subgroup $P^-$ of $G$ such that $T \subseteq P^-$ and $L = P \cap P^-$ is a reductive group. $P^-$ is called the opposite parabolic subgroup of $P$ relative to $T$ and $L$ is called a Levi factor of $P$. If $U = R_u(P)$, then [1, 3], $P = LU$ is a semidirect product. This is called the Levi decomposition of $P$. By Fact 1.1, we have

FACT 1.2. Let $G = G_1 G_2$ where $G_1, G_2$ are closed connected normal subgroups of $G$. Let $P$ be a parabolic subgroup of $G$. Then $P_i = P \cap G_i$ is a parabolic subgroup of $G_i$ ($i = 1, 2$) and $P = P_1 P_2$. If $P = LU$ is a Levi decomposition of $P$, then $P_1 = L_1 U_1$, $L = L_1 L_2$, $U = U_1 U_2$, where $L_i = L \cap G_i$, $U_i = U \cap G_i$, $i = 1, 2$.

The following result follows from [1, Theorem 28.7].

FACT 1.3. Let $P, Q$ be parabolic subgroups of $G$ with Levi decompositions, $P = L_1 U_1$, $Q = L_2 U_2$ such that $T \subseteq L_1 \cap L_2$. Then

$$P \cap Q = (U_1 \cap U_2)(U_1 \cap L_2)(L_1 \cap U_2)(L_1 \cap L_2).$$

By a (linear) algebraic monoid, we mean a monoid $M$ such that the underlying set is an affine variety and the product map is a morphism. The identity component of $M$ is denoted by $M^e$. We will assume that $M$ is connected (i.e. $M = M^e$) and that $M$ has a zero. We will further assume that the group of units $G$ of $M$ is reductive. Then by [6, 10], $M$ is unit regular, i.e. $M = E(M) G$. Here $E(M)$ is the idempotent set of $M$. We fix a maximal torus $T$ of $G$. We fix a maximal torus $T$ of $G$. If $\Gamma \subseteq E(T)$, then we let

$$C_G^\Gamma(\Gamma) = \{a \in G|ae = eae \text{ for all } e \in \Gamma\},$$

$$C_G^\Gamma(\Gamma) = \{a \in G|ea = eae \text{ for all } e \in \Gamma\}.$$

Then $C_G(\Gamma) = C_G^\Gamma(\Gamma) \cap C_G^\Gamma(\Gamma)$ is a reductive group. If $e \in E(T)$, then by [5, 7], $C_G^E(e) \cap C_G^L(e)$ are opposite parabolic subgroups of $G$. We let

$$G^e_c = \{a \in G|ae = e\}^c,$$

$$G^l_e = \{a \in G|ea = e\}^c.$$

$$G^e_c = \{a \in G|ae = ea\} = G^e_c \cap C_G(e),$$

$$G^l_e = \{a \in G|ae = ea\} = \epsilon.$$

Since $G^e \triangleleft C_G(e)$, we have by Fact 1.1,

$$C_G(e) = G^e C_G(G^e).$$
In particular, $\hat{G}_e = G_e \hat{T}_e$. Now $eC_G(e)$ is the group of units of $eMe$ by [4]. In particular, $eC_G^l(e) = eC_G(e)$. Hence we have the surjective homomorphism: $a \to ea$ from $C^l_G(e)$ onto the reductive group $eC_G(e)$. Thus

$$Ru(C^l_G(e)) \subseteq G_e^r \triangleleft C^l_G(e).$$

Similarly

$$Ru(C^r_G(e)) \subseteq G^r_e \triangleleft C^r_G(e).$$

Since $C^r_G(e) = Ru(C^r_G(e))C_G(e)$, we get

$$G^r_e = Ru(C^r_G(e))G_e.$$

**LEMMA 1.4.** Let $e, f \in E(T)$. Then

$$C_G(e, f) = (G_e \cap G_f)(G_f \cap C_G(T_e))(G_e \cap C_G(T_f))C_G(T_e, T_f).$$

**PROOF.** Now $C_G^l(f) \cap C_G(e)$ is a parabolic subgroup of $C_G(e)$ with Levi factor $C_G(e, f)$. Since $C_G(e) = G_eC_G(G_e)$, we have by Fact 1.2,

$$C_G(e, f) = [C_G(f) \cap G_e][C_G(f) \cap C_G(G_e)].$$

Similarly

$$C_G(e, f) = [C_G(e) \cap G_f][C_G(e) \cap C_G(G_f)].$$

Since $C_G(G_e) \subseteq C_G(T_e)$, $C_G(G_f) \subseteq C_G(T_f)$, we are done by Fact 1.1.

**LEMMA 1.5.** Let $e, f \in E(T)$. Then

(i) $G^r_e \cap C^r_G(f) = [G^r_e \cap C_G(T_f)][G^r_e \cap G^r_f],$

(ii) $G^r_e \cap C^l_G(f) = [G^r_e \cap C_G(T_f)][G^r_e \cap G^l_f].$

**PROOF.** We prove (i), as the proof of (ii) is similar. By Fact 1.3,

$$C^r_G(e) \cap C^r_G(f) = [Ru(C^r_G(e)) \cap C^r_G(f)][C_G(e) \cap C_G(f)].$$

Since $Ru(C^r_G(e)) \subseteq G^r_e$, we obtain

$$G^r_e \cap C^r_G(f) = [Ru(C^r_G(e)) \cap C^r_G(f)][G_e \cap C^r_G(f)].$$

By Facts 1.2, 1.3,

$$Ru(C^r_G(e)) \cap C^r_G(f) = [Ru(C^r_G(e)) \cap C^r_G(f)][Ru(C^r_G(e)) \cap C^r_G(f)]$$

$$\subseteq [G^r_e \cap G^r_f][Ru(C^r_G(e)) \cap G_f][Ru(C^r_G(e)) \cap C_G(G_f)]$$

$$\subseteq [G^r_e \cap G^r_f][G^r_e \cap C_G(T_f)].$$

Now $C_G(e) \cap C^r_G(f)$ is a parabolic subgroup of $C_G(e)$ with Levi decomposition

$$[C_G(e, f)][Ru(C^r_G(f)) \cap C_G(e)].$$

So by Fact 1.2,

$$G_e \cap C^r_G(f) = [G_e \cap C_G(f)][G_e \cap Ru(C^r_G(f))]$$

$$\subseteq [G_e \cap C_G(f)][G^r_e \cap G^r_f].$$

By Lemma 1.4,

$$C_G(e, f) = (C_G(e) \cap G_f)(C_G(e) \cap C_G(T_f)).$$

Since $G_e \cap C_G(f) \triangleleft C_G(e, f)$ and since the radical of $G_e \cap C_G(f)$ is contained in $T_e \subseteq G_e \cap C_G(T_f)$, we obtain

$$G_e \cap C_G(f) = (G_e \cap G_f)(G_e \cap C_G(T_f)).$$

Since $G^r_e \cap G^r_f \triangleleft G^r_e \cap C^r_G(f)$, the result follows.
LEMMA 1.6. Let $e \in E(\overline{T})$. Then $C_G(T_e) = T_e C_G(G_e)$.

PROOF. Since $G_e \triangleleft C_G(e)$, $C_G(G_e) \subseteq C_G(T_e) \subseteq C_G(e) = G_e C_G(G_e)$.

So $C_G(T_e) = C_G(G_e)[G_e \cap C_G(T_e)] = C_G(G_e) T_e$.

LEMMA 1.7. Let $e, f \in E(\overline{T})$. Then $C_G^r(e) \cap C_G^l(f) = [G^r_e \cap C_G(T_f)] [C_G(T_e, T_f)] [G^l_f \cap C_G(T_e)] [G^r_e \cap G^l_f]$.

PROOF. By Fact 1.3, $C_G^r(e) \cap C_G^l(f) = [G^r_e \cap G^l_f] [G^r_e \cap C_G(f)] [C_G(e) \cap G^l_f] C_G(e, f)$.

Now $G^r_e \cap G^l_f \triangleleft C_G^r(e) \cap C_G^l(f)$. Also if $a \in G^l_f \cap C_G(T_e)$, $b \in G^r_e \cap C_G(T_f)$, then $a^{-1} b^{-1} a b \in G^r_e \cap G^l_f$. Moreover $C_G(T_e, T_f)$ normalizes $G^r_e \cap C_G(T_f)$ and $G^l_f \cap C_G(T_e)$. So we are done by Lemmas 1.4, 1.5.

LEMMA 1.8. Let $e, f \in E(\overline{T})$, $a \in G^r_e$, $b \in C_G(T_e)$. If $ab \in C_G^r(f)$, then $a, b \in C_G^r(f)$. If $ab \in C_G^l(f)$, then $a, b \in C_G^l(f)$.

PROOF. Suppose $ab \in C_G^r(f)$. Now $a = a_1 b_2$ for some $a_1 \in R_u(C_G^r(e))$, $a_2 \in G^r_e$. Then $a_2 b \in C_G(e)$. So by Fact 1.3, $a_1, a_2 b \in C_G^r(e)$. Then $a_2 b \in C_G(e) \cap C_G^r(f)$. So by Fact 1.2, $u^{-1} a_2 = v b^{-1} \in G^r_e \cap C_G(T_e) = T_e \subseteq T \subseteq C_G^r(f)$. So $b \in C_G^r(f)$. Hence $a \in C_G^r(f)$.

PROPOSITION 1.9. Let $Y \subseteq E(\overline{T})$, $e_1, \ldots, e_{k+1} = f \in \Gamma$. Let $V = C_G(\Gamma)$, $Y_0 = G^l_f$, $Y_1 = G^r_{e_1}$, $Y_i = C_G(e_1, \ldots, e_{i-1}) \cap G^r_{e_i}$ for $i = 2, \ldots, k + 1$. Then $Y_0 \cdots Y_{k+1} \cap V = \prod_{i=1}^{k+1} V_{e_i}$.

PROOF. We prove by induction on $k$. So first let $k = 0$, $a \in G^l_f$, $b \in G^r_f$ such that $ab \in V \subseteq C_G(f)$. Then $a f = a b f = f a b = f$. So $a \in G_f$. Similarly $b \in G_f$. So $ab \in G_f \cap V = G^l_f \cap V = V_f = V_f (G_f \cap T) = V_f$.

So let $k > 0$, $a \in Y_0 \cdots Y_{k+1} \cap V$. Then $a = y_0 \cdots y_{k+1}, y_i \in Y_i$. Now $y_1, \ldots, y_{k+1}$, $a \in C_G^r(e_1)$. Thus $y_0 \in C_G^r(e_1) \cap C_G^l_f$. By Lemma 1.5, there exist $y_0 \in G^l_f \cap C_G(T_{e_1}), u \in G^r_f$, such that $y_0 = y_0 u$. So $y_1 = u y_1 \in G^r_{e_1}$ and $a = y_0 y_1 y_2 \cdots y_{k+1}$. Thus without loss of generality, we may assume that $y_0 \in C_G(T_{e_1}) \cap G^r_{e_1}$. For $i = 2, \ldots, k + 1$, we can factor by Lemma 1.5.

$y_i = c_i y_i^\prime$, $c_i \in G_{e_i}$, $y_i^\prime \in C_G(e_1, \ldots, e_{i-1}) \cap C_G(T_{e_i}) \cap G^r_{e_i}$.

Let $d_i = y_2 \cdots y_{i-1} c_i (y_2 \cdots y_{i-1})^{-1}$, $i = 3, \ldots, k + 1$.

Then $y_i^\prime = y_1 d_{k+1} \cdots d_3 c_2 \in G^r_{e_1}$, $y_i^\prime = y_0 y_i y_0^{-1} \in C_G^r_{e_1}$.

Clearly $a = y_0 y_1^\prime y_2^\prime \cdots y_{k+1} = y_0^\prime y_0 y_2^\prime \cdots y_{k+1}$. 

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Moreover \( y_0 y'_2 \cdots y'_{k+1} \in C_G(T_{e_1}) \). By Lemma 1.8, \( y'_2, y_0 y'_2 \cdots y'_{k+1} \in V \). So \( y'_1 \in V_{e_1} \). By the induction hypothesis \( y_0 y'_2 \cdots y'_{k+1} \in V_{e_2} \cdots V_{e_{k+1}} \). This completes the proof.

2. Main section. We fix a connected linear algebraic monoid \( M \) with zero 0 and a reductive group of units \( G \). As usual two elements \( a, b \in M \) are conjugate \((a \sim b)\) if \( x^{-1} ax = b \) for some \( x \in G \). Note that for \( a \in M \), \( g \in G \), \( ag \sim ga \). We fix a maximal torus \( T \) of \( G \). Let \( W = N_G(T)/T \) denote the Weyl group of \( G \). We let \( \mathcal{R}, \mathcal{L}, \mathcal{K} \) denote the usual Green's relations on \( M \) [2]. If \( a, b \in M \), then \( a \mathcal{R} b \) means \( aM = bM \), \( a \mathcal{L} b \) means \( Ma = Mb \), \( a \mathcal{K} b \). Let \( e \in E(T) \), \( \sigma = nT \in W \). Then we let

\[
e^\sigma = e^{-1} e^\sigma = n^{-1} \epsilon \in E(T).
\]

We let

\[
M_{e, \sigma} = C_G(e^\sigma \theta \in \langle \sigma \rangle) \sigma.
\]

Our first result is that every element of \( M \) is conjugate to an element of some \( M_{e, \sigma} \).

In preparation, we prove

**Lemma 2.1.** Let \( e \in E(T) \), \( \sigma = nT \in W \), \( k \in \mathbb{Z}^+ \), \( x, y \in C_G(e^\sigma | 0 \leq j \leq k-1) \), \( x \in C^{\mathcal{L}}_{e^k} \). Then \( exyn \sim eyxn \).

**Proof.** We prove by induction on \( k \). First let \( k = 1 \). Then

\[
exyn = xyen \sim ynenx = yne^\sigma x
= yne^\sigma = yen = eyxn.
\]

In general let \( k > 1 \). Then

\[
exyn = xeyn \sim eyxn = eyxn^{-1} y^{-1} yn = ex'y
\]

where \( x' = ynxn^{-1} y^{-1} \in C_G(e^\sigma | 0 \leq j \leq k-2) \cap G^{\mathcal{L}}_{e^{k-1}} \). So by the induction hypothesis \( ex'y \sim eyn \).

**Theorem 2.2.** Every element of \( M \) is conjugate to an element of some \( M_{e, \sigma} \).

**Proof.** Let \( a \in M \). By [8, Corollary 2.3], there exists a maximal torus \( T_1 \) of \( G \), \( e, f \in E(T_1) \) such that \( e \mathcal{R} f \). Since all maximal tori of \( G \) are conjugate, we can assume that \( T = T_1 \). There exists \( \theta = mT \in W \) such that \( e^\theta = f \). Thus \( e \mathcal{R} em \mathcal{L} f \). So \( em \not\mathcal{K} a \). Since \( eC_G(e) \) is the \( \mathcal{K} \)-class of \( e \), we see that \( a \in eC_G(e)m = eC_G(e) \theta \). Suppose inductively that \( a \in eC_G(e^j \theta | j = 0, \ldots, k) \theta \). Let \( H = C_G(e^j \theta | j = 0, \ldots, k) \). So there exists \( x \in H \) such that \( a = exm \). By [5], \( C_H(e^{\theta k+1}) \), \( C_H(\theta e \theta^{-1}) \) are parabolic subgroups of \( H \) containing \( T \). By the Bruhat decomposition there exists \( \pi = n_1 T \in W(H) \), \( x_1 \in C_H(e^{\theta k+1}) \), \( x_2 \in C_H(\theta e \theta^{-1}) \) such that \( x = x_1 n_1 x_2 \). So

\[
exm = ex_1 n_1 x_2 m \sim (m^{-1} x_2 m) ex_1 n_1 m.
\]

Now \( m^{-1} x_2 m \in C_G(e^j \theta | j = 1, \ldots, k + 1) \cap C_G(e) \). So

\[
m^{-1} x_2 m e = ze \quad \text{for some} \ z \in C_G(e^j \theta | j = 0, \ldots, k + 1).
\]

Thus

\[
a \sim ezz_1 n_1 m, \quad z \in C_H^t(e^{\theta k+1}).
\]
Let \( \lambda = \pi \theta = n_1 m T \in W \). We claim that \( e^{\lambda j} = e^{\theta j} \) for \( j = 0, \ldots, k + 1 \). For \( j = 0 \), this is obvious. So assume \( e^{\theta j} = e^{\lambda j}, j \leq k \). Then \( \pi \in C_W(e^{\theta j}) \). So
\[
e^{\lambda j+1} = (e^{\theta j})^\lambda = (e^{\theta j})^{\pi \theta} = (e^{\theta j})^\theta = e^{\theta j+1}.
\]
Thus \( y = zx_1 \in C^l_H(e^{\lambda k+1}) \). Hence \( y = y_1 y_2 \) for some \( y_1 \in H^l_{e^{\lambda k+1}}, y_2 \in C_H(e^{\lambda k+1}) \). By Lemma 2.1,
\[
a \sim e y_1 y_2 n_1 m \sim e y_2 n_1 m, \quad y_2 \in C_G(e^{\lambda j} | j = 0, \ldots, k + 1).
\]
Continuing this process, we see that there exist \( \sigma = n T \in W \) and \( u \in C_G(e^{\sigma j} | 0 \leq j \leq |W|) = C_G(e^\gamma | \gamma \in \langle \sigma \rangle) \) such that \( a \sim e u n \). Then clearly \( e u n \in M_{e, \sigma} \). This completes the proof of the theorem.

Schein [13] has shown that the full transformation semigroup on any set is a union of its inverse subsemigroups. The corresponding result, for the full matrix semigroup over a field, follows from the Fitting decomposition.

**Theorem 2.3.** (i) If \( F \) is a commutative, idempotent submonoid of \( M \), then \( FNG(F) \) is the maximal unit regular inverse submonoid of \( M \) with idempotent set \( F \).

(ii) If \( F \) is a subsemilattice of \( E(T) \) with \( 1 \in F \), then
\[
FNG(F) = FC_G(F)N_W(F).
\]

(iii) If \( e \in E(T), \sigma \in W, F = \{1, e^{\theta} | \theta \in \langle \sigma \rangle \} \), then \( M_{e, \sigma} \subseteq FNG(F) \).

(iv) \( M \) is a union of its unit regular inverse submonoids.

**Proof.** (i) That \( FNG(F) \) is a submonoid of \( M \) is obvious. Let \( a \in FNG(F), a^2 = a \). So \( a = fu \) for some \( f \in F, u \in N_G(F) \). Then \( fu f = f \). Since \( M \) is a matrix semigroup and \( f, uf u^{-1} \) commute, we see that \( f = uf u^{-1} \). Thus \( a = fu = fu f = f \in F \). So \( F \) is the idempotent set of \( FNG(F) \). It follows that \( FNG(F) \) is the maximal unit regular submonoid of \( M \) with idempotent set \( F \). Since \( F \) is commutative, it follows [2] that \( FNG(F) \) is an inverse semigroup.

(ii) Let \( a \in N_G(F) \). Clearly \( T \subseteq C_G(F) \). So \( a Ta^{-1} \subseteq C_G(aFa^{-1}) = C_G(F) \). So \( T, a Ta^{-1} \) are maximal tori of \( C_G(F) \). Hence \( b^{-1} a Ta^{-1} b = T \) for some \( b \in C_G(F) \). Hence \( b^{-1} a \in N_G(T) \cap N_G(F) \). So \( a = b(b^{-1} a) \in C_G(F)N_W(F) \).

(iii), (iv) follow from (ii) and Theorem 2.2.

Now fix \( e \in E(T), \sigma = n T \in W \). Let \( f = e^\sigma, \alpha + 1 \) the order of \( \sigma \). Let
\[
V = C_G(e^{\theta} | \theta \in \langle \sigma \rangle).
\]
So \( V \) is a reductive group, \( T \subseteq V, V^\sigma = V, M_{e, \sigma} = e V \sigma \). Now \( \hat{V}_e = \{ a \in V | ae = ea = e \} = \hat{T}_e V_e \) is a closed normal subgroup of \( V \). Let
\[
\Omega = \prod_{\theta \in \langle \sigma \rangle} \hat{V}_e^\theta = \prod_{\theta \in \langle \sigma \rangle} \hat{V}_e^{e^\theta}.
\]
Then \( \Omega \) is a closed normal subgroup of \( V \). If \( x \in V \), let \( x^* = n x^{-1} n_1 \in V \). Then \( \Omega^* = \Omega \). So \( \ast \) induces an antiautomorphism \( \ast \) on the reductive group \( G_{e, \sigma} = V/\Omega \). Define \( \xi : M_{e, \sigma} \rightarrow G_{e, \sigma} \) as follows: If \( a = evn \in M_{e, \sigma}, v \in V \), then, \( \xi(a) = v \Omega \in G_{e, \sigma} \). Since \( \hat{V}_e \subseteq \Omega, \xi \) is well defined. Note further that if \( G_{e, \sigma} \) is replaced by \( e V / e \Omega \) (which is isomorphic to \( G_{e, \sigma} \) as an abstract group), then \( \xi \) would also be a morphism of varieties.
THEOREM 2.4. Let \( a, b \in M_{e, \sigma} \). Then \( a \) is conjugate to \( b \) in \( M \) if and only if there exists \( x \in G_{e, \sigma} \) such that \( x \xi(a)x^* = \xi(b) \).

PROOF. For \( a, b \in M_{e, \sigma} \), define \( a \equiv b \) if \( x \xi(a)x^* = \xi(b) \) for some \( x \in G_{e, \sigma} \). We are to show that \( \equiv = \sim \). Let

\[
A = \{ a \in V | eun \sim eau n \text{ for all } u \in V \}.
\]

Clearly \( \tilde{V} \subseteq A \). Let \( a, b \in A \). Then for \( u \in V \), \( eaubu \sim ebuu \sim eun \). So \( A^2 \subseteq A \). Now let \( a \in A, u \in V \). Then

\[
e(na^{-1})un \sim an^{-1}unen = ea(n^{-1}un)n
\sim e(n^{-1}un)n = n^{-1}unen \sim eun.
\]

Thus \( nAn^{-1} \subseteq A \). It follows that \( \Omega \subseteq A \). Now let \( m_1, m_2 \in M_{e, \sigma} \) such that \( m_1 \equiv m_2 \). Let \( m_1 = eun, m_2 = evn \) where \( u, v \in V \). Then there exists \( x \in V \) such that \( v \in \Omega xu nx^{-1}n^{-1} \). Since \( \Omega \subseteq A \),

\[
m_1 = evn \sim exunx^{-1}n^{-1}n = exunx^{-1}
\sim xeunx^{-1} \sim eun \equiv m_2.
\]

This shows that \( \equiv \subseteq \sim \).

Conversely let \( m_1, m_2 \in M_{e, \sigma} \) such that \( m_1 \sim m_2 \). Then there exists \( X \in G \) such that

(1) \( X_1m_1 = m_2X_1 \).

Let \( m_1 = eun, m_2 = evn \) where \( u, v \in V \). Then by (1),

\[
X_1e \in X_1eun = evnX_1e
\]

So \( X_1e = eX_1e \) and \( X_1 \in C_G^e(e) \). Also by (1),

\[
fX_1 = n^{-1}enX_1 \in m_1X_1 \in m_1 \in C_f \in f.
\]

Thus \( X_1 \in C_G^e(e) \cap C_G^f(e) \). By Lemma 1.7, \( X_1 \in X[G_e \cap G_f] \) for some

\[
X \in [C_G(T_e) \cap G_f][C_G(T_e, T_f)][C_G(T_f) \cap G_e^e].
\]

Since \( m_1 = em_1, m_2 = m_2f \), we see by (1) that

(2) \( Xeun = evnX \).

Now \( X = axb \) for some

(3) \( a \in C_G(T_e) \cap G_f, \quad x \in C_G(T_e, T_f), \quad b \in C_G(T_f) \cap G_e^e \).

So by (2), \( eaxun = evnxb \). Then \( eaxu = evnxbn^{-1} \) and \( axu, vnxbn^{-1} \in C_G(e) \). So

(4) \( axu = vnxbn^{-1}z \) for some \( z \in \tilde{G} \).

Now \( nxbn^{-1} \in C_G(T_e) \). So by Lemma 1.6, \( nxbn^{-1} = \eta t \) for some \( t \in T, \eta \in C_G(G_e) \). So \( vt \in V \subseteq C_G(e) \). So \( v' \in V \subseteq C_G(T_e) \), \( v'' \in V \subseteq G_e \). Also \( u = u'u'' \) for some \( u' \in C_V(V_e) \subseteq C_G(T_e), u'' \in V \subseteq G_e \). Then

\[
aux = vnxbn^{-1}z = v't\eta z = v'v''\eta z = v'\eta v''z.
\]

So

(5) \( aux' = v'\eta (v''z(u'')^{-1}) \).
Let $z' = v''z(u'')^{-1}$. Then $z' \in \hat{G}_e$. So $z'h = hz' = h$ for all $h \in E(M)$ with $h \leq e$. Now $axu', v', \eta \in C_G(T_e)$. So by (5), $z' \in C_G(T_e)$. Thus $z'h = hz'$ for all $h \in E(T)$ with $h \geq e$. So for any maximal chain $\Gamma$ of $E(\bar{T})$ with $e \in \Gamma$, $z' \in C_G(\Gamma) = T \subseteq V$. Let $u_1 = u'(z')^{-1}v'' \in V$. Then by (5),

$$axu_1 = v'v'' = v'v''\eta = vtn\eta = vnxbn^{-1}.$$ 

Also $z = (v'')^{-1}z'u'' \in V \cap \hat{G}_e = \hat{V}_e$. So

$$axu_1 = vnxbn^{-1}, \quad u_1, v \in V, \ z \in \hat{V}_e.$$ 

Now $xb \in C^*_G(e)$. So $nxbn^{-1} \in C^*_G(\sigma e \sigma^{-1})$. Thus $ax \in C^*_G(\sigma e \sigma^{-1})$. By (3), Lemma 1.8, $a, x \in C^*_G(\sigma e \sigma^{-1})$. So $x \in C^*_G(\sigma e \sigma^{-1}) \cap C_G(T_e, T_f)$. Hence we can factor

$$x = y_1x_1 \quad \text{for some } y_1 \in C^*_G(\sigma e \sigma^{-1}) \cap C_G(T_e, T_f), \ x_1 \in C_G(T_{\sigma e \sigma^{-1}}, T_e, T_f).$$

Also $a \in C^*_G(\sigma e \sigma^{-1}) \cap C_G(T_e) \cap G_f^l$. So working within $C_G(T_e)$ and applying Lemma 1.5, we can factor

$$a = c_1a_1 \quad \text{for some } c_1 \in C^*_G(\sigma e \sigma^{-1}) \cap G_f^l \cap C_G(T_e), \ a_1 \in C_G(T_e, T_{\sigma e \sigma^{-1}}) \cap G_f^l.$$ 

Now by (6),

$$c_1a_1y_1x_1u_1 = vny_1x_1bn^{-1}.$$ 

So

$$wa_1x_1u_1 = vny_1x_1n^{-1}$$

where

$$w = c_1a_1y_1a_1^{-1}[(a_1x_1u_1)(nb^{-1}n^{-1})(a_1x_1u_1)^{-1}] \in C^*_G(e).$$

Suppose now inductively that

$$x = y_1 \cdots y_kx_k,$$

where

$$y_i \in C_G(T_f, T_{\sigma i e \sigma^{-1}}) \cap G^l_{\sigma i e \sigma^{-1}}, \quad i = 0, \ldots, k, \ n \cap G^l_{\sigma i e \sigma^{-1}},$$

$$x_k \in C_G(T_f, T_{\sigma i e \sigma^{-1}}), \quad j = 0, \ldots, k.$$ 

Further assume that there exist

$$w_i \in C_G(T_{\sigma i e \sigma^{-1}}) \cap G^l_{\sigma i e \sigma^{-1}}, \quad i = 1, \ldots, k,$$

$$a_k \in C_G(T_{\sigma i e \sigma^{-1}}) \cap G^l_f \cap G^l_{\sigma i e \sigma^{-1}}, \quad i = 0, \ldots, k.$$ 

such that

$$w_k \cdots w_1a_kx_ku_1 = vny_kx_kn^{-1}.$$ 

Note that (7)–(9) show (10)–(12) to be valid for $k = 1$. Now

$$ny_kx_kn^{-1} \in C^*_G(\sigma^{k+1} e \sigma^{-k-1}).$$

So by (12), $w_k \cdots w_1a_kx_k \in C^*_G(\sigma^{k+1} e \sigma^{-k-1})$. Repeated use of Lemma 1.8 shows that $w_1, \ldots, w_k, a_k, x_k \in C^*_G(\sigma^{k+1} e \sigma^{-k-1})$. So by Lemma 1.5, we can factor for
$i = 1, \ldots, k,$

\[
\begin{align*}
  w_i &= q_i w'_i, \\
  a_k &= c_{k+1} a_{k+1}, \\
  x_k &= y_{k+1} x_{k+1},
\end{align*}
\]

$w_i' \in C_G(T_{\sigma^j e \sigma^{-j}} | i + 1 \leq j \leq k + 1) \cap G_{\sigma^j e \sigma^{-j}}^r,$

$q_i \in G_{\sigma^j e \sigma^{-j}}^r \cap C_G(T_{\sigma^j e \sigma^{-j}} | i + 1 \leq j \leq k),$

$a_k \in C_G(T_{\sigma^j e \sigma^{-j}} | 0 \leq j \leq k + 1) \cap G_f,$

$x_{k+1} \in C_G(T_f, T_{\sigma^j e \sigma^{-j}} | 0 \leq j \leq k + 1).$

Let

\[
\begin{align*}
  q'_i &= w'_k \cdots w'_{i+1} q_i (w'_k \cdots w'_{i+1})^{-1} \in G_{\sigma^j e \sigma^{-j}}^r, \\
  c'_{k+1} &= w'_k \cdots w'_i c_{k+1} (w'_k \cdots w'_i)^{-1} \in G_{\sigma^j e \sigma^{-j}}^r, \\
  y'_{k+1} &= w'_k \cdots w'_i a_{k+1} y_{k+1} (w'_k \cdots w'_i a_{k+1})^{-1} \in G_{\sigma^j e \sigma^{-j}}^r, \\
  p &= q_k q'_{k-1} \cdots q'_i y'_{k+1} \in G_{\sigma^j e \sigma^{-j}}^r.
\end{align*}
\]

Then

\[
\begin{align*}
  w_k \cdots w_1 a_k x_k &= pw'_k \cdots w'_i a_{k+1} x_{k+1}.
\end{align*}
\]

So by (12),

\[
\begin{align*}
  w'_{k+1} \cdots w'_{i+1} a_{k+1} x_{k+1} u_1 &= v_1 y_{k+1} x_{k+1} n^{-1},
\end{align*}
\]

where

\[
\begin{align*}
  w'_{k+1} &= v n y_k^{-1} n^{-1} v^{-1} p \in G_{\sigma^j e \sigma^{-j}}^r.
\end{align*}
\]

This completes the induction step in (10)–(12). So (10) is valid for all $k \in \mathbb{Z}^+$. In particular it is valid for $k = a$, where $\sigma^{a+1} = 1$. Then

\[
(13) \quad x = y_1 \cdots y_a x_a, \quad x_a \in C_G(e^\theta | \theta \in \langle \sigma \rangle) = V.
\]

Now by (4), (6)

\[
(14) \quad axu = vn x_n z, \quad z \in \hat{V}.
\]

Let $Y_0 = G_f$, $Y_1 = G_{\sigma e \sigma^{-1}}^r$,

\[
Y_i = C_G(\sigma^i e \sigma^{-j} | j = 1, \ldots, i - 1) \cap G_{\sigma^i e \sigma^{-j}}^r, \quad i \geq 2.
\]

Then $Y_j$ normalizes $Y_i$ for $j \geq i \geq 1$. Also $V$ normalizes $Y_i$ for all $i$. By (3), (11) we see that

\[
(15) \quad a \in Y_0, \quad n b n^{-1} \in Y_1, \quad y_i \in Y_i, \quad 1 = 1, \ldots, a.
\]

Also, since $\sigma^{a+1} = 1$, we see by (11) that

\[
(16) \quad n y_a n^{-1} \in Y_{a+1}, \quad i = 1, \ldots, a - 1, \quad n y_a n^{-1} \in V.
\]

Since $x_a, u, v \in V$ and $V^\sigma = V$, we see by (13)–(16),

\[
\begin{align*}
  v(x_\alpha u x_\alpha^*)^{-1} &= v n x_\alpha^{-1} u^{-1} x_\alpha^{-1} \\
  &= ax u z^{-1} n b^{-1} x^{-1} x_\alpha n^{-1} u^{-1} x_\alpha^{-1} \\
  &= a v y_1 \cdots y_a x_\alpha u z^{-1} u^{-1} x_\alpha^{-1} (y_1^{-1} \cdots y_a) (x_\alpha u) \\
  &\quad \times [n b^{-1} x_\alpha^{-1} (y_1^{-1} \cdots y_a^{-1})] (x_\alpha u)^{-1},
\end{align*}
\]

$\in \hat{V} Y_0 Y_1 \cdots Y_a$.

Since $\sigma^a e \sigma^{-a} = f$, we see by Proposition 1.9 that $v(x_\alpha u x_\alpha^*)^{-1} \in \Omega$. Thus $m_1 = e un \equiv evn = m_2$. This completes the proof of the theorem.
The proof of the above theorem shows

**COROLLARY 2.5.** Let \(a, b \in M_{e, \sigma}\). Then \(a \sim b\) if and only if there exists \(x \in V = C_G(e^\theta|\theta \in \langle \sigma \rangle)\) such that \(x^{-1}ax = b\).

**COROLLARY 2.6.** Let \(D = eC_G(e)\) denote the group of units of \(eMe\), \(h \in E(e^T), \theta = mT \in C_W(e)\). Then \(M_{h, \theta} = (eMe)_{h, e}\theta\) and \(G_{h, \theta} \cong D_{h, e}\theta\). If \(a, b \in M_{h, \theta}\), then \(a\) is conjugate to \(b\) in \(M\) if and only if \(a\) is conjugate to \(b\) in \(eMe\).

**PROOF.** Let

\[ V = C_G(h^\gamma|\gamma \in \langle \theta \rangle), \quad Y = C_D(h^\gamma|\gamma \in \langle \theta \rangle) \]

Let \(a \in Y\). Then \(a = ex\) for some \(x \in C_G(e)\). For \(\gamma \in \langle \theta \rangle\),

\[ xh^\gamma = xeh^\gamma = ah^\gamma = h^\gamma a = h^\gamma ex = h^\gamma x \]

So \(x \in C_V(e)\) and \(Y = eC_V(e)\). Now \(V = V_h C_V(V_h) = V_h C_V(e)\). Hence

\[ M_{h, \theta} = hV\theta = hC_V(e)\theta = heC_V(e)\theta = hY\theta = (eMe)_{h, e}\theta. \]

Let \(\Omega = \prod_{\gamma \in \langle \theta \rangle} V_{h, \gamma}\). Since \(V = V_h C_V(e), h \leq e\),

\[ G_{h, \theta} = V/\Omega \cong C_V(e)/C_{\Omega}(e) \cong eV/eC_{\Omega}(e). \]

By Proposition 1.9,

\[ C_{\Omega}(e) = \prod_{\gamma \in \langle \theta \rangle} [V_{h, \gamma} \cap C_G(e)]. \]

It follows that \(eV/eC_{\Omega}(e) = D_{h, e}\theta\). We are now done by Theorem 2.4.

**CONJECTURE 2.7.** Let \(a, b \in eMe\). Then \(a\) is conjugate to \(b\) in \(M\) if and only if \(a\) is conjugate to \(b\) in \(eMe\).

**CONJECTURE 2.8.** Let \(\mathcal{Y} = \{M_{e, \sigma}|e \in E(\bar{T}), \sigma \in W\}, \mathcal{Y}_0\) the set of maximal elements (with respect to inclusion) of \(\mathcal{Y}\). Then if \(Y_1, Y_2 \in \mathcal{Y}_0, a \in Y_1, b \in Y_2, a \sim b\), then \(Y_1^\theta = Y_2\) for some \(\theta \in W\).

Let \(g \in G\). Then the map: \(x \rightarrow gx^{-1}g^{-1}\) is an antiautomorphism of \(G\). We will call such an antiautomorphism an inner antiautomorphism.

**EXAMPLE 2.9.** Let \(n \in Z^+, M = M_n(K)\). Let \(h = \prod_{\theta \in \langle \sigma \rangle} e^\theta, r\) the rank of \(h\). Then \(G_{e, \sigma} \cong GL(r, K)\) and * is an inner antiautomorphism.

**EXAMPLE 2.10.** Let \(M = \{A \otimes B|A, B \in M_2(K)\}\). Then the possibilities for \(G_{e, \sigma}\) are \(G, SL(2, K), PGL(2, K), G_m, \{1\}\). In all cases, * is inner.

**CONJECTURE 2.11.** If the simple components of \(G\) are all of type \(A_t\), then * is necessarily inner.

By [3, Theorem 27.4], an antiautomorphism of a semisimple group is the composition of an inner antiautomorphism and an automorphism determined by an automorphism of the Dynkin diagram of the group.

**CONJECTURE 2.12.** For all \(t \in R(G_{e, \sigma})\), \(t^* = t^{-1}\) and hence * is completely determined by its action on the semisimple group \(G'_{e, \sigma} = (G_{e, \sigma}, G_{e, \sigma})\).

3. Nilpotent elements. We continue from [8] the analysis of conjugacy classes of nilpotent elements of \(M\). It was shown in [8] that the conjugacy classes of minimal nilpotent elements (in the \(J\)-class ordering) is always finite. Renner [12] has introduced the finite fundamental inverse monoid \(\text{Ren}(M) = N_G(\bar{T})/T\) and
used it to generalize the Bruhat decomposition to $M$. We easily have

**Proposition 3.1.** Let $e \in E(T)$, $\sigma = nT \in W$, $k \in \mathbb{Z}^+$. Then the following conditions are equivalent:

(i) $a^k = 0$ for some $a \in M_{e,\sigma}$,

(ii) $M_{e,\sigma}^k = 0$,

(iii) $(\sigma e)^k = 0$ in $\text{Ren}(M)$,

(iv) $e^\sigma \cdots e^\sigma = 0$.

Since $V = C_G(e^\sigma|\theta \in (\sigma))$ is a reductive group, we see that any closed normal subgroup of $V$ containing $T$, must equal $V$. Thus

**Proposition 3.2.** Let $e \in E(T)$, $\sigma \in W$. Then $G_{e,\sigma}$ is trivial if and only if $T = \prod_{\theta \in (\sigma)} T_{e^\theta}$.

In particular, we see that $G_{e,\sigma}$ trivial implies that $e\sigma$ is nilpotent. If the groups $G_{e,\sigma}$ are trivial for all nilpotent $e\sigma$, then by Theorems 2.2, 2.4, the number of conjugacy classes of nilpotent elements in $M$ is finite.

**Conjecture 3.3.** The number of conjugacy classes of nilpotent elements of $M$ is finite if and only if the groups $G_{e,\sigma}$ are trivial for all nilpotent $e\sigma$.

**Example 3.4.** If $M = M_n(K)$, then we see by Example 2.9 that the groups $G_{e,\sigma}$ are trivial for nilpotent $e\sigma$.

**Example 3.5.** Let $G_0 = \{A \otimes (A^{-1})^t|A \in \text{SL}(3,K)\}$, $G = K^*G_0$, $M = \overline{KG_0}$.

Let $S = M \setminus G$. Then

$$E(S) = \{e \otimes f|e^2 = e, f^2 = f \in M_3(K), ef^t = f^te = 0\}.$$  

In particular

$$e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in E(M).$$

Also if

$$\sigma = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in W(G),$$

then $e^\sigma = f$ and $(e^\sigma)^2 = 0$. The group $G_{e,\sigma}$ can be seen to be the one dimensional torus with $*$ being given by $x \rightarrow x^{-1}$. Thus by Theorem 2.4, the number of conjugacy classes of nilpotent elements of $M$ is infinite. However if $C$ denotes the center of $G$, then the number of conjugacy classes of nilpotent elements in $M/C$ is finite.

**Example 3.6.** Suppose char $K \neq 2$, $n \in \mathbb{Z}^+$, $n \geq 2$. For $r \in \mathbb{Z}^+$, let $J_r$ denote the $r \times r$ matrix

$$\begin{bmatrix} 1 & 0 & & & \\ & & & & \\ & & & & \\ & & & & \\ 1 & & & & \end{bmatrix}.$$  

Let $G_0$ consist of all $A \in \text{SL}(2n+1,K)$ such that

$$A^t \begin{bmatrix} 1 & 0 \\ 0 & J_{2n} \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & J_{2n} \end{bmatrix}.$$
Thus [3, §7.2], \( G_0 \) is the special orthogonal group of type \( B_n \). Let \( G = K^*G_0 \), \( M = KG_0 \). Then

\[
e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_n \end{bmatrix} \in E(M).
\]

If

\[
\sigma = \begin{bmatrix} \pm 1 & 0 \\ 0 & J_{2n} \end{bmatrix} \in W(G),
\]

then \( e^\sigma = f \) and \((e\sigma)^2 = 0\). It can be seen that \( G_{e,\sigma} \cong \text{PGL}(n, K) \) with the antiautomorphism * on \( G_{e,\sigma} \) given by \( A \to J_n A^t J_n \). Thus by Theorem 2.4, the number of conjugacy classes of nilpotent elements of \( M \) is infinite. This gives a counterexample to [8, Conjectures 4.5, 4.6]. Note also that for \( n \geq 3 \), * is not inner.

The above examples suggest

**Conjecture 3.7.** Suppose that the center of \( G \) is one dimensional. Then the number of conjugacy classes of nilpotent elements of \( M \) is finite if and only if \( \text{Ren}(M) \) is isomorphic to the symmetric inverse semigroup of some finite set.

**References**


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