ON THE ELLIPTIC EQUATIONS $\Delta u = K(x)u^\sigma$ AND $\Delta u = K(x)e^{2u}$

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Abstract. We give some nonexistence results for the equations $\Delta u = K(x)u^\sigma$ and $\Delta u = K(x)e^{2u}$ for $K(x) > 0$.

1. Introduction. In this paper we study the elliptic equations

$$(1.1) \quad \Delta u = K(x)u^\sigma \quad \text{in } \mathbb{R}^n$$

and

$$(1.2) \quad \Delta u = K(x)e^{2u} \quad \text{in } \mathbb{R}^n,$$

where $\sigma > 1$ is a constant, $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ and $K(\cdot)$ is a bounded Hölder continuous function in $\mathbb{R}^n$. We are concerned with the existence problems of locally bounded and positive solutions for (1.1) and locally bounded solutions for (1.2).

These problems come from geometry. We give a brief description and refer the details to Kazdan and Warner [5] and Ni [13, 14]. Let $(M, g)$ be a Riemannian manifold of dimension $n$, $n \geq 2$, and $K(\cdot)$ be a given function on $M$. We ask the following question: can one find a new metric $g_1$ on $M$ such that $K$ is the scalar curvature of $g_1$ and $g_1$ is conformal to $g$ (i.e., $g_1 = \psi g$ for some function $\psi > 0$ on $M$)? In the case $n \geq 3$, we write $\psi = u^{4/(n-2)}$. Then this problem is equivalent to the problem of finding positive solutions of the equation

$$(1.3) \quad \frac{4(n-1)}{n-2} \Delta u - ku + Ku^{(n+2)/(n-2)} = 0,$$

where $\Delta$, $k$ are the Laplacian and scalar curvature in the $g$ metric, respectively. In the case $M = \mathbb{R}^n$ and $g = (\delta_{ij})$, then $k = 0$ and equation (1.3) reduces to (1.1) with $\sigma = (n + 2)/(n - 2)$, after an appropriate scaling and sign changing of $K(\cdot)$. In the case $n = 2$, we write $\psi = e^{2u}$. Then this problem is equivalent to the problem of finding locally bounded solutions of the equation

$$(1.4) \quad \Delta u - k + Ke^{2u} = 0,$$

where $\Delta$, $k$ are the Laplacian and Gaussian curvature on $M$ in the $g$ metric. In the case $M = \mathbb{R}^2$ and $g = (\delta_{ij})$, we have $k = 0$ and equation (1.4) reduces to (1.2), after a sign changing of $K$. 

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In [13 and 14], Ni makes major contributions to the existence of solutions for (1.1) and (1.2). After these two papers, there are many improved results published, such as McOwen [10, 11], Naito [12], Kawano, Kusano and Naito [3], Kawano and Kusano [4], Kusano and Oharu [7], Ding and Ni [1], Kusano, Swanson and Usami [8] and Lin [9].

In this paper, we consider the case \( K(x) \geq 0 \) in (1.1) and (1.2). We obtain some nonexistence results which make the understanding of the case \( K(x) > 0 \) almost complete. We divide this paper into two parts. In Part I, we consider (1.1). Thus we consider the case (1.1) with \( n > 3 \) in §2, (1.1) with \( n = 2 \) in §3 and (1.1) with \( n = 1 \) in §4. We consider (1.2) in Part II. Thus we consider the case (1.2) with \( n > 3 \) in §5, (1.2) with \( n = 2 \) in §6 and (1.2) with \( n = 1 \) in §7.

We remark that the technique of the proof of the main nonexistence theorem is essentially equivalent to the proof of Keller [6]. We thank the referee for bringing the reference [6] to our attention.

**Part I. \( \Delta u = K(x)u^p \)**

2. The case \( n \geq 3 \). In this case, Ni [13] proves the main existence result: Let \( K \) be bounded. If \( |K(x)| \leq C/|x|^{2+\epsilon} \) at \( \infty \) for some constants \( C > 0 \) and \( \epsilon > 0 \), then equation (1.1) has infinitely many bounded solutions in \( \mathbb{R}^n \) with positive lower bounds. Later on, Naito [12] improves the result: If \( |K(x)| \leq \phi(|x|) \) for all \( x \in \mathbb{R}^n \) and \( \int_0^\infty t\phi(t)\,dt < \infty \), then equation (1.1) has infinite many bounded positive solutions which tend to a positive constant at \( \infty \). On the other hand, when \( K(x) \geq 0 \), Ni [13] proves a nonexistence result: If \( K(x) \geq C/|x|^{2-\epsilon} \) at \( \infty \) for some constants \( C > 0 \) and \( \epsilon > 0 \), then (1.1) does not possess any positive solution in \( \mathbb{R}^n \). Lin [9] proves that it is still true even \( \epsilon = 0 \). In view of Naito’s existence result, we expect that the following conjecture be true.

**Conjecture.** Let \( K(x) \geq \tilde{K}(|x|) \geq 0 \) for all \( x \in \mathbb{R}^n \) and \( \int_0^\infty s\tilde{K}(s)\,ds = \infty \). Then (1.1) does not possess any positive solution in \( \mathbb{R}^n \).

We give three theorems which almost answer this conjecture completely. Following Ni [13], we define the averages of \( u(x) > 0 \) and \( K(x) ^ 0 \) by \( \bar{u}(r) \) and \( \bar{K}(r) \),

\[
\bar{u}(r) = \frac{1}{\omega_n r^{n-1}} \int_{|x|=r} u(x)\,dS,
\]

\[
\bar{K}(r) = \left( \frac{1}{\omega_n r^{n-1}} \int_{|x|=r} \frac{dS}{K(x)^{\sigma/\mu}} \right)^{-\sigma/\mu},
\]

where \( dS \) denotes the volume element in the surface integral, \( \omega_n \) denotes the surface area of the unit sphere in \( \mathbb{R}^n \) and \( 1/\mu + 1/\sigma = 1 \).

For the sake of completeness, we give another proof of Lin’s result of non-existence [9] in the following.

**Theorem 2.1.** Let \( K(x) \) be a locally Hölder continuous function. If \( K(x) \geq 0 \) and \( \bar{K}(r) \geq C/r^2 \) for \( r \) large for some constant \( C > 0 \), then equation (1.1) does not possess any positive solution in \( \mathbb{R}^n \).
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**Proof.** Let $u$ be a positive solution of (1.1) in $\mathbb{R}^n$. Then from Ni [12, Lemma 3.21], we have

$$
\begin{aligned}
\begin{cases}
\bar{u}''(r) + \frac{n-1}{r} \bar{u}'(r) \geq \bar{K}(r) \bar{u}^\sigma(r) & \text{in } (0, \infty), \\
\bar{u}(0) = \alpha > 0, & \bar{u}'(0) = 0.
\end{cases}
\end{aligned}
$$

Hence we have

$$
\bar{u}(r) \geq \alpha + \frac{1}{n-2} \int_0^r s \bar{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] \bar{u}^\sigma(s) \, ds.
$$

Now assume that $\bar{K}(r) \geq C/r^2$ for $r \geq R_0$. Let $r > R_0$. Then from (2.4), we have

$$
\bar{u}(r) \geq \alpha + \frac{1}{n-2} \int_{R_0}^r s \bar{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] \bar{u}^\sigma(s) \, ds
$$

$$
\geq \alpha + \frac{1}{n-2} \int_{R_0}^r s \bar{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] \bar{u}^\sigma(s) \, ds
$$

$$
\geq \alpha + \frac{\alpha^\sigma}{n-2} \cdot C \cdot \left[ 1 - \left( \frac{1}{2} \right)^{n-2} \right] \cdot \int_{R_0}^{r/2} \frac{1}{s} \, ds
$$

$$
\geq C_1 \log r
$$

for some $C_1 > 0$ and $r \geq R_1 > 2R_0$. For $R > R_1$ and $R \leq r \leq 2R$, we have

$$
1/2 \leq s/r \leq 1.
$$

Hence

$$
\left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] = \frac{s}{r^{n-2}} \left[ r^{n-2} - s^{n-2} \right] \geq (n-2) \left( \frac{1}{2} \right)^{n-2} (r-s).
$$

From (2.4), (2.5) and (2.7), we obtain

$$
\bar{u}(r) \geq C_1 \log R + \frac{C_2}{R^2} \int_R^r (r-s) \bar{u}^\sigma(s) \, ds
$$

for $R > R_1$ and $R \leq r \leq 2R$, where $C_2 > 0$ is a constant. Let

$$
g(r) = C_1 \log R + \frac{C_2}{R^2} \int_R^r (r-s) \bar{u}^\sigma(s) \, ds.
$$

Then

$$
g(R) = C_1 \log R, \quad g'(R) = 0,
$$

$$
g'(r) = \frac{C_2}{R^2} \int_R^r \bar{u}^\sigma(s) \, ds \geq 0,
$$

and

$$
g''(r) = \frac{C_2}{R^2} \bar{u}^\sigma(r) \geq \frac{C_2}{R^2} (g(r))^\sigma.
$$
From (2.10) and (2.11), we have

\[ 2g''(r)g'(r) \geq \frac{2C_2}{R^2} \left( g(r) \right)^\sigma g'(r), \]

or

\[ \frac{d}{dr} \left( [g'(r)]^2 \right) \geq \frac{2C_2}{R^2} \frac{1}{\sigma + 1} g^{\sigma+1}(r). \]

Hence

\[ (2.12) \quad [g'(r)]^2 \geq \left( \frac{2C_2}{(\sigma + 1)R^2} \right) \left[ g^{\sigma+1}(r) - g^{\sigma+1}(R) \right]. \]

Let \( \beta = C_1 \log R = g(R) \) and \( \delta = C_2/R^2 \). Then we have

\[ [g'(r)]^2 \geq \frac{2\delta}{\sigma + 1} \left[ g^{\sigma+1}(r) - \beta^{\sigma+1} \right]. \]

Thus

\[ (2.13) \quad \int_\beta^g \frac{dg}{\sqrt{g^{\sigma+1} - \beta^{\sigma+1}}} \geq \left( \frac{2\delta}{\sigma + 1} \right)^{1/2} \int_R^s ds. \]

Let \( g(r) = \beta z \), we have

\[ (2.14) \quad \int_1^z \frac{dz'}{\sqrt{(z')^{\sigma+1} - 1}} \geq \left( \frac{2\delta}{\sigma + 1} \right)^{1/2} \beta^{(\sigma-1)/2} (r - R). \]

Now if we choose \( R \) so large that

\[ (2.15) \quad \left( \frac{2\delta}{\sigma + 1} \right)^{1/2} \beta^{(\sigma-1)/2} \cdot R = \left( \frac{2C_2}{(\sigma + 1)R^2} \right)^{1/2} \left( C_1 \log R \right)^{(\sigma-1)/2} \cdot R \]

\[ = \left( \frac{2C_2}{\sigma + 1} \right)^{1/2} \left( C_1 \log R \right)^{(\sigma-1)/2} \]

\[ > \int_1^\infty \frac{dz}{\sqrt{z^{\sigma+1} - 1}}. \]

Then there is a \( R_c \leq 2R \), such that

\[ (2.16) \quad \lim_{r \to R_c} g(r) = \infty. \]

But \( u(R_c) \geq g(R_c) = \infty \). This is a contradiction. This completes the proof of this theorem.

Now we can state our main nonexistence results.

**Theorem 2.2.** Let \( K(x) \geq 0 \) be a locally Hölder continuous function. If \( K(r) \) satisfies

1. there exist \( \alpha > 0 \), \( R_0 > 0 \) and \( C > 0 \), such that

\[ K(r) \geq C/r^\alpha \quad \text{for} \ r \geq R_0, \]
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(2) there exist $\varepsilon > 0$ and $P > 2$, such that

$$\int_0^{(P-1)R} r\overline{K}(r) \, dr \geq \varepsilon \quad \text{for } R \geq R_0,$$

then equation (1.1) does not possess any positive solution in $\mathbb{R}^n$.

Proof. Assume that (1.1) has a positive solution $u(x)$ in $\mathbb{R}^n$. Then as in the proof of Theorem 2.1, we have

$$\overline{u}(r) \geq \alpha + \frac{1}{n-2} \int_0^r s\overline{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] \overline{u}(s) \, ds.$$  \hfill (2.17)

From assumption (2), we have

$$\int_0^\infty s\overline{K}(s) \, ds = \infty.$$  \hfill (2.18)

Hence

$$\overline{u}(r) \geq \alpha + C \int_0^{r/2} \alpha^s s\overline{K}(s) \, ds$$

and

$$\lim_{r \to \infty} \overline{u}(r) = \infty.$$  \hfill (2.19)

Thus we can choose $R_0$ so large that

$$\overline{u}(R_0) \geq 1.$$  \hfill (2.20)

Now let $R \geq R_0$. From assumption (2), we have

$$\overline{u}(PR) \geq \overline{u}(R) + \frac{1}{n-2} \int_R^{PR} s\overline{K}(s) \left[ 1 - \left( \frac{s}{PR} \right)^{n-2} \right] \overline{u}(s) \, ds$$

$$\geq \overline{u}(R) + \frac{1}{n-2} \cdot \overline{u}(R) \cdot \left[ 1 - \left( \frac{P-1}{P} \right)^{n-2} \right] \cdot \int_R^{(P-1)R} s\overline{K}(s) \, ds$$

$$\geq \overline{u}(R) + C_1 \overline{u}(R),$$

where $1 > C_1 > 0$ and $C_1$ is a constant.

From (2.20), (2.21) and the fact that $\sigma > 1$, we have

$$\overline{u}(P^mR) \geq (1 + C_1)^m$$

for all $R \geq R_0$ and $m \geq 1$.

Choose $\alpha_1 > 0$ so small that

$$\log(1 + C_1) \geq \alpha_1 [\log P + \log(PR_0)].$$  \hfill (2.23)

Then

$$m \log(1 + C_1) \geq \alpha_1 [m \log P + \log(PR_0)].$$  \hfill (2.24)

Hence $(1 + C_1)^m \geq (P^mR)^{\alpha_1}$ for all $m \geq 1$ and $PR_0 \geq R \geq R_0$. This means that

$$\overline{u}(P^mR) \geq (P^mR)^{\alpha_1}$$

for all $m \geq 1$ and $PR_0 \geq R \geq R_0$. Hence

$$\overline{u}(r) \geq r^{\alpha_1}$$

for $r \geq R_0$.

Now we return to (2.21). We have for $R \geq R_0$

$$\overline{u}(P^mR) \geq C_1 \overline{u}^\sigma(P^{m-1}R) \geq C_1^{1+\sigma+\cdots+\sigma^{m-1}} \cdot \overline{u}^{\sigma m}(R)$$

$$= C_1^{(\sigma^{m-1})/(\sigma-1)} \cdot \overline{u}^{\sigma m}(R), \quad m \geq 1.$$
Hence
\[(2.27)\quad \log(\bar{u}(P^mR)) \geqslant \sigma^m \left[ \log \bar{u}(R) + \frac{1 - 1/\sigma^m}{\sigma - 1} \log C_1 \right] \geqslant \sigma^m \left[ \alpha_1 \log R - \frac{1}{\sigma - 1} |\log C_1| \right].\]

Choose $C_2 > 0$ and $R_1$ sufficiently large, such that
\[(2.28)\quad \alpha_1 \log R_1 \geqslant \frac{1}{\sigma - 1} |\log C_1| + C_2.\]

Then
\[(2.29)\quad \log(\bar{u}(P^mR)) \geqslant C_2 \sigma^m\]
for $R \geqslant R_1$ and $m \geqslant 1$.

Now we can choose $\alpha_2$ sufficiently small, such that
\[
\log \sigma \geqslant \alpha_2 \left( \log P + \log PR_1 \right).
\]

Then
\[
\begin{align*}
m \log \sigma & \geqslant \alpha_2 \left( m \log P + \log PR_1 \right), \quad m \geqslant 1. \\
\end{align*}
\]
Hence $\sigma^m \geqslant (P^mR)^{\alpha_2}$ for $m \geqslant 1$ and $PR_1 \geqslant R \geqslant R_1$. Hence from (2.29), we have
\[
\bar{u}(P^mR) \geqslant \exp \left[ C_2 (P^mR)^{\alpha_2} \right]
\]
for $m \geqslant 1$ and $PR_1 \geqslant R \geqslant R_1$. That is,
\[(2.30)\quad \bar{u}(r) \geqslant \exp \left[ C_2 r^{\alpha_2} \right]\]
for $r \geqslant R_1$. Hence from (2.17), for $r \geqslant R_1$, we have
\[
\bar{u}(r) \geqslant \bar{u}(R_1) + \frac{1}{n - 2} \int_{R_1}^r s \bar{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] \bar{u}(s) \, ds
\]
\[
= \bar{u}(R_1) + \frac{1}{n - 2} \int_{R_1}^r s \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] \left[ \bar{K}(s) \cdot \bar{u}^{(\sigma-1)/2}(s) \right] \bar{u}^{(\sigma+1)/2}(s) \, ds.
\]
Now from (2.30) and the assumption (1), we can choose $R_2 \geqslant R_1$ so large that
\[
\bar{K}(s) \bar{u}^{(\sigma-1)/2}(s) \geqslant C_3/s^2
\]
for $s \geqslant R_2$ for some constant $C_3 > 0$. Hence we have
\[(2.31)\quad \bar{u}(r) \geqslant \bar{u}(R_1) + \frac{1}{n - 2} \int_{R_2}^r s \left[ \bar{K}(s) \cdot \bar{u}^{(\sigma-1)/2}(s) \right] \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] \bar{u}^{(\sigma+1)/2}(s) \, ds.
\]
But from the proof of Theorem 2.1, this is impossible. Hence we complete the proof of this theorem.

**Theorem 2.3.** Let $K(x) \geqslant 0$ be a locally Hölder continuous function. If $\bar{K}(r)$ satisfies
1. $\int_0^s \bar{K}(s) \, ds$ is strictly increasing in $[0, \infty)$ and $\int_0^\infty \bar{K}(s) \, ds = \infty$,
2. $(s/r)^m \leqslant \int_0^{t\bar{K}(t)} dt / \int_0^{t\bar{K}(t)} dt$ for some finite $m > 0$ and for all $r \geqslant s \geqslant R_0 > 0$,
then equation (1.1) does not possess any positive solution in $\mathbb{R}^n$. 

In particular, if $\overline{K}(r)$ satisfies (1) and $0 \leq \overline{K}(r) \leq C/r^2$ for $r \geq R_1$ for some constants $C > 0$ and $R_1 > 0$, then $\overline{K}(r)$ also satisfies (2) and hence (1.1) does not possess any positive solution in $\mathbb{R}^n$.

Proof. Assume that (1.1) has a positive solution $u(x)$ in $\mathbb{R}^n$. Then as in the proof of Theorem 2.2, we have (2.17). Let

$$f(r) = \int_0^r sK(s) \, ds = \eta.$$ 

Then $f: [0, \infty) \to [0, \infty)$ is one-one and onto. Hence $f^{-1}$ exists and let it be denoted by $g$. Let

$$t = f(s), \quad \eta = f(r), \quad \tilde{u}(g(\eta)) = v(\eta).$$

Then from (2.17), we have

$$v(\eta) \geq \alpha + \frac{1}{n-2} \int_0^\eta \left[ 1 - \left( \frac{g(t)}{g(\eta)} \right)^{(n-2)/m} \right] v^a(t) \, dt.$$ 

From the assumption (2), we have

$$g(t)/g(\eta) \leq (t/\eta)^{1/m} \text{ for all } \eta \geq t \geq f(R_0).$$

Hence from (2.32) and (2.33), we have

$$v(\eta) \geq \tilde{u}(R_0) + \frac{1}{n-2} \int_{f(R_0)}^\eta \left[ 1 - \left( \frac{t}{\eta} \right)^{(n-2)/m} \right] v^a(t) \, dt.$$ 

But from Theorem 2.1, this is impossible. Hence (1.1) does not possess any positive solution.

If in addition to condition (1), $\overline{K}(r)$ also satisfies $0 \leq \overline{K}(r) \leq C/r^2$ for $r \geq R_1$. Then we have

$$\frac{d}{dr} \left( \frac{\int_0^r t\overline{K}(t) \, dt}{r} \right) = \frac{r^2\overline{K}(r) - \int_0^r t\overline{K}(t) \, dt}{r^2} \leq \frac{C - \int_0^r t\overline{K}(t) \, dt}{r^2}.$$ 

for $r \geq R_1$. Thus we can choose $R_2 \geq R_1$ so large that

$$C - \int_0^r t\overline{K}(t) \, dt \leq 0 \text{ for } r \geq R_2.$$ 

Hence $\int_0^r t\overline{K}(t) \, dt/r$ is monotonically decreasing for $r \geq R_2$. Thus $\overline{K}(r)$ satisfies condition (2) for $r \geq s \geq R_2$.

This completes the proof of this theorem.

Theorem 2.4. Let $K(x) \geq 0$ be a locally Hölder continuous function in $\mathbb{R}^n$ and $\overline{K}(t)$ be a locally Hölder continuous function in $[0, \infty)$.

Let the average $\overline{K}(r)$ of $K(x)$ in the sense of (2.2) satisfy:

$$\overline{K}(r) \geq \overline{K}(r - \beta_i) \text{ if } \alpha_i + \beta_i \leq r \leq \alpha_{i+1} + \beta_i,$$

$$\overline{K}(r) \geq 0 \text{ if } \alpha_{i+1} + \beta_i < r < \alpha_{i+1} + \beta_{i+1}.$$
for \( i = 0, 1, 2, \ldots \), where \( \{ a_i \}_{i=0}^{\infty} \) is a strictly increasing sequence satisfying \( a_0 = 0 \) and \( \lim_{n \to \infty} a_n = \infty \) and \( \{ \beta_i \}_{i=0}^{\infty} \) is a nondecreasing sequence satisfying \( \beta_0 = 0 \) and \( \beta_i / a_i \leq M \) for some constant \( M > 0 \) and \( i = 1, 2, \ldots \). If

\[
\begin{aligned}
\left\{ \begin{array}{ll}
u''(r) + \frac{n-1}{r} u'(r) = \bar{K}(r) u^\alpha(r) & \text{in } (0, \infty), \\
u(0) = \alpha > 0, & u'(0) = 0 
\end{array} \right.
\end{aligned}
\]

(2.35)

does not possess any solution in \([0, \infty)\) for all \( \alpha > 0 \), then (1.1) does not possess any positive solution in \( \mathbb{R}^n \).

**Proof.** Assume that (1.1) has a positive solution \( u(x) \) in \( \mathbb{R}^n \). Then as in the proof of Theorem 2.2, we have

\[
\bar{u}(r) \geq \alpha + \frac{1}{n-2} \int_0^r s \bar{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] \bar{u}^\alpha(s) \, ds.
\]

(2.36)

Now we define the function \( v \) by

\[
v(r) = \bar{u}(r + \beta_i) \quad \text{if } \alpha_i < r < \alpha_{i+1}
\]

for \( i = 0, 1, 2, \ldots \). We shall prove that

\[
v(r) \geq \alpha + \frac{A}{n-2} \int_0^r s \bar{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] v^\alpha(s) \, ds,
\]

(2.38)

where \( A \) is a positive constant depending only on the constant \( M \). To prove (2.38), let \( \alpha_i \leq r \leq \alpha_{i+1} \). Then from (2.36), we have

\[
\bar{u}(r + \beta_i) \geq \alpha + \frac{1}{n-2} \int_{0}^{r+\beta_i} s \bar{K}(s) \left[ 1 - \left( \frac{s}{r + \beta_i} \right)^{n-2} \right] \bar{u}^\alpha(s) \, ds
\]

\[
\geq \alpha + \frac{1}{n-2} \int_{0}^{\alpha_i} s \bar{K}(s) \left[ 1 - \left( \frac{s}{r + \beta_i} \right)^{n-2} \right] \bar{u}^\alpha(s) \, ds
\]

\[
+ \frac{1}{n-2} \int_{\alpha_i}^{\alpha_{i+1}} s \bar{K}(s) \left[ 1 - \left( \frac{s}{r + \beta_i} \right)^{n-2} \right] \bar{u}^\alpha(s) \, ds
\]

\[+ \cdots
\]

\[
+ \frac{1}{n-2} \int_{r+\beta_i}^{r} s \bar{K}(s) \left[ 1 - \left( \frac{s}{r + \beta_i} \right)^{n-2} \right] \bar{u}^\alpha(s) \, ds
\]

\[= \alpha + \frac{1}{n-2} \int_{0}^{\alpha_i} s \bar{K}(s) \left[ 1 - \left( \frac{s}{r + \beta_i} \right)^{n-2} \right] \bar{u}^\alpha(s) \, ds
\]

\[+ \frac{1}{n-2} \int_{\alpha_i}^{\alpha_{i+1}} (s + \beta_i) \bar{K}(s + \beta_i) \left[ 1 - \left( \frac{s + \beta_i}{r + \beta_i} \right)^{n-2} \right] \bar{u}^\alpha(s + \beta_i) \, ds
\]

\[+ \cdots
\]

\[+ \frac{1}{n-2} \int_{r+\beta_i}^{r} (s + \beta_i) \bar{K}(s + \beta_i) \left[ 1 - \left( \frac{s + \beta_i}{r + \beta_i} \right)^{n-2} \right] \bar{u}^\alpha(s + \beta_i) \, ds.
\]
But for \( 1 \leq j \leq i, \)
\[
1 - \left( \frac{s + \beta_j}{r + \beta_i} \right)^{n-2} \geq 1 - \left( \frac{s + \beta_i}{r + \beta_i} \right)^{n-2} = \frac{(1 + \beta_i/r)^{n-2} - (s/r + \beta_i/r)^{n-2}}{(1 + \beta_i/r)^{n-2}} \geq \frac{1 - (s/r)^{n-2}}{(1 + \beta_i/\alpha_i)^{n-2}} \geq A \left[ 1 - (s/r)^{n-2} \right].
\]
Hence we have
\[
\bar{u}(r + \beta_i) \geq \alpha + \frac{A}{n-2} \int_0^{\alpha_1} s \tilde{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] \bar{u}^\sigma(s) \, ds \\
+ \frac{A}{n-2} \int_{\alpha_1}^{\alpha_2} s \tilde{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] \bar{u}^\sigma(s + \beta_i) \, ds \\
+ \ldots \\
+ \frac{A}{n-2} \int_{\alpha_i}^{r} s \tilde{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] \bar{u}^\sigma(s + \beta_i) \, ds.
\]
Hence (2.38) is true for all \( r \in [0, \infty) \). Let \( \bar{v} = A^{1/(\sigma - 1)} \bar{v} \) and \( \bar{\alpha} = A^{1/(\sigma - 1)} \alpha \). Then (2.38) becomes
\[
\bar{v}(r) \geq \bar{\alpha} + \frac{1}{n-2} \int_0^{r} s \tilde{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] \bar{v}^\sigma(s) \, ds.
\]
Now let \( X \) denote the locally convex space of all continuous functions on \([0, \infty)\) with the usual topology and consider the set
\[
Y = \{ y \in X : \bar{\alpha} \leq y(r) \leq \bar{v}(r) \text{ for } r \geq 0 \},
\]
where \( \bar{v} \) is defined above. Clearly, \( Y \) is a closed convex subset of \( X \). Define the mapping \( T \) by
\[
(2.39) \quad Ty(r) = \bar{\alpha} + \frac{1}{n-2} \int_0^{r} s \tilde{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] y^\sigma(s) \, ds.
\]
If \( y \in Y \), then \( \bar{\alpha} \leq y(r) \leq \bar{v}(r) \). Hence we have
\[
Ty(r) = \bar{\alpha} + \frac{1}{n-2} \int_0^{r} s \tilde{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] y^\sigma(s) \, ds \geq \bar{\alpha}
\]
and
\[
Ty(r) \leq \bar{\alpha} + \frac{1}{n-2} \int_0^{r} s \tilde{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] \bar{v}^\sigma(s) \, ds \leq \bar{v}(r).
\]
Thus \( T \) maps \( Y \) into itself. Let \( \{ y_m \}_{m=1}^{\infty} \subset Y \) be a sequence which converges to \( y \) in \( X \). Then \( \{ y_m \} \) converges uniformly to \( y \) on any compact interval of \([0, \infty)\). Since
\[
(2.40) \quad |Ty_m(r) - Ty(r)| \leq \frac{1}{n-2} \int_0^{r} s \tilde{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] |y_m^\sigma(s) - y^\sigma(s)| \, ds,
\]
we have \( \{ Ty_m \} \) converges uniformly to \( Ty \) on any compact interval of \([0, \infty)\). Hence \( T \) is a continuous mapping from \( Y \) into \( Y \). On the other hand, we have
\[
(2.41) \quad (Ty)'(r) = \int_0^{r} \left( \frac{s}{r} \right)^{n-1} \tilde{K}(s) y^\sigma(s) \, ds.
\]
Hence for any fixed \( R > 0 \), \( TY \) is a uniformly bounded and equicontinuous family of functions defined on \([0, R]\). Hence \( TY \) is relatively compact. Thus we can use the Schauder-Tychonoff fixed point theorem (see Edwards [2, p. 161]) to conclude that \( T \) has a fixed point \( y \in Y \). This fixed point \( y \) satisfies the integral equation
\[
y(r) = \tilde{\alpha} + \frac{1}{n-2} \int_0^r s \tilde{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] y^a(s) \, ds.
\]
Hence (2.35) has a solution for this \( \tilde{\alpha} \). This is a contradiction. The theorem is proved. Q.E.D.

3. The case \( n = 2 \). In this case, we consider only the situation \( K(x) \geq 0 \) in (1.1). Kawano, Kusano and Naito [3] obtain the following existence result: Let \( K(x) \geq 0 \) be a locally Hölder continuous function which is positive in some neighborhood of the origin. If
\[
K(x) \leq \tilde{K}(|x|) \quad \text{for all } x \in \mathbb{R}^2
\]
and
\[
\int_1^\infty s (\log s)^a \tilde{K}(s) \, ds < \infty.
\]
Then equation (1.1) has infinitely many positive solutions in \( \mathbb{R}^2 \) with logarithmic growth at infinity.

To our knowledge, there seems no known nonexistence result. Our nonexistence results are

**Theorem 3.1.** Let \( K(x) \geq 0 \) be a locally Hölder continuous function in \( \mathbb{R}^2 \). Let the average \( \bar{K}(r) \) of \( K(x) \) in the sense of (2.2) satisfy
\[
\bar{K}(r) \geq C/r^2 (\log r)^{a+1} \quad \text{for } r \geq R_0.
\]
Then equation (1.1) does not possess any positive solution in \( \mathbb{R}^2 \).

**Proof.** Assume that (1.1) has a positive solution \( u(x) \) in \( \mathbb{R}^2 \). Then we have
\[
\begin{align*}
\bar{u}''(r) + \frac{\bar{u}'(r)}{r} &\geq \bar{K}(r) \bar{u}^a(r), \\
\bar{u}(0) &= \alpha > 0, \quad \bar{u}'(0) = 0,
\end{align*}
\]
where \( \bar{u} \) and \( \bar{K} \) are defined in (2.1) and (2.2). From (3.2), \( \bar{u}(r) \) satisfies the integral equation
\[
\bar{u}(r) \geq \alpha + \int_0^r s \log \left( \frac{r}{s} \right) \bar{K}(s) \bar{u}^a(s) \, ds.
\]
Without loss of generality, we assume that \( K(0) > 0 \) and hence \( \bar{K}(0) > 0 \). Thus we have from (3.3)
\[
\begin{align*}
\bar{u}(r) &\geq \alpha + \int_0^1 s \log \left( \frac{r}{s} \right) \bar{K}(s) \bar{u}^a(s) \, ds + \int_1^r s \log \left( \frac{r}{s} \right) \bar{K}(s) \bar{u}^a(s) \, ds \\
&\geq \alpha + \int_0^1 s \log r \bar{K}(s) \bar{u}^a(s) \, ds \\
&\geq \alpha + \alpha^a \cdot \log r \cdot \int_0^1 s \bar{K}(s) \, ds \\
&\geq \alpha + C_1 \log r
\end{align*}
\]
for \( r \geq 1 \) and a constant \( C_1 > 0 \).
Now consider \( r \geq e \). We have
\[
\bar{u}(r) \geq \alpha + \int_0^1 s \log \left( \frac{r}{s} \right) \bar{K}(s) \bar{u}^\sigma(s) \, ds \\
+ \int_e^r s \log \left( \frac{r}{s} \right) \bar{K}(s) \bar{u}^\sigma(s) \, ds \\
\geq C_1 \log r + \int_e^r s \log \left( \frac{r}{s} \right) \bar{K}(s) \bar{u}^\sigma(s) \, ds.
\]
Let \( v(r) = \bar{u}(r)/\log r \) for \( r \geq e \). Then from (3.5), we have
\[
v(r) \geq C_1 + \int_e^r s \left( 1 - \frac{\log s}{\log r} \right) \bar{K}(s) (\log s)^\alpha v^\sigma(s) \, ds.
\]
Let \( t = \log s \), \( \eta = \log r \) and \( v(e^n) = v(r) = \bar{v}(\eta) \). Then (3.6) becomes
\[
\bar{v}(\eta) \geq C_1 + \int_1^n t \left( 1 - \frac{t}{\eta} \right) e^{2t} \bar{K}(e^t) e^{(\alpha - 1) \bar{v}^\sigma(t)} \, dt.
\]
Let \( \bar{K}(t) = e^{2t} \bar{K}(e^t) t^{\alpha - 1} \). Then from (3.1), we have
\[
\bar{K}(t) \geq C/t^2 \quad \text{for } t \geq \exp(R_0)
\]
and
\[
\bar{v}(\eta) \geq C_1 + \int_1^n t \left( 1 - \frac{t}{\eta} \right) \bar{K}(t) \bar{v}^\sigma(t) \, dt.
\]
Using a similar argument as in the proof of Theorem 2.1, we obtain a contradiction.
This completes the proof of this theorem. Q.E.D.

**Theorem 3.2.** Let \( K(x) > 0 \) be a locally Hölder continuous function in \( \mathbb{R}^2 \). Let the average \( \bar{K}(r) \) of \( K(x) \) in the sense of (2.2) satisfy
\[
\text{There exist } e > 0, P > 2 \text{ and } R_0 > 0, \text{ such that } \\
\int_e^{e^{(P-1)R}} s \bar{K}(s) (\log s)^\alpha \, ds \geq e \text{ for all } R \geq R_0.
\]
\[
\text{There exist } \alpha > 0, R_1 > 0 \text{ and } C > 0, \text{ such that } \\
\bar{K}(s) \geq C/s^2 (\log s)^{(\alpha + \alpha)} \text{ for all } s \geq R_1.
\]
Then equation (1.1) does not possess any positive solution in \( \mathbb{R}^2 \).

**Proof.** Assume that (1.1) has a positive solution \( u(x) \) in \( \mathbb{R}^2 \). As in the proof of Theorem 3.1, we have (3.3)-(3.7). Hence
\[
\bar{v}(\eta) \geq C_1 + \int_1^n t \left( 1 - \frac{t}{\eta} \right) \bar{K}(t) \bar{v}^\sigma(t) \, dt.
\]
But from (3.9) and (3.10), \( \bar{K}(t) \) satisfies
\[
\int_R^{(P-1)R} t \bar{K}(t) \, dt \geq e \text{ for all } R \geq R_0,
\]
\[
\bar{K}(s) \geq C/t^{(1+\alpha)} \text{ for all } t \geq \log R_1.
\]
Using a similar argument as in the proof of Theorem 2.2, we obtain a contradiction.
This completes the proof. Q.E.D.
Theorem 3.3. Let $K(x) \geq 0$ be a locally Hölder continuous function in $\mathbb{R}^2$. Let the average $\overline{K}(r)$ of $K(x)$ in the sense of (2.2) satisfy

\begin{equation}
\int_{0}^{\infty} s\overline{K}(s)(\log s)^{\gamma} \, ds \text{ is strictly increasing on } [0, \infty) \text{ and}
\end{equation}

\begin{equation}
\int_{0}^{\infty} s\overline{K}(s)(\log s)^{\gamma} \, ds = \infty,
\end{equation}

\begin{equation}
\left( \frac{\log s}{\log r} \right)^{m} \leq \int_{0}^{s} t\overline{K}(t)(\log t)^{\gamma} \, dt / \int_{0}^{r} t\overline{K}(t)(\log t)^{\gamma} \, dt
\end{equation}

for some $m > 0$ and for all $r \geq s \geq R_0 > 0$. Then equation (1.1) does not possess any positive solution in $\mathbb{R}^2$. In particular, if $\overline{K}(r)$ satisfies (3.14) and $0 \leq \overline{K}(r) \leq C/r^2(\log r)^{\gamma+1}$ for $r \geq R_1$ for some constants $C > 0$ and $R_1 > 0$, then $\overline{K}(r)$ also satisfies (3.15) and hence (1.1) does not possess any positive solution in $\mathbb{R}^2$.

Proof. Assume that (1.1) has a positive solution $u(x)$ in $\mathbb{R}^2$. As in the proof of Theorem 3.1, we have (3.3)–(3.7). Hence we obtain (3.8) or (3.11). But now $\overline{K}(t)$ satisfies

\begin{equation}
\int_{1}^{\infty} t\overline{K}(t) \, dt \text{ is strictly increasing in } [1, \infty) \text{ and}
\end{equation}

\begin{equation}
\int_{1}^{\infty} t\overline{K}(t) \, dt = \infty,
\end{equation}

\begin{equation}
\left( \frac{s}{\eta} \right)^{m} \leq \int_{1}^{s} t\overline{K}(t) \, dt / \int_{1}^{\eta} t\overline{K}(t) \, dt
\end{equation}

for some $m > 0$ and for all $\eta \geq s \geq \log R_0$. Using a similar argument as in the proof of Theorem 2.3, we obtain a contradiction. This completes the proof. Q.E.D.

Theorem 3.4. Let $K(x) \geq 0$ be a locally Hölder continuous function in $\mathbb{R}^2$ and $\overline{K}(t)$ be a locally Hölder continuous function in $[0, \infty)$. Let the average $\overline{K}(r)$ of $K(x)$ in the sense of (2.2) satisfy

\begin{equation}
\overline{K}(r) \geq 0 \quad \text{if } \alpha_i + 1 + \beta_i < r < \alpha_{i+1} + 1 + \beta_{i+1},
\end{equation}

\begin{equation}
\overline{K}(r) \geq \overline{K}(r - \beta_i) \quad \text{if } \alpha_i + \beta_i \leq r \leq \alpha_{i+1} + \beta_i,
\end{equation}

for $i = 0, 1, 2, \ldots$, where $\{\alpha_i\}_{i=0}^{\infty}$ is a strictly increasing sequence satisfying $\alpha_0 = 0$ and $\lim_{n \to \infty} \alpha_n = \infty$ and $\{\beta_i\}_{i=0}^{\infty}$ is a nondecreasing sequence satisfying $\beta_0 = 0$ and $\beta_i/\alpha_i \leq M$ for some $M > 0$ for all $i \geq 1$. If

\begin{equation}
\begin{cases}
u''(r) + u'(r)/r = \overline{K}(r)u^\alpha(r) & \text{in } (0, \infty), \\
u(0) = \alpha > 0, \quad \nu'(0) = 0
\end{cases}
\end{equation}

does not possess any solution in $[0, \infty)$ for all $\alpha > 0$, then (1.1) does not possess any positive solution in $\mathbb{R}^2$.

Proof. The proof is very similar to that of Theorem 2.4. Hence we only sketch the proof. Assume that (1.1) has a positive solution in $\mathbb{R}^2$. Then we have

\begin{equation}
\bar{u}(r) \geq \alpha + \int_{0}^{r} s\log \left( \frac{r}{s} \right) \overline{K}(s) \bar{u}^\alpha(s) \, ds.
\end{equation}
Let
\[ v(r) = \bar{u}(r + \beta_i) \quad \text{if } \alpha_i \leq r < \alpha_{i+1} \]
for \( i = 0, 1, 2, \ldots \). Then
\[ (3.19) \quad v(r) \geq \alpha + A \cdot \int_0^r s \log \left( \frac{r}{s} \right) \tilde{K}(s) v^\sigma(s) \, ds. \]

Let \( X \) denote the locally convex space of all continuous function on \([0, \infty)\) with the usual topology and consider the set
\[ Y = \{ y \in X : \tilde{\alpha} < y(r) < \bar{v}(r) \text{ for } r \geq 0 \}. \]

Define the mapping \( T \) by
\[ (3.20) \quad (Ty)(r) = \tilde{\alpha} + \int_0^r s \log \left( \frac{r}{s} \right) \tilde{K}(s) y^\sigma(s) \, ds. \]

We can prove that \( TY \subset Y \) and \( T \) is continuous. Furthermore \( TY \) is relatively compact. Hence \( T \) has a fixed point in \( Y \). Thus \( (3.17) \) has a solution for this given \( \tilde{\alpha} > 0 \). This is a contradiction. The proof is complete. Q.E.D.

4. The case \( n = 1 \). In this case, we also consider only the situation \( K(x) \geq 0 \) in (1.1). We give a main existence result which have an extension to the higher-dimensional case. We also give some nonexistence results which may have applications.

**THEOREM 4.1.** Let \( K(x) \geq 0 \) be a Hölder continuous (actually only continuous is sufficient) function in \( \mathbb{R} \). If \( K(0) > 0 \)
\[ \int_{-\infty}^{\infty} |x|^\sigma K(x) \, dx < \infty, \]
then (1.1) has infinitely many positive solutions in \( \mathbb{R} \) with linear growth at \( |x| = \infty \).

**PROOF.** We shall seek solutions \( u \) such that \( u(0) = \alpha > 0 \) and \( u'(0) = 0 \). Consider now \( x > 0 \). Then equation (1.1) with \( u(0) = \alpha > 0 \) and \( u'(0) = 0 \) is equivalent to the integral equation
\[ (4.2) \quad u(x) = \alpha + \int_0^x (x - t) K(t) u^\sigma(t) \, dt, \quad x > 0. \]

Now choose \( \alpha \) so small that
\[ (4.3) \quad 2^\sigma \alpha^{(\sigma-1)} \int_0^1 K(t) \, dt \leq \frac{1}{2}, \]
\[ (4.4) \quad 2^\sigma \alpha^{(\sigma-1)} \int_1^\infty K(t) t^\sigma \, dt \leq \frac{1}{2}. \]

Let
\[ A(x) = \begin{cases} 2\alpha & \text{if } 0 \leq x \leq 1, \\ 2\alpha x & \text{if } 1 \leq x. \end{cases} \]

As in the proofs of Theorems 2.4 and 3.4, we let \( X \) denote the locally convex space of all continuous functions on \([0, \infty)\) with the usual topology and consider the set
\[ Y = \{ y \in X : \alpha \leq y(x) \leq A(x) \text{ for } x \geq 0 \}. \]
Clearly, $Y$ is a closed convex subset of $X$. Let the mapping $T$ be defined by

$$ (Ty)(x) = \alpha + \int_0^x (x - t)K(t)y^\sigma(t)\,dt, \quad x \geq 0. \quad (4.5) $$

If $y \in Y$, then $\alpha \leq y(x) \leq A(x)$. Hence we have

$$ (Ty)(x) = \alpha + \int_0^x (x - t)K(t)y^\sigma(t)\,dt \geq \alpha + \int_0^x (x - t)K(t)\alpha^\sigma\,dt \geq \alpha. \quad (4.6) $$

On the other hand, for $0 \leq x \leq 1$, we have

$$ (Ty)(x) = \alpha + \int_0^x (x - t)K(t)y^\sigma(t)\,dt \leq \alpha + \int_0^1 K(t)(2\alpha)^\sigma\,dt = \alpha \left[1 + 2\alpha(\sigma - 1)\int_0^1 K(t)\,dt\right] \leq \alpha \left[1 + \frac{1}{2}\right] \leq 2\alpha = A(x). \quad (4.7) $$

For $1 \leq x$, we have

$$ (Ty)(x) = \alpha + \int_0^1 (x - t)K(t)y^\sigma(t)\,dt + \int_1^x (x - t)K(t)y^\sigma(t)\,dt \leq \alpha + x\int_0^1 K(t)(2\alpha)^\sigma\,dt + x\int_1^\infty K(t)(2\alpha t)^\sigma\,dt \leq ax + ax \left[2\alpha(\sigma - 1)\int_0^1 K(t)\,dt\right] + ax \left[2\alpha(\sigma - 1)\int_1^\infty K(t)t^\sigma\,dt\right] \leq ax \left[1 + \frac{1}{2} + \frac{1}{2}\right] \leq 2ax = A(x). \quad (4.8) $$

Thus $T$ maps $Y$ into itself. Now let $\{y_m\}_m=1^\infty \subset Y$ be a sequence which converges to $y$ in $X$. Then $\{y_m\}$ converges uniformly to $y$ on any compact interval of $[0, \infty)$. But

$$ |Ty_m(x) - Ty(x)| \leq \int_0^x |(x - t)K(t)|y_m^\sigma(t) - y^\sigma(t)|\,dt, \quad (4.9) $$

we conclude that $\{Ty_m\}$ converges uniformly to $Ty$ on any compact interval of $[0, \infty)$. Hence $T$ is a continuous mapping from $Y$ into $Y$. As in the proof of Theorem 2.4, the precompactness of $T$ can be verified by

$$ \left|(Ty)'(x)\right| \leq \int_0^x K(t)y^\sigma(t)\,dt \leq \int_0^\infty K(t)(2\alpha)^\sigma t^\sigma\,dt < \infty. \quad (4.10) $$

Thus $T$ has a fixed point $y \in Y$. This fixed point $y$ is a solution of equation (1.1) for $x \geq 0$ with $y(0) = \alpha$ and $y'(0) = 0$.

Similarly, we can find a solution of equation (1.1) for $x \leq 0$ with $y(0) = \alpha$ and $y'(0) = 0$ if $\alpha$ is sufficiently small. Now let $y(x)$ be the solution of (1.1) in $\mathbb{R}$ with
\[ y(0) = \alpha, \ y'(0) = 0. \] Then

\[ 2\alpha x \geq y(x) = \alpha + \int_0^x (x-t)K(y)y'(t)\,dt \]
\[ \geq \alpha + \int_0^1 (x-1)K(t)\alpha\,dt \]
\[ \geq \alpha + k_1(x-1) \geq k_2 x \]

for \( x \) large. Hence \( y \) grows linearly at \( |x| = \infty \). Now we can choose a smaller \( y(0) \), such as \( y(0) = \alpha/2 \) to obtain another solution. This completes the proof of this theorem. Q.E.D.

We can apply this theorem to the higher-dimensional case as used in Ni [13, 14] and Kawano, Kusano and Naito [3].

**Theorem 4.2.** Let \( K(x) \geq 0 \) be a locally Hölder continuous function in \( \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} \). Let \( \phi^*(x) \) and \( \phi^*(x) \) be two locally Hölder continuous function in \( \mathbb{R} \). If

\[ 0 \leq \phi^*(x) \leq K(x) \leq \phi^*(x) \quad \text{for all} \quad x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}, \]
\[ \int_{-\infty}^{\infty} |x_1|^n \phi^*(x) \, dx_1 < \infty, \]

then equation (1.1) has infinitely many positive solutions in \( \mathbb{R}^n \) which are unbounded.

**Proof.** Consider the equations

\[ d^2v/dx_1^2 = \phi^*(x) v, \]
\[ d^2w/dx_1^2 = \phi^*(x) w. \]

From the proof of Theorem 4.1 we see that (4.14) and (4.15) have unbounded solutions (linear growth at \( \infty \)) \( \hat{v} \) and \( \hat{w} \). We can choose \( \hat{v} \) and \( \hat{w} \) such that \( \hat{v}(x_1) \leq \hat{w}(x_1) \) for all \( x_1 \in \mathbb{R} \). Now let

\[ v(x_1, x') = \hat{v}(x_1) \quad \text{and} \quad w(x_1, x') = \hat{w}(x_1). \]

Then from (4.12), we have

\[ \Delta v - K(x) v = \frac{d^2\hat{v}(x_1)}{dx_1^2} - K(x) \hat{v}(x_1) = 0, \]
\[ \Delta w - K(x) w = \frac{d^2\hat{w}(x_1)}{dx_1^2} - K(x) \hat{w}(x_1) = 0. \]

in \( \mathbb{R}^n \). Hence \( v(x_1, x') \) and \( w(x_1, x') \) are, respectively, a subsolution and a supersolution of (1.1) in \( \mathbb{R}^n \). Since \( v(x_1, x') \leq w(x_1, x') \) in \( \mathbb{R}^n \), from Theorem 2.10 of Ni [13], it follows that (1.1) has a positive solution \( u(x) \) in \( \mathbb{R}^n \) such that \( \hat{v}(x_1) \leq u(x_1, x') \leq \hat{w}(x_1) \). It is easy to see that \( k_1|x_1| \leq u(x_1, x') \leq k_2|x_1| \) for \( |x_1| \) large for some positive constants \( k_1 \) and \( k_2 \). This completes the proof of the theorem. Q.E.D.
Now let \( u \) be a positive function in \( \mathbb{R} \) and \( K(x) \geq 0 \) in \( \mathbb{R} \). Define for \( r > 0 \)
\[
\overline{u}(r) = \left( u(r) + u(-r) \right)/2, \\
\overline{K}(r) = \left[ \frac{1}{2} \left( K(r)^{-\sigma/a} + K(-r)^{-\sigma/a} \right) \right]^{-a/\sigma} 
\]
where \( 1/\sigma + 1/\sigma' = 1 \). It is easy to see that
\[
\overline{u}(0) = u(0) \quad \text{and} \quad \overline{u}'(0) = 0
\]
if \( u \) is also continuously differentiable.

**Theorem 4.3.** Let \( K(x) \geq 0 \) be a continuous function in \( \mathbb{R} \). If the average \( \overline{K}(r) \) of
\( K(x) \) in the sense (4.18) satisfies
\[
\overline{K}(r) > C/(r^{(\sigma+1)})
\]
for \( r > R_0 \) for some constant \( C > 0 \), then equation (1.1) does not possess any positive
solution in \( \mathbb{R} \).

**Proof.** Assume that \( u(x) \) is a positive solution of (1.1) in \( \mathbb{R} \). Then we have
\[
\overline{u}''(r) = \frac{u''(r) + u''(-r)}{2} = \frac{1}{2} \left[ K(r)\overline{u}^a(r) + K(-r)\overline{u}^a(-r) \right].
\]
But
\[
\overline{u}(r) = \left[ \frac{1}{2} \left( K(r)\overline{u}(r) + K(-r)\overline{u}(-r) \right) \right]^{1/\sigma} \\
\quad \cdot \left[ \frac{1}{2} \left( K^{-\sigma/a}(r) + K^{-\sigma/a}(-r) \right) \right]^{1/\sigma'}
\]
Hence
\[
\frac{1}{2} \left( K(r)\overline{u}^a(r) + K(-r)\overline{u}^a(-r) \right) \geq \overline{K}(r)\overline{u}^a(r).
\]
Thus we have
\[
\begin{cases}
\overline{u}''(r) \geq \overline{K}(r)\overline{u}^a(r) & \text{for } r > 0, \\
\overline{u}(0) = \alpha > 0, & \overline{u}'(0) = 0.
\end{cases}
\]
Hence \( \overline{u} \) satisfies
\[
\overline{u}(r) \geq \alpha + \int_0^r (r-t)\overline{K}(t)\overline{u}^a(t) \, dt.
\]
Without loss of generality, we may assume that \( K(0) > 0 \) and hence \( \overline{K}(0) > 0 \). Thus
for \( r \geq 2 \), we have
\[
\begin{align*}
\overline{u}(r) & \geq \alpha + \int_0^1 (r-t)\overline{K}(t)\overline{u}^a(t) \, dt + \int_1^r (r-t)\overline{K}(t)\overline{u}^a(t) \, dt \\
& \geq \alpha + \left( \alpha^a \cdot \int_0^1 \left( 1 - \frac{t}{r} \right)\overline{K}(t) \, dt \right) \cdot r + \int_1^r (r-t)\overline{K}(t)\overline{u}^a(t) \, dt \\
& \geq C_1 \cdot r + \int_1^r (r-t)\overline{K}(t)\overline{u}^a(t) \, dt,
\end{align*}
\]
where
\[ C_1 = \alpha \cdot \int_0^1 \left(1 - \frac{1}{2}\right) \overline{K}(t) \, dt = \alpha \cdot \frac{1}{2} \cdot \int_0^1 \overline{K}(t) \, dt > 0. \]

Now let \( \bar{u}(r) = v(r) \cdot r \) for \( r \geq 2 \). We obtain
\[
(4.27) \quad v(r) \geq C_1 + \int_1^r t \left(1 - \frac{t}{r}\right) \overline{K}(t) \, dt.
\]
Letting \( \tilde{K}(t) = \overline{K}(t) t^{(\alpha-1)} \). Then from (4.20), we have
\[
(4.28) \quad \tilde{K}(t) \geq C \cdot t^{-2} \quad \text{for} \quad t \geq R_0
\]
and
\[
(4.29) \quad v(r) \geq C_1 + \int_1^r t \tilde{K}(t) \left(1 - \frac{t}{r}\right) v^\alpha(t) \, dt.
\]
From the proof of Theorem 2.1, we see that it is impossible to have a function \( v \) defined in \([2, \infty)\) satisfying (4.29). This completes the proof. Q.E.D.

**Theorem 4.4.** Let \( K(x) \geq 0 \) be a continuous function in \( \mathbb{R} \). If the average \( \overline{K}(r) \) of \( K(r) \) in the sense (4.18) satisfies
\[
(4.30) \quad \text{there exist } \alpha > 0, R_0 > 0 \text{ and } C > 0 \text{ such that } \overline{K}(r) \geq C/r^{(\alpha+\alpha)} \quad \text{for } r \geq R_0,
\]
\[
(4.31) \quad \text{there exist } \epsilon > 0 \text{ and } P > 2 \text{ such that } \int_{R_0}^{(P-1)R} r^\alpha \overline{K}(r) \, dr \geq \epsilon \quad \text{for } R \geq R_0.
\]
Then equation (1.1) does not possess any positive solution in \( \mathbb{R} \).

**Proof.** Assume on the contrary that (1.1) has a positive solution \( u(x) \) in \( \mathbb{R} \). Then as in the proof of Theorem 4.3, we have (4.24)–(4.27). But now \( \tilde{K}(r) = r^{(\alpha-1)} \overline{K}(r) \) satisfies
\[
(4.32) \quad \tilde{K}(r) \geq C/r^{(1+\alpha)} \quad \text{for } r \geq R_0,
\]
\[
(4.33) \quad \int_{R_0}^{(P-1)R} r \tilde{K}(r) \, dr \geq \epsilon \quad \text{for } R \geq R_0.
\]
But from the proof of Theorem 2.2, there is no positive function \( v \) satisfying (4.27). This contradiction proves the theorem. Q.E.D.

**Theorem 4.5.** Let \( K(x) \geq 0 \) be a continuous function in \( \mathbb{R} \). Let the average \( \overline{K}(r) \) of \( K(x) \) in the sense (4.18) satisfy
\[
(4.34) \quad \int_0^s s^r \overline{K}(s) \, ds \text{ is strictly increasing in } [0, \infty) \text{ and }
\]
\[
(4.35) \quad \int_0^\infty s^r \overline{K}(s) \, ds = \infty,
\]
\[
(4.36) \quad \left( \frac{s}{r} \right)^m \leq \int_0^s t^r \overline{K}(t) \, dt \int_0^r t^s \overline{K}(t) \, dt \text{ for some } m > 0 \text{ and }
\]
\[
(4.37) \quad \text{for all } r \geq s \geq R_0 > 0.
\]
Then equation (1.1) does not possess any positive solution in \( \mathbb{R} \). In particular, if \( \bar{K}(r) \) satisfies (4.34) and \( 0 \leq \bar{K}(r) \leq C/r^{(\sigma-1)} \) for \( r \geq R_1 \) for some constants \( C > 0 \) and \( R_1 > 0 \), then \( \bar{K}(r) \) also satisfies (4.35) and hence (1.1) does not possess any positive solution in \( \mathbb{R} \).

**Proof.** Assume on the contrary that (1.1) has a positive solution \( u(x) \) in \( \mathbb{R} \). Then as in the proof of Theorem 4.3, we have (4.24)–(4.27). Now the function \( \tilde{K}(r) = r^{(\sigma-1)}\bar{K}(r) \) satisfies the assumptions of Theorem 2.3. Hence there is no positive function \( v \) satisfying (4.27). This contradiction proves the theorem. \( \Box \).

**Theorem 4.6.** Let \( K(x) > 0 \) be a continuous function in \( \mathbb{R} \) and \( k(r) \) be a continuous function in \( [0, \infty) \). Let the average \( \bar{K}(r) \) of \( K(x) \) in the sense (4.18) satisfy

\[
\bar{K}(r) \geq 0 \quad \text{if} \quad \alpha_{i+1} + \beta_i < r < \alpha_{i+1} + \beta_{i+1}, \\
\bar{K}(r) \geq \bar{K}(r - \beta_i) \quad \text{if} \quad \alpha_i + \beta_i \leq r \leq \alpha_{i+1} + \beta_i
\]

for \( i = 0, 1, 2, \ldots, \) where \( \{\alpha_i\}_{i=0}^{\infty} \) is a strictly increasing sequence satisfying \( \alpha_0 = 0 \) and \( \lim_{n \to \infty} \alpha_n = \infty \), and \( \{\beta_i\}_{i=0}^{\infty} \) is a nondecreasing sequence satisfying \( \beta_0 = 0 \) and \( \beta_i/\alpha_i \leq M \) for some \( M > 0 \) and for \( i \geq 1 \). \( \mu' = \bar{K}(r)u(t) \) in \( (0, \infty) \),

\[
u(t) = v(r + \beta_i) \quad \text{if} \quad \alpha_i < r < \alpha_{i+1}
\]

for \( i = 0, 1, 2, \ldots, \) As in the proof of Theorem 2.4, we have

\[
u(r) \geq \alpha + \int_0^r (r - t)\bar{K}(t)v(t) \, dt.
\]

Now we can let \( X \) denote the locally convex space of all continuous functions on \( [0, \infty) \) with the usual topology and consider the set

\[
Y = \{ y \in X: \alpha \leq y(r) \leq v(r) \text{ for } r \geq 0 \},
\]

where \( v \) is defined in (4.38). Clearly, \( Y \) is a closed convex subset of \( X \). We define the mapping \( T \) by

\[
(Ty)(r) = \alpha + \int_0^r (r - t)\bar{K}(t)y(t) \, dt.
\]

Then it is easy to verify that (i) \( TY \subset Y \), (ii) \( T \) is continuous and (iii) \( TY \) is precompact. Hence \( T \) has a fixed point in \( Y \). Thus (4.36) has a solution for this \( \alpha \). This contradiction completes the proof. \( \Box \).
PART II. \( \Delta u = K(x)e^{2u} \)

5. The case \( n \geq 3 \). In this case, the existence results are very similar to that of §2. Ni [14] proves that, if \( |K(x)| \leq C/|x|^{l} \) for \( |x| \) large and uniformly in \( x_{2} \) for some \( l > 2 \), then equation (1.2) possesses infinitely many bounded solutions in \( \mathbb{R}^{n} = \mathbb{R}^{m} \times \mathbb{R}^{n-m} \), where \( x = (x_{1}, x_{2}) \) and \( m \geq 3 \). Later on, Kusano and Oharu [7] extend the result to the case where \( |K(x)| \leq K(|x_{1}|) \) for all \( x \in \mathbb{R}^{m} \times \mathbb{R}^{n-m} \) and \( \int_{0}^{\infty} tK(t) \, dt < \infty \). On the other hand, when \( K(x) \geq 0 \) in (1.2), Oleinik [15] shows that if \( K(x) \geq C/|x|^{p} \) at infinity for some \( p < 2 \), then (1.2) has no solution in \( \mathbb{R}^{n} \). The case when \( K(x) \) behaves like \( C/|x|^{2} \) at infinity is left unsettled for \( n \geq 3 \). In this section, we give several theorems to settle the nonexistence question of (1.2), in particular we settle the case when \( K(x) \) behaves like \( C/|x|^{2} \) at infinity.

We need some notations first. Let \( u \) be a smooth function in \( \mathbb{R}^{n} \) and \( K(x) \geq 0 \) be a continuous function in \( \mathbb{R}^{n} \). Following Ni [13] and Sattinger [16], we define the averages of \( u \) and \( K \) by \( \bar{u}(r) \) and \( \bar{K}(r) \),

\[
\bar{u}(r) = \frac{1}{\omega_{n} r^{n-1}} \int_{|x|=r} u(x) \, dS,
\]

\[
\bar{K}(r) = \left( \frac{1}{\omega_{n} r^{n-1}} \int_{|x|=r} \frac{dS}{K(x)} \right)^{-1}.
\]

We have

**Lemma 5.1.** Let \( u(x) \) be a solution of (1.2) in \( \mathbb{R}^{n} \) and \( K(x) \geq 0 \). Then \( \bar{u}(r) \) satisfies

\[
\begin{cases}
\bar{u}''(r) + \frac{n-1}{r} \bar{u}'(r) \geq K(r) e^{2\bar{u}(r)}, & r \in (0, \infty), \\
\bar{u}(0) = u(0), & \bar{u}'(0) = 0.
\end{cases}
\]

**Proof.** From the definition of \( \bar{u} \), we have

\[
\bar{u}'(r) = \frac{1}{\omega_{n} r^{n-1}} \int_{|x|=r} \nabla u(x) \cdot \xi \, dS = \frac{1}{\omega_{n} r^{n-1}} \int_{|x|=r} \sum_{i} u_{i} \xi_{i} \, dS.
\]

Thus,

\[
\omega_{n} \left( r^{n-1} \bar{u}'(r) - R^{n-1} \bar{u}'(R) \right)
\]

\[
= \int_{D} \Delta u \, dx = \int_{R} \int_{|x|=r} \Delta u \, dS \, dt
\]

where \( D = \{ x \in \mathbb{R}^{n}: R < |x| < r \} \). Hence we have

\[
\omega_{n} \left( r^{n-1} \bar{u}'(r) \right)' = \int_{|x|=r} \Delta u \, dS = \int_{|x|=r} K(x) e^{2u(x)} \, dS.
\]

Now Jensen’s and Cauchy-Schwarz’s inequalities give

\[
e^{2\bar{u}(r)} = \left( e^{\bar{u}(r)} \right)^{2} \leq \left( \frac{1}{\omega_{n} r^{n-1}} \int_{|x|=r} e^{u(x)} \, dS \right)^{2}
\]

\[
\leq \left( \frac{1}{\omega_{n} r^{n-1}} \int_{|x|=r} K(x) e^{2u(x)} \, dS \right) \left( \frac{1}{\omega_{n} r^{n-1}} \int_{|x|=r} \frac{dS}{K(x)} \right).
\]
Hence
\begin{equation}
\frac{1}{\omega_r r^{n-1}} \int_{|x|=r} K(x) e^{2u(x)} dS \geq \bar{K}(r) e^{2\bar{u}(r)}.
\end{equation}

Combining (5.5) and (5.7), we obtain the first equation of (5.3). \( \bar{u}(0) = u(0) \) and \( \bar{u}'(0) = 0 \) can also be easily obtained. This completes the proof. Q.E.D.

Now we can state our main nonexistence theorems.

**Theorem 5.1.** Let \( K(x) \geq 0 \) be a locally Hölder continuous function in \( \mathbb{R}^n \). If \( \bar{K}(r) \), as defined in (5.2), satisfies
\begin{equation}
\bar{K}(r) \geq C/r^2
\end{equation}
for \( r \geq R_0 \) for some constant \( C > 0 \), then equation (1.2) does not possess any locally bounded solution in \( \mathbb{R}^n \).

**Proof.** Assume that \( u \) is a locally bounded solution of (1.2) in \( \mathbb{R}^n \). Then the average \( \bar{u} \) satisfies (5.3) from Lemma 5.1. Let \( \bar{u}(0) = u(0) = \alpha \). Then \( \bar{u} \) also satisfies
\begin{equation}
\bar{u}'(r) \geq \int_0^r \left( \frac{s}{r} \right)^{n-2} \bar{K}(s) e^{2\bar{u}(s)} ds,
\end{equation}
\begin{equation}
\bar{u}(r) \geq \alpha + \frac{1}{n-2} \int_0^r \bar{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] e^{2\bar{u}(s)} ds.
\end{equation}
Hence
\begin{equation}
\bar{u}(r) \geq \alpha + \frac{1}{n-2} \int_0^{r/2} \bar{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] e^{2\bar{u}(s)} ds
= \alpha + \frac{1}{n-2} \cdot e^{2\alpha} \cdot \left[ 1 - \left( \frac{1}{2} \right)^{n-2} \right] \cdot \int_0^{r/2} \bar{K}(s) ds.
\end{equation}
Thus there exists a constant \( R_0 \), such that \( \bar{u}(R_0) \geq 1 \). For \( r \geq R_0 \), we have
\begin{equation}
\bar{u}(r) \geq 1 + \frac{1}{n-2} \int_{R_0}^r \bar{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] e^{2\bar{u}(s)} ds
\geq 1 + \frac{1}{n-2} \int_{R_0}^r \bar{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] \bar{u}^2(s) ds.
\end{equation}
In view of (5.8) and the proof of Theorem 2.1, we conclude that no function \( \bar{u} \) can satisfy (5.12) in \( [R_0, \infty) \). This completes the proof. Q.E.D.

**Theorem 5.2.** Let \( K(x) \geq 0 \) be a locally Hölder continuous function in \( \mathbb{R}^n \). If \( \bar{K}(r) \), as defined in (5.2), satisfies
\begin{equation}
\bar{K}(r) \geq C/r^a \quad \text{for} \quad r \geq R_0,
\end{equation}
\begin{equation}
\text{there exist} \ \alpha > 0, \ R_0 > 0 \ \text{and} \ C > 0, \ \text{such that}
\end{equation}
\begin{equation}
\int_R^{(P-1)R} r\bar{K}(r) \ dr \geq \epsilon \quad \text{for} \ R \geq R_0,
\end{equation}
then equation (1.2) does not possess any locally bounded solution in \( \mathbb{R}^n \).
Proof. Assume that \( u \) is a locally bounded solution of (1.2) in \( \mathbb{R}^n \). Then as in the proof of Theorem 5.1, we have (5.9)-(5.12). But from (5.13), (5.14) and Theorem 2.2, there is no function \( \bar{u}(r) \) defined on \([R_0, \infty)\) satisfying (5.12). This contradiction proves the theorem. Q.E.D.

Theorem 5.3. Let \( K(x) \geq 0 \) be a locally Hölder continuous function. If \( \bar{K}(r) \), as defined in (5.2), satisfies

\[
\int_0^r s \bar{K}(s) \, ds \text{ is strictly increasing in } [0, \infty) \quad \text{and} \\
\int_0^\infty s \bar{K}(s) \, ds = \infty,
\]

then equation (1.2) does not possess any locally bounded solution in \( \mathbb{R}^n \). In particular, if \( \bar{K}(r) \) satisfies (5.15) and \( 0 \leq \bar{K}(r) \leq C/r^2 \) for \( r \geq R_1 \) for some constants \( C > 0 \) and \( R_1 > 0 \), then \( \bar{K}(r) \) also satisfies (5.16) and hence (1.2) does not possess any locally bounded solution in \( \mathbb{R}^n \).

Proof. Using the proofs of Theorems 5.1 and 2.3, we can easily obtain a proof. We omit the details. Q.E.D.

Theorem 5.4. Let \( K(x) \geq 0 \) be a locally Hölder continuous function in \( \mathbb{R}^n \) and \( \bar{K}(t) \) be a locally Hölder continuous function on \([0, \infty)\). Let the average \( \bar{K}(r) \) of \( K(x) \) in the sense of (5.2) satisfy

\[
K(r) \geq 0 \quad \text{if } \alpha_{i+1} + \beta_i < r < \alpha_{i+1} + \beta_{i+1}, \\
K(r) \geq \bar{K}(r-\beta_i) \quad \text{if } \alpha_i + \beta_i \leq r \leq \alpha_{i+1} + \beta_i
\]

for \( i = 0, 1, 2, \ldots \), where \( \{\alpha_i\}_{i=0}^\infty \) and \( \{\beta_i\}_{i=0}^\infty \) are two sequences satisfying the same conditions as in Theorem 2.4. If

\[
\begin{cases} 
\frac{3}{3-n} u''(r) + \frac{n-1}{r} u'(r) = \bar{K}(r) e^{2u(r)} & \text{in } (0, \infty), \\
u(0) = \alpha, \quad u'(0) = 0
\end{cases}
\]

does not possess any locally bounded solution in \([0, \infty)\) for any real number \( \alpha \), then (1.2) does not possess any locally bounded solution in \( \mathbb{R}^n \).

Proof. The proof is similar to that of Theorem 2.4. Hence we omit the details. Q.E.D.

6. The case \( n = 2 \). In the case \( n = 2 \) and \( K(x) \geq 0 \), Ni [14] shows that: If \( K(x) \neq 0 \) and \( K(x) \leq C/|x|^l \) at infinity for some \( l > 2 \), then for every \( \alpha \in (0, \beta) \) where \( \beta = \min(8, (l-2)/3) \), there exists a solution \( u \) of (1.2) such that

\[
\log |x|^{\alpha} - C' \leq u(x) \leq \log |x|^{\alpha} + C''
\]

for \(|x| \) large, where \( C' \) and \( C'' \) are two constants.
Later, McOwen [10, 11] improves this result by giving a sharp bound on $\beta$ and sharp behavior of $u$ at infinity. For the nonexistence results, Sattinger [16] proves that $K$ be a smooth function on $\mathbb{R}^2$. If $K \geq 0$ on $\mathbb{R}^2$ and $K(x) \geq C/|x|^2$ at infinity, then (1.2) has no solution on $\mathbb{R}^2$. Ni [14] improves Sattinger’s result to include the $K$ such as $K = (1 + \sin r)/r^2$.

In this section, we give an existence result which overlaps parts of the results of Ni [14] and McOwen [10, 11] but with different method. We also give some nonexistence results improving Ni’s result.

**Theorem 6.1.** Let $K(x) \geq 0$ be a locally Hölder continuous function on $\mathbb{R}^2$. Let $K_1(r)$ and $K_2(r)$ be two locally Hölder continuous functions on $[0, \infty)$. If

\begin{align}
K_1(0) &> 0, \\
0 &\leq K_1(|x|) \leq K(x) \leq K_2(|x|) \quad \text{for all } x \in \mathbb{R}^2, \\
\text{there exists } \alpha > 0 \text{ such that } \int_0^\infty s^{(1+2\alpha)}K_2(s)\,ds < \infty,
\end{align}

then (1.2) has infinitely many solutions on $\mathbb{R}^2$ with logarithmic growth at infinity.

**Proof.** Consider the equations

\begin{align}
\Delta v &= K_1(|x|)e^{2v}, \quad x \in \mathbb{R}^2, \\
\Delta w &= K_2(|x|)e^{2w}, \quad x \in \mathbb{R}^2.
\end{align}

From (6.2), it is easy to see that a solution $v$ of (6.4) is a supersolution of (1.2) and a solution $w$ of (6.5) is a subsolution of (1.2) in $\mathbb{R}^2$. It is natural to seek solutions of $v$ and $w$ depending only on $|x|$. Consider now (6.5). We try to find a solution $w(|x|)$ of (6.5) with $w(0) = \beta$ and $w'(0) = 0$. Then (6.5) is equivalent to the following integral equation

\begin{equation}
w(r) = \beta + \int_0^r s\log\left(\frac{r}{s}\right)K_2(s)e^{2w(s)}\,ds.
\end{equation}

Now we choose $0 < \alpha' < \alpha$ and $\beta$ such that

\begin{align}
\int_0^e s\log\left(\frac{e}{s}\right)K_2(s)e^{2(\beta+1)}\,ds &< \frac{1}{2}, \\
\int_0^e sK_2(s)e^{2(\beta+1)}\,ds &< \frac{\alpha'}{2}, \\
\int_e^\infty s^{(1+2\alpha')}K_2(s)e^{2(\beta+1)}\,ds &< \frac{\alpha'}{2}, \\
\int_e^\infty s^{(1+2\alpha')}\log\left(\frac{e}{s}\right)K_2(s)e^{2(\beta+1)}\,ds &< \frac{1}{2}.
\end{align}

Define the function $A_\beta(r)$ by

\begin{align}
A_\beta(r) &= (\beta + 1) \quad \text{if } 0 \leq r \leq e, \\
A_\beta(r) &= (\beta + 1) + \alpha'\log(r/e) \quad \text{if } e \leq r.
\end{align}
Now let $X$ denote the locally convex space of all continuous functions on $[0, \infty)$ with the usual topology and consider the set

$$Y = \{ w \in X: \beta \leq w(r) \leq A_{\beta}(r), \ r \in [0, \infty) \}.$$  

It is easy to see that $Y$ is a closed convex subset of $X$. Let $T$ be the mapping

$$(T w)(r) = \beta + \int_0^r s \log \left( \frac{r}{s} \right) K_2(s) e^{2w(s)} \, ds.$$  

We shall prove that $T$ is a continuous mapping from $Y$ into itself such that $TY$ is relatively compact.

First, we verify that $TY \subset Y$. Assume $w \in Y$. Hence we have

$$\beta \leq w(r) \leq A_{\beta}(r) \quad \text{for} \quad r \in [0, \infty).$$  

It is easy to see that $T w$ is also continuous and $\beta \leq T w(r)$ for $r \in [0, \infty)$. Now for $0 \leq r \leq e$, we have

$$(T w)(r) = \beta + \int_0^r s \log \left( \frac{r}{s} \right) K_2(s) e^{2w(s)} \, ds \leq \beta + \int_0^e s \log \left( \frac{e}{s} \right) K_2(s) e^{2w(s)} \, ds \leq \beta + (\beta + 1) = A_{\beta}(r).$$

For $e \leq r$, we have

$$(T w)(r) = \beta + \int_0^e s \log \left( \frac{r}{s} \right) K_2(s) e^{2w(s)} \, ds + \int_e^r s \log \left( \frac{r}{s} \right) K_2(s) e^{2w(s)} \, ds \leq \beta + \int_0^e s \log \left( \frac{e}{s} \right) K_2(s) e^{2A_{\beta}(s)} \, ds + \int_e^r s \log \left( \frac{r}{s} \right) K_2(s) e^{2A_{\beta}(s)} \, ds \leq \beta + \log \left( \frac{r}{e} \right) \int_0^e s K_2(s) e^{2(\beta + 1)} \, ds + \int_0^e s \log \left( \frac{e}{s} \right) K_2(s) e^{2(\beta + 1)} \, ds + \log \left( \frac{r}{e} \right) \int_e^{\infty} s^{(1+2\alpha')} K_2(s) e^{2(\beta + 1)} \, ds + \int_e^{\infty} s^{(1+2\alpha')} \log \left( \frac{e}{s} \right) K_2(s) e^{2(\beta + 1)} \, ds \leq \beta + \alpha' \log \left( \frac{r}{e} \right) + \frac{1}{2} + \alpha' \log \left( \frac{r}{e} \right) + \frac{1}{2} = (\beta + 1) + \alpha' \log \left( \frac{r}{e} \right) = A_{\beta}(r).$$

This verifies that $TY \subset Y$.  

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Now let \( \{ w_m \}_{n=1}^{\infty} \subset Y \) be a sequence converges to \( w \in Y \) in the space \( X \). Then \( \{ w_m \} \) converges to \( w \) uniformly on any compact interval on \([0, \infty)\). Now

\[
(6.17) \quad |T w_m(r) - T w(r)| \leq \int_0^r s \log \left( \frac{r}{s} \right) K_2(s) \left| e^{2w_m(s)} - e^{2w(s)} \right| ds
\]

But

\[
(6.18) \quad s \log \left( \frac{r}{s} \right) K_2(s) \left| e^{2w_m(s)} - e^{2w(s)} \right| \leq s \log \left( \frac{r}{s} \right) K_2(s) \left( e^{2A_\beta(s)} - e^{2\beta} \right)
\]

and \( s \log(r/s) K_2(s) e^{2A_\beta(s)} \) is integrable. Hence from (6.17) and the uniform convergence of \( w_m \) to \( w \) on any compact interval, we conclude that \( T w_m \) converges to \( T w \) in \( X \). This verifies that \( T \) is continuous in \( Y \). We can easily compute that

\[
(6.19) \quad (T w)'(r) = \int_0^r \left( \frac{s}{r} \right) K_2(s) e^{2w(s)} ds
\]

Hence, on any compact interval of \([0, \infty)\), \( TY \) is uniformly bounded and equicontinuous. This proves that \( TY \) is relatively compact in \( Y \). Thus we can apply the Schauder-Tychonoff fixed point theorem to conclude that \( T \) has a fixed point \( w \) in \( Y \). This fixed point \( w \) is a solution of (6.6) and hence a solution of (6.5). Note that, when we have a solution \( w \) of (6.6) with a given \( \beta \), then we also have a solution \( w \) of (6.6) with \( \beta \) replaced by smaller \( \beta \)’s.

Similarly, we can construct solution \( v(|x|) \) of (6.4) such that \( v(0) = \beta' \) and \( v'(0) = 0 \). For a given \( \beta' \), since \( K_1(0) > 0 \), we can choose \( \beta < \beta' \), such that (6.6) has a solution \( w \) and \( w(r) < v(r) \) for all \( r \in [0, \infty) \). Using Theorem 2.10 of Ni [13], we conclude that (1.2) has a solution \( u(x) \) between \( w(|x|) \) and \( v(|x|) \). Now we can choose another \( \beta' \) smaller than this \( \beta \) to repeat the arguments. This completes the proof of this theorem.

**Q.E.D.**

**Theorem 6.2.** Let \( K(x) > 0 \) be a locally Hölder continuous function in \( \mathbb{R}^2 \). If \( \bar{K}(r) \), as defined in (5.2), satisfies

\[
(6.20) \quad \bar{K}(r) \geq C/r^2(\log r)^a
\]

for \( r \geq R_0 \) for some constants \( C > 0 \) and \( a > 0 \), then equation (1.2) does not possess any locally bounded solution in \( \mathbb{R}^2 \).

**Proof.** Assume that \( u \) is a locally bounded solution of (1.2) in \( \mathbb{R}^2 \). Then the average \( \bar{u} \) satisfies (5.3) for \( n = 2 \). Letting \( \bar{u}(0) = \beta = u(0) \), we have

\[
(6.21) \quad \bar{u}'(r) \geq \int_0^r \left( \frac{s}{r} \right) \bar{K}(s) e^{2\bar{u}(s)} ds,
\]

\[
(6.22) \quad \bar{u}(r) \geq \beta + \int_0^r s \log \left( \frac{s}{r} \right) \bar{K}(s) e^{2\bar{u}(s)} ds.
\]

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Without loss of generality, we may assume that \(K(0) > 0\) and hence \(\bar{K}(0) > 0\). For \(r > e\), we have

\[
(6.23) \quad \bar{u}(r) \geq \beta + \int_0^r s \log\left(\frac{r}{s}\right) \bar{K}(s) e^{2\bar{u}(s)} ds
\]

\[
+ \int_1^r s \log\left(\frac{r}{s}\right) \bar{K}(s) e^{2\bar{u}(s)} ds
\]

\[
\geq \beta + \int_0^r s \log r \bar{K}(s) e^{2\beta ds} + \int_1^r s \log\left(\frac{r}{s}\right) \bar{K}(s) e^{2\bar{u}(s)} ds
\]

\[
\geq \beta + C_1 \log r + \int_e^r s \log\left(\frac{r}{s}\right) \bar{K}(s) e^{2\bar{u}(s)} ds.
\]

Thus there exists a constant \(R_0\) such that, for \(r \geq R_0\),

\[
(6.24) \quad \bar{u}(r) \geq C_2 \log r + \int_r^{R_0} s \log\left(\frac{r}{s}\right) \bar{K}(s) e^{2\bar{u}(s)} ds
\]

\[
\geq C_2 \log r + \int_{R_0}^r s \log\left(\frac{r}{s}\right) \bar{K}(s) e^{2\bar{u}(s)} ds
\]

for some \(C_2 > 0\). Let

\[
(6.25) \quad \bar{u}(r) = \frac{1}{2} C_2 \log r + v(r) \quad \text{for } r \geq R_0.
\]

From (6.24), we have

\[
(6.26) \quad v(r) \geq \frac{1}{2} C_2 \log r + \int_{R_0}^r s \log\left(\frac{r}{s}\right) \bar{K}(s) s^{c_2} e^{2v(s)} ds
\]

\[
\geq \frac{1}{2} C_2 \log r + \int_{R_0}^r s \log\left(\frac{r}{s}\right) \bar{K}(s) s^{c_2} v^2(s) ds.
\]

But from assumption (6.20), we have

\[
(6.23) \quad \bar{K}(s) s^{c_2} \geq C/s^{2-c_2}(\log s)^\alpha \geq C/s^2
\]

for \(s \geq R_1 > R_0\). Hence from Theorem 3.1, there is no \(v\) in \([R_0, \infty)\) satisfying (6.26). This completes the proof of this theorem.

**Theorem 6.3.** Let \(K(x) \geq 0\) be a locally Hölder continuous function in \(\mathbb{R}^2\). If \(\bar{K}(r)\), as defined in (5.2), satisfies

\[
(6.24) \quad \int_0^r s^{1+\alpha} \bar{K}(s) ds \text{ is monotonically strictly increasing in } [0, \infty) \text{ for all } \alpha > 0.
\]

\[
(6.25) \quad \text{For given any } \alpha > 0, \text{ there exists an } R_\alpha > 0 \text{ such that }
\]

\[
\left(\frac{\log s}{\log r}\right)^m \leq \int_0^s t^{1+\alpha} \bar{K}(t) dt / \int_0^r t^{1+\alpha} \bar{K}(t) dt
\]

for some \(m > 0\) and for all \(r \geq s \geq R_\alpha\), then (1.2) does not possess any locally bounded solution in \(\mathbb{R}^2\).
PROOF. Assume that $u$ is a locally bounded solution of $(1.2)$ in $\mathbb{R}^2$. Then as in the proof of Theorem 6.2, we have $(6.21)-(6.26)$. Now we can let $w(r)\log r = v(r)$ for $r > R_0$. Then from $(6.26)$, we have

\begin{equation}
(6.27) \quad w(r) \geq \frac{1}{2}C_2 + \int_{R_0}^r s \left( 1 - \frac{\log s}{\log r} \right) \tilde{K}(s) s^{C_2} v^2(s) \, ds.
\end{equation}

Now using a similar argument as in the proof of Theorem 3.3, we conclude that there is no function $w$ satisfying $(6.27)$. This contradiction proves the theorem. Q.E.D.

THEOREM 6.4. Let $K(x) \geq 0$ be a locally Hölder continuous function in $\mathbb{R}^2$ and $\tilde{K}(t)$ be a locally Hölder continuous function on $[0, \infty)$. Let the average $\tilde{K}(r)$ of $K(x)$ in the sense of $(5.2)$ satisfy the same assumptions as in Theorem 5.4. If

\begin{equation}
(6.28) \begin{cases}
\frac{u''(r)}{r} + \frac{u'(r)}{r} = \tilde{K}(r) e^{2u(r)} \quad \text{in } (0, \infty), \\
u(0) = \alpha, \quad u'(0) = 0
\end{cases}
\end{equation}

does not possess any locally bounded solution in $[0, \infty)$ for any real number $\alpha$, then $(1.2)$ does not possess any locally bounded solution in $\mathbb{R}^2$.

PROOF. The proof is similar to that of Theorem 2.4. Hence we omit the details. Q.E.D.

7. The case $n = 1$. In this case, we consider only the situation $K(x) \geq 0$ in $(1.2)$. We give a main existence result which has an extension to the higher-dimensional case. We also give some nonexistence results.

THEOREM 7.1. Let $K(x) \geq 0$ be a Hölder continuous function in $\mathbb{R}$. If $K(0) > 0$ and there exists an $\alpha > 0$, such that

\begin{equation}
(7.1) \quad \int_{-\infty}^{\infty} e^{2|x|} K(x) \, dx < \infty,
\end{equation}

then $(1.2)$ has infinitely many locally bounded solutions in $\mathbb{R}$ with linear growth at $|x| = \infty$.

PROOF. We shall seek solution $u$ such that $u(0) = \beta$ and $u'(0) = 0$. Consider now $x \geq 0$. In this situation, $(1.2)$ is equivalent to the integral equation

\begin{equation}
(7.2) \quad u(x) = \beta + \int_0^x (x - t) K(t) e^{2u(t)} \, dt, \quad x \geq 0.
\end{equation}

Now choose $\beta \in \mathbb{R}$ so that

\begin{equation}
(7.3) \quad \int_0^1 K(t) e^{2(\beta + 1)} \, dt \leq \min \left\{ \frac{\alpha}{2}, 1 \right\},
\end{equation}

\begin{equation}
(7.4) \quad \int_1^\infty K(t) e^{2\alpha e^{2(\beta + 1)}} \, dt \leq \frac{\alpha}{2}.
\end{equation}

Let

\begin{equation}
A(x) = \begin{cases}
(\beta + 1) & \text{if } 0 \leq x \leq 1, \\
(\beta + 1) + \alpha x & \text{if } 1 < x.
\end{cases}
\end{equation}
As in the proofs of Theorems 2.4 and 3.4, we let $X$ denote the locally convex space of all continuous functions on $[0, \infty)$ with the usual topology and consider the set

$$Y = \{ y \in X : \beta \leq y(x) \leq A(x) \text{ for } x \geq 0 \}.$$ 

Clearly, $Y$ is a closed convex subset of $X$. Now define the mapping $T$ by

$$(Ty)(x) = \beta + \int_0^x (x - t) K(t) e^{2y(t)} dt.$$ 

If $y \in Y$, then $\beta \leq y(x) \leq A(x)$. Hence we have

$$(Ty)(x) = \beta + \int_0^x (x - t) K(t) e^{2y(t)} dt \geq \beta.$$ 

On the other hand, for $0 \leq x \leq 1$, we have

$$(Ty)(x) = \beta + \int_0^1 (x - t) K(t) e^{2y(t)} dt \leq \beta + \int_0^1 K(t) e^{2(\beta + 1)} dt \leq \beta + 1 = A(x).$$ 

For $1 < x$, we have

$$(Ty)(x) = \beta + \int_0^1 (x - t) K(t) e^{2y(t)} dt + \int_1^x (x - t) K(t) e^{2y(t)} dt \leq \beta + x \cdot \int_0^1 K(t) e^{2(\beta + 1)} dt + x \cdot \int_1^\infty K(t) e^{2\alpha't} e^{2(\beta + 1)} dt \leq \beta + \frac{\alpha}{2} \cdot x + \frac{\alpha}{2} x \leq (\beta + 1) + ax = A(x).$$ 

Hence $T$ maps $Y$ into itself. As in the proofs of Theorems 2.4, 3.4 and 4.1, we can easily verify that $T$ is continuous and $TY$ is precompact. Hence $T$ has a fixed point $y \in Y$. This fixed point $y$ is a solution of (1.2) for $x \geq 0$ with $y(0) = \beta$ and $y'(0) = 0$.

Similarly, we can find a solution of (1.2) for $x \leq 0$ with $y(0) = \beta$ and $y'(0) = 0$ provided that $\beta \in \mathbb{R}$ is properly selected. It is also easy to see that if $y$ is a solution of (1.2) with $y(0) = \beta$ and $y'(0) = 0$, then there is also solution $y$ with $y(0) = \beta'$ and $y'(0) = 0$ provided that $\beta' < \beta$. The linear growth of solutions at $|x| = \infty$ can be easily established as in the proof of Theorem 4.1. This completes the proof of this theorem. Q.E.D.

We can apply this theorem to the higher-dimensional case as used in Ni [13, 14] and Kawano, Kusano and Naito [3].

**Theorem 7.2.** Let $K(x) \geq 0$ be a locally Hölder continuous function in $\mathbb{R} = \mathbb{R} \times \mathbb{R}^{-1}$. Let $\phi\ast(x_1)$ and $\phi\ast(x_1)$ be two locally Hölder continuous function in $\mathbb{R}$. If

$$0 \leq \phi\ast(x_1) \leq K(x) \leq \phi\ast(x_1) \text{ for all } x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{-1},$$

$$\phi\ast(0) > 0 \text{ and } \int_{-\infty}^\infty e^{2\alpha|x_1|} \phi\ast(x_1) dx_1 < \infty \text{ for some } \alpha > 0,$$

then equation (1.2) has infinitely many locally bounded solutions in $\mathbb{R}$. 

PROOF. The proof is actually similar to that of Theorem 4.2. We omit the details.

Q.E.D.

Now let \( u \) be a smooth function on \( \mathbb{R} \) and \( K(x) \geq 0 \) be a continuous function on \( \mathbb{R} \). We define the averages \( \bar{u} \) and \( \bar{K} \) by

\[
\bar{u}(r) = \frac{1}{2} [u(r) + u(-r)], \quad r \geq 0, \\
\bar{K}(r) = \left[ \frac{1}{2} (K(r)^{-1} + K(-r)^{-1}) \right]^{-1}, \quad r \geq 0.
\]

Our nonexistence results are

**Theorem 7.3.** Let \( K(x) \geq 0 \) be a locally Hölder continuous function on \( \mathbb{R} \). If the average \( \bar{K}(r) \) of \( K(x) \) in the sense of (7.12) satisfies

\[
\bar{K}(r) > C/r^a
\]

for \( r \geq R_0 \) and for some constants \( C > 0, a > 0 \), then equation (1.2) does not possess any locally bounded solution on \( \mathbb{R} \).

PROOF. Assume that \( u(x) \) be a solution of (1.2) in \( \mathbb{R} \). Then we have

\[
\bar{u}''(r) = \frac{1}{2} [u''(r) + u''(-r)]
\]

\[
= \frac{1}{2} [K(r) e^{2u(r)} + K(-r) e^{2u(-r)}].
\]

But we have

\[
e^{2\bar{u}(r)} = \left( e^{\bar{u}(r)} \right)^2 \leq \left[ \frac{1}{2} (e^{u(r)} + e^{u(-r)}) \right]^2
\]

\[
\leq \left[ \frac{1}{2} (K(r) e^{2u(r)} + K(-r) e^{2u(r)}) \right] \cdot \left[ \frac{1}{2} (K(r)^{-1} + K(-r)^{-1}) \right].
\]

Hence we have

\[
\bar{u}''(r) \geq \bar{K}(r) e^{2\bar{u}(r)}, \quad r \geq 0.
\]

It is also easy to see that \( \bar{u}(0) = u(0) \) and \( \bar{u}'(0) = 0 \). From (7.16), we have

\[
\bar{u}'(r) \geq \int_0^r \bar{K}(t) e^{2\bar{u}(t)} dt,
\]

\[
\bar{u}(r) \geq \beta + \int_0^r (r-t) \bar{K}(t) e^{2\bar{u}(t)} dt.
\]

Without loss of generality, we may assume that \( K(0) > 0 \) and hence \( \bar{K}(0) > 0 \). For \( r \geq 1 \), we have

\[
\bar{u}(r) \geq \beta + \int_0^1 (r-t) \bar{K}(t) e^{2\bar{u}(t)} dt + \int_1^r (r-t) \bar{K}(t) e^{2\bar{u}(t)} dt
\]

\[
\geq \beta + r \int_0^1 (1-t) \bar{K}(t) e^{2\beta} dt + \int_1^r (r-t) \bar{K}(t) e^{2\bar{u}(t)} dt
\]

\[
\geq 2C_1 \cdot r + \int_{R_1}^r (r-t) \bar{K}(t) e^{2\bar{u}(t)} dt.
\]
for $r \geq R_1 > 1$ and for some $C_1 > 0$. Now let $v(r) = \tilde{u}(r) + C_1 \cdot r$. We have from (7.19)

$$v(r) \geq C_1 \cdot r + \int_{R_1}^{r} (r - t) \tilde{K}(t) e^{2C_1t} \cdot e^{2v(t)} \, dt.$$  

Let $v(r) = w(r) \cdot r$, we have

$$w(r) \geq C_1 + \int_{R_1}^{r} (1 - \frac{t}{r}) \tilde{K}(t) e^{2C_1t} \cdot e^{2w(t)} \, dt.$$  

Now let $\tilde{K}(t) = t^{-1} \tilde{K}(t) e^{2C_1t}$. We have from (7.13)

$$\tilde{K}(t) \geq C/t^2$$  

for $t \geq R_2 > R_1$ for some $C > 0$. But (7.21) becomes

$$w(r) \geq C_1 + \int_{R_1}^{r} t \left(1 - \frac{t}{r} \right) \tilde{K}(t) w^2(t) \, dt.$$  

From Theorem 2.1, there is no function $w$ satisfying (7.23). This contradiction proves the theorem. Q.E.D.

**Theorem 7.4.** Let $K(x) > 0$ be a locally Hölder continuous function on $\mathbb{R}$. If the average $\overline{K}(r)$ of $K(x)$ in the sense of (7.12) satisfies

$$\int_{0}^{r} e^{\alpha s} \overline{K}(s) \, ds \text{ is strictly increasing and } \int_{0}^{\infty} e^{\alpha s} \overline{K}(s) \, ds = \infty$$  

for all $\alpha > 0$.

For any given $\alpha > 0$, there exists $R_{\alpha} > 0$, such that

$$\left( \frac{s}{r} \right)^{m} \leq \int_{0}^{s} e^{\alpha s} \overline{K}(t) \, dt / \int_{0}^{r} e^{\alpha s} \overline{K}(t) \, dt$$  

for some $m > 0$ and for $r \geq s \geq R_{\alpha}$, then equation (1.2) does not possess any locally bounded solution in $\mathbb{R}$.

**Proof.** Using the proofs of Theorems 7.3 and 2.3, we can easily prove this theorem. We omit the details. Q.E.D.

**Theorem 7.5.** Let $K(x) > 0$ be a locally Hölder continuous function in $\mathbb{R}$ and $\tilde{K}(t)$ be a locally Hölder continuous function in $[0, \infty)$. Let the average $\overline{K}(r)$ of $K(x)$ in the sense of (7.12) satisfy the same assumptions as in Theorem 5.4. If

$$\begin{cases} u''(r) = \tilde{K}(r) e^{2u(r)} \quad \text{in } (0, \infty), \\ u(0) = \beta, \quad u'(0) = 0 \end{cases}$$  

does not possess any locally bounded solution in $[0, \infty)$ for any real number $\beta$, then equation (1.2) does not possess any locally bounded solution in $\mathbb{R}$.

**Proof.** The proof is quite similar to that of Theorem 2.4. Hence we omit it. Q.E.D.

**References**


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