OPERATOR THEORETICAL REALIZATION
OF SOME GEOMETRIC NOTIONS

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ABSTRACT. This paper studies the realization of certain geometric constructions in Cowen-Douglas operator class. Through this realization, some operator theoretical phenomena are easily seen from the corresponding geometric phenomena. In particular, we use this technique to solve the first-order equivalence problem and introduce a new operation among certain operators.

The nature of Cowen-Douglas theory is to identify operators of a certain type with certain geometric objects.

Based on this idea, we work on certain geometric constructions, holomorphic curves in Gr(n, C^{2n}) (the Grassmannian of n-dim subspaces of C^{2n}) in Part 1 and tensor product of vector bundles in Part 2, and seek their operator theoretical realization.

Our realization of holomorphic curves in Gr(n, C^{2n}) will preserve important relations, and can be informally viewed as the imbedding of holomorphic curves in Gr(n, C^{2n}) into the Cowen-Douglas operator class B_n(\Omega). Using this realization, we solve the first-order equivalence problem by explicitly exhibiting two operators T_1, T_2 \in B_n(D) such that T_1 is not unitarily equivalent to T_2 but T_1 and T_2 have identical curvatures.

The realization of tensor product of vector bundles gives a natural operation among Cowen-Douglas operators. Using this operation, certain operator theoretical phenomena have been clarified naturally. E.g., for certain g \in H^\infty, the corresponding Bergman operator B^*_g is the “square” of the corresponding Toeplitz operator T^*_g.

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PART 1. OPERATOR THEORETICAL REALIZATION OF HOLOMORPHIC CURVES IN Gr(n, C^{2n})

1.1. Introduction. We will state only the main point of Cowen-Douglas theory here, and refer the reader to [C-D, 1] for further details.

If H is a separable Hilbert space, and \Omega is an open connected subset of C, then the operator class \text{B}_n(\Omega) is by definition

\[ \{ T \in \mathcal{L}(H) : \begin{array}{l}
1. \quad \text{range}(T-w) = H, \text{ if } w \in \Omega; \\
2. \quad \dim \ker(T-w) = n, \text{ if } w \in \Omega; \\
3. \quad \bigvee_{w \in \Omega} \ker(T-w) = H \text{ (spanning property)} \} \].

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We call the operators in $B_n(\Omega)$ Cowen-Douglas operators.

The fundamental relation between $T \in B_n(\Omega)$ and the associated $n$-dim holomorphic Hermitian vector bundle over $\Omega$ defined by

$$E_T : \ker(T - w)$$

is the following identification:

**THEOREM A** [C-D, 1]. Two operators $T$ and $\tilde{T}$ in $B_n(\Omega)$ are unitarily equivalent $\iff E_T$ and $E_{\tilde{T}}$ are equivalent as holomorphic Hermitian vector bundles. We write $E_T \cong E_{\tilde{T}}$.

On the other hand, the Calabi Rigidity Theorem gives a perfect identification of a holomorphic curve (with spanning property) $\gamma : \Omega \to \Gr(n, \mathbb{C}^{2n})$ with its pull-back of the universal subbundle (i.e. $\gamma^*(\mathcal{S}(n, \mathbb{C}^{2n}))$).

Our philosophy is they are related through their identified vector bundles.

**DEFINITION 1.1.1.** Let $F = (f_{ij})_{i,j}$ be a $n \times n$ matrix with $H^\infty$ entries and let $g \in H^\infty$; we define the operator $S(g, F)$ by

$$S(g, F) \overset{\text{def}}{=} (T_g^*\otimes I_n \oplus T_g^*\otimes I_n)|_{\text{Graph}(T_g^*)}$$

where $T_g^*$ and $T_g^*\otimes I_n$ are matrix Toeplitz operators acting on row vectors in $H^2 \otimes \mathbb{C}^n$.

It turns out that $S(g, F)$ is the right operator realization of the holomorphic curve, $\text{span}(f^*) : \mathcal{D} \to \Gr(n, \mathbb{C}^{2n})$, where $\mathcal{D}$ is the open unit disk.

The geometric nature of $S(g, F)$ will be discussed in §1.2 and its operator theoretical nature will be discussed in §1.4.

(In this paper, the geometric part of a Cowen-Douglas operator or of a holomorphic curve means the geometric part of its corresponding vector bundle.)

**1.2. Geometric aspects of this realization.** In this section, we will show that for certain $g \in H^\infty$, the associated operator $S(g, F)$ belongs to $B_n(\Omega)$, and that through this realization, i.e. from the pull-back of the universal subbundle by $\text{span}(f^*) : \mathcal{D} \to \Gr(n, \mathbb{C}^{2n})$ to $E_{S(g, F)}$, the important geometric relations are preserved.

In order to do this, we need to recall the definitions of some geometric invariants.

Let $E, \tilde{E}$ denote two $n$-dim Hermitian holomorphic vector bundles over an open connected set $\Omega \subset \mathbb{C}$. Let $D_E$ denote the canonical connection of $E$ and $D^2_E = K_E \, dz \, d\bar{z}$ be its curvature tensor. We sometimes write $K_E$ as $K$ when no confusion arises.

It is well known that $K$ is a $C^\infty$ selfadjoint bundle map of $E$ to $E$.

**DEFINITION 1.2.1.** If $\phi : E \to E$ is a $C^\infty$ bundle map, then define $\phi_x$ and $\phi_{\bar{z}}$ by

$$[D_E, \phi] = D_E \phi - \phi D_E = \phi_x dz + \phi_{\bar{z}} d\bar{z};$$

$[D_E, \phi]$ is a $C^\infty$ bundle map of $E$ to $E \otimes \mathcal{E}'(\Omega)$, where $\mathcal{E}'(\Omega)$ denotes the set of $C^\infty$ 1-forms over $\Omega$.

\(^1\) A holomorphic curve $\gamma$ in $\Gr(n, \mathbb{C}^{2n})$ has the spanning property if $\sum_{z \in \Omega} \gamma(z) = \mathbb{C}^{2n}$. In this paper, we only consider holomorphic curves with this property.
Here $\phi_x$, $\phi_z$ are clearly bundle maps of $E \to E$; they are called the first covariant derivatives for $\phi$.

Taking the first covariant derivatives of $\phi_x$, $\phi_z$ and taking covariant derivatives of their covariant derivatives, etc., we get higher order covariant derivatives of $\phi$.

For details see [C-D, 2].

** Remark.** Relative to any $C^\infty$ frame $S$ of $E$, write $\Gamma(S)\, dz + \tilde{\Gamma}(S)\, d\bar{z}$ for the connection 1-form matrix of $D_E$ and $\phi(S)$ for the matrix representation of $\phi$ relative to $S$. Then

$$\phi_x(S) = [\Gamma(S), \phi(S)] + \partial \phi(S) / \partial z, \quad \phi_z(S) = [\tilde{\Gamma}(S), \phi(S)] + \partial \phi(S) / \partial \bar{z}.$$  

The covariant derivatives of the curvature bundle map $K_E$ give the important geometric invariants of $E$.

** Definition 1.2.2.** Let $E$, $\tilde{E}$ be $n$-dim Hermitian holomorphic vector bundles and let $k$ be a positive integer. We say $E$ is equivalent to order $k$ with $\tilde{E}$, if for each $z \in \Omega$, there is an isometry $\phi_z : E_z \to \tilde{E}_z$ such that $\phi_z \circ \chi = \tilde{\chi} \circ \phi_z$, where $\chi$ is a covariant derivative of $K$ with total order $\leq k$, but bi-order $(p, q) \neq (0, k)$ or $(k, 0)$, and $\tilde{\chi}$ is the corresponding covariant derivative for $\tilde{K}$. (We shall say $\chi$ has total order $\leq k$ and satisfies the bi-order condition.)

For example, $E$ is equivalent to order 1 with $\tilde{E} \Leftrightarrow$ for each $z \in \Omega$, there is an isometry $\phi_z : E_z \to \tilde{E}_z$ such that $\phi_z \circ K = \tilde{K} \circ \phi_z$. (We say $E$ and $\tilde{E}$ have identical curvatures.)

** Theorem B [C-D, 2].** If $\dim E = \dim \tilde{E} = n$, then $E \cong \tilde{E} \Leftrightarrow E$ and $\tilde{E}$ are equivalent to order $n$.

We list two simple facts related to Definition 1.2.2:

1. If $\tilde{\Omega}, \Omega \subset \mathbb{C}$, $g : \tilde{\Omega} \to \Omega$ is an analytic function, then $E_1$ and $E_2$ are equivalent to order $k$, so are $g_*(E_1)$ and $g_*(E_2)$.

2. If $E_1$ and $\tilde{E}_1$, $E_2$ and $\tilde{E}_2$ are both equivalent to order $k$ respectively, so are $E_1 \otimes E_2$ and $\tilde{E}_1 \otimes \tilde{E}_2$.

For an explanation of this, see [L].

If $T_1$ and $T_2$ are in $B_\mu(\Omega)$, the relation of $E_{T_1}$ and $E_{T_2}$ being equivalent to order $k$ is directly reflected in the relation of $T_1$ and $T_2$.

** Theorem C [C-D, 1].** If $T_1, T_2 \in B_\mu(\Omega)$, then $E_{T_1}$ and $E_{T_2}$ are equivalent to order $k \Leftrightarrow T_1|_{\ker(T_w^{k+1})}$ and $T_2|_{\ker(T_w^{k+1})}$ are unitarily equivalent for each $w \in \Omega$.

In this situation, we will say $T_1$ and $T_2$ are equivalent to order $k$.

** Notation.** We will use $\tilde{\Omega}$ to denote the conjugate of a subset $\Omega$ of $\mathbb{C}$ and $\text{bd}(D)$ to denote the boundary of $D$.

The following lemma is a characterization of $T_g^* \in B_1(\Omega)$ for $g \in H^\infty$.

** Lemma 1.2.1.** If $g \in H^\infty$, $\Omega$ connected open in $\mathbb{C}$, then $T_g^* \in B_1(\Omega) \Leftrightarrow$ the map $g : g^{-1}(\tilde{\Omega}) \to \tilde{\Omega}$ is onto and is a conformal equivalence.

** Proof.** Recall

$$(*): \quad (T_g^* - \bar{g}(z)) k_z = 0,$$

where $z \in D$ and $k_z(\zeta) = 1/(1 - z \zeta)$ for $\zeta \in D$. 

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"⇒" The mapping \( g: g^{-1}(\tilde{\Omega}) \to \tilde{\Omega} \) has to be injective, because \( z_1 \neq z_2 \) in \( \mathcal{D} \) implies \( k_{z_1} \) and \( k_{z_2} \) are linearly independent.

The fact that \( g = g^{-1}(\tilde{\Omega}) \to \tilde{\Omega} \) is surjective follows from:

1. \( \tilde{\Omega} \subset \sigma(T_g^*) = \sigma(T_g) = \text{clos}(g(\mathcal{D})) \);
2. \( \tilde{\Omega} \cap \sigma_e(T_g) = \tilde{\Omega} \cap \sigma_e(T_g^*) \) is empty;
3. \( \sigma_e(T_g) \supset \text{bd}(g(\mathcal{D})) \). (See [D].)

"⇐" Step 1. Fix any \( w = g(z_0) \in \tilde{\Omega} \), then \( g(z) - w = (z - z_0)h(z) \).

We claim \( h \) is invertible in \( H^\infty \).

It is trivial to see \( h \in H^\infty \) and \( h \) is nowhere zero in \( \mathcal{D} \).

The invertibility of \( h \) in \( H^\infty \) follows from observing that for any \( z_n \in \mathcal{D} \), with \( g(z_n) \to w \) (assume \( g(z_n) \in \tilde{\Omega} \)), we have \( g^{-1}(g(z_n)) = z_n - z_0 \) (because \( g: g^{-1}(\tilde{\Omega}) \to \tilde{\Omega} \) is a conformal equivalence).

Step 2. Let \( w = g(z_0) \in \tilde{\Omega} \) as above. Notice that \( \ker(T_g - w) = 0 \).

We claim \( \text{range}(T_g - w) = \{ f \in H^2 : f(z_0) = 0 \} \).

This follows from two facts:

1. \( [(T_g - w)(f)](z) = (g(z) - w)f(z) = (z - z_0)h(z)f(z) \);
2. if \( f(z) = (z - z_0)l(z) \) with \( l \in H^2 \), then

\[
[(z - z_0)h(z)](l(z)/h(z)) = [(T_g - w)(l/h)](z).
\]

Thus \( f - (f(z_0)/k_{z_0}(z_0))k_{z_0} \in \text{range}(T_g - w) \). Thus \( T_g - w \) has closed range and using formula (\( \ast \)), we have

\[
\text{span}(k_{z_0}) \oplus \text{range}(T_g - w) = H^2.
\]

So \( \dim(\ker(T_g^* - \tilde{w})) = \dim[\text{coker}(T_g - w)] = 1 \), and \( T_g^* - \tilde{w} \) is Fredholm of index 1.

This shows \( T_g^* \in B_1(\Omega) \).

From now on, \( E_F \) will denote the holomorphic Hermitian vector bundle

\[
\text{span}
\begin{pmatrix}
I \\
F(z)
\end{pmatrix},
\]

where \( F = (f_{ij})_{i,j} \) is an \( n \times n \) matrix of analytic functions on \( \Omega \subset \mathbb{C} ; (I_{F(z)}) \) is always viewed as a collection of column vectors in \( \mathbb{C}^{2n} \).

**THEOREM 1.2.2.** If \( g \in H^\infty \) and \( T_g^* \in B_1(\Omega) \), then \( S(g, F) \in B_n(\Omega) \) for any \( F = \{f_{ij}\}_{i,j} \) with each \( f_{ij} \in H^\infty \). Moreover

1. \( E_{S(g,F)} \cong E_{T_g^*} \otimes E_f \), where \( f(z) = F(g^{-1}(z)) \);
2. \( E_F \) and \( E_G \) are equivalent to order \( k \) if \( S(g,F) \) and \( S(g,G) \) are equivalent to order \( k \) (where \( G \) has entries in \( H^\infty \)).

**PROOF.** It is trivial to see that \( S(g,F) \in B_n(\Omega) \), since \( B_n(\Omega) \) is closed under similarity and \( S(g,F) \sim T_g^*I_n \) (by the graph mapping \( x \mapsto (x,T_g^*x) \)). We go directly to 1 and 2.
1. Observe that \( k_{g^{-1}(z)} \) is a holomorphic frame of \( E_{T_g}(k_w(\zeta) = 1/(1 - \zeta w)) \), that \( (k_{g^{-1}(z)} \otimes I_n) \oplus \mathcal{F}(z)(k_{g^{-1}(z)} \otimes I_n) \) is a holomorphic frame of \( E_S(g,F) \), and that \( (I_{\mathcal{F}(z)}) \) is a holomorphic frame of \( E_T \). Therefore

\[
(k_{g^{-1}(z)} \otimes I_n) \oplus \mathcal{F}(z)(k_{g^{-1}(z)} \otimes I_n) \rightarrow k_{g^{-1}(z)} \otimes (I_{\mathcal{F}(z)})
\]

is the desired holomorphic isometric bundle map. (In the expression above, both sides are thought of as collections of \( n \)-vectors.)

2. Using the fact (2) following Definition 1.2.2, we see \( E_T \) is equivalent to order \( k \) with \( E_g \equiv E_S(g,F) \) and \( E_S(g,G) \) are equivalent to order \( k \), where \( g(z) = G(g^{-1}(z)) \).

Once we prove “\( E_F \) and \( E_G \) are equivalent to order \( k \) \( E_T \) and \( E_g \) are equivalent to order \( k \) for \( g(z) \equiv z \),” then the rest follows directly from fact (1).

NOTE. Over the holomorphic frame \( (F(z)) \), the connection 1-form matrix of \( E_F \) is \( \{(I + F^*(z)F(z))^{-1}F^*(z)F'(z)\} \, dz \) and so the matrix of its curvature bundle map \( K_{E_F} \) is

\[
-(I + F^*(z)F(z))^{-1}(F'(z))^*(I + F(z)F^*(z))^{-1}F'(z)
\]

(see [C-D, 1]).

So by the remark following Definition 1.2.1, over the holomorphic frame \( (F(z)) \) the matrix representations of the covariant derivatives of \( K_{E_F} \) on \( E_F \) are all noncommutative polynomials in \( F^{(i)}(z), F^G(z) \) and \( (I + F^*(z)F(z))^{-1}, (I + F(z)F^*(z))^{-1} \), \( i,j \geq 0 \). Also such polynomials are canonical in the sense that they are independent of the choice of \( F \). So for a covariant derivative of \( E_F \), the conjugate of its matrix (relative to the frame \( (\mathcal{F}) \)) at \( \bar{z} \) is exactly the corresponding one for \( E_T \) (relative to the frame \( (\mathcal{F}) \)) at \( z \).

From Definition 1.2.2, the rest of the proof is quite straightforward. □

Notice that \( E_F \) is the pull-back of the universal bundle under the holomorphic mapping \( z \rightarrow \text{span} \left( F(z) \right) \in G(n, \mathbb{C}^{2n}) \); in view of the Calabi Rigidity Theorem, Theorem 1.2.2 above says the geometry of these realization operators mirrors the geometry of holomorphic curves in \( G(n, \mathbb{C}^{2n}) \).

COROLLARY 1.2.3. If there is a \( z_0 \) in \( D \) (unit disk) with \( F(z_0) = G(z_0) = 0 \), then \( S(g,F) \equiv S(g,G) \) \( \Rightarrow \) there are constant unitary matrices \( V, W \) such that \( VF(z)W \equiv G(z) \) on \( D \).

PROOF. From Theorems A, B and 1.2.2, this corollary really says that \( E_F \equiv E_G \) \( \Rightarrow \) \( \exists \) constant unitary matrices \( V, W \) such that \( VF(z)W \equiv G(z) \).

By the Calabi Rigidity Theorem, \( E_F \equiv E_G \) \( \Rightarrow \) \( \exists \) constant \( 2n \times 2n \) unitary matrix

\[
\mathcal{U} = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix},
\]

where each \( U_j \) is an \( n \times n \) matrix, such that

\[
\begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} \begin{pmatrix} I \\ F(z) \end{pmatrix} = \begin{pmatrix} I \\ G(z) \end{pmatrix} A(z),
\]

where \( A(z) \) is an \( n \times n \) invertible matrix for each \( z \in D \).

“ \( \Rightarrow \)” We have the identity

\[
(I,0)\mathcal{U}^*\mathcal{U} \begin{pmatrix} I \\ F(z) \end{pmatrix} = A^*(z_0)(I,0) \begin{pmatrix} I \\ G(z) \end{pmatrix} A(z),
\]
if \( z \in \mathcal{D} \) which implies \( I \equiv A^*(z_0)A(z) \) and therefore \( A(z) \equiv A(z_0) \) is unitary.

But \( U_1 + U_2 F(z) \equiv A(z_0) \) and \( F(z_0) = 0 \) implies \( A(z_0) = U_1 \) is unitary, hence \( U_2 = U_3 = 0 \) and \( U_4 \) is unitary.

So \( U_4 F(z) \equiv G(z) U_1 \) in \( \mathcal{D} \).

\[
\left( \begin{array}{c c}
I \\
F(z)
\end{array} \right) \equiv \left( \begin{array}{c c}
I \\
G(z)
\end{array} \right) W^* \text{ on } \mathcal{D} \text{ gives } E_F \cong E_G. \quad \square
\]

1.3. The first-order equivalence problem. We seek two operators \( T_1, T_2 \in B_n(\Omega) \) such that \( T_1 \not\sim T_2 \) but \( T_1|_{\ker(T_1 - w)^2} \cong T_2|_{\ker(T_2 - w)^2} \) for each \( w \in \Omega \).

Using Theorem C and Theorem 1.2.2, this problem is reduced to a geometric problem on \( \text{Gr}(n, \mathbb{C}^{2n}) \), namely “Find two holomorphic curves \( f_1, f_2 \) in \( \text{Gr}(n, \mathbb{C}^{2n}) \) such that \( f_1^*(S(n, \mathbb{C}^{2n})) \) and \( f_2^*(S(n, \mathbb{C}^{2n})) \) have the same curvature, but are inequivalent.” Recall first that the Calabi Rigidity Theorem says \( f_1^*(S(n, \mathbb{C}^{2n})) = f_2^*(S(n, \mathbb{C}^{2n})) \) if \( f_1 \) and \( f_2 \) are identical up to a unitary action of \( \mathbb{C}^{2n} \).

Second, fix an orthonormal basis of \( \mathbb{C}^{2n} \), say \( e_1, \ldots, e_{2n} \); then \( (e_1, \ldots, e_{2n}) X, (e_1, \ldots, e_{2n}) Y \rightarrow Y^T X \) is a nondegenerated bilinear form. It is not hard to see that it induces an automorphism of \( \text{Gr}(n, \mathbb{C}^{2n}) \). Call this kind of automorphism a correlation of \( \text{Gr}(n, \mathbb{C}^{2n}) \).

Recall the Plücker imbedding of \( \text{Gr}(n, \mathbb{C}^{2n}) \rightarrow \mathbb{P}(\wedge^n \mathbb{C}^{2n}) \) is the mapping \( \text{span}\{Z_1, \ldots, Z_n\} \rightarrow \text{span}\{Z_1 \wedge Z_2 \wedge \cdots \wedge Z_n\} \).

If \( \mathbb{P}(\wedge^n \mathbb{C}^{2n}) \) carries the Fubini-Study metric, then the canonical Kähler structure of \( \text{Gr}(n, \mathbb{C}^{2n}) \) is induced by this holomorphic imbedding. (See [Chern].)

With this metric on \( \text{Gr}(n, \mathbb{C}^{2n}) \), every correlation of \( \text{Gr}(n, \mathbb{C}^{2n}) \) is an isometry and is in fact the unique nontrivial isometric automorphism of \( \text{Gr}(n, \mathbb{C}^{2n}) \) up to the action of \( U(2n) \) on \( \text{Gr}(n, \mathbb{C}^{2n}) \). (See [Chow], [Cowen].)

Fix a correlation composed with a unitary \( \left( \begin{array}{c c} 0 & I \end{array} \right) \) action:

\[
\phi: \left( \begin{array}{c c}
I \\
F
\end{array} \right) \rightarrow \left( \begin{array}{c c}
-I^T \\
F^T
\end{array} \right) \rightarrow \left( \begin{array}{c c}
0 & I \\
-I^T & I
\end{array} \right) \rightarrow \left( \begin{array}{c c}
-I^T \\
F^T
\end{array} \right),
\]

where \( F \) is an \( n \times n \) matrix, \( I \) is the identity \( n \times n \) matrix and \( F^T \) is the transpose of \( F \). We shall show that for any holomorphic curve \( f: \Omega \rightarrow \text{Gr}(n, \mathbb{C}^{2n}) \), \( f^*(S(n, \mathbb{C}^{2n})) \) and \( (\phi \circ f)^*(S(n, \mathbb{C}^{2n})) \) have the same curvature, but there is an \( f \) such that we cannot get \( \phi \circ f \) by any unitary action on \( f \).

**LEMMA 1.3.1.** The vector bundles \( E_1, E_2 \) are equivalent to order one \( \Leftrightarrow \) for any \( C^\infty \) frame \( S_j \) on \( E_j \) \( (j = 1, 2) \), \( K_1(S_1) \) is similar to \( K_2(S_2) \) pointwise.

**PROOF.** Notice that the matrix representation of curvature changes by similarity under change of frame and the curvature of the canonical connection is selfadjoint.

So \( E_1 \) is equivalent to order one with \( E_2 \Leftrightarrow \) the eigenvalues of \( K_1 \) and \( K_2 \) are the same. \( \square \)

In the following two lemmas, we write \( F = (f_{ij})_{i,j}, \tilde{F} = (\tilde{f}_{ij})_{i,j}, \) where all \( f_{ij}, \tilde{f}_{ij} \) are analytic functions on \( \Omega \subset \mathbb{C} \).

**LEMMA 1.3.2.** \( E_F \) and \( E_{F^T} \) are equivalent to order one.

**PROOF.** Over the holomorphic frame \( \left[ \begin{array}{c c}
F^I(x) \\
F^T(x)
\end{array} \right] \), \( K_E \) has matrix

\[
\left( \begin{array}{c c}
I + F^*F \end{array} \right)^{-1} \left( \begin{array}{c c}
\left( F' \right)^* \\
(I + FF^*)^{-1} F'
\end{array} \right).
\]
Using the elementary fact if $A$, $B$ are invertible matrices, then $AB \sim BA$ and $A \sim A^T$, it is obvious that (**) is similar to

$$-\{F'(I + F^*F)^{-1}(F')^*(I + FF^*)\}^T$$
$$= -(I + (F^T)^*F^T)^{-1}[(F^T)']^*(I + F^T(F^T)^*)^{-1}(F^T)' \quad \square$$

**Lemma 1.3.3.** Fix $z_0 \in \Omega$ and suppose that $F(z_0) = \tilde{F}(z_0) = 0$, and

$$F'(z_0) = \tilde{F}'(z_0) = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \quad \text{with } |\lambda_i| \neq |\lambda_j| \quad \text{(if } i \neq j)\text{.}$$

Then $E_F \cong E_{\tilde{F}}$ implies $|f_{ij}(z)| \equiv |\tilde{f}_{ij}(z)|$ for all $i, j$ and $z \in \Omega$.

**Proof.** By Corollary 1.2.3,

$$E_F \cong E_{\tilde{F}} \iff \exists \text{ constant unitary } n \times n \text{ matrices } V, W \text{ such that } V F(z) W \cong \tilde{F}(z) \text{ on } \Omega$$
$$V F'(z_0) W = \tilde{F}'(z_0)$$
$$\Rightarrow W^* \begin{pmatrix} |\lambda_1|^2 & |\lambda_2|^2 \\ & \ddots \\ & & |\lambda_n|^2 \end{pmatrix} W$$
$$= \begin{pmatrix} |\lambda_1|^2 & |\lambda_2|^2 \\ & \ddots \\ & & |\lambda_n|^2 \end{pmatrix}$$
$$V^* \begin{pmatrix} |\lambda_1|^2 & |\lambda_2|^2 \\ & \ddots \\ & & |\lambda_n|^2 \end{pmatrix} V$$
$$\Rightarrow W, V \text{ are both diagonal.} \quad \square$$

**Corollary 1.3.4.** Take any $F = (f_{ij})_{i,j}$ with

1. $f_{ij} \in H^\infty$ and $f_{ij}(0) = 0$ for all $i, j$;
2. $f''(z_0) = (\ldots, \frac{\partial^2 f_{ij}}{\partial z_i \partial z_j}(z_0))$ for some $i, j$.

Then $E_F \not\cong E_{F^T}$.

We can now summarize the solution of the first-order equivalence problem as follows.
THEOREM 1.3.5. If \( n > 1 \), and \( F \) is as in Corollary 1.3.4, then \( S(z,F) \), \( S(z,F^T) \in B_n(D) \), \( S(z,F) \) and \( S(z,F^T) \) are equivalent to order one, but they are not unitarily equivalent.

1.4. The operator theoretical aspect of this realization. We begin with a powerful theorem of Brown-Douglas-Fillmore [BDF].

THEOREM D [BDF]. Two essentially normal operators \( T_1 \) and \( T_2 \) are unitarily equivalent modulo compact operators \( \Leftrightarrow \sigma_e(T_1) = \sigma_e(T_2) = X \) and \( \text{ind}(T_1 - \lambda) = \text{ind}(T_2 - \lambda) \) whenever \( \lambda \in \mathbb{C} - X \).

Notice that similarity of the operators \( T_1 \) and \( T_2 \) already implies the conditions on the right of Theorem D.

LEMMA 1.4.1. If \( T, S \) are two bounded linear operators on \( H \) such that \( T \) is essentially normal and \( [T, S] = 0 \), then

1. Graph(\( S \)) is invariant under \( T \oplus T^* \);
2. \( T \oplus T|_{\text{Graph}(S)} = T_s \) is unitarily equivalent to a compact perturbation of \( T \).

PROOF. 1 is trivial. For 2, note \( T_s \sim T \) via the map \( \phi: x \rightarrow (x, Sx) \). In view of [BDF], it suffices to show \( [T_s, T_s^*] \) is compact.

Let \( P \) be the orthogonal projection of \( H \oplus H \) onto Graph(\( S \)), then if \( x \in H \),

\[
T_s^*(\phi(x)) = P(T^*x \oplus T^*Sx) = \phi(T^*x) + P(0 \oplus [T^*, S]|x).
\]

Define \( K: \text{Graph}(S) \rightarrow \text{Graph}(S) \) by \( K(\phi(x)) = P(0 \oplus [T^*, S]|x) \).

Note that by Fuglede's theorem in the Calkin algebra (i.e., \( ts = st, t^*t = tt^* \Rightarrow t^*s = st^* \)) \([T^*, S]| \) is compact, thus \( K \) is a compact operator. Then

\[
T_s^*(\phi(x)) = \phi(T^*x) + K(\phi(x)),
\]

and

\[
T_s \circ T_s^*(\phi(x)) = \phi(T \circ T^*x) + T_s \circ K(\phi(x)),
\]

\[
T_s^* \circ T_s(\phi(x)) = T_s^*(\phi(Tx)) = \phi(T^* \circ Tx) + K \circ T_s(\phi(x)).
\]

Thus \([T_s, T_s^*] = \phi \circ ([T, T^*]) \circ \phi^{-1} + [T_s, K] \). \( \square \)

THEOREM 1.4.2. If \( g \in H^\infty \cap QC \), then \( S(g,F) \cong (T_{g \otimes I_n} + K) \sim T_{g \otimes I_n} \), where \( K \) is a compact operator

\[
QC = (H^\infty + C(S')) \cap (H^\infty + C(S')).
\]

PROOF. Since \( g \) is quasi-continuous, \( T_{g \otimes I_n} \) is essentially normal (see [D]); the lemma above can then be applied. \( \square \)

We know very little about the compact operator \( K \) in Theorem 1.4.2. One situation in which we do have some information is that of the following theorem, here stated without proof.

THEOREM 1.4.3. If \( 0 < |a| < 1 \), then \( S(z, (z - a)/(1 - za)) \cong U_+^* + K \), where \( U_+ \) is the unilateral shift on the orthonormal basis \( \{e_n\}_{n=0}^\infty \) and \( K(e_j) = 0, j \geq 2, K(e_0) = a e_0, K(e_1) = b e_0 \) with \( 2|1 + b|^2 + |a|^2 = 1 \).
PART 2. OPERATOR THEORETICAL REALIZATION OF TENSOR PRODUCT OF VECTOR BUNDLES

We will use the definitions and notations introduced in Part 1. Besides, we shall use $\alpha = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(n)})$ to denote an ordered $n$-tuple of vectors in the Hilbert space $H$ (i.e. $\alpha^{(j)} \in H$), and define $\alpha^* \beta = (\langle \beta^{(1)}, \alpha^{(i)} \rangle)_{i,j}$, the $n \times n$ Gramian matrix of $\alpha$ and $\beta$, and $||\alpha||^2 \overset{\text{def}}{=} \text{tr}(\alpha^* \alpha) = \sum_{i=1}^{n} ||\alpha^{(i)}||^2$. We shall say $\alpha \perp \beta$, if $\alpha^* \beta = 0$.

Moreover, let $\alpha_j = (\alpha^{(1)}_j, \alpha^{(2)}_j, \ldots, \alpha^{(n)}_j)$ be a sequence of ordered $n$-tuple vectors, $j = 1, 2, \ldots$. We shall say $\{\alpha^{(k)}_j\}_{j=1}^{\infty}$ is a linearly independent set, if $\{\alpha^{(k)}_j : 1 \leq k \leq n, j = 1, 2, \ldots\}$ is a linearly independent set in $H$. We write

$$\text{span}\{\alpha_j\}_{j=1}^{\infty} \overset{\text{def}}{=} \text{span}\{\alpha^{(k)}_j : 1 \leq k \leq n, j = 1, 2, \ldots\}.$$ 

Notice that if $\gamma_1$ is an ordered $m$-tuple vector and $\gamma_2$ is an ordered $n$-tuple vector, then $\gamma_1 \otimes \gamma_2$ is an ordered $mn$-tuple vector.

2.1. The operator realization of tensor product of vector bundles.

Theorem A in Part 1 says, for $T \in B_n(\Omega)$, $T$ and $E_T$ are identified. Now, it is natural to ask: if $T_1 \in B_m(\Omega)$ and $T_2 \in B_n(\Omega)$, is $E_{T_1} \otimes E_{T_2}$ identified with some operator in $B_{mn}(\Omega)$?

The answer is affirmative, and hence we get a natural operation: $B_m(\Omega) \times B_n(\Omega) \rightarrow B_{mn}(\Omega)$. We shall call this operation the "geometric tensor product."

**DEFINITION 2.1.1.** Let $T_1 \in B_m(\Omega)$, $T_2 \in B_n(\Omega)$, where $T_j$ is defined on $H_j$ (separable Hilbert space), $j = 1, 2$. We define the subspace $H(T_1) \otimes H(T_2)$ of $H_1 \otimes H_2$ by

$$H(T_1) \otimes H(T_2) = \text{span}\{\ker(T_1 - z) \otimes \ker(T_2 - z) : z \in \Omega\}$$

and the operator $T_1 \otimes T_2$ by

$$T_1 \otimes T_2 \overset{\text{def}}{=} (T_1 \otimes I)|_{H(T_1) \otimes H(T_2)}.$$

Observe that $H(T_1) \otimes H(T_2)$ is a common invariant subspace of $T_1 \otimes I$ and $I \otimes T_2$. Thus $T_1 \otimes T_2$ is well defined. Moreover, $\langle T_1 \otimes I \rangle|_{H(T_1) \otimes H(T_2)} = (I \otimes T_2)|_{H(T_1) \otimes H(T_2)}$ and $||T_1 \otimes T_2|| \leq \text{min}(||T_1||, ||T_2||)$.

**LEMMA 2.1.1.** Let $W$ be any open subset of $\Omega$, then

$$H(T_1) \otimes H(T_2) = \text{span}\{\ker(T_1 - z) \otimes \ker(T_2 - z) : z \in W\},$$

where $T_1$, $T_2$ are as in Definition 2.1.1.

**PROOF.** Let $\gamma_j$ be a global holomorphic frame of $E_{T_j}$ (cf. [G]), $j = 1, 2$. By Definition 2.1.1,

$$H(T_1) \otimes H(T_2) = \text{span}\{\gamma_1(z) \otimes \gamma_2(z) : z \in \Omega\}.$$

Also,

$$\text{span}\{\ker(T_1 - z) \otimes \ker(T_2 - z) : z \in W\} = \text{span}\{\gamma_1(z) \otimes \gamma_2(z) : z \in W\}.$$

The lemma then follows from the application of the Identity Theorem in complex analysis. \square
THEOREM 2.1.2. Let \( T_1 \in B_m(\Omega) \) and \( T_2 \in B_n(\Omega) \), then \( T_1 \ast T_2 \in B_mn(\Omega) \) and \( E_{T_1} \ast E_{T_2} = E_{T_1} \otimes E_{T_2} \).

PROOF. Step 1. We claim that for each \( z \in \Omega \),
\[
\ker(T_1 \ast T_2 - z) = \ker(T_1 - z) \otimes \ker(T_2 - z).
\]
Fix \( z \in \Omega \), since \( (T_1 \ast T_2) - z = [(T_1 - z) \otimes I]H(T_1) \ast H(T_2) \), clearly \( \ker((T_1 \ast T_2) - z) \supset \ker(T_1 - z) \otimes \ker(T_2 - z) \).

Conversely, for any \( x \in H_1 \otimes H_2 \) write \( x = x_1 + x_2 + y \), where
\[
x_1 \in \ker(T_1 - z) \otimes \ker(T_2 - z), \quad x_2 \in \ker(T_1 - z) \otimes [\ker(T_2 - z)]^\perp,
\]
\[
y \in [\ker(T_1 - z)]^\perp \otimes H_2.
\]
Notice that since \( (T_1 - z) \otimes I \) and \( I \otimes (T_2 - z) \) are both onto, as linear mappings \( [(T_1 - z) \otimes I]\mid_{\ker(T_1 - z)} \otimes H_2 \) and \( [I \otimes (T_2 - z)]\mid_{H_1 \otimes [\ker(T_2 - z)]^\perp} \) are both invertible.

Now, if \( x \in \ker(T_1 \ast T_2 - z) \), then \( [(T_1 - z) \otimes I]y = [(T_1 \ast T_2) - z]x = 0 \), so \( y = 0 \); moreover \( [I \otimes (T_2 - z)]x_2 = [(T_1 \ast T_2) - z]x = 0 \), so \( x_2 = 0 \). Thus \( x = x_1 \in \ker(T_1 - z) \otimes \ker(T_2 - z) \).

Step 2. We claim \( T_1 \ast T_2 - z \) is onto for each \( z \in \Omega \). (Notice that \( T_1 \ast T_2 \in B_mn(\Omega) \) will then follow from Step 1 and Step 2.)

Since \( T_1 \ast T_2 - z = (T_1 - z) \ast (T_2 - z) \) (by Definition 2.1.1), it will be enough if we assume \( 0 \in \Omega \) and get a right inverse of \( T_1 \ast T_2 \).

Let \( S_j = T_j^*(T_j T_j^*)^{-1}, \ j = 1,2 \). It is easy to see (1) \( T_j S_j = \text{id} \) for \( j = 1,2 \); (2) if \( e_j \) is an orthonormal basis of \( \ker T_j \) (\( j = 1,2 \)), then \( (1 - z S_j)^{-1} e_j \) is a local holomorphic frame of \( E_{T_j} \) near \( 0 \) (\( j = 1,2 \)) and \( e_j \perp S_k e_j \) for all \( k \geq 1 \).

Let \( \{|z| < r\} \subset \Omega \) such that the two series \( \sum_{k=0}^{\infty} z^k (S_j^k e_j), \ j = 1,2, \) both converge in this disk. Then by Lemma 2.1.1,
\[
\text{span}_{0 < |z| < r} \left\{ \left[ \sum_{k=0}^{\infty} z^k (S_1^k e_1) \right] \otimes \left[ \sum_{k=0}^{\infty} z^k (S_2^k e_2) \right] \right\} = H(T_1) \ast H(T_2).
\]

If \( P \) is the orthogonal projection of \( H_1 \) onto \( \ker T_1 \), then the following computation shows \( H(T_1) \ast H(T_2) \) is invariant under \( (S_1 \otimes I) + (P \otimes S_2) \):
\[
[(S_1 \otimes I) + (P \otimes S_2)] \left\{ \left( \sum_{k=0}^{\infty} z^k S_1^k e_1 \right) \otimes \left( \sum_{k=0}^{\infty} z^k S_2^k e_2 \right) \right\}
\]
\[
= \sum_{k=0}^{\infty} z^k S_1^k S_1^{k+1} e_1 \otimes \sum_{k=0}^{\infty} z^k S_2^k e_2 + e_1 \otimes \sum_{k=0}^{\infty} z^k S_2^k e_2^k e_2
\]
\[
= \frac{1}{z} \left\{ \left[ \sum_{k=1}^{\infty} z^k S_1^k e_1 \right] \otimes \left[ \sum_{k=0}^{\infty} z^k S_2^k e_2 \right] \right\} + \frac{1}{z} \left\{ e_1 \otimes \left[ \sum_{k=1}^{\infty} z^k S_2^k e_2 \right] \right\}
\]
\[
= \frac{1}{z} \left\{ \left[ \sum_{k=0}^{\infty} z^k S_1^k e_1 \right] \otimes \left[ \sum_{k=0}^{\infty} z^k S_2^k e_2 \right] \right\} - \frac{1}{z} \left[ e_1 \otimes e_2 \right],
\]
for all \( 0 < |z| < r \).

Now, it becomes clear that \( (S_1 \otimes I) + (P \otimes S_2) \) is a right inverse of \( T_1 \ast T_2 \).

Finally, \( T_1 \ast T_2 \) is identified with \( E_{T_1} \otimes E_{T_2} \), i.e. \( E_{T_1} \ast E_{T_2} = E_{T_1} \otimes E_{T_2} \) is a trivial consequence of Step 1. \( \square \)
2.2. Partial transformation induced by geometric tensor product. Fix $S \in B_1(\Omega_1)$ and consider the transformation

$$T \rightarrow S * T$$

($B_n(\Omega_2)$ to $B_n(\Omega_1 \cap \Omega_2)$). We shall prove that if $S$ is almost the backward unilateral shift then $S * T \sim S \otimes i d_n$.

The philosophy of this proof is that “we can read $T \in B_n(\Omega)$ in a different way by reading $E_T$ in a different way.”

**Definition 2.2.1.** Let $T \in B_n(\Omega)$, where $\Omega$ is a component of $C - \sigma_e(T)$. A holomorphic frame $\gamma$ of $E_T$ (defined near $z_0 \in \Omega$) will be said to be normalized at $z_0$ if $\gamma^*(z_0)\gamma(z) = I_n$ wherever $\gamma$ is defined.

Notice that if $\gamma$ is as in the definition above, then $\gamma(z_0) \perp (\partial^k_\gamma/\partial z^k)(z_0)$ for all $k \geq 1$.

**Definition 2.2.2.** If $T \in B_n(\Omega)$ and $\Omega$ is a component of $C - \sigma_e(T)$, then for each $z_0 \in \Omega$, we define the number $R(T, z_0)$ by

$$\max\{r \leq d(z_0, \sigma_e(T)) : \text{for each } |z - z_0| < r, \text{ there is no nontrivial subspace of ker}(T - z) \text{ perpendicular to ker}(T - z_0)\}.$$

(Notice: there is no nontrivial subspace of ker$(T - z)$ perpendicular to ker$(T - z_0) \iff$ for any basis $\gamma_z$ of ker$(T - z)$ and basis $\gamma_{z_0}$ of ker$(T - z_0)$, we have det$[\gamma_z^* \gamma_{z_0}] \neq 0$.)

It is easy to see that $R(T, z_0)$ is invariant under unitary equivalence of $T$ and $R(T, z_0) > 0$ always.

**Lemma 2.2.1.** Let $T \in B_n(\Omega)$ and assume $\Omega$ is a component of $C - \sigma_e(T)$. Fix $z_0 \in \Omega$. Then there is a holomorphic frame defined on the entire disk $\{z : |z - z_0| < R(T, z_0)\}$ which is normalized at $z_0$.

**Proof.** Let $\tilde{\gamma}$ be a global holomorphic frame of $E_T$ on $\Omega$; then

$$\gamma(z) = \tilde{\gamma}(z)[\tilde{\gamma}^*(z_0)\tilde{\gamma}(z)]^{-1} [\tilde{\gamma}^*(z_0)\tilde{\gamma}(z)]^{1/2}$$

works. □

**Lemma 2.2.2.** Let $0 \in \Omega$, $T \in B_n(\Omega)$ and $0 < r < R(T, 0)$. Let $\gamma$ be the holomorphic frame of $E_T$ as defined in the lemma above. Then there is an orthonormal basis $\{e_j^{(k)} : j \geq 0, 1 \leq k \leq n\}$ (where $\gamma(z) = e_0 + \sum_{j=1}^{\infty} e_j B_j(z)$, where the $B_j(z)$'s are $n \times n$ matrices of analytic functions on $|z| < R(T, 0)$). Moreover, if $\|B_j\|_r^2 \overset{\text{def}}{=} \sup_{|z| \leq r} \|T^r(B_j^*(z)B_j(z))\|$, then

$$\sum_{j=1}^{\infty} \|B_j\|_r^2 < +\infty.$$

**Proof.** Notice that $\{(\partial^{k}_\gamma/\partial z^k)(0)\}_{k=0}^{\infty}$ is a linearly independent set spanning $H$ (see [C-D, §1]). Also $\gamma(0) \perp (\partial^{k}_\gamma/\partial z^k)(0)$ for all $k \geq 1$ and $\gamma(0)$ is an orthonormal set.

We shall show that Gram-Schmidt orthonormalization $\{e_j\}_{j=0}^{\infty}$ of the set $\{(\partial^{k}_\gamma/\partial z^k)(0)\}_{k=0}^{\infty}$ is a right choice of our orthonormal basis.

Let $H_j = \text{span}\{(\partial^{k}_\gamma/\partial z^k)(0) : 0 \leq k \leq j\}$ for each $j \geq 0$.

Let $e_0 = \gamma(0)$ and, for each $j \geq 1$, let $e_j = (e_j^{(1)}, \ldots, e_j^{(n)})$ be an orthonormal basis of $H_j \ominus H_{j-1}$.
Thus \( \{ e_j^{(l)} : 1 \leq l \leq n, j = 0, 1, 2, \ldots \} \) is an orthonormal basis of \( H \) and for each \( k \geq 1 \) we have
\[
\frac{1}{k!} \left( \frac{\partial^k}{\partial z^k} \gamma \right)(0) = \sum_{i=1}^k e_i A_{ik},
\]
where \( A_{ik} \) is an \( n \times n \) constant matrix.

Now \( \gamma(z) = \sum_{j=1}^\infty (\sum_{i=1}^j e_i A_{ij}) z^j + e_0 \) on \( |z| < R(T,0) \) and \( \sum_{j=1}^\infty A_{ij} z^j = e_i^* \gamma(z) \) is convergent in \( M(n, \mathbb{C}) \).

Let \( \sum_{j=1}^\infty A_{ij} z^j = z^i \tilde{A}_i(z) = B_i(z) \). Then
\[
\gamma(z) = e_0 + \sum_{i=1}^\infty e_i B_i(z) \quad \text{for } |z| < R(T,0).
\]

For \( A \in M(n, \mathbb{C}) \), write \( \|A\|^2 = \text{tr}(A^* A) \).

Assume \( 0 < r < \delta < R(T,0) \) and \( \|\gamma(z)\| = \sum_{j=0}^\infty \|z^j \tilde{A}_i(z)\|^2 \leq M \) for \( |z| \leq \delta \).

By the maximum principle, on \( |z| \leq \delta \), we have
\[
\|\delta^i \tilde{A}_i(z)\|^2 \leq \sup_{|\xi| = \delta} \|\xi^i \tilde{A}_i(\xi)\|^2 \leq \sup_{|\xi| = \delta} \|\gamma(\xi)\|^2 \leq M.
\]

Thus
\[
\sum_{i=1}^\infty \|B_i\|^2 = \sum_{i=1}^\infty \sup_{|z| \leq r} \|z^i \tilde{A}_i(z)\|^2 \leq \sum_{i=1}^\infty \sup_{|z| \leq \delta} \|z^i \tilde{A}_i(z)\|^2 \left( \frac{r}{\delta} \right)^{2i} \\
\leq M \sum_{i=1}^\infty \left( \frac{r}{\delta} \right)^{2i} < +\infty.
\]

Our next lemma is a generalization of Lemma 1.1.1.

If \( A \in H^\infty \otimes M(n,C) \), let \( ||A||_2^2 \) \( \text{def} = \text{tr}(A^* A) \). It is easy to see \( ||T_A|| < ||A||_\infty \), where \( T_A \) is the matrix analytic Toeplitz operator acting on \( H^2 \otimes \mathbb{C}^n = \hat{H} \) (as row vectors).

Now, if \( \{F_k\}_{k=1}^\infty \) is a sequence of \( n \times n \) matrices, each of them having all entries in \( H^\infty \), with \( \sum_{k=1}^\infty \|F_k\|^2 < \infty \), then \( \phi: \hat{H} \rightarrow \hat{H} \oplus \hat{H} \oplus \cdots \) defined by
\[
f \rightarrow f \oplus T_F^* f \oplus T_{F_2}^* f \oplus \cdots
\]
is clearly a bounded linear and bounded below mapping; denote its range by \( R \).

**Lemma 2.2.3.** If \( g \in H^\infty \cap QC \), let \( S_R(g) \) denote \( (T_{g \otimes I_n}^* \oplus T_{g \otimes I_n}^* \oplus \cdots)|R \). Then \( S_R(g) \equiv (T_{g \otimes I_n}^* + K) \sim T_{g \otimes I_n}^* \), where \( K \) is a compact operator.

**Proof.** The operator \( S_R(g) \) is similar to \( T_{g \otimes I_n}^* \) via \( \phi \). By [BDF], it suffices to show \( S_R(g) \) is essentially normal. Let \( P \) be the orthogonal projection of \( \hat{H} \oplus \hat{H} \oplus \cdots \) onto \( R \), then
\[
(S_R(g))^* (\phi(f)) = P \{ T_{g \otimes I_n}(f) \oplus (T_{g \otimes I_n} \circ T_{F_1}(f)) \oplus (T_{g \otimes I_n} \circ T_{F_2}(f)) \oplus \cdots \} \\
= \phi(T_{g \otimes I_n}(f)) + P(0 \oplus L_1(f) \oplus L_2(f) \oplus \cdots),
\]
where \( L_j = [T_{g \otimes I_n}, T_{F_j}] \), which is compact.

Notice that \( \|L_j\| \leq 2\|g\|_\infty \circ \|F_j\|_\infty, j \geq 1 \).
Denote the bounded linear mapping \( f \to 0 \oplus L_1(f) \oplus L_2(f) \oplus \cdots \) by \( 0 \oplus L_1 \oplus L_2 \oplus \cdots (\tilde{H} \text{ into } \tilde{H} \oplus \tilde{H} \oplus \cdots) \).

Set \( K_\infty = P \circ (0 \oplus L_1 \oplus L_2 \oplus \cdots) \circ \phi^{-1} \) and set
\[
K_j = P \circ (0 \oplus L_1 \oplus \cdots \oplus L_{j-1} \oplus L_j \oplus 0 \oplus 0 \oplus \cdots) \circ \phi^{-1}.
\]
Then \( K_j \) is compact and \( K_j \to K_\infty \) follows from
\[
\| (K_\infty - K_j) \phi(f) \|^2 \leq \sum_{k=j+1}^\infty \| L_k(f) \|^2 \leq \left\{ 4 \| g \|^2_\infty \left( \sum_{k=j+1}^\infty \| F_k \|^2 \right) \right\} \| f \|^2.
\]
Therefore \( K_\infty \) is compact.

Finally, using \( (S_R(g))^* \phi(f) = \phi(T_g \otimes I_0(f)) + K_\infty(f) \), we can directly check
\[
[S_R(g), (S_R(g))^*] = \phi \circ [T_g \otimes I_0, T_g \otimes I_0] \circ \phi^{-1} + [S_R(g), K_\infty]. \tag*{\Box}
\]

Now, we are ready to prove the main result of this section.

**Theorem 2.2.4.** For any \( T \in B_n(\Omega) \) (where \( \Omega \) is a component of \( C - \sigma_e(T) \)) and \( z_0 \in \Omega \), let \( 0 < r < R(T, z_0) \) and \( g(z) = z_0 + rz \). Then
\[
T_g^* T \cong (T_g^* I_0 + K) \sim T_g^* I_0,
\]
where \( K \) is a compact operator.

**Proof.** Without loss of generality, we assume \( z_0 = 0 \in \Omega \). Let \( \gamma(z), \{ B_j(z) \}_{j=1}^\infty \) be as in Lemma 2.2.2, and write \( B_j(z) = \overline{B_j(z)} \). Then by Lemma 2.2.2, the linear mapping
\[
\phi: f \to f \oplus T_{B_1 \circ g}(f) \oplus T_{B_2 \circ g}(f) \oplus \cdots
\]
is bounded and clearly bounded below. Let \( R_T = \text{range}(\phi) \) and
\[
S_{R_T}(g) = \left( T_{g \otimes I_0} \oplus T_{g \otimes I_0} \oplus \cdots \right)|_{\text{range}(\phi)}.
\]
By Lemma 2.2.3, the theorem will be true once we show \( S_{R_T}(g) \cong T_g^* T \). This last equivalence is achieved by the holomorphic isometric bundle map from \( E_{S_{R_T}(g)} \) to \( E_{T_g^* T} \) (both over the disk \( rD \))
\[
[k_{g(z)} \otimes I_0] \oplus B_1(z) k_{g(z)} \oplus B_2(z) k_{g(z)} \oplus \cdots \to k_{g(z)} \otimes \gamma(z)
\]
where \( \gamma(z) = \overline{g^{-1}(z)} \) and \( k_z(\xi) = 1/(1 - \xi z) \) is the reproducing kernel of the Hardy space. \( \Box \)

**Corollary 2.2.5.** If \( z_0 = 0 \) and \( T \), \( \Omega \), \( g \), \( r \) are all as above, then \( ||T_g^* T|| = ||T_g^*|| = r \).

**Proof.** By Definition 2.2.1, \( ||T_g^* T|| \leq ||T^*|| = r \); also by Weyl’s theorem about the spectrum of a compact perturbation (see [H]),
\[
||T^* g^* I_0 + K|| = ||T^* z^* I_0 + K/r|| \geq r. \tag*{\Box}
\]

**Corollary 2.2.6.** If \( z_0 = 0 \) and \( T \in B_n(\Omega) \), \( R(T, 0) > 1 \), then
\[
T^* z^* T \cong (T^* z^* I_0 + K) \sim T^* z^* I_0,
\]
where \( K \) is a compact operator.
**Definition 2.2.3.** Let \( J \) be the set of Cowen-Douglas operators \( T \) with \( 0 \in C - \sigma_e(T) \) and \( R(T, 0) > 1 \). Then we define the transformation \( \psi : J \rightarrow \bigcup_{n=1}^{\infty} B_n(D) \) by \( \psi(T) = T_e^* \ast T \).

What is this transformation good for? In addition to Corollaries 2.2.5, 2.2.6, we have "\( T_1 \) and \( T_2 \) are equivalent to order \( k \leftrightarrow \psi(T_1) \) and \( \psi(T_2) \) are equivalent to order \( k \) (in particular, \( T_1 \cong T_2 \leftrightarrow \psi(T_1) \cong \psi(T_2) \))", "\( T \) and \( \psi(T) \) have the same reducibility." Some other properties of this transformation were discussed in [L].

Before we studied the transformation, we had the Fourier and Laplace transformations in mind. But it turns out the flavor is quite different. Further modification of the transformation or a new way of studying this transformation is expected.

### 2.3. The square transformation induced by geometric tensor product.

**Definition 2.3.1.** We define the square transformation \( \Gamma : B_1(\Omega) \rightarrow B_1(\Omega) \) by \( \Gamma(T) = T \ast T \).

**Theorem 2.3.1.** If \( T_1, T_2 \in B_1(\Omega) \) and \( T_1 \sim T_2 \), then \( \Gamma(T_1) \sim \Gamma(T_2) \).

**Proof.** Let \( T_2 = Q^{-1} T_1 Q \), where \( Q \) is an invertible operator. Then \( T_2 \otimes I = (Q^{-1} \otimes Q^{-1})(T_1 \otimes I)(Q \otimes Q) \).

Notice that

\[
H(T_2) \ast H(T_2) = (Q^{-1} \otimes Q^{-1})[H(T_1) \ast H(T_1)].
\]

Now it is easy to check that the following diagram commutes:

\[
\begin{array}{ccc}
H(T_2) \ast H(T_2) & \xrightarrow{T_2 \otimes I} & H(T_2) \ast H(T_2) \\
Q^{-1} \otimes Q^{-1} & & Q^{-1} \otimes Q^{-1} \\
H(T_1) \ast H(T_1) & \xrightarrow{T_1 \otimes I} & H(T_1) \ast H(T_1). & \square
\end{array}
\]

Since the curvature of \( T \) is exactly one half of the curvature of \( \Gamma(T) \), so we have

**Theorem 2.3.2.** If \( T_1, T_2 \in B_1(\Omega) \), then \( T_1 \cong T_2 \Leftrightarrow \Gamma(T_1) \cong \Gamma(T_2) \).

**Theorem 2.3.3.** If \( g \in H^\infty \) and \( T_g^* \in B_1(\Omega) \), then \( \Gamma(T_g^*) \cong B_g^* \), the corresponding Toeplitz operator on the Bergman space.

**Proof.** From Lemma 1.2.1, we know \( g : g^{-1}(\bar{\Omega}) \rightarrow \bar{\Omega} \) is a conformal equivalence.

For \( B_g^* \), if we repeat the proof of Lemma 1.2.1 word by word, it is easy to see \( B_g^* \in B_1(\Omega) \).

Let \( \beta_z(\xi) = \pi^{-1}(1 - \xi z)^{-2} \) be the Bergman kernel. Recall \( B_g^*(\beta_z) = g(\bar{z}) \beta_z \), so the mapping

\[
k_{g^{-1}(\bar{z})} \otimes k_{g^{-1}(\bar{z})} \rightarrow \beta_{g^{-1}(\bar{z})} \quad (z \in \Omega)
\]

gives a holomorphic isometric bundle map from \( E_{T_g^*} \) to \( E_{B_g^*} \). \( \square \)

It is well known that for each \( \psi \in C(\text{clos}(D)) \), \( T_{\phi|_{S^1}} \), and \( B_\phi \) are unitarily equivalent up to compact perturbation (see [CO]). We pose a problem here (related to Theorem 2.3.3):

Let \( \psi \in C(S^1) \) with harmonic extension \( \hat{\phi} \) to \( D \). If \( T_\phi \in B_1(\Omega) \), do we have \( \Gamma(T_\phi) \cong B_\phi? \)

We hope this approach can lead to a rediscovery of the fact: \( B_g^* \) is not unitarily equivalent to any Toeplitz operator on Hardy space.
SOME GEOMETRIC NOTIONS

REFERENCES


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