CORRECTION TO "MEROMORPHIC FUNCTIONS THAT SHARE FOUR VALUES"

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It has been brought to my attention by Norbert Steinmetz that the symmetry reasoning that is used in the proof of Theorem 2 in [1] to conclude that \( \beta_1 \) is symmetric in \( a \) and \( b \) in the third line from the bottom on page 564, and also to conclude that (123) holds on page 565, is not valid.

We note that the symmetry reasoning that is used in the sentence in lines 8 and 9 from the bottom on page 564 is valid, because this sentence is meant in the sense of the following sentence: By considering a \( b \)-point that is simple for \( f \) and double for \( g \), a similar argument to the above argument (with (61) instead of (60)) will produce

\[
\alpha \equiv \frac{2bw'}{b-a} + \frac{w''}{w'} + w'.
\]

We also note that in the proof that (113) holds on pages 563–564, we showed that \( C = 1/2 \) is impossible, and the proof that \( C = 2 \) is impossible is completely analogous.

Thus we will now finish the proof of Theorem 2 by starting from line 5 from the bottom on page 564, where we have just completed showing that (113) holds.

We note that our original assumption is that either \( a \) or \( b \) is shared by DM, and that this implies \( w' \neq 0 \) from (82).

Consider the following function:

\[
(A) \quad \mu = \frac{f'(f-g)^2g'}{f(f-a)(f-b)g(g-a)(g-b)}.
\]

It is easy to see that \( \mu \) is an entire function. By making use of partial fractions (e.g., see (49)) and (2), it is easy to deduce that \( m(r,\mu) = S(r,f) \), i.e.

\[
(B) \quad T(r,\mu) = S(r,f).
\]

Now let \( z_0 \) be either an \( a \)-point or a \( b \)-point of order \( k \) for \( f \) and of order \( m \) for \( g \). Then from (82) and (A), we can obtain that

\[
(C) \quad \mu(z_0) = \frac{2km(w'(z_0))^2}{(a-b)^2}.
\]

Now suppose that \( \mu \neq 4(w')^2(a-b)^{-2}, \mu \neq 6(w')^2(a-b)^{-2}, \) and \( \mu \neq 8(w')^2(a-b)^{-2} \). If \( N_\alpha(r,h,a) \) refers only to those \( a \)-points of \( h \) in \( N(r,h,a) \) that have order...
at least five, then from (C), (B), (84), (17), (62), and (2), we can deduce that
\[
\begin{align*}
\overline{N}(r, a) & \leq \overline{N}(r, \mu - 4(w')^2(a - b)^{-2}, 0) + \overline{N}(r, \mu - 6(w')^2(a - b)^{-2}, 0) \\
& \quad + \overline{N}(r, \mu - 8(w')^2(a - b)^{-2}, 0) + \overline{N}(r, f, a) + \overline{N}(r, g, a) + S(r, f) \\
& \leq 3T(r, \mu) + 6T(r, w') + \frac{1}{5} N(r, f, a) + \frac{1}{5} N(r, g, a) + S(r, f) \\
& \leq \frac{2}{5} T(r, f) + S(r, f),
\end{align*}
\]
which contradicts (60). Therefore, either \( \mu \equiv 4(w')^2(a - b)^{-2} \), or \( \mu \equiv 6(w')^2(a - b)^{-2} \), or \( \mu \equiv 8(w')^2(a - b)^{-2} \). Then it follows from (C) that one of the following two cases (D) or (E) must occur:

(D) An a-point or b-point of \( f \) and \( g \) is simple for one of \( f, g \), and double for the other.

(E) An a-point or b-point of \( f \) and \( g \) is either (i) simple for one of \( f, g \), and of order three or four for the other, or (ii) double for both \( f \) and \( g \).

First we suppose that case (D) holds. We now make the following three observations (F), (G), and (J), each of which can be derived in the same manner that (97) (with (98) and (99)) was derived:

If \( z_0 \) is a simple b-point of \( f \) and a double b-point of \( g \), then \( \alpha_1 \) in (88) is analytic at \( z_0 \) and

\[
\alpha_1'(z_0) = \beta_2(z_0)
\]

where \( \beta_2 \) is \( \beta_1 \) in (98) with “a” and “b” interchanged in (98) and (99).

If \( z_0 \) is a simple a-point of \( g \) and a double a-point of \( f \), then \( \alpha_2 \) in (89) is analytic at \( z_0 \) and

\[
\alpha_2'(z_0) = \beta_3(z_0)
\]

where

\[
\begin{align*}
\beta_3 = \frac{w''}{w'} + 4H_1 - \frac{8}{w'} + \left( \frac{a}{b-a} - \frac{3}{2} \right) w'' + \left( \frac{3}{4} + \frac{ab - 2a^2}{(a-b)^2} \right) \left( w' \right)^2 \\
- \frac{3}{4} \left( 2\alpha_2 + \frac{b - 3a}{a-b} \frac{w''}{w'} \right)^2 - \frac{2aw'}{b-a} \frac{\alpha_2}{w'} - \frac{1}{4} \left( \frac{w''}{w'} \right)^2,
\end{align*}
\]

for

\[
\begin{align*}
H_1 = -w'\alpha_2 + \frac{a(w')^2}{a-b} + w'' \quad \text{and} \quad H_2 = \alpha_1 - \frac{1}{2} \frac{w''}{w'} - \frac{1}{2} w'.
\end{align*}
\]

If \( z_0 \) is a simple b-point of \( g \) and a double b-point of \( f \), then \( \alpha_2 \) is analytic at \( z_0 \) and

\[
\alpha_2'(z_0) = \beta_4(z_0)
\]

where \( \beta_4 \) is \( \beta_3 \) in (H) with “a” and “b” interchanged in (H) and (I).

From (98), (99), (F), (H), (I), (J), (84), (94), and (95), we obtain that

\[
\begin{align*}
T(r, \beta_i) = S(r, f) \quad \text{for } i = 1, 2, 3, 4.
\end{align*}
\]
Now suppose that both \( \alpha'_1 \neq \beta_1 \) and \( \alpha'_2 \neq \beta_3 \). Then from (D), (97), (G), (K), (94), and (95), we can deduce that
\[
\overline{N}(r, a) \leq \overline{N}(r, \alpha'_1 - \beta_1, 0) + \overline{N}(r, \alpha'_2 - \beta_3, 0) \leq S(r, f),
\]
which contradicts (60). Thus either \( \alpha'_1 \equiv \beta_1 \) or \( \alpha'_2 \equiv \beta_3 \). A similar argument with (F), (J), and (61) will show that either \( \alpha'_1 \equiv \beta_2 \) or \( \alpha'_2 \equiv \beta_4 \).

Now suppose that \( \beta_1 \equiv \beta_2 \). Since \( \beta_2 \) is \( \beta_1 \) with "a" and "b" interchanged (see (F)), a calculation will show that the identity \( \beta_1 \equiv \beta_2 \) reduces to the identity \( (a+b)w'(\alpha_1 - \alpha_2) \equiv 0 \), which is a contradiction (see pages 564–565). Thus \( \beta_1 \neq \beta_2 \).

Since \( \beta_3 \) is \( \beta_4 \) with "a" and "b" interchanged (see (J)), a similar calculation will show that the identity \( \beta_3 \equiv \beta_4 \) is impossible. Thus \( \beta_3 \neq \beta_4 \).

It therefore follows that exactly one of the following two cases (L) or (M) must occur:

\[(L) \quad \alpha'_1 \equiv \beta_1 \neq \beta_2 \quad \text{and} \quad \alpha'_2 \equiv \beta_4 \neq \beta_3;\]
\[(M) \quad \alpha'_1 \equiv \beta_2 \neq \beta_1 \quad \text{and} \quad \alpha'_2 \equiv \beta_3 \neq \beta_4.\]

Suppose case (L) holds. Consider the following function:
\[
\beta = 2f'' - 3f' - 2g'' + 3g' - 4g' + 4\frac{g'}{f - a} - 4\frac{g'}{f - a} + 4\frac{g'}{f - a}.
\]
We see that \( \beta \) is analytic (i) at poles of \( f \) and \( g \), (ii) at a-points that are simple for \( f \) and double for \( g \), and (iii) at b-points that are double for \( f \) and simple for \( g \). The a-points that are double for \( f \) and simple for \( g \), and the b-points that are simple for \( f \) and double for \( g \), are zeros of \( \alpha'_2 - \beta_3 \) and \( \alpha'_1 - \beta_2 \) respectively, from (G) and (F). Hence from (N), (D), (L), (94), (95), (K), (22), and (2), it follows that
\[
N(r, \beta) \leq \overline{N}(r, \alpha'_2 - \beta_3, 0) + \overline{N}(r, \alpha'_1 - \beta_2, 0) + S(r, f) \leq S(r, f).
\]
Since \( m(r, \beta) = S(r, f) \) from (N) and (2), we have
\[(O) \quad T(r, \beta) = S(r, f).
\]
Now suppose that \( z_0 \) is a simple a-point of \( f \) and a double a-point of \( g \). Then from (N), (24), and (102), we obtain that \( \beta(z_0) = 3aw'(z_0)/(b - a) \). Now if \( \beta \neq 3aw'(b - a)^{-1} \), then from (D), (G), (O), (84), (95), and (K),
\[
\overline{N}(r, a) \leq \overline{N}(r, \beta - 3aw'(b - a)^{-1}, 0) + \overline{N}(r, \alpha'_2 - \beta_3, 0) \leq S(r, f),
\]
which contradicts (60). Thus \( \beta \equiv 3aw'(b - a)^{-1} \). By considering a b-point that is double for \( f \) and simple for \( g \), and also (F) and (61), a similar argument will show that \( \beta \equiv 3bw'(a - b)^{-1} \). Hence \( (a + b)w' \equiv 0 \), which is a contradiction.

Therefore, case (L) cannot hold. A similar argument will show that case (M) cannot hold. It follows that case (D) cannot hold.

Next we assume that case (E) holds. We now make the following three observations (P), (Q), and (R), each of which can be derived in the same manner that (100) was derived:

If \( z_0 \) is a b-point that is simple for \( g \) and is of multiplicity at least three for \( f \), then \( \alpha_2 \) is analytic at \( z_0 \) and
\[(P) \quad \alpha_2(z_0) = w''(z_0)/w'(z_0) + bw'(z_0)/(b - a).\]
If $z_0$ is a $b$-point that is simple for $f$ and is of multiplicity at least three for $g$, then $\alpha_1$ is analytic at $z_0$ and
\[(Q) \quad \alpha_1(z_0) = w''(z_0)/w'(z_0) + aw'(z_0)/(a - b).\]

If $z_0$ is an $a$-point that is simple for $g$ and is of multiplicity at least three for $f$, then $\alpha_2$ is analytic at $z_0$ and
\[(R) \quad \alpha_2(z_0) = w''(z_0)/w'(z_0) + aw'(z_0)/(a - b).\]

Now suppose that $\alpha_1 \neq w''/w' + bw'/(b - a)$ and $\alpha_2 \neq w''/w' + aw'/(a - b)$. Then from (E), (100), (R), (94), (95), (84), and (17), we obtain
\[
\begin{align*}
\overline{N}(r, a) & \leq \overline{N}\left(r, \alpha_1 - \frac{w''}{w'} - \frac{bw'}{b - a}, 0\right) + \overline{N}\left(r, \alpha_2 - \frac{w''}{w'} - \frac{aw'}{a - b}, 0\right) \\
& + S(r, f) \leq S(r, f),
\end{align*}
\]
which contradicts (60). Therefore,
\[(S) \quad \text{either } \alpha_1 \equiv w''/w' + bw'/(b - a) \text{ or } \alpha_2 \equiv w''/w' + aw'/(a - b).\]

By considering (P), (Q), and (61), a similar argument will show that
\[(T) \quad \text{either } \alpha_1 \equiv w''/w' + aw'/(a - b) \text{ or } \alpha_2 \equiv w''/w' + bw'/(b - a).\]

Since $\alpha_1 \neq \alpha_2$ from (113), and $a + b \neq 0$, (T) and (S) give a contradiction. Thus case (E) cannot hold.

Since we have shown that both (D) and (E) cannot hold, this means that our original assumption that either $a$ or $b$ is shared by DM must be false. This proves Theorem 2.

REFERENCES