BRANCHED COVERINGS OF 2-COMPLEXES
AND DIAGRAMMATIC REDUCIBILITY

S. M. GERSTEN

Abstract. The condition that all spherical diagrams in a 2-complex be reducible is shown to be equivalent to the condition that all finite branched covers be aspherical. This result is related to the study of equations over groups. Furthermore large classes of 2-complexes are shown to be diagrammatically reducible in the above sense; in particular, every 2-complex has a subdivision which admits a finite branched cover which is diagrammatically reducible.

A consequence of the classical Riemann-Hurwitz formula for Riemann surfaces [6, p. 301] is the fact that a finite branched cover of an aspherical Riemann surface is also aspherical. However the property of asphericity of a 2-complex implying asphericity of finite branched covers fails for general 2-complexes. In fact we shall prove (Theorem 4.5 below) that this property holds precisely for those 2-complexes which are diagrammatically reducible.

A 2-complex $X$ is said to be diagrammatically reducible (abbreviated DR($X$)) if for every combinatorial map $f: S^2 \to X$ of the 2-sphere into $X$ (a "spherical diagram" in $X$) there is a pair of faces in the domain with an edge in common which are mapped mirrorwise by $f$ across that edge. A closely related notion was introduced by Lyndon and Schupp [13] but this notion appears in its present form in a paper by Sieradski [14]. In that same paper Sieradski asked a question which would have implied that all classical knot complements in the 3-sphere are homotopy equivalent to diagrammatically reducible 2-complexes. We shall prove this latter assertion (Theorem 6.5) making use of our amalgamation theorem for diagrammatic reducibility (Theorem 5.4).

The characterization of diagrammatic reducibility in terms of branched covers follows from a lifting theorem which may be of general interest. If $f: X \to Y$ is a combinatorial map of combinatorial 2-complexes, then $f$ induces a map of the link complexes of vertices $L_f: L_X \to L_Y$ (the "star graphs" or "coinitial graph" in group theoretical language). The map $f$ is said to be reduced if $L_f$ is an immersion. We prove (Theorem 4.2) that if $f: X \to Y$ is a reduced map of 2-complexes with $X$ finite, then there is a commutative diagram (see below) where $\pi: Z \to Y$ is a finite branched cover and where $g$ is injective off the zero skeleton of $X$. This result is
itself a consequence of a theorem of Marshall Hall's on free groups. To our knowledge this is the first application of M. Hall's theorem to 2-complexes.

The organization of the paper is as follows. In §1 we review what is known about diagrammatic reducibility, its relation to solving equations over groups, and the weight test and its relation to the conjugacy problem for hyperbolic 2-complexes. In §2 we introduce the basic notions related to branched covers and give examples related to group theory. The examples (Fibonacci groups, Higman presentations) have a long history in the literature, but our way of viewing them in terms of branched covers is new. In §3 we prove that every finite 2-complex has a subdivision which admits hyperbolic branched covers and in §4 we establish the lifting theorem quoted above and prove our characterization of diagrammatic reducibility in terms of branched covers. In §§5 we prove amalgamation theorems for constructing new diagrammatically reducible 2-complexes out of old ones. The applications are contained in §6. Among these are a strengthening of the characteristic property of the Higman-Neumann-Neuman imbedding of a free group of countable rank in one of rank 2 and the application to classical knots mentioned earlier.

An appendix has been included to relate our notion of diagrammatic reducibility to the notion of diagrammatic asphericity [21]. It is appropriate to say here that showing a 2-complex is diagrammatically reducible is very much stronger than proving it is diagrammatically aspherical and has implications about equations over groups that the latter does not.

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Notation. If $A$ is a set, let $\bar{A}$ be a set disjoint from $A$, in 1-1 correspondence with $A$ by the map $a \mapsto \bar{a}$. A presentation $\mathcal{P}$ is a symbol $\langle A \mid S \rangle$ where $S$ is a subset of the free monoid on the alphabet $A \cup \bar{A}$. The symbol $K(\mathcal{P})$ denotes the 2-complex canonically associated to $\mathcal{P}$ and $G(\mathcal{P})$ denotes $\pi_1(K(\mathcal{P}))$, the group of the presentation.

If $G$ is a group and $x \in G$, $\bar{x}$ (or $x^{-1}$ if space permits) denotes the inverse of $x$. If $S$ is a subset of $G$, then $\langle S \rangle$ denotes the subgroup generated by $S$ and $\langle \langle S \rangle \rangle_G$ (or $\langle \langle S \rangle \rangle$, if $G$ is clear) denotes the normal closure of $S$ in $G$. If $x \in F$, $x^G$ denotes the conjugacy class of $x$ in $G$.

1. Diagrammatic reducibility.

1.1 We work in the category of combinatorial 2-complexes and combinatorial maps. A cellular map $f \colon X \to Y$ of two CW complexes is combinatorial if the restriction of $f$ to each open cell is a homeomorphism onto its image. A 2-complex $X$ is called combinatorial if the attaching map $S^1 \xrightarrow{e_\alpha} X\langle 1 \rangle$ of each 2-cell $e_\alpha$ of $X$ is
combinatorial for suitable subdivision $S^1_\alpha$ of $S^1$ (depending on $e_\alpha$). The number of vertices in this subdivision $S^1_\alpha$ is denoted $d(e_\alpha)$ and is called the degree of $e_\alpha$. The face $e_\alpha$ has $d(e_\alpha)$ (unoriented) corners [5, §2], where each corner is incident with one vertex of $f^{-1}(X^{(1)})$; thus each corner of a 2-complex $X$ is incident with one face and with one vertex of $X$.

Each unoriented corner represents two oriented corners with opposite orientations. Corners are invariant under combinatorial maps and give rise to a functor $X \mapsto F_X$ from combinatorial 2-complexes and combinatorial maps to free groups, where $F_X$ is the free group with free basis consisting of a choice of one oriented corner from each pair of oppositely oriented corners.

From now on we shall understand 2-complex to mean combinatorial 2-complex.

1.2 With each 2-complex $X$ is associated a graph $L_X$, the link of the zero skeleton $X^{(0)}$ in $X$. Thus $L_X$ can be defined as the boundary of a regular neighborhood of $X^{(0)}$ in $X$. Since we have need of a specific cell structure on $L_X$, we recall the following construction of it due to J. H. C. Whitehead [17]. It has come to be known as the “star graph” or even “coinitial graph”. Let $E$ denote the set of oriented edges of $X$. Thus $E$ is equipped with a fixed point free involution $e \mapsto \bar{e}$ associating to the edge $e$ the edge $\bar{e}$ with same carrier but opposite orientation. We take $E$ as the vertex set for the graph $L_X$. Each attaching map $f_\alpha$ of a 2-cell $e_\alpha$ of $X$ determines a cyclic word $w_\alpha$ (not necessarily reduced) in the free semigroup in the alphabet $E$. If the syllable $ab$ occurs in the spelling of $w_\alpha$, we join the vertex $a$ to the vertex $b$ of $L_X$ by an edge corresponding to this syllable. Hence the unoriented edges connecting $a$ to $b$ in $L_X$ correspond 1-1 to occurrences of syllables $ab$ or $\bar{b}a$ in the cyclic words $w_\alpha$ determined by attaching maps of 2-cells of $X$. In particular, unoriented edges of $L_X$ correspond 1-1 to unoriented corners of $X$.

From the definition, it is clear that the assignment $X \mapsto L_X$ is a functor from 2-complexes and combinatorial maps to graphs and nondegenerate morphisms of graphs.

1.3 A combinatorial map $f: X \to Y$ of 2-complexes is reduced if the induced map $L_f: L_X \to L_Y$ is an immersion [16]. Recall that a nondegenerate map $g: \Gamma \to \Gamma'$ of graphs is an immersion if for each vertex $v$ of $\Gamma$ the induced map $g: \text{Star}_\Gamma(v) \to \text{Star}_{\Gamma'}(g(v))$ is injective, where $\text{Star}_\Gamma(v)$ is the set of oriented edges of $\Gamma$ with initial vertex $v$.

The only way a nondegenerate morphism of graphs $g: \Gamma \to \Gamma'$ can fail to be an immersion is for two distinct edges of $\Gamma$ to be folded [16]. That is there exist oriented edges $e \neq e'$ of $\Gamma$ with the same initial vertex and such that $ge = ge'$. License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
A combinatorial map \( f: C \to X \) is called a spherical diagram if \( C \) is some cell structure on \( S^2 \). We have the following criterion.

1.4 Lemma. The spherical diagram \( f: C \to X \) is not reduced iff there is a pair of distinct open 2-cells \( \alpha \) and \( \beta \) of \( C \) with an edge \( e \) in the closure of each and an involutory homeomorphism \( g \) of \( \alpha \cup \beta \cup e \) such that \( g \) interchanges \( \alpha \) and \( \beta \), \( g \mid e \) is the identity of \( e \), and \( f \mid \beta = f \cdot g \mid \alpha \).

Of necessity \( g \) maps \( \alpha \) in an orientation reversing manner onto \( \beta \). We abbreviate the criterion 1.4 by saying there is a pair of faces mapped mirrorwise by \( f \) across a common face. In this case we can cut out \( \alpha, \beta, \) and \( e \) and paste along the boundary to obtain a finite collection of 2-spheres glued together at vertices and a combinatorial map to \( X \).

1.5 The 2-complex \( X \) is called diagrammatically reducible (abbreviated DR(\( X \))) if no spherical diagram \( f: C \to X \) is reduced. In this case the cut-and-paste operation indicated above produces an explicit null homotopy of the map \( f \). Since a transversality argument shows that \( \pi_2(X) \) is generated as a \( \pi_1(X) \) module by classes of spherical diagrams, it follows that diagrammatic reducibility implies asphericity for 2-complexes. The converse however is false. The simplest counterexample is a reduced spherical diagram in the dunce hat \( K(\mathcal{P}) \) where \( \mathcal{P} = \langle t \mid tt^{-1} \rangle \), sketched below:

(The second cell is at infinity, the unbounded region in the plane. The author has become accustomed to call this diagram “my favorite diagram” [4].)

1.6 There is an intimate relation between diagrammatic reducibility and solving equations over groups [3, 5]. Rather than recall the procedure in general how a 2-complex determines systems of equations over groups [5, 2.7] we give a specific example which should make the process clear. The 2-complex \( X = K(\mathcal{P}) \) can be seen to be diagrammatically reducible, where \( \mathcal{P} = \langle x, y, z, w \mid xyx^{-1}y^{-1}z^{-1}w^{-1} \rangle \). Consider the “equation”

\[
E = a_1xa_2ya_3xa_4ya_5za_6wa_7za_8w = 1
\]

where \( a_i \) are elements of some group \( A \) (1 \( \leq i \leq 8 \)) and where \( x, y, z \) and \( w \) are variables. Then it follows from DR(\( X \)) that equation (1.6.1) can be solved in an overgroup of \( A \). In fact much more is true. The natural homomorphism \( \phi: A \to A \langle x, y, z, w \rangle / \langle \langle E \rangle \rangle \) (where \( A \langle x, y, z, w \rangle \) is the free product of \( A \) with the free group freely generated by \( x, y, z \) and \( w \)) satisfies “Property G” [5, 2.8.3]. Here a homomorphism \( \phi: A \to B \) of groups satisfies “Property G” if, for any number \( n \) and elements \( a_1, a_2, \ldots, a_n \) in \( A \) satisfying \( 1 \in \prod_{i=1}^n \phi(a_i)^{a_i} \), one has \( 1 \in \prod_{i=1}^n a_i^A \).

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Briefly one says that DR(A) implies that all systems of equations over an arbitrary coefficient group A modeled on X satisfy “Property G”, and hence are solvable in an overgroup of A. The reverse implication is invalid: The real projective plane is not aspherical, but the associated equation satisfies “Property G” [5, 5.4]. We shall need the connection of DR(X) with “Property G” only in §6 in our discussion of the HNN imbedding. However it was this connection which motivated us at first to study diagrammatic reducibility.

There is a simple “weight test” for diagrammatic reducibility which we recall since we need it to describe hyperbolic 2-complexes. A weight on a 2-complex S is a real valued function on the corners of X. We write γ < A to indicate the corner γ of X is incident with the 2-cell A of X and γ < v to indicate that γ is incident at the vertex v of X.

1.7 Theorem [5, 4.7]. The 2-complex X is diagrammatically reducible if there is a weight w on X satisfying

1.7.1 Curvature Condition. For each 2-cell A of X one has \( \sum_{\gamma < A} w(\gamma) \leq d(A) - 2 \), and

1.7.2 Link Condition. For each nontrivial cyclically reduced circuit \( \gamma_1 \gamma_2 \cdots \gamma_n \) in the link complex \( L_X \) one has \( \sum_{i=1}^{n} w(\gamma_i) \geq 2 \). Observe here that corners of X correspond 1-1 to edges of \( L_X \). The condition that the circuit be cyclically reduced means \( \gamma_{i+1} \neq \gamma_i \) for \( 1 \leq i \leq n - 1 \), and \( \gamma_n \neq \gamma_1 \).

1.8 On the basis of this weight test we define a hyperbolic 2-complex to be a pair (X, w), where X is a 2-complex and w is a nonnegative weight on X satisfying the link condition 1.7.2 and satisfying the curvature condition 1.7.1 with strict inequality for each 2-cell A. In [5, Appendix], we proved an isoperimetric inequality for finite hyperbolic 2-complexes, leading to a solution of the conjugacy problem for their fundamental groups. To relate these notions to the theory of Riemann surfaces, we observe that surfaces of nonpositive Euler characteristic admit weights satisfying 1.7.1 and 1.7.2, whereas surfaces of negative Euler characteristic admit weights satisfying 1.7.2 and admitting 1.7.1 with strict inequalities.

1.9 Finally we recall a result of Howie, whose proof is sketched in [5, 6.10], that a 2-complex X is diagrammatically reducible iff every subdivision of X is diagrammatically reducible. Thus the notion of diagrammatic reducibility is a combinatorial invariant for 2-complexes.

2. Branched covers.

2.1 Definition. A combinatorial map \( f: Y \to X \) of 2-complexes is a branched covering if \( f_0: Y_0 \to X_0 \) is a covering map. Here \( X_0 = X - X(0) \), \( Y_0 = Y - Y(0) \) and \( f_0 = f | Y_0 \). We shall only be concerned with finite branched coverings, where \( f_0: Y_0 \to X_0 \) is a finite sheeted covering map.

To construct branched covers we need to reformulate the definition slightly. Let \( N \) be a regular neighborhood of \( X(0) \) in \( X \), so the boundary of \( N \) can be identified with the link complex \( L_X \). Let \( X_1 = X - \bar{N} \), so the inclusion \( X_1 \subset X_0 \) is a homotopy equivalence. Thus coverings of \( X_1 \) are the same as those of \( X_0 \). Given a finite
covering $U \xrightarrow{\pi} X_1$ there are in general several ways to extend it to a branched cover of $X$. One way is to attach a cone to each connected component $L'$ of $\pi^{-1}(L_X)$ (recall $L_X \subset X_1$) and extend $\pi$ conically. This works since $L'$ is a finite cover of $\pi(L')$, a connected component of $L_X$. We call the branched cover of $X$ constructed in this way the “conical extension of $\pi$: $U \to X_1$”. If on the other hand $X$ has only one vertex, then $X = K(\mathcal{P})$ for some presentation $\mathcal{P}$, then it is more natural to extend $\pi$: $U \to X_1$ to a branched cover $Y$ of $X$ in a different way, by identifying all cone points of the conical extension to a single vertex. In this case $Y = K(\mathcal{D})$ for some presentation $\mathcal{D}$. In particular, if $\pi^{-1}(L_X)$ is connected, then the extension of $\pi$ to a branched cover of $X$ is unique up to isomorphism and is of the form $K(\mathcal{D})$. Since $X_1$ has an explicit cell structure whose edges correspond 1-1 to the union of the edges of $X$ with those of $L_X$, this often makes the construction of explicit branched covers very easy.

2.2 Example. Let $X = K(\mathcal{P})$, with $\mathcal{P} = \langle t | tt \rangle$, the dunce hat. We label the oriented corners $a, b, c$ as shown above. Then $X_1$ has two vertices $P$ and $Q$, four edges $a, b, c, t_1$ and one 2-cell with attaching map $at_1 bt_1 ct_1$. Since the edge “$t_1$” is a maximal tree of $X_1^{(1)}$, to construct an abelian cover of $X_1$ it suffices to choose elements $\phi(a), \phi(b), \phi(c)$ in an abelian group $A$ such that $\phi(a) + \phi(b) + \phi(c) = 0$. If the elements $\phi(a), \phi(b), \phi(c)$ generate $A$, the cover $U \xrightarrow{\pi} X_1$ of $X_1$ will be connected and $\pi^{-1}(L_X)$ will also be connected here.

As a specific example, if $A = \mathbb{Z}_n$ and $\phi(b) = \phi(c) = 1$ while $\phi(a) = n - 2$, then the branched cover $Y$ of $X$ is uniquely determined and $Y = K(\mathcal{D})$, with

$$\mathcal{D} = \langle t_1, t_2, \ldots, t_n | t_i t_{i+1} t_{i+2}, i \pmod{n} \rangle.$$ 

The group $G(\mathcal{D})$ of this presentation is the Fibonacci group $F(2, n)$ due originally to Conway [1] when $n = 5$.

2.3 The general Fibonacci groups $F(r, n)$ [12] occur as $G(\mathcal{D})$ where

$$\mathcal{D} = \langle t_1, t_2, \ldots, t_n | t_i t_{i+1} \cdots t_{i+r} t_{i+r+1}, i \pmod{n} \rangle.$$ 

The complex $K(\mathcal{D})$ is easily seen to be a branched cyclic cover of $K(\mathcal{P})$, where $\mathcal{P} = \langle t | t t \rangle$.

2.4 There is no need however to restrict oneself to the covers of $X_1$ in 2.2 giving rise to the Fibonacci groups. For example, the complex $Y = K(\mathcal{D})$ occurs as a branched cyclic cover of the dunce hat, where

$$\mathcal{D} = \langle t_1, t_2, \ldots, t_n | t_1 t_2 t_{i+1}, i \pmod{n} \rangle.$$
If \( n = 5 \), this is a presentation of the binary icosahedral group, as John Stallings pointed out to me. This group does not appear in the list of finite Fibonacci groups \( F(2, n) \).

2.5 We discussed the "Higman presentations" \( \mathcal{H}_n, n \geq 2 \), in [5, 4.20], where
\[
\mathcal{H}_n = \langle x_1, x_2, \ldots, x_n | x_{i+1}x_i^{-1}x_j^{-1}x_{i+1}x_i^{-1}x_j^{-1}, i \equiv j \pmod{n} \rangle.
\]
For \( n = 2 \) or \( 3 \), \( G(\mathcal{H}_n) \) is trivial, while for \( n \geq 4 \), \( G(\mathcal{H}_n) \) is an infinite perfect group having no proper subgroups of finite index [7]. It can be checked that \( K(\mathcal{H}_n) \) is a branched \( n \)-fold cyclic covering of \( K(P) \), where \( P = \langle t | t^4 \rangle \). The complex \( K(\mathcal{H}_n) \) was shown to be diagramatically reducible for \( n \geq 4 \) [5].

We shall next analyze the geometry of \( X, X_0, \) and \( X_1 \) more carefully for \( X \) an arbitrary 2-complex.

2.6 The dual 1-skeleton of a 2-complex \( X \) is the largest subgraph \( D_X \) of the first barycentric subdivision \( X' \) of \( X \) whose vertices are barycenters of 1-cells and 2-cells of \( X \). Thus each face \( A \) of \( X \) contributes \( d(A) \) edges to \( D_X \). The dual 1-skeleton is a functor from 2-complexes and combinatorial maps to graphs and nondegenerate morphisms of graphs.

There is a canonical deformation retraction \( \eta: X_1 \to D_X \) obtained by joining a vertex \( v \) of \( X \) linearly in a face \( F \) containing \( v \) to a point of \( D_X \).

In particular \( \eta \) induces by restriction a morphism of graphs \( \eta_X: L'_X \to D_X \) where \( L'_X \) is the first barycentric subdivision of \( L_X \), the link complex. The following three properties of \( \eta_X \) are easily checked.

2.7 The map \( \eta_X \) is natural in combinatorial maps \( X \to Y \), so one has a commutative diagram of graphs:

\[
\begin{array}{ccc}
L'_X & \xymatrix{\eta_X} & D_X \\
\ar[d]^f & & \ar[d]^f \\
L'_Y & \eta_Y & D_Y
\end{array}
\]

2.8 If \( X \) has no monogons (so \( d(A) > 1 \) for each face \( A \) of \( X \)), then \( \eta_X \) is an immersion of graphs.

2.9 \( X_1 \) is isomorphic to the mapping cylinder of the map \( \eta_X: L'_X \to D_X \).

In particular, 2.9 has an important consequence for the sequel.

2.10 Proposition. Covering spaces of \( X_1 \) correspond 1-1 to those of \( D_X \).
Thus one has a construction for all finite branched covers of $X$. First construct the finite covers of $D_X$, a graph. This is a problem about subgroups of finite index in a free group. Next use the deformation retraction $\eta: X_1 \to D_X$ to construct all finite covers of $X_1$. Take the conical extensions of such covers (2.1) and finally make identifications ad libidum in the fibers over $X^{(0)}$.

2.11 Proposition. Let $f: Y \to X$ be a branched covering.
(a) If $X$ is diagrammatically reducible, so is $Y$.
(b) If $X$ satisfies the curvature and link conditions for weight $w$, then so does $Y$ for the pull-back weight $f^*(w)$.
(c) If $(X,w)$ is hyperbolic, then so is $(Y,f^*w)$.

Proof. All of these assertions are consequences of the fact that $f$ induces an immersion $L_Y \to L_X$ of link complexes, which in turn preserves reduced paths [16]. Thus, for example in 2.11(a), if $g: S^2 \to Y$ is a reduced spherical diagram, then $f \circ g: S^2 \to X$ will also be reduced. The remaining statements are proved similarly.

2.12 It is worth giving an example of a finite branched cover $Y$ of an aspherical 2-complex $X$ which is not itself aspherical. One may take $X$ to be the dunce hat and $Y = K(2)$ where $\mathcal{L} = \langle x, y | xx\bar{y}, yy\bar{x} \rangle$. It is easily checked that $Y$ is a 2-fold branched cover of $X$. But $\pi_1(Y) = \mathbb{Z}_3$ so (by P. A. Smith’s theorem) $Y$ cannot be aspherical. This phenomenon will be explained in §4, when reduced diagrams will be related to branched covers.

Here is a curious consequence of 2.9.

2.13 Proposition. Let $X$ be a 2-complex and let $Y$ be a triangulable subset of $X_0 = X - X^{(0)}$. Then $Y$ is aspherical.

Proof. If $Y$ is compact, then $Y$ may be assumed to be a subset of $X_1$. Now $X_1$ collapses onto $D_X$, a graph, whence $X_1$ is diagrammatically reducible. Since diagrammatic reducibility is invariant under subdivision and is inherited by subcomplexes, it follows that $Y$ is diagrammatically reducible and hence aspherical.

In the general case, $Y$ may be triangulated; so every finite subcomplex of $Y$ is aspherical, by the preceding paragraph. It follows that $Y$ is aspherical.

3. Existence of hyperbolic covers

3.1 Theorem. Suppose that $X$ is a finite 2-complex such that $d(F) \geq 3$ for each face $F$ of $X$. Then $X$ has a finite branched cover which admits a hyperbolic structure.

Observe that any cell structure $C$ on $S^2$ with only two vertices and no monogons provides a counterexample to extending 3.1, for any branched cover of $C$ is itself the 2-sphere. Observe also that for any 2-complex $X$ the first barycentric subdivision $X'$ satisfies $d(F) = 3$ for all faces $F$ of $X'$.

3.2 Corollary. If $X$ is a finite 2-complex then $X'$, the first barycentric subdivision of $X$, has a finite branched cover which admits a hyperbolic structure.
**Proof of 3.1.** We may assume $X$ is connected. We can find a positive number $\varepsilon$ such that for any face $F$ of $X$, $\varepsilon < (d(F) - 2)/d(F)$. This is true because $X$ is finite and $d(F) \geq 3$ for all faces $F$. We set $w(\gamma) = \varepsilon$ for all corners $\gamma$ of $X$. Pick a number $N$ such that $N\varepsilon \geq 2$ and list all the finite number of nontrivial cyclically reduced circuits of the link complex $L_X$ of length $< N$, say $C_1, C_2, \ldots, C_k$.

Since $\eta_X$ is an immersion by 2.8 it follows that each of $\eta_X(C_1), \eta_X(C_2), \ldots, \eta_X(C_k)$ is a nontrivial cyclically reduced circuit in $D_X$. Choose a base point $v$ for $D_X$. By connecting some vertex in the circuit $\eta_X(C_i)$ to $v$ in $D_X$ by an arc $\alpha_i$, we construct elements $x_1, x_2, \ldots, x_k$ of $\pi_1(D_X, v) = G$. These elements depend on the choice of $\alpha_i$, but their conjugacy classes in $G$ are independent of choices. By M. Hall's theorem [16, 6.3], we can find a subgroup $M$ of finite index in $G$ with $x_i \notin M$, $1 \leq i \leq k$. Furthermore, by taking the intersection of finitely many conjugate subgroups, we may assume in addition that $M \triangleleft G$. Then no conjugate of $x_i$ lies in $M$.

Let $Y_1 \to X_1$ be the finite regular cover of $X_1$ (see 2.1 for notation) corresponding to $M \triangleleft G$ and extend $\pi$ to a finite branched cover $\pi: Y \to X$, by the conical extension, for definiteness. We claim that $(Y, \pi^*w)$ is a hyperbolic 2-complex. The curvature condition is automatic, since $(X, w)$ satisfies the curvature condition. It remains to check the link condition.

Suppose then that $Z$ is a nontrivial cyclically reduced circuit in $L_Y$. Then $\pi(Z)$ is a nontrivial cyclically reduced circuit in $X$. On the other hand $\pi(Z) \neq C_i$ for $1 \leq i \leq k$, for none of the circuits $C_i$ lift to closed paths in $Y_1$; otherwise some conjugate of some $x_i$ would lie in $M$, contrary to choice of $M$. Thus the length of $\pi(Z)$ is greater than or equal to $N$. If $Z = \gamma_1\gamma_2 \cdots \gamma_n$ with $\gamma_i$ corners, $n \geq N$, it follows that $\sum_{i=1}^n w(\gamma_i) \geq N\varepsilon \geq 2$. Thus the link condition is satisfied and $(Y, \pi^*w)$ is a hyperbolic structure. This completes the proof of Theorem 3.1.

**4. Lifting theorem.**

4.1 **Theorem.** Let $f: \Gamma \to \Delta$ be an immersion of graphs, where $\Gamma$ is finite. Then there is a finite covering $\Delta_1 \to \Delta$ and a commutative diagram

\[ \begin{array}{ccc} 
\Gamma & \xrightarrow{f_1} & \Delta_1 \\
\downarrow f & & \downarrow \pi \\
\Delta & \xrightarrow{\pi} & \Delta 
\end{array} \]

where $f_1$ is injective.

This is a graphical form of Marshall Hall's theorem on free groups. The proof for the special case when $\Delta$ is a 1-vertex graph can be found in [16, 6.1]. The reduction of the general case to the 1-vertex case is given in [19, Appendix].
4.2. Theorem. Let \( f: X \to Y \) be a reduced combinatorial map of the 2-complexes \( X \) and \( Y \) with \( X \) finite. Then there is a finite branched cover \( \pi: Z \to Y \) and a commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{\pi} & Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y \\
\end{array}
\]

where \( g \mid (X - X^{(0)}) \) is injective.

Proof. Since \( f \) is reduced it follows that \( f \) induces immersions of graphs \( L_X \xrightarrow{f} L_Y \) and \( D_X \xrightarrow{f} D_Y \). Since \( D_X \) is finite, it follows from 4.1 that there is a commutative diagram

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\pi} & D_Y \\
\downarrow & & \downarrow \\
D_X & \xrightarrow{f} & D_Y \\
\end{array}
\]

where \( \pi \) is a finite covering and where \( g \) is injective. Since coverings of \( D_Y \) are the same as coverings of \( Y_1 \) by 2.9 it follows we can thicken up the diagram (4.2.1) to obtain a commutative diagram

\[
\begin{array}{ccc}
Z_1 & \xrightarrow{\pi} & Y_1 \\
\downarrow & & \downarrow \\
X_1 & \xrightarrow{f} & Y_1 \\
\end{array}
\]

where \( \pi: Z_1 \to Y_1 \) is a finite covering. Next extend \( \pi \) by conical extension (2.1) to a finite branched cover \( \pi : Z \to Y \) and extend the map \( g : X_1 \to Z_1 \) also conically to a map \( g : X \to Z \) so that the diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{\pi} & Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y \\
\end{array}
\]

is commutative. It remains to check that \( g \mid (X - X^{(0)}) \) is injective. But this will follow if we can show that \( g : X_1 \to Z_1 \) is injective. Since \( X_1 \) and \( Z_1 \) are mapping cylinders over \( D_X \) and \( \Gamma \) respectively, it suffices to show that \( g : X_1 \to Z_1 \) is injective on some neighborhood of \( D_X \) in \( X_1 \). However \( f : X_1 \to Y_1 \) is locally injective, so it follows from (4.2.2) that \( g : X_1 \to Z_1 \) is also locally injective. Since \( g : D_X \to \Gamma \) is

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injective, it follows that $g: X_1 \to Z_1$ is injective on some neighborhood of $D_X$ in $X_1$, as required.

4.3 Lemma. Let $f: C \to X$ be a combinatorial map of 2-complexes where $C$ is a cell structure on $S^2$. If $f$ is injective on $C - C^{(0)}$, then the Hurewicz map $\pi_2(X) \to H_2(X)$ is nonzero.

Proof. One sees that $f$ induces a nonzero map $H_2(C) \to H_2(X)$, so the class of $f$ in $\pi_2(X)$ is detected in $H_2(X)$.

4.4 Definition. We recall that a connected 2-complex $X$ is called Cockcroft if the Hurewicz map $\pi_2(X) \to H_2(X)$ is zero.

4.5 Theorem. The following assertions are equivalent for a connected 2-complex $X$:

4.5.1 $X$ is diagrammatically reducible.

4.5.2 Every finite branched cover of $X$ is aspherical.

4.5.3 Every finite branched cover of $X$ is Cockcroft.

Proof. 4.5.1 $\Rightarrow$ 4.5.2 follows from 2.11(a).

4.5.2 $\Rightarrow$ 4.5.3 follows since aspherical 2-complexes are Cockcroft.

4.5.3 $\Rightarrow$ 4.5.1: Suppose that every finite branched cover of $X$ is Cockcroft and let $f: C \to X$ be a spherical diagram. If $f$ were reduced, it would follow from 4.2 that there is a commutative diagram

\[
\begin{array}{ccc}
C & \xrightarrow{f} & X \\
g \downarrow & & \pi \\
Y & \xrightarrow{h} & X
\end{array}
\]

where $\pi: Y \to X$ is a finite branched cover and where $g: C \to Y$ is injective when restricted to $C - C^{(0)}$. It follows from Lemma 4.3 that $Y$ is not Cockcroft, contrary to hypothesis.

Thus every spherical diagram in $X$ is reduced, and $X$ is diagrammatically reducible.

5. Amalgamation theorems. Theorem 4.5 is a powerful tool which can be exploited to prove certain 2-complexes are diagrammatically aspherical. The arguments also use the next result which is a generalized version of a theorem of J. H. C. Whitehead [18].

5.1 Theorem. Let $Z$ be a connected CW complex which is the union of two subcomplexes $X$ and $Y$. Assume that each connected component of $X$ and of $Y$ is aspherical. Assume also that each connected component $C_0$ of $X \cap Y$ is aspherical and the inclusions $C_0 \subseteq X_0$ and $C_0 \subseteq Y_0$ induce injections in $\pi_1$; here $X_0$ and $Y_0$ are connected components of $X$ and $Y$ respectively. Then $Z$ is aspherical.

The proof makes use of the Mayer-Vietoris homology sequence for the universal cover $\tilde{Z}$ of $Z$ together with the fact that the universal covers of $X_0$, $Y_0$, and $C_0$ all are imbedded as subcomplexes of $\tilde{Z}$. One deduces that $\tilde{H}_*(\tilde{Z}) = 0$, so by Whitehead's theorem $\tilde{Z}$ is contractible.
5.2 Example. A naive attempt to generalize 5.1, replacing “aspherical” everywhere by “diagrammatically reducible” fails as the next example shows. Let $X$ be the dunce hat:

$$X = K(\mathcal{P}), \quad \mathcal{P} = \langle t \mid tt \rangle.$$

Bore out the interior of a disc $D$ from the 2-cell of $X$ as shown below to get $A = X - D$.

Thus $A = K(\langle t, b \mid tbt' \rangle)$. Let $Z$ be the double of $A$ along the closure of the edge $b$; $Z = A \cup_B A'$, where $A \equiv A'$ and where $B = K(\langle b \rangle)$, a circle. One sees that $A$ (and hence $A'$) collapses onto a graph; so $A$ is diagrammatically reducible, as in $B$. In addition the inclusions $B \subset A$ and $B \subset A'$ are injective on $\pi_1$. However $Z$ is not diagrammatically reducible.

To see this, observe that $Z$ is a subdivision of $W = K(\langle t, u \mid t^2 = u^2 \rangle)$, as the next two diagrams indicate:

However here is a reduced spherical diagram in $W$:

This example was also discovered independently by Sieradski (private communication).
Thus some additional conditions are needed to obtain an amalgamation theorem for diagrammatic reducibility.

5.3 Definition. The subgraph $A$ of the 2-complex $X$ is said to be 2-sided in $X$ if $A$ has a neighborhood $U$ homeomorphic to $A \times I$, $I = [0, 1]$. If $A_0$ is a connected component of $A$, identified with $A \times \{1/2\}$ under the homeomorphism, and if $X_0$ is a connected component of $X - A$ containing $A_0 \times \{\delta\}, \delta = 0$ or $1$, then the composite map $A_0 \rightarrow A_0 \times \{\delta\} \subset X_0$ is called “pushing $A_0$ to one side in $X$”.

5.4 Theorem. Let $A$ be a subgraph of $X^{(1)}$ where $X$ is a connected 2-complex. Assume that all components of $X - A$ are diagrammatically reducible and $A$ is two-sided in $X$. Assume further that all maps obtained by pushing connected components of $A$ to one side in $X$ induce injections on $\pi_1$. Then $X$ is diagrammatically reducible.

Proof. Let $\rho: Y \rightarrow X$ be a finite connected branched cover of $X$. By 4.5 it suffices to prove that $Y$ is aspherical. We identify a neighborhood of $A$ in $X$ with $A \times I$ and identify $A$ with $A \times \{1/2\}$. By subdividing we may assume $A \times I$ is a subcomplex of $X$ and by passing to a smaller product neighborhood of $A$, we may assume that $\pi$ is unbranched over $A_\delta = A \times \{\delta\}; \delta = 0, 1$. Let $B_\delta = \pi^{-1}(A_\delta)$, a 2-sided subgraph of $Y$, and let $B = B_0 \cup B_1$. Observe that all connected components of $Y - B$ are aspherical, by 4.5 (the connected components over $A \times I$ are aspherical since $A \times I$ is diagrammatically aspherical, $A$ being a graph). Let $\mathcal{C}$ be a connected component of $B$ and let $Y_0$ be a connected component of $Y$ containing $C_0$, the push of $C$ to one side in $Y$. Let $X_0 = \rho(H_0)$ and $D_0 = \rho(C_0)$ and consider the following commutative diagram of fundamental groups (for suitable choice of base points):

$$
\begin{array}{c}
\pi_1(C_0) & \rightarrow & \pi_1(Y_0) \\
\rho_* & 1-1 & \rho_* \\
\pi_1(D_0) & \rightarrow & \pi_1(X_0)
\end{array}
$$

(5.4.1)

The left vertical arrow is injective, since $C_0$ can be assumed to avoid all branch points, whence $\rho|_{C_0}: C_0 \rightarrow D_0$ is a covering. The bottom arrow is injective by the hypothesis of the theorem if $X_0$ is homeomorphic to one of the components of $X - A$, and it is trivially so if $X_0$ is a component of $A \times I$. It follows that the top arrow in 5.4.1 is injective. Thus $Y$ is aspherical by Theorem 5.1.

5.5 Example. Let $X$ be a 2-complex with no monogons. Let $X_1 = X - \hat{N}$, with $N$ a regular neighborhood of $X^{(0)}$ in $X$, so $X_1$ contains $L_X$, the link complex. Let $Z$ be the double of $X_1$ along $L_X$ obtained by gluing two isomorphic copies of $X_1$ along $L_X$. Then $Z$ is diagrammatically reducible. To see this observe that $X_1$ collapses onto $D_X$, a graph. Obviously $L_X$ is 2-sided in $Z$. Also since $X$ has no monogons, the canonical map $\eta_X: L_X \rightarrow D_X$ of 2.8 is an immersion, so induces on each connected component an injection on $\pi_1$. It follows from 5.4 that $Z$ is diagrammatically reducible.
This result is certainly false in general if $X$ possesses monogons. For one can take $X$ to be a cell structure on $S^2$ with one vertex, one edge, and two monogons. The complex $Z$ constructed above is homeomorphic to $S^2$ in this example.

Another hypothesis on $X$ which will guarantee that $Z = X_1 \sqcup L_X$, $X_1$ is diagrammatically reducible is to assume that $H_2(X) = 0$. In this case, the map of a connected component of $L_X$ into $X_1$ induces an injection on $\pi_1$ by the Kervaire Conjecture, which is valid for free groups. We omit the details.

Theorem 5.4 is very powerful, as the next result shows.

5.6 Theorem. Let $F$ be a free group with free basis $S$. Let $\phi: G_1 \rightarrow G_2$ be an isomorphism of two subgroups $G_1$ and $G_2$ of $F$. Let $\{a_i, i \in I\}$ be a free basis for $G_1$, where $a_i$ is a word, not necessarily reduced, in the alphabet $S \cup \bar{S}$, and let $b_i$ be a representative word of $\phi(a_i)$, not necessarily reduced. Then the 2-complex $K(\mathcal{P})$ is diagrammatically reducible, where $\mathcal{P} = \langle t, s; s \in S \mid ta_i t^{-1} = b_i, i \in I \rangle$.

Proof. Let $X = K(\langle S, S \rangle)$ and $Y = K(\langle y_i, i \in I \rangle)$, both bouquets of circles. Form the identification space $Z$ of $X \sqcup (Y \times I)$ by attaching $Y \times \{0\}$ to $X$ by subdividing the circle $y_i$ and attaching it by the word $a_i$, and by attaching $Y \times \{1\}$ to $X$ by subdividing the circle $y_i$ and attaching it by the word $b_i$. The space $Z$ is diagrammatically reducible by 5.4. If we identify all vertices of $Z$ to a single vertex, the resulting space $W$ is still diagrammatically reducible. Furthermore $W = K(\mathcal{Q})$, where

$$\mathcal{Q} = \langle S, u, v, y_i, i \in I \mid uy_iu^{-1} = a_i, vy_i v^{-1} = b_i, i \in I \rangle$$

(5.6.1)

Clearly $K(\mathcal{Q})$ is obtained from a subdivision of $K(\mathcal{P})$, by pinching vertices as one sees by inspecting (5.6.1). It follows that $K(\mathcal{P})$ is diagrammatically reducible.

5.7 Example. It follows from 5.6 that the 2-complex $K(\mathcal{P})$ is diagrammatically reducible, where $\mathcal{P} = \langle x, y, t \mid txxxt = yyy \rangle$. Compare this with Example 5.2. From this result and from 1.6 it follows that the equation over the group $G$,

$$a_1 a_2 a_3 a_4 a_5 a_6 \bar{a}_7 y a_8 \bar{y} = 1$$

has a solution in an overgroup of $G$. Here $a_i, 1 \leq i \leq 8$, are elements of $G$ and $t, x, \text{ and } y$ are indeterminates.

5.8 Remark. Theorem 5.6 says that certain HNN presentations are diagrammatically aspherical. There is an analogous result for free-product-with-amalgamation presentations (indeed, for graphs of free groups) which we leave to the reader to formulate with a caution: the amalgamation relations must have the form of a conjugation by new stable letter.

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5.9 We want to discuss another amalgamation result, the connected sum of two 2-complexes, since this appeared in Sieradski’s paper [14].

Let $X$ be a connected 2-complex and let $F$ be a face (= closed 2-cell) of $X$. Imbed a closed disc $D$ in $F$ so that $D \cap X^{(1)}$ is precisely one point on the boundary $\partial D$ of $D$. See (5.2.1) for a picture of this. Let us call the pair $(X, D)$, “suitable”. If $(Y, D)$ is another suitable pair we may amalgamate by boring out $D$ and identifying along $\partial D$:

$$Z = (X - \bar{D}) \coprod_{\partial D} (Y - \bar{D})$$

Let us denote $Z$ by $X \# Y$, an abuse of notation, since $Z$ depends on the pairs $(X, D)$ and $(Y, D)$.

5.10 Proposition. If $X$ and $Y$ are diagrammatically reducible, then so is $X \# Y$.

Proof. Let $X_1 = X - \bar{D}$ and $Y_1 = Y - \bar{D}$ and let $A = \partial D$. Since diagrammatic reducibility is preserved under subdivision and under taking subcomplexes, it follows that $X_1$ and $Y_1$ are diagrammatically reducible, as is the graph $A$. Let $\{v\} = \partial D \cap X^{(1)} = \partial D \cap Y^{(1)}$, the unique vertex of $A$. Let $p: \hat{Z} \to Z$ be a finite branched cover of $Z = X \# Y$. We may assume that $p^{-1}(v)$ consists of precisely one point, $\hat{v}$, if necessary by identifying together all points of the fiber $p^{-1}(v)$. Then $p^{-1}(A) = B$ is a bouquet of $n$ circles all attached together at $\hat{v}$, where $n$ is the generic degree of the cover. Theorem 5.1 will imply that $\hat{Z}$ is aspherical provided we can show that the inclusions $B \subset \hat{X}_1 = \text{def} p^{-1}(X_1)$ and $B \subset \hat{Y}_1 = \text{def} p^{-1}(Y_1)$ are injective on $\pi_1$. Thus by symmetry we can forget about $Y$ and consider only the inclusion $B \subset \hat{X}_1$.

Observe that $p: \hat{X}_1 \to X_1$ can be extended to a branched cover $p: \hat{X} \to X$. To do this, just make each of the $n$-circles of $B$ the boundary of a disc and map that disc homeomorphically onto $D$.

If $\pi_1(B) \to \pi_1(\hat{X_1})$ were not injective there would be a reduced planar connected simply connected diagram $K$ in $\hat{X}_1$ with boundary label a cyclically reduced nontrivial word in $\pi_1(B)$. Choosing $K$ to have the minimal number of faces among all such diagrams it follows we may assume $K$ is a reduced disc diagram $f: K \to \hat{X}_1$ with boundary label $f|\partial K$ a cyclically reduced nontrivial word in $\pi_1(B)$. Now each letter of the boundary label $f|\partial K$ bounds a unique disc in $p^{-1}(D)$. Thus $f: K \to \hat{X}_1$ can be extended to a reduced spherical diagram in $\hat{X}$. But $\hat{X}$ is a branched cover of $X$ and $X$ is diagrammatically reducible. By 2.11, so is $\hat{X}$ diagrammatically reducible. This contradiction shows that $\pi_1(B) \to \pi_1(\hat{X_1})$ is injective, and the proof is complete.

6. Applications.

6.1 We recall some terminology. A group homomorphism $\phi: G \to H$ is said to be normally convex [15] if for any subset $S$ of $G$ one has

$$\phi^{-1}(\langle \langle \phi(S) \rangle \rangle_H) = \langle \langle S \rangle \rangle_G.$$ 

In particular this implies that $\phi$ is injective and for any $N \triangleleft G$, the induced homomorphism $G/N \to H/\langle \langle \phi(N) \rangle \rangle_H$ is injective. If $\phi: G \to H$ satisfies “Property G” (see 1.6) then it is normally convex.

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In [9], Higman, Neumann and Neumann produce a normally convex imbedding of a free group $F_\infty$ of countable rank into a free group $F_2$ of rank 2. This imbedding plays an important role in other group theoretic imbedding theorems, e.g. [8]. We shall establish below a strengthening of the property of normal convexity and give a geometric interpretation for the HNN imbedding.

6.2 Theorem. There is an imbedding of $F_\infty$ in $F_2$ which satisfies “Property G.”

Proof. Let $F = F(a, b)$ be the free group with free basis \{a, b\} and consider the subgroups $G_1 = \langle b^{-n}ab^n, n \geq 0 \rangle$ and $G_2 = \langle a^{-n}ba^n, n \geq 0 \rangle$ of $F$. Both are free on the indicated set of generators, so let $\phi$: $G_1 \rightarrow G_2$ be given by $\phi(b^{-n}ab^n) = a^{-n}ba^n$; $n \geq 0$. By Theorem 5.6 the 2-complex $K(\mathcal{P})$ is diagrammatically reducible, where $\mathcal{P} = \langle a, b, t \mid t^{-1}b^{-n}ab^n t = a^{-n}ba^n; n \geq 0 \rangle$. This implies that the corresponding system of equations of groups satisfies “Property G”. We shall give an explicit specialization of the corner variables in these equations as shown in the following diagrams of 2-cells of $K(\mathcal{P})$:

\[ n = 0: \]

\[ n > 0: \]

It follows that the natural homomorphism of groups

\[ F_\infty = \text{def} \langle x_n, n \geq 1 \rangle \]

\[ \rightarrow \langle a, b, t, x_n; n \geq 1 \mid t^{-1}at = b, t^{-1}b^{-n}ab^n t = x_n a^{-n}ba^n; n \geq 1 \rangle \]

satisfies “Property G”. But using Tietze transformations we see that $H = \langle a, t \rangle = F(a, t)$. The indicated homomorphism $F_\infty \rightarrow F(a, t)$ is exactly the HNN imbedding.

6.3 Remark. I know of essentially only one example of normally convex homomorphism $\phi$: $G \rightarrow H$ that does not satisfy Property G. If $F(a, b)$ is free on \{a, b\}, the natural homomorphism $F(a, b) \rightarrow F(a, b, t)/\langle \langle at^2bt^{-1} \rangle \rangle$ is normally convex, by a theorem of Howie’s [11] but does not satisfy “Property G” by a result of this author’s [4]. Why this should be so is a great mystery.

6.4 We now come to the application to knot complements in the 3-sphere. From the work of Haken [22] it follows that if $M$ is a bounded Haken manifold there is a sequence of (possibly disconnected) 3-manifolds $M = M_0, M_1, \ldots, M_n$ such that $M_j$
is obtained from $M_{j-1}$ by splitting $M_{j-1}$ along a bounded properly imbedded two-sided incompressible surface $F_j$ and such that each connected component of $M_n$ is a ball. Using our Theorem 5.4, the fact the bounded surfaces have graphs as spines, and an obvious induction we deduce

6.5 **Theorem.** Every bounded Haken manifold has the homotopy type of a diagrammatically reducible 2-complex.

6.6 **Remark.** Since tame knot complements are bounded Haken manifolds, Theorem 6.5 applies to show they are homotopy equivalent to DR 2-complexes. This can be viewed as a solution to a question raised by Sieradski [14]. Sieradski asked whether it was possible to find a presentation $\mathcal{P}$ of the fundamental group of a tame knot which satisfied his “coloring test;” that is, so that $K(\mathcal{P})$ possessed a weight (1.7) satisfying the link and curvature conditions and so that $w$ takes values in $\{0, 1\}$. This conjecture implies all tame knot complements have the homotopy type of DR 2-complexes; since it is DR that Sieradski is interested in, it is in this sense that 6.5 answers his question.

6.7 **Remark.** Howie has brought to my attention that the argument for 6.5 is the same argument used in [21] to prove that bounded Haken manifolds are diagrammatically aspherical. This latter is a very much weaker result and has no implications about equations over groups.

**Appendix.** I had at one time considered including an appendix relating the notion of diagrammatic reducibility (DR) to the notion of diagrammatic asphericity (DA) discussed in [21]. However I have recently shown conclusively that there is no relationship between DA and the solution of equations over groups, my main interest here (compare 1.6 above). Thus I believe it is more important to inform the reader of new developments in equations over groups rather than to include here a scholarly survey of another attempt to formulate a notion of combinatorial homotopy which is demonstrably irrelevant to the problem at hand.

To state my result let me recall the notion of a Kervaire complex due to S. Brick [20]. A 2-complex $X$ is called “Kervaire” if all equations over all coefficient groups, whose words in the variable letters are the attaching maps of the 2-cells of $X$, are solvable in an overgroup of the coefficient group. Thus, for example, the dunce hat $K(x | x^3)$ is Kervaire since Howie has shown that the equation $axbxcx = 1$ is solvable in an overgroup of $G$, where $a, b, c \in G$, the coefficient group [11]. Recall also that a 2-complex $X$ is called DA [21] if, given a spherical diagram in $X$, some sequence of diamond moves exists which splits off a 2-sphere component with precisely two faces. My result is

**Theorem.** There is a 2-complex $Z$ which is both Cockcroft and DA, but not Kervaire.

The example, $Z$, I construct is given as $Z = X \sqcup \Gamma X$, where $X = K(x, y, z | x^2, y^2, t = xy)$ and $\Gamma = K(t \mid )$. The details will appear in my forthcoming article, *Amalgamations and the Kervaire problem.*
To complete the justification of my assertion that DA is unrelated to solving equations over groups, it suffices to exhibit a Kervaire complex which is not DA. For this one can take the example $X = K(P)$, where $P = \langle x, y | y^{-2}xyx, x \rangle$. Chiswell has shown that $X$ is not DA [21]. However $X$ is Kervaire, as one sees without difficulty by reducing to the fact that the dunce hat is Kervaire.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY, UTAH 84112