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PARACOMMUTATORS—BOUNDEDNESS AND SCHATTEN-VON NEUMANN PROPERTIES

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ABSTRACT. A very general class of operators, acting on functions in $L^2(\mathbf{R}^d)$, is introduced. The name "paracommutator" has been chosen because of the similarity with the paramultiplication of Bony and also because paracommutators comprise as a special case commutators of Calderón-Zygmund operators, as well as many other interesting examples (Hankel and Toeplitz operators etc.). The main results, extending previous results by Peller and others, express boundedness and Schatten-von Neumann properties of a paracommutator in terms of its symbol.

0. Introduction. In this paper we study the following type of operators (called *paracommutators*):

$$\widehat{T_b f}(\xi) = (2\pi)^{-d} \int_{\mathbf{R}^d} \hat{b}(\xi - \eta) A(\xi, \eta) \hat{f}(\eta)\, d\eta. \tag{0.1}$$

Detailed motivation for this will be given in the next section. Let us however notice right away that paracommutators contain Hankel and Toeplitz operators as special cases, and besides that many other operators as well. The word paracommutator itself is coined by analogy with Bony's notion of paramultiplication (see Bony (**1981**); cf. Strichartz (**1982**) for a more "popular" account). Note also that (0.1) can be rewritten as

$$T_b f(x) = (2\pi)^{-2d} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} e^{ix(\xi+\eta)} \sigma(\xi, \eta) \hat{b}(\xi) \hat{f}(\eta)\, d\xi\, d\eta \tag{0.2}$$

where $\sigma(\xi, \eta) = A(\xi + \eta, \eta)$. Hence these operators are bilinear pseudodifferential operators (see Coifman and Meyer (**1978**)).

Our philosophy, indicated in the notation, is that A is a fixed function on $\mathbf{R}^d \times \mathbf{R}^d$ and that f varies over $L^2(\mathbf{R}^d)$, while the function b (*the symbol*) is more variable than A but not as variable as f. For instance, if $A(\xi, \eta) \equiv 1$, then T_b is just a multiplication operator, $T_b f = bf$. Thus, in the general case, a paracommutator can be viewed as a multiplication operator perturbed by a Schur multiplier A on the Fourier side.

In some connections, when we consider several choices of A simultaneously, the above notation is inadequate and we will then denote the paracommutator by $T_b(A)$. A further extension of the notation is introduced in §5.

We will address ourselves to the following problems: Given A, for which functions b is T_b a bounded operator in $L^2(\mathbf{R}^d)$? When does T_b belong to the Schatten-von Neumann class S_p of compact operators $(1 \leq p < \infty)$? We will also consider: When is T_b compact?

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For Hankel operators, results of this type are associated with the names Nehari, Hartman, Peller, Rochberg, Semmes etc. (see Nikol′skiĭ (**1986**, Appendix 4), Power (**1982**), Sarason (**1978**)). Our results formally contain those of most of our predecessors in this special case.

Recall that S_2 is the class of Hilbert-Schmidt operators and that S_1 is the trace class (= the nuclear operators); we refer e.g. to Simon (**1979**) or McCarthy (**1967**) for the definition and properties of S_p. We let S_∞ denote the class of all bounded linear operators on L^2.

We will not consider S_p, $0 < p < 1$, but refer to Peng (**1986**) and Timotin (**1985**) for similar results in that case.

We now explain the plan of the paper. §1 contains the above and several other examples of paraproducts. The questions above have been studied for most of our examples and one of our objectives is to give a unified treatment of them. In some cases the present proofs indeed turn out to be simpler than the earlier ones.

§§2 and 3 contain various preliminaries. Especially in the latter, we have assembled those—rather dull—results about Schur multipliers which we are going to use in this paper.

The class of all operators of the form (0.1) is presumably too large to be of much interest, hence we will impose various restrictions on A. We will mainly study functions A that are bounded and in a suitable sense vanish on the "diagonal" $\{(\xi, \eta) : \xi = \eta\}$. In §4 we present systematically the various assumptions (labelled A0–A8) on A used in the paper.

The main results are stated in §5 and the reader is urged to familiarize himself with the contents of that section at an early stage, and afterwards perhaps look at §4 too. Under some conditions we prove that T_b is bounded iff $b \in \mathrm{BMO}$, and, provided A also vanishes at a sufficiently high order at the diagonal, that $T_b \in S_p$ iff $b \in B_p^{d/p}$ $(1 \le p < \infty)$. However, if A furthermore vanishes along the "axes" $\{(\xi, 0)\}$ and $\{(0, \eta)\}$, we may instead obtain T_b bounded iff $b \in B_\infty^0$. These results seem to contain all previous "trace ideal" and boundedness criteria. As a contrast we also briefly discuss a case when A does not vanish on the diagonal. In that case T_b is bounded iff $b \in L^\infty$ and is never compact (except when it vanishes). Obviously this may be conceived as a generalization of the standard Toeplitz result (see e.g. Nikol′skiĭ (**1986**, Appendix 4)).

In §6 we apply the theorems to the examples in §1. This gives both old and new results and (which maybe is more important) it illustrates the use of our results and conditions.

§§7–13 contain proofs of the theorems stated in §5 as well as some additional results.

§14 is devoted to some additional examples of paracommutators. These can only partly be treated by the methods in this paper and we show how results in the range $2 < p < \infty$ can be obtained for some of them by an entirely different method.

We will not (except in Remark 7.2) consider the action of paracommutators on spaces other than L^2, although we expect that it is possible to find general results. Such problems for specific examples of paracommutators have been studied by many authors (see for example Calderón (**1965**) (bounded commutators in L^p), Coifman, Rochberg and Weiss (**1976**) (bounded commutators in L^p), Uchiyama

(**1978**) (bounded and compact commutators in L^p), Janson (**1978**) (bounded commutators between L^p-spaces and into Orlicz spaces), Janson, Peetre and Semmes (**1984**) (bounded Hankel operators in various spaces), Peller (**1984**) (nuclear Hankel operators between L^p-spaces)).

Instead of T_b we may study the corresponding sesquilinear form

$$\langle T_b f, g\rangle = (2\pi)^{-d} \iint \hat{b}(\xi - \eta) A(\xi, \eta) \hat{f}(\eta) \overline{\hat{g}(\xi)}\, d\xi\, d\eta \tag{0.3}$$

or the bilinear form

$$(2\pi)^{-d} \iint \hat{b}(\xi + \eta) A(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta)\, d\xi\, d\eta. \tag{0.4}$$

(In (0.4) we have substituted $A(\xi, \eta) \to A(\xi, -\eta)$; hence the diagonal corresponds to $\{(\xi, \eta) : \xi + \eta = 0\}$.) In particular (0.4) suggests that one ultimately should consider analogous multilinear forms (see Peetre (**1985a**)).

Our calculations will be formal and we will ignore the technical problem of assigning a meaning to the integrals (0.1)–(0.4). A careful definition has to impose conditions on f (g) and perhaps on b, and then take an appropriate limit. Note that in all the examples below $A \in C^\infty((\mathbf{R}^d \backslash \{0\}) \times (\mathbf{R}^d \backslash \{0\}))$, hence (0.3) is well defined if $\hat{f}$ and $\hat{g}$ are test functions with compact supports disjoint from the origin and b is any tempered distribution. Furthermore, when our conditions are satisfied, the proofs in §7 that T_b is bounded implicitly yield definitions of the operators using decompositions of $\mathbf{R}^d$.

A preliminary mention of some of the theorems of this paper was given by one of the authors at the NATO Advanced Study Institute on Operators and Function Theory in Lancaster, July 1984 (Peetre (**1985a**)) and then again on the occasion of the XIXth Nordic Congress of Mathematicians in Reykjavik, August 1984.

1. Examples.

1. *Products.* If $A(\xi, \eta) \equiv 1$, then $\widehat{T_b f} = (2\pi)^{-d} \hat{b} * \hat{f}$ and thus $T_b f = bf$. Obviously T_b is bounded on $L^2(\mathbf{R}^d) \Leftrightarrow b \in L^\infty$. Furthermore, T_b is never compact unless $b = 0$.

2. *Toeplitz operators.* Let $H^2 = H^2(\mathbf{R}) = \{f \in L^2(\mathbf{R}) \colon \operatorname{supp} \hat{f} \subset [0, \infty)\}$. Let P denote the orthogonal projection of L^2 onto H^2. The Toeplitz operator with symbol b is the operator $f \to P(bf)$, $f \in H^2$. Obviously, we may just as well study $f \to P(bPf)$ as an operator $L^2 \to L^2$. This is T_b as defined in (0.1) with $d = 1$ and

$$A(\xi, \eta) = I(\xi > 0 \text{ and } \eta > 0). \tag{1.1}$$

(Here and in the sequel, $I(\cdots)$ denotes the indicator function which is 1 when the condition in the parenthesis holds and 0 otherwise.)

In this case, it is well known that T_b is bounded iff $b \in L^\infty$, and that T_b is never compact unless $b = 0$. The theory of Toeplitz operators (usually studied on the unit circle) is well developed (see e.g. Douglas (**1972**), Nikol′skiĭ (**1986**), Sarason (**1978**)).

3. *Hankel operators.* With the same notation as in the preceding example, $\overline{P} = I - P$ is the orthogonal projection onto $\overline{H}^2$, and H_b, the Hankel operator with symbol b, is defined as the operator $f \to \overline{P}(bf)$, $f \in H^2$. This is essentially the same as $f \to \overline{P}(bPf)$, $f \in L^2$, which is the paracommutator T_b with $d = 1$ and

$$A(\xi, \eta) = I(\xi < 0 \text{ and } \eta > 0). \tag{1.2}$$

General references are e.g. Power (**1982**) and Nikol′skiĭ (**1986**).

By Nehari's theorem, H_b is bounded iff $\overline{P}b \in \mathrm{BMO}$. (Note that $H_{Pb} = 0$, thus only the anti-analytic part $\overline{P}b$ is important.) Furthermore, $H_b \in S_p$ iff $\overline{P}b \in B_p^{1/p}$ $(0 < p < \infty)$ (see Peller (**1980**), (**1982**), Coifman and Rochberg (**1980**), Rochberg (**1982**), Semmes (**1984**)).

4. *Commutators of singular integral transforms.* Let K denote a Calderón-Zygmund transform, i.e. the principal value convolution with a kernel that is homogeneous of degree $-d$ and, for simplicity, C^∞ outside the origin (see e.g. Stein (**1970**)). The commutator of K and multiplication by b is

$$[b, K]f = b(Kf) - K(bf).$$

Since $\widehat{Kf}(\xi) = m(\xi)\hat{f}(\xi)$, where the multiplier m is homogeneous of degree 0 and C^∞ outside the origin,

$$\begin{aligned}\widehat{[b,K]f} &= (2\pi)^{-d}(\hat{b} * \widehat{Kf} - m(\hat{b} * \hat{f})) \\ &= (2\pi)^{-d} \int \hat{b}(\xi - \eta)(m(\eta) - m(\xi))\hat{f}(\eta)\, d\eta.\end{aligned}$$

Thus the commutator is T_b with

$$A(\xi, \eta) = m(\eta) - m(\xi). \tag{1.3}$$

When $d = 1$, K is a scalar multiple of the Hilbert transform which has kernel $1/\pi x$ and multiplier $-i \operatorname{sign} \xi$. For commutators with the Hilbert transform we obtain

$$\begin{aligned}A(\xi, \eta) &= i \operatorname{sign} \xi - i \operatorname{sign} \eta \\ &= 2iI(\xi > 0 > \eta) - 2iI(\xi < 0 < \eta).\end{aligned} \tag{1.4}$$

This is almost the same as in Example 3; in particular it follows from (1.4) and (1.2), since $\widehat{\overline{P}b}(\xi) = 0$ for $\xi > 0$, that

$$[\overline{P}b, K] = -2iH_{\overline{P}b} = 2iH_b. \tag{1.5}$$

In fact, the commutator $[b, K]$ decomposes, apart from constant factors, into the two Hankel operators H_{Pb} and $H_{\overline{P}b}$ (see e.g. Rochberg (**1982**) for details).

When $d \geq 2$, the commutator is bounded on L^2 iff $b \in \mathrm{BMO}$ (see Coifman, Rochberg and Weiss (**1976**), Uchiyama (**1978**) and Janson (**1978**)). Janson and Wolff (**1982**) proved that $[b, K] \in S^p$ iff $b \in B_p^{d/p}$, provided $p > d$, while $[b, K]$ never belongs to S^p when $p \leq d$, unless it vanishes.

Note that $[b, K]$ is a (singular) integral operator with kernel $(b(x) - b(y))k(x - y)$, where k is the kernel of K. In particular, if $d = 1$, the kernel is a constant times the difference quotient $(b(x) - b(y))/(x - y)$.

5. *Higher commutators.* We may also study the second commutator $[[b, K], K]$ where K is as in Example (1.4), and more generally, the Nth order commutator $[\dots[b, K], \dots, K]$ $(N \geq 1)$, which is T_b with

$$A(\xi, \eta) = (m(\eta) - m(\xi))^N. \tag{1.6}$$

Furthermore, it is possible to use different Calderón-Zygmund transforms; $[\dots[b, K_1], \dots, K_N]$ is given by (with $m_j = \widehat{K_j}$)

$$A(\xi, \eta) = \prod_1^N (m_j(\eta) - m_j(\xi)). \tag{1.7}$$

Provided a certain nondegeneracy condition is satisfied, T_b is bounded iff $b \in \mathrm{BMO}$, and $T_b \in S_p$ iff $b \in B_p^{d/p}$ and $P > d/N$ $(1 \le p < \infty)$ (see Janson and Peetre (**1984**)).

6. *Higher-dimensional Hankel and Toeplitz operators.* The commutators of Example 4 may be regarded as generalizations of Hankel operators to $\mathbf{R}^d$. Another, more straightforward, generalization is obtained as follows.

Let Γ_1 and Γ_2 be two closed cones in $\mathbf{R}^d$, i.e. $t\Gamma_j = \Gamma_j$ for every $t > 0$, $j = 1, 2$. Let P_j denote the orthogonal projection of L^2 onto the set of functions whose Fourier transforms are supported in Γ_j, and study the operator $f \to P_2(bf)$: $P_1 L^2 \to P_2 L^2$, or equivalently $f \to P_2(bP_1 f)$ on L^2. This is the paracommutator defined by

$$A(\xi, \eta) = I(\xi \in \Gamma_2 \text{ and } \eta \in \Gamma_1). \tag{1.8}$$

We say this is a generalized Hankel operator if $\Gamma_1 \cap \Gamma_2 = \{0\}$, and a generalized Toeplitz operator if $\operatorname{int}(\Gamma_1) \cap \operatorname{int}(\Gamma_2) \neq \varnothing$.

7. *The Calderón commutators.* Calderón (**1965**) studied the commutators

$$[b, K]\frac{d}{dx} \quad \text{and} \quad \left[b, K\frac{d}{dx}\right] \quad \text{on } L^2(\mathbf{R}),$$

where K is the Hilbert transform (see Example 4). (Both operators have been called "the Calderón commutator" in the literature.) The first commutator is the paracommutator given by, cf. (1.4),

$$\begin{aligned} A(\xi, \eta) &= (i \operatorname{sign} \xi - i \operatorname{sign} \eta) i\eta \\ &= 2|\eta|(I(\xi > 0 > \eta) + I(\xi < 0 < \eta)). \end{aligned} \tag{1.9}$$

Obviously, it is bounded on L^2 iff $[b, K]$ maps the Sobolev space H^{-1} into L^2. The second commutator is, since the multiplier corresponding to Kd/dx is $-i \operatorname{sign}(\xi) i\xi = |\xi|$, the paracommutator given by

$$A(\xi, \eta) = |\eta| - |\xi|. \tag{1.10}$$

(This is a singular integral operator with kernel $-\pi^{-1}(b(x) - b(y))/(x - y)^2$.) Hence the difference of the two commutators is T_b with

$$A(\xi, \eta) = (\xi - \eta) \operatorname{sign}(\xi), \tag{1.11}$$

which is seen to equal $f \to K((db/dx)f)$. Thus this difference is bounded on L^2 iff $db/dx \in L^\infty$.

Calderón (**1965**) proved that this condition is sufficient for the two commutators to be bounded. Coifman and Meyer (**1980**) proved that $[b, K]d/dx$ is bounded when $db/dx \in \mathrm{BMO}$.

8. *Commutators with fractional integration or differentiation.* Similarly we can treat commutators between a multiplication and any multiplier transform. For example, if I^s is defined as in (2.6) below, then $[b, I^s]$ is T_b with

$$A(\xi, \eta) = |\eta|^{-s} - |\xi|^{-s}. \tag{1.12}$$

Note that the special case $d = 1$, $s = -1$ gives (1.10), i.e. a Calderón commutator.

Murray (**1985**) has shown that when $d = 1$ and $-1 < s < 0$, $[b, I^s]$ is bounded on L^2 iff $I^s b \in \mathrm{BMO}$. She remarks that this could also be obtained from the theorem by David and Journé (**1984**), which also applies when $d > 1$. The S_p-results that we obtain (in §6) seem to be new.

9. *Paraproducts.* The name "paraproduct" denotes an idea rather than a unique definition; several versions exist and can be used for the same purposes. The name was coined by Bony (**1981**), who used paraproducts as a tool in the study of singularities of solutions of semilinear partial differential equations; the paraproducts help to "linearize" the problem (see e.g. the article by Strichartz (**1982**)).

Earlier examples of paraproducts are studied in Calderón (**1965**) and Coifman and Meyer (**1978**) (see also Peetre (**1976**)).

One version is the paracommutator (0.1) with $A \in C^\infty(\mathbf{R}^{2d}\setminus\{0\})$, A homogeneous of degree 0, $A = 0$ in neighborhoods of the diagonal $\{(\xi,\xi)\}$ and the axis $\{(0,\eta)\}$, and $A = 1$ in a neighborhood of the axis $\{(\xi,0)\}$ (omitting (0,0) each time). For example, we may take

$$A(\xi,\eta) = \varphi\left(\frac{|\eta|}{|\xi|}\right) \tag{1.13}$$

or

$$A(\xi,\eta) = \varphi\left(\frac{|\eta|}{|\xi-\eta|}\right) \tag{1.14}$$

where $\varphi \in C^\infty(0,\infty)$, $\varphi = 1$ on $(0,\delta)$ and $\varphi = 0$ on $(1-\delta,\infty)$ for some $\delta > 0$, or

$$A(\xi,\eta) = \varphi\left(\frac{|\xi-\eta|}{|\xi|}\right) \tag{1.15}$$

where $\varphi \in C_0^\infty(0,\infty)$ and $\varphi = 1$ on $(1-\delta, 1+\delta)$.

Coifman and Meyer (**1978**) studied e.g. operators of the type

$$T_b f = \int_0^\infty (\psi_t * b)(\varphi_t * f)\, dt/t \tag{1.16}$$

where $\hat\varphi, \hat\psi \in \mathcal{S}(\mathbf{R}^d)$, $\hat\psi(0) = 0$ and $\hat\varphi_t(\xi) = \hat\varphi(t\xi)$, $\hat\psi_t(\xi) = \hat\psi(t\xi)$.

A simple calculation shows that this is the paracommutator given by

$$A(\xi,\eta) = \int_0^\infty \hat\psi(t(\xi-\eta))\hat\varphi(t\eta)\, dt/t. \tag{1.17}$$

If $\hat\varphi$ and $\hat\psi$ are radial and have compact supports with $\operatorname{supp}\hat\varphi \subset \{\xi\colon |\xi| < 1\}$ and $\operatorname{supp}\hat\psi \subset \{\xi\colon |\xi| > 1\}$, $\hat\varphi = 1$ in a neighborhood of 0, and $\int_0^\infty \hat\psi(t\xi)\, dt/t = 1$, then (1.17) yields a kernel of the type (1.14), i.e. (1.16) defines a paraproduct of the above type.

A related version of the paraproduct is

$$T_b f = \sum_{j,k\,:\,k<j-N} b_j f_k, \tag{1.18}$$

where $b = \sum_{-\infty}^\infty b_j$ and $f = \sum_{-\infty}^\infty f_k$ are dyadic decompositions as in (2.5) below and N is a sufficiently large positive integer. This is the paracommutator with

$$A(\xi,\eta) = \sum_{k<j-N} \hat\psi_j(\xi-\eta)\hat\psi_k(\eta), \tag{1.19}$$

where ψ_k is as in (2.2). This A is not truly homogeneous, but it satisfies $A(2\xi, 2\eta) = A(\xi,\eta)$.

Coifman and Meyer (**1978**, Theorem 33, p. 144) proved that T_b (defined by (1.16)) is bounded if $b \in \mathrm{BMO}$. Peng (**1984**) proved the converse and that $T_b \in S_p$ iff $b \in B_p^{d/p}$ $(1 < p < \infty)$.

10. *A smooth.* Suppose that $A \in C^\infty(\mathbf{R}^{2d}\backslash\{0\})$, and that, for each multi-index α and some constants C_α,

$$|D^\alpha A(\xi,\eta)| \le C_\alpha(|\xi|+|\eta|)^{-|\alpha|}. \tag{1.20}$$

The paraproducts in the preceding example are included in this. In fact, it is easy to see that (1.20) holds for every A that is C^∞ outside the origin and satisfies $A(r\xi, r\eta) = A(\xi,\eta)$ for some fixed $r > 1$ (in particular, if A is homogeneous of degree 0), a case studied by Timotin (**1984**). Coifman and Meyer (**1978**, Proposition 2, p. 154) proved that if A satisfies (1.20), then T_b is bounded for $b \in L^\infty$, and if furthermore $A(\xi,\xi) = 0$, then T_b is bounded for $b \in \mathrm{BMO}$.

We obtain converses and additional results in §6.

2. Preliminaries and some notation. C, and sometimes C_1 etc., denote positive constants, changing from formula to formula.

A dyadic decomposition of $\mathbf{R}^d\backslash\{0\}$ will be important, and for $k \in \mathbf{Z}$ we define

$$\begin{aligned} \Delta_k &= \{\xi \in \mathbf{R}^d\colon\ 2^k \le |\xi| < 2^{k+1}\},\\ \tilde\Delta_k &= \{\xi\colon\ 2^{k-1} \le |\xi| < 2^{k+2}\} = \Delta_{k-1}\cup\Delta_k\cup\Delta_{k+1}. \end{aligned} \tag{2.1}$$

The Besov space can be defined as follows; we refer to Peetre (**1976**) or Bergh and Löfström (**1976**) for further details. (We consider only the homogeneous Besov spaces.) Let $\hat\psi$ be a test function with support in some "annulus" $\{\xi\colon\ r < |\xi| < \mathbf{R}\}$ such that $\inf\{|\hat\psi(\xi)|\colon\ \xi \in \Delta_0\} > 0$, and define ψ_k by

$$\hat\psi_k(\xi) = \hat\psi(2^{-k}\xi), \qquad k \in \mathbf{Z}. \tag{2.2}$$

(Thus $\hat\psi_k$ is nonzero on Δ_k.) The Besov space B_p^{sq} $(-\infty < s < \infty,\ 1 \le p \le \infty,\ 1 \le q \le \infty)$ is defined as $\{b \in S'\colon\ \{2^{ks}\|\psi_k * b\|_{L^p}\}_{-\infty}^{\infty} \in l^q\}$. The norm in B_p^{sq} is defined in the natural way. Different choices of ψ give the same space and equivalent norms.

We will only be interested in the "diagonal" case $q = p$, and we write B_p^s for B_p^{sp}. Thus

$$b \in B_p^s \Leftrightarrow \{2^{ks}\|\psi_k * b\|_{L^p}\} \in l^p. \tag{2.3}$$

$B_\infty^0 = \{b\colon \sup_k \|\psi_k * b\|_{L^\infty} < \infty\}$ is the real variable version of the Bloch space.

It is sometimes convenient to add the additional requirement

$$\sum_{-\infty}^{\infty} \psi_k(\xi) = 1, \qquad \xi \ne 0. \tag{2.4}$$

A distribution b then has the dyadic decomposition (modulo polynomials)

$$b = \sum_{-\infty}^{\infty} b_k, \quad \text{with } b_k = \psi_k * b. \tag{2.5}$$

Definitions of the space BMO can be found a.e. in the literature. Note that $L^\infty \subset \mathrm{BMO} \subset B_\infty^0$ with strict inclusions.

Fractional integration (and differentiation) is defined by

$$(I^s f)^\wedge(\xi) = |\xi|^{-s}\hat{f}(\xi), \qquad -\infty < s < \infty. \tag{2.6}$$

I^s is an isomorphism of B_p^t onto B_p^{t+s} for every s, t and p.

Let U and V be two subsets of $\mathbf{R}^d$. We write $k \in S_p(U \times V)$ if k is a function (distribution) on $U \times V$ such that the linear operator $f \to \int_V k(\xi,\eta)f(\eta)$ defines an operator $L^2(V) \to L^2(U)$ that belongs to S_p, and we denote the S_p-norm of that operator by $\|k\|_{S_p(U\times V)}$. In particular, $\|k\|_{S_\infty}$ is the operator norm. (We omit U and V from the notation when no confusion may arise.)

The problems posed in the introduction may be reformulated (using the Plancherel theorem): When does $\hat{b}(\xi-\eta)A(\xi,\eta) \in S_p(\mathbf{R}^d \times \mathbf{R}^d)$?

For future reference, we state the simple result for Example 1.

LEMMA 2.1. $(2\pi)^{-d}\|\hat{b}(\xi-\eta)\|_{S_\infty(\mathbf{R}^d\times\mathbf{R}^d)} = \|b\|_{L^\infty(\mathbf{R}^d)}$. □

It is obvious that $\|k\|_{S_p(U_1\times V_1)} \le \|k\|_{S_p(U\times V)}$ whenever $U_1 \subset U$, $V_1 \subset V$. For $p=\infty$, we have the following elementary converse results.

LEMMA 2.2. *If $\{U_n\}_1^\infty$ and $\{V_n\}_1^\infty$ are partitions of U and V respectively, then*

$$\left\| k(\xi,\eta)\sum_1^\infty \chi_{U_n}(\xi)\chi_{V_n}(\eta)\right\|_{S_\infty(U\times V)} = \sup_n \|k\|_{S_\infty(U_n\times V_n)}. \quad \square$$

In the next lemma G is an arbitrary group (written additively). We will later use the cases $G = \mathbf{Z}$ and $G = \mathbf{Z}^d$.

LEMMA 2.3. *Suppose that $\{U_n\}_{n\in G}$ and $\{V_n\}_{n\in G}$ are partitions of U and V respectively. Suppose that $\|k\|_{S_\infty(U_m\times V_n)} \le a(m-n)$ for some $a \in l^1(G)$. Then $\|k\|_{S_\infty(U\times V)} \le \sum_n a(n)$.*

PROOF (assuming for simplicity that G is countable). By Lemma 2.2,

$$\|k\|_{S_\infty} \le \sum_m \left\| k(\xi,\eta)\sum_n \chi_{U_{n+m}}(\xi)\chi_{V_n}(\eta)\right\|_{S_\infty} \le \sum_m a(m). \quad \square$$

3. Schur multipliers. The following subalgebra of L^∞ plays an important role in the theorems and proofs in this paper.

DEFINITION. $M(U\times V)$ denotes the set of all $\varphi \in L^\infty(U\times V)$ that admit the representation

$$\varphi(\xi,\eta) = \int_X \alpha(\xi,x)\beta(\eta,x)\,d\mu(x) \tag{3.1}$$

for some σ-finite measure space (X,μ) and measurable functions α on $U\times X$ and β on $V\times X$ with

$$\int \|\alpha(\cdot,x)\|_{L^\infty(U)}\|\beta(\cdot,x)\|_{L^\infty(V)}\,d\mu(x) < \infty. \tag{3.2}$$

It is easily seen that we may restrict μ to finite measures, or let μ be a complex measure (taking $d|\mu|$ in (3.2)), and that we may demand $\alpha \in L^\infty(U\times X)$, $\beta \in L^\infty(V\times X)$ and replace (3.2) by

$$\|\alpha\|_{L^\infty(U\times X)}\|\beta\|_{L^\infty(V\times X)}\|\mu\| < \infty. \tag{3.3}$$

$M(U\times V)$ is a Banach algebra with the norm given by the minimum of the left-hand side of (3.2) (or (3.3)) over all representations. (The product property follows by

$$\int \alpha_1(\xi, x_1)\beta_1(\eta, x_1)\, d\mu_1(x_1) \int \alpha_2(\xi, x_2)\beta_2(\eta, x_2)\, d\mu_2(x_2)$$
$$= \int \alpha(\xi, x)\beta(\eta, x)\, d(\mu_1 \times \mu_2)(x)$$

with $x = (x_1, x_2)$, $\alpha(\xi, x) = \alpha_1(\xi, x_1)\alpha_2(\xi, x_2)$, $\beta(\eta, x) = \beta_1(\eta, x_1)\beta_2(\eta, x_2)$.)

REMARK 3.1. $M(U \times V)$ contains the tensor product $L^\infty(U) \hat{\otimes} L^\infty(V)$ (which is obtained if μ above is restricted to discrete measures), but M is, except in trivial cases, strictly larger. A simple example in the discrete case is given by $\{\delta_{mn}\}$ which belongs to $M(\mathbf{Z} \times \mathbf{Z})$ by the argument in Lemma 3.3 below, but not to $l^\infty \hat{\otimes} l^\infty$ because the corresponding operator $l^1 \to l^\infty$, viz. the identity mapping, is not compact.

Peller (**1985**) has shown that $M(U \times V)$ is the algebra of bounded Schur multipliers, i.e.

$$\varphi \in M(U \times V) \Leftrightarrow \|\varphi k\|_{S_1(U\times V)} \le C\|k\|_{S_1(U\times V)}$$
$$\Leftrightarrow \|\varphi k\|_{S_\infty(U\times V)} \le C\|k\|_{S_\infty(U\times V)}.$$

(The discrete case is studied by Bennett (**1977**) and Haagerup (personal communication).)

We will only need the easy part:

LEMMA 3.1. *If $\varphi \in M(U \times V)$ and $k \in S_p(U \times V)$, then*

$$\|\varphi k\|_{S_p} \le \|\varphi\|_M \|k\|_{S_p}, \qquad 1 \le p \le \infty.$$

PROOF. Fix $x \in X$, let K: $L^2(V) \to L^2(U)$ denote the operator corresponding to k and let M_α, M_β denote the bounded operators $f(\xi) \to \alpha(\xi, x)f(\xi)$, $g(\eta) \to \beta(\eta, x)g(\eta)$ on $L^2(U)$ and $L^2(V)$, respectively. Then

$$\|\alpha(\xi, x)\beta(\eta, x)k(\xi, \eta)\|_{S_p(U\times V)} = \|M_\alpha K M_\beta\|_{S_p} \le \|M_\alpha\|\,\|K\|_{S_p}\|M_\beta\|$$
$$= \|\alpha(\cdot, x)\|_{L^\infty(U)}\|k\|_{S_p(U\times V)}\|\beta(\cdot, x)\|_{L^\infty(V)}.$$

Now integrate over X, using (3.1). □

Alternatively we could use interpolation between $p = 1$ and $p = \infty$.

LEMMA 3.2. *If $k \in S_\infty(U \times V)$ defines a compact operator on L^2 and $\varphi \in M(U \times V)$, then φk defines a compact operator.*

PROOF. There exist $k_n \in S_2(U \times V)$ with $k_n \to k$ in S_∞ as $n \to \infty$. Consequently, $\varphi k_n \in S_2$ and $\varphi k_n \to \varphi k$ in S_∞. □

The remainder of this section is devoted to methods of showing that certain functions belong to M. Note first that (by taking μ as a point mass in (3.1))

$$\|f(\xi)g(\eta)\|_{M(U\times V)} = \|f\|_{L^\infty(U)}\|g\|_{L^\infty(V)}, \tag{3.4}$$

in particular, $g \equiv 1$ yields

$$\|f(\xi)\|_{M(U\times V)} = \|f\|_{L^\infty(U)}. \tag{3.5}$$

The Fourier representation is often an efficient tool to show that functions belong to M.

LEMMA 3.3. *If* $b \in L^1(\mathbf{R}^d)$ *then*

$$\hat{b}(\xi-\eta) \in M(\mathbf{R}^d \times \mathbf{R}^d) \quad \text{and} \quad \|\hat{b}(\xi-\eta)\|_M \leq \|b\|_{L^1}.$$

PROOF. $\hat{b}(\xi-\eta) = \int e^{-i\xi\cdot x} e^{i\eta\cdot x} b(x)\,dx.$ □

The same argument applies to other groups as well; in the next lemma we implicitly use it for the multiplicative group $\mathbf{R}_+$.

LEMMA 3.4. *If* $\varphi \in C_0^\infty(0,\infty)$, *then* $\varphi(|\eta|/|\xi|) \in M(\mathbf{R}^d \times \mathbf{R}^d)$.

PROOF. Let $\psi(t) = \varphi(e^t)$. Then $\psi \in C_0^\infty(\mathbf{R})$ and

$$\varphi\left(\frac{|\eta|}{|\xi|}\right) = \psi(\log|\eta| - \log|\xi|) = \frac{1}{2\pi}\int e^{it(\log|\eta|-\log|\xi|)}\hat{\psi}(t)\,dt$$
$$= \frac{1}{2\pi}\int |\xi|^{-it}|\eta|^{it}\hat{\psi}(t)\,dt. \quad \square$$

LEMMA 3.5. *If* $s \geq 0$ *and* $\delta > 0$, *then there exists* F *with* $\|F\|_{M(\mathbf{R}^d\times\mathbf{R}^d)} = \delta^s$ *such that* $F(\xi,\eta) = (|\eta|/|\xi|)^s$ *when* $|\eta| \leq \delta|\xi|$.

PROOF. If $s = 0$ take $F = 1$. Otherwise, let

$$\psi(t) = e^{-s|t-\log\delta|} \quad \text{and} \quad \varphi(t) = \psi(\log t) = \min\left(\left(\frac{t}{\delta}\right)^s, \left(\frac{\delta}{t}\right)^s\right), \qquad t > 0.$$

We obtain, as in Lemma 3.4,

$$\left\|\varphi\left(\frac{|\eta|}{|\xi|}\right)\right\|_{M(\mathbf{R}^d\times\mathbf{R}^d)} \leq \frac{1}{2\pi}\|\hat{\psi}\|_{L^1(\mathbf{R}^d)} = 1.$$

Now take

$$F(\xi,\eta) = \delta^s\varphi\left(\frac{|\eta|}{|\xi|}\right) = \min\left(\left(\frac{|\eta|}{|\xi|}\right)^s, \left(\delta^2\frac{|\xi|}{|\eta|}\right)^s\right). \quad \square$$

LEMMA 3.6. *If* $\delta < (2d+1)^{-1}$ *and* $-\infty < s < \infty$, *then there exists* $G \in M(\mathbf{R}^d \times \mathbf{R}^d)$ *such that* $G(\xi,\eta) = (|\xi|/|\xi-\eta|)^s$ *when* $|\eta| \leq \delta|\xi|$.

PROOF. Let F be the function given by Lemma 3.5 with $s = 1$, and let

$$H(\xi,\eta) = 2\sum_{j=1}^d \frac{\xi_j}{|\xi|}\frac{\eta_j}{|\eta|}F(\xi,\eta) - F(\xi,\eta)^2.$$

Thus

$$\|H\|_{M(\mathbf{R}^d\times\mathbf{R}^d)} \leq 2d\|F\|_M + \|F\|_M^2 = (2d+\delta)\delta < 1.$$

Since $(1-z)^{-s/2}$ is an analytic function in the unit disc and M is a Banach algebra, $(1-H)^{-s/2} \in M$. This is the sought function, because, when $|\eta| \leq \delta|\xi|$,

$$1 - H(\xi,\eta) = 1 - 2\sum_{j=1}^d \frac{|\xi_j|}{|\xi|}\frac{|\eta_j|}{|\eta|}\frac{|\eta|}{|\xi|} + \left(\frac{|\eta|}{|\xi|}\right)^2$$
$$= \frac{|\xi|^2 - 2\xi\cdot\eta + |\eta|^2}{|\xi|^2} = \left(\frac{|\xi-\eta|}{|\xi|}\right)^2. \quad \square$$

The full Fourier transform in $\mathbf{R}^{2d}$ also is useful.

LEMMA 3.7. *If $D^\alpha f \in L^2(\mathbf{R}^d \times \mathbf{R}^d)$ for every multi-index α with $|\alpha| \le d+1$, then $f \in M(\mathbf{R}^d \times \mathbf{R}^d)$ and $\|f\|_M \le C\sum_{|\alpha|\le d+1} \|D^\alpha f\|_{L^2}$.*

PROOF.

$$f(\xi,\eta) = (2\pi)^{-2d} \iint e^{ix\cdot\xi} e^{iy\cdot\eta} \hat{f}(x,y)\, dx\, dy.$$

Hence, by standard estimates,

$$\|f\|_M \le \|\hat{f}\|_{L^1(\mathbf{R}^{2d})} \le C \sum_{\alpha \le d+1} \|D^\alpha f\|_{L^2}. \quad \square$$

(The proof actually shows that $B_2^{d1}(\mathbf{R}^{2d}) \subset M(\mathbf{R}^d \times \mathbf{R}^d)$.)

By localization, we obtain e.g.

LEMMA 3.8. *If $f \in C^{d+1}(B(\xi_0, 2r) \times B(\xi_0, 2r))$, then*

$$\|f\|_{M(B(\xi_0,r)\times B(\xi_0,r))} \le C \sup_{|\alpha|\le d+1} r^{|\alpha|} \sup_{\xi,\eta\in B(\xi_0,2r)} |D^\alpha f(\xi,\eta)|.$$

PROOF. By homogeneity and translation invariance it suffices to consider $\xi_0 = 0$ and $r = 1$. Let $\varphi \in C_0^\infty(\mathbf{R}^{2d})$ have support in $B(0,2) \times B(0,2)$ and be 1 on $B(0,1) \times B(0,1)$. Then, by Lemma 3.7,

$$\begin{aligned}\|f\|_{M(B(0,1)\times B(0,1))} &\le \|f\varphi\|_{M(\mathbf{R}^d\times\mathbf{R}^d)} \le C \sup_{|\alpha|\le d+1} \|D^\alpha(f\varphi)\|_{L^2} \\ &\le C_1 \sup_{|\alpha|\le d+1} \sup_{|\xi|,|\eta|<2} |D^\alpha f(\xi,\eta)|. \quad \square\end{aligned}$$

LEMMA 3.9. *If $f \in C^{d+1}(\tilde{\Delta}_j \times \tilde{\Delta}_k)$, then*

$$\|f\|_{M(\Delta_j\times\Delta_k)} \le C \sup_{|\alpha|+|\beta|\le d+1} \sup_{\substack{\xi\in\tilde{\Delta}_j \\ \eta\in\tilde{\Delta}_k}} |\xi|^{|\alpha|} |\eta|^{|\beta|} |D_\xi^\alpha D_\eta^\beta f(\xi,\eta)|.$$

PROOF. By homogeneity it suffices to consider $j = k = 0$. The result then follows from Lemma 3.7 by considering $f(\xi,\eta)\psi(\xi)\psi(\eta)$, where $\psi \in C_0^\infty(\tilde{\Delta}_0)$ with $\psi = 1$ on Δ_0. $\square$

Taylor's formula yields the following version of Lemma 3.8.

LEMMA 3.10. *Suppose that $k \ge 1$ and $m \ge \max(d+1, k)$. Suppose further that $r \le |\xi_0|$ and $f \in C^m(B(\xi_0, 2r) \times B(\xi_0, 2r))$ with $D^\alpha f(\xi_0, \xi_0) = 0$ when $|\alpha| \le k-1$. Then*

$$\begin{aligned}\|f\|_{M(B(\xi_0,r)\times B(\xi_0,r))} &\le C \sup_{k\le|\alpha|\le m} r^{|\alpha|} \sup_{\xi,\eta\in B(\xi_0,2r)} |D^\alpha f(\xi,\eta)| \\ &\le C \left(\frac{r}{|\xi_0|}\right)^k \sup_{|\alpha|\le m} \sup_{\xi,\eta\in B(\xi_0,2r)} |\xi_0|^{|\alpha|} |D^\alpha f(\xi,\eta)|. \quad \square\end{aligned}$$

Finally, we give results corresponding to Lemmas 2.2 and 2.3.

LEMMA 3.11. *If $\{U_n\}_1^\infty$ and $\{V_n\}_1^\infty$ are partitions of U and V respectively, then*

$$\left\| f(\xi,\eta) \sum_1^\infty \chi_{U_n}(\xi) \chi_{V_n}(\eta) \right\|_{M(U\times V)} = \sup_n \|f\|_{M(U_n\times V_n)}.$$

PROOF. Let $g(\xi,\eta) = f(\xi,\eta)\sum_1^\infty \chi_{U_n}(\xi)\chi_{V_n}(\eta)$. It is obvious that

$$\|f\|_{M(U_n\times V_n)} = \|g\|_{M(U_n\times V_n)} \leq \|g\|_{M(U\times V)}.$$

Conversely, suppose that $\|f\|_{M(U_n\times V_n)} \leq 1$ for every n. Then there exist representations $f(\xi,\eta) = \int_{X_n} \alpha_n(\xi,x)\beta_n(\eta,x)\,d\mu_n(x)$, where $\mu_n(X_n) = 1$ and

$$\|\alpha_n\|_{L^\infty(U_n\times X_n)},\ \|\beta_n\|_{L^\infty(V_n\times X_n)} \leq 1.$$

Let $(X,\mu) = \prod_1^\infty (X_n,\mu_n)$ and define, with $x = (x_n)_1^\infty$,

$$\begin{aligned} \alpha(\xi,x,t) &= e^{2\pi int}\alpha_n(\xi,x_n), \qquad \xi \in U_n,\\ \beta(\eta,x,t) &= e^{-2\pi int}\beta_n(\eta,x_n), \qquad \eta \in V_n. \end{aligned}$$

Then $\int_X\int_0^1 \alpha(\xi,x,t)\beta(\eta,x,t)\,d\mu(x)\,dt = g(\xi,\eta)$, whence

$$\|g\|_M \leq \|\alpha\|_{L^\infty}\|\beta\|_{L^\infty}\|\mu\| \leq 1. \quad \square$$

Let G be a group as in Lemma 2.3.

LEMMA 3.12. *Suppose that $\{U_n\}_{n\in G}$ and $\{V_n\}_{n\in G}$ are partitions of U and V respectively. Suppose further that $\|f\|_{M(U_m\times V_n)} \leq a(m-n)$ for some $a \in l^1(G)$. Then $f \in M(U\times V)$, with $\|f\|_M \leq \sum_G a(n)$.*

PROOF. As for Lemma 2.3, using Lemma 3.11 (and an extended version thereof when G is uncountable, a case does not appear in our applications). $\square$

4. Assumptions on A.

For easy reference, we list here in five groups nine conditions on A that in various combinations will be used in the theorems below.

Homogeneity.

A0. There exists an $r > 1$ such that $A(r\xi, r\eta) = A(\xi,\eta)$.

In fact, Examples 1–6 satisfy this condition for all $r > 0$. The slightly weaker version that we use, following Timotin (**1984**), allows e.g. a dyadic structure on A as in (1.18).

While this homogeneity assumption simplifies some of the results, it is not essential and we will also give results without it.

Boundedness.

A1. $\|A\|_{M(\Delta_j\times\Delta_k)} \leq C$ for all $j,k \in \mathbf{Z}$.

In some theorems we need a stronger assumption near the axes.

A2. There exist $A_1, A_2 \in M(\mathbf{R}^d \times \mathbf{R}^d)$ and $\delta > 0$ such that

$$\begin{aligned} A(\xi,\eta) &= A_1(\xi,\eta) \qquad \text{for } |\eta| < \delta|\xi|,\\ A(\xi,\eta) &= A_2(\xi,\eta) \qquad \text{for } |\xi| < \delta|\eta|. \end{aligned}$$

REMARK 4.1. Both A1 and A2 are satisfied if $A \in M(\mathbf{R}^d\times\mathbf{R}^d)$ as in Examples 1–6, but that hypothesis would be unnecessarily restrictive.

REMARK 4.2. In Theorems 5.2 and 7.2 we use a stronger version of A1 (which implies A2). However, this stronger version requires A to vanish at the axes and is too restrictive for our main applications.

Zero on the diagonal.

A3. There exist $\gamma > 0$ and $\delta > 0$ such that if $B = B(\xi_0, r)$ with $r < \delta|\xi_0|$, then $\|A\|_{M(B\times B)} \leq C(r/|\xi_0|)^\gamma$.

While the value of δ is uninteresting (if A1 holds and A3 holds for some $\delta > 0$, then A3 holds for every $\delta < 1$), the value of γ is of utmost importance and we often use the notation A3(γ) (always assuming $\gamma > 0$). We use A3(∞) to signify that A3(γ) holds for all $\gamma > 0$. Since $M \subset L^\infty$, the condition A3 implies that, if $|\xi - \eta| < \delta|\xi|$, $|A(\xi,\eta)| \le C|\xi|^{-\gamma}|\xi-\eta|^\gamma$. Hence γ can be interpreted as a measure of the order of the zero of A at the diagonal (cf. also Lemma 3.10 and the examples in §6, and note that A3 fails to hold for the Toeplitz operators).

Conditions A1–A3 are the only ones needed to obtain "direct results," i.e. sufficient conditions for $T_b \in S_p$. In order to obtain converse results, we need conditions saying that A is not too small.

Nondegeneracy.

A4. There exists no $\xi \neq 0$ such that $A(\xi+\eta,\eta) = 0$ for a.e. η.

A4 says that T_b really depends on $\hat{b}(\xi)$ for every $\xi \neq 0$, and is obviously necessary for any converse result. The next condition concerns A close to the axis $\eta = 0$.

A5. For every $\xi_0 \neq 0$ there exist $\delta > 0$ and $\eta_0 \in \mathbf{R}^d$ such that, with $U = \{\xi\colon\ |\xi/|\xi| - \xi_0/|\xi_0|| < \delta \text{ and } |\xi| > |\xi_0|\}$ and $V = B(\eta_0, \delta|\xi_0|)$, $1/A(\xi,\eta) \in M(U \times V)$.

A4 and A5 will be used in the homogeneous case (A0 holds). In that case A5$\Rightarrow$A4. Furthermore, for every $t > 1$ of the form r^k,

$$|A(\xi_0, t^{-1}\eta_0)| = |A(t\xi_0,\eta_0)| \ge ||A^{-1}||^{-1}_{M(U\times V)}.$$

Hence A0 + A5 implies that $A(\xi,\eta)$ does not converge to 0 as $\eta \to 0$.

In the nonhomogeneous case we need the following uniform version of A5.

A6. There exist C_1 and C_2 such that A5 holds with δ independent of ξ_0, $|\eta_0| < C_1|\xi_0|$ and $||A^{-1}||_{M(U\times V)} < C_2$.

We do not know any simple uniform version of A4 that works for our purposes. We will instead use the following much stronger condition.

A7. There exist δ_1 and δ_2 with $0 \le \delta_1 < \delta_2$ and $A_1 \in M(\mathbf{R}^d \times \mathbf{R}^d)$ such that $A(\xi,\eta)^{-1} = A_1(\xi,\eta)$ for $\delta_1 \le |\eta|/|\xi| \le \delta_2$.

Note that if A7 holds with $\delta_1 = 0$, then A6 (and thus A5) holds (we may take $\delta = \delta_2$ and $\eta_0 = 0$).

Not zero at the diagonal.

A8. There exists a sequence $\{\xi_n\}$ such that, with $B_n = B(\xi_n, n)$,

$$||A^{-1}||_{M(B_n\times B_n)} \le C, \qquad n = 1,2,\dots.$$

Obviously, A8 contradicts A3. If A0 is satisfied, it is sufficient that $A^{-1} \in M(B\times B)$ for some ball B.

REMARK 4.3. Some of these conditions (e.g. A5), are highly asymmetric in ξ and η and consequently the theorems which result from them (e.g. Theorem 5.3) will be asymmetric too. Since any result that holds for $A(\xi,\eta)$ has to hold for $A(\eta,\xi)$ as well (take the adjoint of T_b and exchange b for $b(-x)$ and the parameters s and t (see §5) for each other), this is a defect and should ultimately be remedied for.

To help the reader to get a quicker grasp of all the various assumptions we schematically summarize them once more in the form of a table (s = symmetric, as = asymmetric):

A0$^{\text{s}}$	homogeneity	A4$^{\text{s}}$, A5$^{\text{as}}$, A6$^{\text{as}}$, A7$^{\text{s}}$	nondegeneracy (not too small)
A1$^{\text{s}}$, A2$^{\text{s}}$	boundedness		
A3$(\gamma)^{\text{s}}$	vanishing on the diagonal	A8$^{\text{s}}$	nonvanishing on the diagonal

5. Main results. The results will be proved for the more general operator defined by

$$(T_b^{st}f)\hat{}(\xi) = (2\pi)^{-d}\int \hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\hat{f}(\eta)\,d\eta. \tag{5.1}$$

Here s and t are two real parameters. (In §8 we will also consider complex s and t.) Although $T_b^{st}(A)$ equals $T_b(A(\xi,\eta)|\xi|^s|\eta|^t)$, the present notation gives added technical convenience.

Note further that $T_b^{st} = I^{-s}T_bI^{-t}$, with I^s defined by (2.6). Hence T_b^{st} is bounded on L^2 iff T_b: $H^{-t} \to H^s$, where H^s is the Sobolev space $I^s(L^2) = B_2^s$.

The following theorem combines Theorems 8.1, 9.1 and 9.2.

THEOREM 5.1. *Suppose that A satisfies* A0, A1, A3(γ), A4 (*or* A1, A3(γ), A7). *Suppose further that $1 \le p \le \infty$, $s+t+d/p < \gamma$ and $s,t > \max(-d/2, -d/p)$. Then*

$$T_b^{st} \in S_p \Leftrightarrow b \in B_p^{s+t+d/p}. \quad \square$$

Taking $s = t = 0$ we obtain

COROLLARY 5.1. *Suppose that A satisfies* A0, A1, A3(γ), A4 (*or* A1, A3(γ), A7). *If $p \ge 1$ and $d/\gamma < p < \infty$, then $T_b \in S_p \Leftrightarrow b \in B_p^{d/p}$.* $\square$

We show in §11 that, in general, the condition $s+t+d/p < \gamma$ is necessary for Theorem 5.1 to hold. In fact, in typical cases $T_b^{st} \in S_p$ with $s+t+d/p \ge \gamma$ only if $T_b^{st} = 0$. In particular, the conclusion of Corollary 5.1 fails in general for $p \le d/\gamma$ (see e.g. Example 4 (with $\gamma = 1$) for $d \ge 2$ (Janson and Wolff (**1982**))).

The corollary also may fail for $p = \infty$, although Theorems 7.2, 9.1 and 9.2 imply the following more restrictive result.

THEOREM 5.2. *Suppose that A satisfies* A0, A3, A4 (*or* A3, A7) *and furthermore the following sharpening of* A1 :

$$\|A\|_{M(\Delta_j\times\Delta_k)} \le a(j-k), \quad \text{with} \quad \sum_{-\infty}^{\infty} a(n) < \infty. \tag{5.2}$$

Then

$$T_b \in S_\infty \Leftrightarrow b \in B_\infty^0.$$

More generally, if $0 \le s < \gamma$, then $T_b^{s0} \in S_\infty \Leftrightarrow T_b^{0s} \in S_\infty \Leftrightarrow b \in B_\infty^s$. $\square$

Assumption (5.2) says that A vanishes in a specific way at the axes. This is not true in most of our examples; for them we have (by Theorems 7.3, 10.1 and 10.2) a different complement to Corollary 5.1.

THEOREM 5.3. *Suppose that A satisfies* A0, A1, A2, A3, A5 (*or* A1, A2, A3, A6, A7). *Then*

$$T_b \text{ is bounded} \Leftrightarrow b \in \mathrm{BMO}.$$

More generally, if $0 \leq s < \gamma$,

$$\begin{aligned} T_b^{s0} \text{is bounded on } L^2 &\Leftrightarrow T_b \text{ is bounded } L^2 \to H^s \\ &\Leftrightarrow b \in I^s(\mathrm{BMO}) \Leftrightarrow I^{-s}b \in \mathrm{BMO}. \quad \square \end{aligned}$$

The corresponding result for T_b^{0t} holds by duality provided A5 and A6 are replaced by their mirror images (cf. Remark 4.3).

When A does not vanish on the diagonal, as in the Toeplitz case, we obtain (by Theorems 7.4, 12.1 and 12.2) quite different results.

THEOREM 5.4. *Suppose that A satisfies* A1, A2, A8. *Then*

$$T_b \text{ is bounded} \Leftrightarrow b \in L^\infty, \qquad T_b \text{ is compact} \Leftrightarrow b = 0. \quad \square$$

6. The examples revisited. In this section we apply the results in §5 (and occasionally additional results from later sections) to the examples in §1, using the lemmas of §3 to verify conditions A0–A8 defined in §4.

Examples 1 (product) and 2 (Toeplitz) satisfy A0, A1, A2 and A8. Theorem 5.4 yields well-known results.

Example 3 (Hankel) satisfies A0, A1 and A2. Furthermore, since $A(\xi,\eta)$ vanishes when ξ and η have the same sign, A3(∞) holds. However, A4 and A5 are not satisfied, which corresponds to the fact that the analytic part of b does not affect T_b at all and hence may be arbitrary. Although thus the theorems in §5 are not directly applicable, partial results follow from the implications proved in §7, and the complete results ($H_b \in S_\infty$ iff $\overline{P}b \in \mathrm{BMO}$ and $H_b \in S_p$ iff $\overline{P}b \in B_p^{1/p}$ $(1 \leq p < \infty)$, see §1) follow easily using Example 4 below.

Example 4 (commutators) satisfies A0, A1, A2, A3(1), A4, A5. A1 and A2 hold by Remark 4.1, and A3 follows by (with $B = B(\xi_0, r)$)

$$\begin{aligned} \|A(\xi,\eta)\|_{M(B\times B)} &= \|m(\eta) - m(\xi_0) - (m(\xi) - m(\xi_0))\|_{M(B\times B)} \\ &\leq \|m(\eta) - m(\xi_0)\|_{L^\infty(B)} + \|m(\xi) - m(\xi_0)\|_{L^\infty(B)} \leq Cr/|\xi_0|, \end{aligned} \tag{6.1}$$

while A5 follows by taking η_0 with $m(\eta_0) \neq m(\xi_0)$ and choosing δ such that

$$\begin{aligned} &\|(m(\eta_0) - m(\xi_0)) - A(\xi,\eta)\|_{M(U\times V)} \\ &\qquad \leq \|m(\xi) - m(\xi_0)\|_{L^\infty(U)} + \|m(\eta) - n(\eta_0)\|_{L^\infty(V)} \\ &\qquad \leq C\delta < |m(\eta_0) - m(\xi_0)|, \end{aligned} \tag{6.2}$$

which implies that A is invertible in $M(U \times V)$.

When $d \geq 2$, A3(1) is best possible; A3(γ) does not hold for any $\gamma > 1$. Theorem 5.3 and Corollary 5.1 give the results $[b,K] \in S_\infty \Leftrightarrow b \in \mathrm{BMO}$ and $[b,K] \in S_p \Leftrightarrow b \in B_p^{d/p}$ $(d < p < \infty)$ stated in §1. Recall that it is impossible to obtain $[b,K] \in S_p$ with $p \leq d$ (except when $[b,K] = 0$); this follows by Theorem 11.1.

When $d = 1$, $A(\xi,\eta)$ vanishes when ξ and η have the same sign, and A3(∞) holds. Thus we obtain $[b,K] \in S_\infty \Leftrightarrow b \in \mathrm{BMO}$ and $[b,K] \in S_p \Leftrightarrow b \in B_p^{1/p}$ $(1 \leq p < \infty)$, which imply the corresponding results for Hankel operators using

(1.5). Theorem 5.3 yields also that $[b,K]$ maps L^2 into H^s iff $b \in I^s(\mathrm{BMO})$, and thus that the Hankel operator H_b maps L^2 into H^s iff $\overline{P}b \in I^s(\mathrm{BMO})$ $(0 \le s < \infty)$. This confirms a conjecture by Vladimir Peller (personal communication).

Example 5 (higher commutators) satisfies A0, A1, A2, A3(N) and, except in degenerate cases, A4 and A5. This follows by the corresponding conditions for Example 4 and the fact that M is a Banach algebra. For example, by (6.1), if $B = B(\xi_0, r)$,

(6.3)

$$\left\|\prod_1^N (m_j(\eta) - m_j(\xi))\right\|_{M(B\times B)} \le \prod_1^N \|(m_j(\eta) - m_j(\xi))\|_{M(B\times B)} \le C(r/|\xi_0|)^N.$$

We recover the results stated in §1.

Example 6 (higher-dimensional Hankel and Toeplitz) satisfies A0, A1 and A2. In the Toeplitz case, i.e. if $\operatorname{int}(\Gamma_1) \cap \operatorname{int}(\Gamma_2) \ne 0$, A8 holds and thus Theorem 5.4 shows that T_b is bounded iff $b \in L^\infty$ and that T_b never is compact.

In the Hankel case, i.e. if $\Gamma_1 \cap \Gamma_2 = \{0\}$, $A(\xi,\eta)$ vanishes in a neighborhood of the diagonal, whence A3(∞) holds. However, A5 never holds and A4 holds only if $\Gamma_1 - \Gamma_2 = R^d$, a rather peculiar case, which nevertheless is possible (let e.g. $d = 2$ and $\Gamma_1 = \{(\xi_1,\xi_2)\colon |\xi_2| \le \frac{1}{2}|\xi_1|\}$, $\Gamma_2 = \{(\xi_1,\xi_2)\colon |\xi_1| \le \frac{1}{2}|\xi_2|\}$). In particular, A4 does not hold if the cones Γ_1 and Γ_2 are convex. Consequently, Theorem 5.1 and Corollary 5.1 apply only in exceptional cases and Theorem 5.3 does not apply at all. (Nor does Theorems 5.2 and 5.4.)

Theorems 7.3 and 8.1 yield the implications $b \in \mathrm{BMO} \Rightarrow T_b \in S_\infty$ and $b \in B_p^{d/p} \Rightarrow T_b \in S_p$ $(1 \le p < \infty)$, but we have no results in the converse direction. In the one-dimensional case (Example 3) it was possible to obtain converses by extending the operator (Example 4) and projecting the symbol, but that method fails in higher dimensions as is seen by the following example. Let $d = 2$ and $\Gamma_1 = \{(\xi_1,\xi_2)\colon \xi_1 < 0,\ \xi_2 < 0\}$, $\Gamma_2 = -\Gamma_1$. Then

(6.4)

$$\begin{aligned}\|T_b\|_{S_2}^2 &= \iint |(2\pi)^{-d}\hat{b}(\xi-\eta)A(\xi,\eta)|^2\,d\xi\,d\eta = (2\pi)^{-2d}\int_{\Gamma_1}\int_{\Gamma_2} |\hat{b}(\xi-\eta)|^2\,d\xi\,d\eta \\ &= (2\pi)^{-2d}\int_0^\infty\int_0^\infty |\hat{b}(\xi_1,\xi_2)|^2\xi_1\xi_2\,d\xi_1\,d\xi_2.\end{aligned}$$

This is not equivalent to

$$\|P_2 b\|_{B_2^1}^2 = \int_{\Gamma_2} |\hat{b}(\xi)|^2|\xi|^2\,d\xi = \int_0^\infty\int_0^\infty |\hat{b}(\xi_1,\xi_2)|^2(\xi_1^2+\xi_2^2)\,d\xi_1\,d\xi_2$$

and $T_b \in S_2 \not\Rightarrow P_2 b \in B_2^1$.

In the next examples A does not have the right homogeneity. This is compensated for by introducing powers of ξ, η or $\xi - \eta$.

Example 7 (Calderón commutators). The commutator $[b,K]\,d/dx$ equals T_b with A given by (1.9); hence it equals T_b^{01} with

(6.5) $$A(\xi,\eta) = 2I(\xi > 0 > \eta) + 2I(\xi < 0 < \eta).$$

It is easily seen that A defined by (6.5) satisfies A0, A1, A2, A3(∞), A4, A5. Hence, by Theorem 5.3 and symmetry, $[b, K]\, d/dx = T_b^{01}$ is bounded $\Leftrightarrow b \in I^1(\mathrm{BMO}) \Leftrightarrow$ $I^{-1}b \in \mathrm{BMO} \Leftrightarrow db/dx \in \mathrm{BMO}$. By Theorem 5.1,

$$[b, K]d/dx \in S_p \quad \text{iff} \quad b \in B_p^{1+1/p} \qquad (1 \leq p < \infty).$$

The commutator $[b, K d/dx]$ equals T_b with A given by (1.10), and we rewrite it as $T_{db/dx}$ with

$$A(\xi, \eta) = (i(\xi - \eta))^{-1}(|\eta| - |\xi|) = i\frac{|\xi| - |\eta|}{\xi - \eta}. \tag{6.6}$$

This A satisfies A0, A1, A2, A8. (A1 and A2 may e.g. be shown by the argument for Example 10 below, applied to the quadrants separately.) Consequently, by Theorem 5.4, $[b, K\, d/dx]$ is bounded iff $db/dx \in L^\infty$, and this commutator is never compact.

Example 8 (fractional integration commutators). Let us first consider the case $s < 0$, i.e. commutators with fractional differentiation. For convenience, we change the notation to $[b, I^{-s}]$, $s > 0$. This equals, by (1.12) and (2.6), the paracommutator $T_{I^{-s}b}$ with

$$A(\xi, \eta) = \frac{|\eta|^s - |\xi|^s}{|\xi - \eta|^s}. \tag{6.7}$$

While it is possible to prove directly that, if $0 < s < 1$, A satisfies A1, A2 and A3$(1 - s)$, it is simpler and more instructive to decompose the operator into two parts as follows. Let $\varphi \in C_0^\infty(0, \infty)$ with $\varphi(t) = 1$ in a neighborhood of 1, and define

$$A_1(\xi, \eta) = \left(1 - \varphi\left(\frac{|\eta|}{|\xi|}\right)\right) \frac{|\eta|^s - |\xi|^s}{|\xi - \eta|^s}, \tag{6.8}$$

$$A_2(\xi, \eta) = \varphi\left(\frac{|\eta|}{|\xi|}\right) \left(\frac{|\eta|^s}{|\xi|^s} - 1\right). \tag{6.9}$$

It is easily seen that

$$[b, I^{-s}] = T_{I^{-s}b}(A_1) + T_b^{s0}(A_2) \tag{6.10}$$

and, by Lemmas 3.4 and 3.1, $[b, I^{-s}] \in S_p$ iff both $T_{I^{-s}b}(A_1)$ and $T_b^{s0}(A_2) \in S_p$. We treat the two pieces separately.

It follows from Lemmas 3.4, 3.5 and 3.6 that A_1 satisfies A0, A1, A2, A3(∞), A4 and A5 (A7 holds with $\delta_1 = 0$). Thus, by Theorem 5.3 and Corollary 5.1,

$$T_{I^{-s}b}(A_1) \in S_\infty \Leftrightarrow I^{-s}b \in \mathrm{BMO} \Leftrightarrow b \in I^s(\mathrm{BMO})$$

and

$$T_{I^{-s}b}(A_1) \in S_p \Leftrightarrow I^{-s}b \in B_p^{d/p} \Leftrightarrow b \in B_p^{s+d/p} \qquad (1 \leq p < \infty).$$

Similarly, by Lemmas 3.4 and 3.10, A_2 satisfies A0, A1, A3(1), A4. Since A_2 vanishes on $\Delta_j \times \Delta_k$ when $|j - k|$ is large, (5.2) holds and Theorems 5.1 and 5.2 show that, provided $s + d/p < 1$,

$$T_b^{s0}(A_2) \in S_p \Leftrightarrow b \in B_p^{s+d/p} \qquad (1 \leq p \leq \infty). \tag{6.11}$$

Theorem 11.1 shows that if $T_b^{s0}(A_2) \in S_p$ and $s + d/p > 1$, or $s + d/p = 1$ and $p < \infty$, then b is polynomial and it follows that b is a constant and $T_b^{s0}(A_2) = 0$. Summarizing, we have found (cf. Murray (**1985**))

$$[b, I^{-s}] \text{ is bounded } \Leftrightarrow \begin{cases} b \in I^s(\mathrm{BMO}), & 0 < s < 1, \\ b \text{ constant}, & s > 1, \end{cases}$$

$$[b, I^{-s}] \in S_p \ (d < p < \infty) \Leftrightarrow \begin{cases} b \in B^{s+d/p}, & 0 < s < 1 - d/p, \\ b \text{ constant}, & 1 - d/p \le s, \end{cases}$$

$$[b, I^{-s}] \in S_p \ (1 \le p \le d) \Leftrightarrow b \text{ constant}, \qquad s > 0.$$

The only remaining case is $p = \infty$, $s = 1$, i.e. the problem: When is $[b, I^{-1}]$ bounded? The answer for $d = 1$ ($db/dx \in L^\infty$) was obtained in Example 7; the referee has remarked that the same result ($db/dx_i \in L^\infty$, i.e. b Lipschitz) holds for $d \ge 2$ too, see Coifman, McIntosh and Meyer (**1982**, Theorem IX) for the sufficiency. We obtain less complete results for $s < 0$. The decomposition (6.10) is still valid and we obtain by Theorem 5.2 that (6.11) holds provided $s < 1 - d/p$, and that $T_b^{s0}(A_2) \in S_p$ only when b is constant for $s \ge 1 - d/p$. However, A_1 is not bounded near the axes and our theorems are not applicable to it. Instead we employ a different decomposition. Let $\psi \in C^\infty(0, \infty)$ with $\psi = 1$ on (0,1) and $\psi = 0$ on $(2, \infty)$, and define

$$A_3(\xi, \eta) = \psi\left(\frac{|\eta|}{|\xi|}\right)\left(1 - \left(\frac{|\eta|}{|\xi|}\right)^{-s}\right), \tag{6.12}$$

$$A_4(\xi, \eta) = \left(1 - \psi\left(\frac{|\eta|}{|\xi|}\right)\right)\left(\left(\frac{|\xi|}{|\eta|}\right)^{-s} - 1\right). \tag{6.13}$$

Then

$$[b, I^{-s}] = T_b^{0s}(A_3) + T_b^{s0}(A_4). \tag{6.14}$$

Since A_3 and A_4 satisfy A0, A1, A3(1), Theorem 5.1 shows that if

$$\max(-d/2, -d/p) < s < 1 - d/p$$

and $b \in B_p^{s+d/p}$ then $T_b^{0s}(A_3)$ and $T_b^{s0}(A_4) \in S_p$, and thus $[b, I^{-s}] \in S_p$. Consequently, changing the sign again,

$$[b, I^s] \in S_p \ (d \le p < \infty) \Leftrightarrow b \in B_p^{d/p-s}, \qquad 0 < s < \min(d/p, d/2),$$

$$[b, I^s] \in S_p \ (1 \le p < d) \Leftrightarrow \begin{cases} b \text{ constant}, & 0 < s \le d/p - 1, \\ b \in B_p^{d/p-s}, & d/p - 1 < s < \min(d/p, d/2). \end{cases}$$

For $s \ge \min(d/p, d/2)$ we obtain only the partial result that $b \in B_p^{d/p-s}$ is necessary. For $p = \infty$ this may be improved to $b \in I^{-s}(\mathrm{BMO})$, because of Theorem 10.2 applied to $T_{I^s b}(A_1)$. The referee, however, has pointed out that these conditions cannot be sufficient (even for boundedness) when $s > d/p$.

Example 9 (paraproducts) satisfies A0, A1, A2, A3(∞), A4, A5 (for all versions given in §1). Hence we obtain the results T_b bounded iff $b \in \mathrm{BMO}$ and $T_b \in S_p$ iff $b \in B_p^{d/p}$ $(1 \le p < \infty)$. However, this is only in part a true application of our theory since we use the result that T_b is bounded for $b \in$ BMO in the proof of Theorem 7.3.

Example 10 (A smooth) does not have to satisfy A0, but the estimates (1.20) have the right homogeneity. A1 follows immediately by Lemma 3.9. In order to prove A2, we define

$$A_0(\xi,\eta) = A(\xi,\eta) - A(\xi,0). \tag{6.15}$$

By (1.20)

$$|\xi|^\alpha |D_\xi^\alpha A_0(\xi,\eta)| \le |\xi|^\alpha |\eta| \sup_{\eta'} |\nabla_\eta D_\xi^\alpha A_0(\xi,\eta')| \le C_\alpha |\eta|\,|\xi|^{-1},$$

and, if $|\beta| > 0$,

$$|\xi|^\alpha |\eta|^\beta |D_\xi^\alpha D_\eta^\beta A_0(\xi,\eta)| = |\xi|^\alpha |\eta|^\beta |D_\xi^\alpha D_\eta^\beta A(\xi,\eta)| \le C_{\alpha\beta} |\eta|^\beta |\xi|^{-\beta}.$$

Hence, by Lemma 3.9,

$$||A_0||_{M(\Delta_j\times\Delta_k)} \le C2^{k-j} \quad \text{for } j \ge k. \tag{6.16}$$

Lemma 3.12 yields

$$A_0(\xi,\eta)\sum_{j>k} \chi_{\Delta_j}(\xi)\chi_{\Delta_k}(\eta) \in M(\mathbf{R}^d \times \mathbf{R}^d),$$

and A2 follows with $A_1(\xi,\eta) = A(\xi,0) + A_0(\xi,\eta)I((\xi,\eta) \in \bigcup_{j>k} \Delta_j \times \Delta_k)$, and A_2 obtained by symmetry.

If furthermore $A(\xi,0)^{-1} \in L^\infty$, i.e. $\inf |A(\xi,0)| > 0$, then (6.16) and Lemma 3.12 shows that if N is large enough, then $A(\xi,0)+A_0(\xi,\eta)I((\xi,\eta) \in \bigcup_{j>k+N} \Delta_j \times \Delta_k)$ is invertible in $M(\mathbf{R}^d \times \mathbf{R}^d)$, whence A7 holds with $\delta_1 = 0$, and thus A6 holds too.

If f is zero with multiplicity k on the diagonal, i.e. $D^\alpha f(\xi,\xi) = 0$ for $|\alpha| \le k-1$, then Lemma 3.10 shows that A3(k) is satisfied. On the other hand, A8 holds iff $f(\xi,\xi)$ does not converge to 0 as $|\xi| \to \infty$ (because of Lemma 3.8).

Theorems 7.4 and 7.3 yield the results by Coifman and Meyer (**1978**)

$$b \in L^\infty \Rightarrow T_b \text{ bounded}, \tag{6.17}$$

$$\text{if } A(\xi,\xi) \equiv 0, \quad \text{then } b \in \mathrm{BMO} \Rightarrow T_b \text{ bounded}. \tag{6.18}$$

Furthermore, Theorem 7.2 yields (by (6.16))

$$\text{if } A(\xi,0) \equiv A(\xi,\xi) \equiv A(0,\xi) \equiv 0, \quad \text{then } b \in B_\infty^0 \Rightarrow T_b \text{ bounded}. \tag{6.19}$$

We obtain also by Theorems 12.1, 10.2, 9.1, 9.2,

$$\text{if } \limsup_{|\xi|\to\infty} |A(\xi,\xi)| > 0, \quad \text{then } T_b \text{ bounded} \Rightarrow b \in L^\infty, \tag{6.20}$$

$$\text{if } \inf |A(\xi,0)| > 0, \quad \text{then } T_b \text{ bounded} \Rightarrow b \in \mathrm{BMO}, \tag{6.21}$$

$$\text{if A7 (or A0+A4) holds}, \quad \text{then } T_b \text{ bounded} \Rightarrow b \in B_\infty^0. \tag{6.22}$$

Similar results for S_p are left for the reader.

7. Boundedness.

LEMMA 7.1. *Suppose that A satisfies* A1 *and* A3(γ) *and that $s+t<\gamma$. If $\|b\|_{B^{s+t}_\infty} \le 1$, then*

$$\|\hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\|_{S_\infty(\Delta_j\times\Delta_k)} \le \begin{cases} C2^{s(j-k)}, & j\le k, \\ C2^{t(k-j)}, & j\ge k. \end{cases} \tag{7.1}$$

PROOF. By symmetry, it suffices to consider $j\le k$. We consider two cases separately.

(i) $j\le k-2$. Let $\hat{\psi}$ be a test function with support in $\{\xi\colon \frac14<|\xi|<8\}$ that equals 1 on $\tilde{\Delta}_0$, and define ψ_k by (2.2). Since $\xi\in\Delta_j$ and $\eta\in\Delta_k$ implies $|\xi|\le 2^{j+1}\le 2^{k-1}$ and thus $2^{k-1}\le|\xi-\eta|<2^{k+2}$, Lemma 2.1 and the definition of B^{s+t}_∞ yield

$$\begin{aligned}\|\hat{b}(\xi-\eta)\|_{S_\infty(\Delta_j\times\Delta_k)} &= \|\hat{\psi}_k(\xi-\eta)\hat{b}(\xi-\eta)\|_{S_\infty(\Delta_j\times\Delta_k)} \\ &= \|\widehat{\psi_k * b}(\xi-\eta)\|_{S_\infty(\Delta_j\times\Delta_k)} \\ &\le (2\pi)^d\|\psi_k * b\|_{L^\infty(\mathbf{R}^d)} \le C2^{-(s+t)k}.\end{aligned} \tag{7.2}$$

Further, by A1,

$$\begin{aligned}\|A(\xi,\eta)|\xi|^s|\eta|^t\|_{M(\Delta_j\times\Delta_k)} &\le \|A(\xi,\eta)\|_{M(\Delta_j\times\Delta_k)}\|\,|\xi|^s\|_{L^\infty(\Delta_j)}\|\,|\eta|^t\|_{L^\infty(\Delta_k)} \\ &\le C2^{sj}2^{tk}.\end{aligned} \tag{7.3}$$

Hence (7.1) follows by Lemma 3.1.

(ii) $j=k-1$ or $j=k$. Let $\hat{\psi}$ be a test function with support in $\tilde{\Delta}_0$ such that (2.4) holds. Fix $l\le k$ and suppose that $2^{l-k}<\delta(4+2d)^{-1}$, where δ is the constant in A3. Let Q_n be the cube with center $2^l n$ and side 2^l for $n\in\mathbf{Z}^d$. Thus $\{Q_n\}$ is a partition of $\mathbf{R}^d$ into cubes. Note that if $m,n\in\mathbf{Z}^d$ and $|m-n|>4+d$, then $\hat{\psi}_l(\xi-\eta)=0$ for $\xi\in Q_m$, $\eta\in Q_n$. Furthermore, if $|m-n|\le 4+d$ and $Q_n\cap\Delta_k\neq\varnothing$, let $\xi_0\in Q_n\cap\Delta_k$. Then $Q_m,Q_n\subset B(\xi_0,(4+2d)2^l)$ and thus, by A3,

$$\|A|_{M(Q_m\times Q_n)} \le C\left(\frac{2^l}{|\xi_0|}\right)^\gamma \le C2^{(l-k)\gamma}, \tag{7.4}$$

whence Lemmas 3.1 and 2.1 yield

$$\begin{aligned}\|\hat{\psi}_l(\xi-\eta)\hat{b}(\xi-\eta)A(\xi,\eta)\|_{S_\infty(Q_m\times Q_n)} &\le C2^{(l-k)\gamma}\|\psi_l * b\|_{L^\infty} \\ &\le C2^{(l-k)\gamma-l(s+t)}.\end{aligned} \tag{7.5}$$

Consequently, since $\{Q_n\}_{n\in\mathbf{Z}^d}$ is a partition of $\mathbf{R}^d$ and $\{Q_n\cap\Delta_k\}_{n\in\mathbf{Z}^d}$ a partition of Δ_k, Lemma 2.3 applies with $a(n)=C2^{l(\gamma-s-t)-k\gamma}$ for $|n|\le 4+d$ and $a(n)=0$ for $|n|>4+d$ and we obtain

$$\|\hat{\psi}_l(\xi-\eta)\hat{b}(\xi-\eta)A(\xi,\eta)\|_{S_\infty(\mathbf{R}^d\times\Delta_k)} \le C2^{l(\gamma-s-t)-k\gamma}, \tag{7.6}$$

provided $l-k$ is small enough.

On the other hand, for every j, A1 and Lemmas 3.1 and 2.1 yield

$$\|\hat{\psi}_l(\xi-\eta)\hat{b}(\xi-\eta)A(\xi,\eta)\|_{S_\infty(\Delta_j\times\Delta_k)} \le C\|\hat{\psi}_l * b\|_{L^\infty(\mathbf{R}^d)} \le C2^{-l(s+t)}. \tag{7.7}$$

Since $\hat{\psi}_l(\xi-\eta)=0$ on $\Delta_j\times\Delta_k$ when $l\geq k+3$, we obtain by (7.6) and (7.7)

(7.8)
$$\begin{aligned}||\hat{b}(\xi-\eta)A(\xi,\eta)||_{S_\infty(\Delta_j\times\Delta_k)} &\leq \sum_{l=-\infty}^{k+2}||\hat{\psi}_l(\xi-\eta)\hat{b}(\xi-\eta)A(\xi,\eta)||_{S_\infty(\Delta_j\times\Delta_k)}\\ &\leq C2^{-k(s+t)},\end{aligned}$$

and (7.1) follows. □

THEOREM 7.1. *Suppose that A satisfies* A1 *and* A3(γ) *and that $s>0$, $t>0$ and $s+t<\gamma$. Then*

$$||T_b^{st}||\leq C||b||_{B_\infty^{s+t}}. \tag{7.9}$$

PROOF. An immediate consequence of Lemmas 7.1 and 2.3. □

This theorem does not apply when s or t equals 0, and in particular not to T_b itself. However, with stronger assumptions on A the theorem extends. The appropriate condition is analogous to A3 and implies that A vanishes at the axis: $A(\xi,0)=A(\eta,0)=0$.

THEOREM 7.2. *Suppose that A satisfies* A3(γ) *and the following sharpening of* A1*:*

$$||A||_{M(\Delta_j\times\Delta_k)}\leq a(j-k) \tag{7.10}$$

with

$$\sum_{-\infty}^{\infty}a(n)<\infty. \tag{7.11}$$

If $s\geq 0$, $t\geq 0$ and $s+t<\gamma$, then

$$||T_b^{st}||\leq C||b||_{B_\infty^{s+t}}. \tag{7.12}$$

PROOF. Lemma 7.1 and its proof yield

$$||\hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t||_{S_\infty(\Delta_j\times\Delta_k)}\leq C||b||_{B_\infty^{s+t}}a(j-k). \tag{7.13}$$

Hence Lemma 2.3 applies. □

REMARK 7.1. More generally, (7.12) holds whenever A3(γ) and (7.10) hold, $s+t<\gamma$ and $\sum_1^\infty 2^{-sn}a(-n)+\sum_1^\infty 2^{-tn}a(n)<\infty$.

In most cases of interest, A does not vanish at the axes as prescribed in Theorem 7.2. As a substitute we prove the following BMO result, which is sharp in many applications. We may assume that $t=0$, since the case $s=0$ is the same by symmetry (and the case $s>0$, $t>0$ is covered by Theorem 7.1). We take both the decomposition technique and a crucial special case from Coifman and Meyer (**1978**).

REMARK 7.2. Another substitute, in which L^2 is abandoned and replaced by a pair of Besov spaces, follows immediately from Lemma 7.1: If A satisfies A1 and A3 and $b\in B_\infty^0$, then T_b maps B_2^{01} into $B_2^{0\infty}$. For Hankel operators (Example 3), this (and a converse) is Janson, Peetre and Seemes (**1984**, Theorem 1).

THEOREM 7.3. *Suppose that A satisfies* A1, A2 *and* A3(γ) *and that* $0 \leq s < \gamma$. *Then*

$$||T_b^{s0}|| \leq C||b||_{I^s(\mathrm{BMO})} = C||I^{-s}b||_{\mathrm{BMO}}. \tag{7.14}$$

PROOF. Let δ, A_1, A_2 be as in condition A2. We may assume that $\delta < (2d+1)^{-1}$. Choose $\varphi \in C_0^\infty(\mathbf{R})$ with support in $(-\delta, \delta)$ and $\varphi = 1$ on $(0, \delta/2)$. Then $A = A_3 + A_4 + A_5$, where

$$A_3(\xi,\eta) = \varphi\left(\frac{|\eta|}{|\xi|}\right) A(\xi,\eta) = \varphi\left(\frac{|\eta|}{|\xi|}\right) A_1(\xi,\eta), \tag{7.15}$$

$$A_4(\xi,\eta) = \varphi\left(\frac{|\xi|}{|\eta|}\right) A(\xi,\eta) = \varphi\left(\frac{|\xi|}{|\eta|}\right) A_2(\xi,\eta), \tag{7.16}$$

$$A_5(\xi,\eta) = \psi\left(\frac{|\eta|}{|\xi|}\right) A(\xi,\eta), \quad \text{where } \psi(t) = 1 - \varphi(t) - \varphi(1/t). \tag{7.17}$$

We study the three pieces separately, thus treating the three problematic sets, viz. the diagonal and the axes, one at a time.

Since $\psi \in C_0^\infty(0,\infty)$, Lemma 3.4 implies that A_5 satisfies A1 and A3(γ). Furthermore, $A_5(\xi,\eta) = 0$ on $\Delta_j \times \Delta_k$ when $|j-k|$ is large enough ($> 2 - {}^2\log\delta$). Consequently, Theorem 7.2 yields

$$||T_b^{s0}(A_5)||_{S_\infty} \leq C||b||_{B_\infty^s} \leq C_1||b||_{I^s(\mathrm{BMO})}. \tag{7.18}$$

Next, let $G \in M(\mathbf{R}^d \times \mathbf{R}^d)$ be the function constructed in Lemma 3.6. Then

$$\hat{b}(\xi-\eta)A_3(\xi,\eta)|\xi|^s = |\xi-\eta|^s\hat{b}(\xi-\eta)G(\xi-\eta)\varphi(|\eta|/|\xi|)A_1(\xi,\eta). \tag{7.19}$$

Furthermore, let A_6 be as in (1.17) and satisfying (1.14) (the different φ have different meanings) (see §1, Example 9). Then, provided δ is chosen small enough,

$$\varphi(|\eta|/|\xi|) = \varphi(|\eta|/|\xi|)A_6(\xi,\eta) = (1-\psi(|\eta|/|\xi|))A_6(\xi,\eta). \tag{7.20}$$

Consequently, with $\beta = I^{-s}b$,

$$\hat{b}(\xi-\eta)A_3(\xi,\eta)|\xi|^s = \hat{\beta}(\xi-\eta)A_6(\xi,\eta)(1-\psi(|\eta|/|\xi|))G(\xi,\eta)A_1(\xi,\eta). \tag{7.21}$$

Thus, by Lemma 3.2 and Coifman and Meyer (**1978**, Theorem 33, p. 144),

$$\begin{aligned} ||T_b^{s0}(A_3)||_{S_\infty} &\leq ||1-\psi(|\eta|/|\xi|)||_M||G||_M||A_1||_M||T_\beta(A_6)||_{S_\infty} \\ &\leq C||\beta||_{\mathrm{BMO}} = C||b||_{I^s(\mathrm{BMO})}. \end{aligned} \tag{7.22}$$

A_4 is treated similarly using Lemmas 3.5, 3.6 and

$$\begin{aligned} &\hat{b}(\xi-\eta)A_4(\xi,\eta)|\xi|^s \\ &= |\xi-\eta|^s\hat{b}(\xi-\eta)\varphi\left(\frac{|\xi|}{|\eta|}\right)A_2(\xi,\eta)\left(\frac{|\xi|}{|\eta|}\right)^s\left(\frac{|\eta|}{|\xi-\eta|}\right)^s \\ &= \hat{\beta}(\xi-\eta)A_6(\eta,\xi)\left(1-\psi\left(\frac{|\xi|}{|\eta|}\right)\right)A_2(\xi,\eta)F(\eta,\xi)G(\eta,\xi). \quad \square \end{aligned} \tag{7.23}$$

The final theorem in this group does not require A3, and thus applies to Toeplitz-type operators.

THEOREM 7.4. *Suppose that A satisfies* A1 *and* A2. *Then*

$$\|T_b\| \le C\|b\|_{L^\infty}. \tag{7.24}$$

PROOF. We proceed as in the proof of Theorem 7.3, and make the same decomposition $A = A_3 + A_4 + A_5$. The argument above yields

$$\|T_b(A_3)\|_{S_\infty} + \|T_b(A_4)\|_{S_\infty} \le C\|b\|_{\mathrm{BMO}} \le C\|b\|_{L^\infty}. \tag{7.25}$$

Furthermore, since A_5 satisfies A1 and vanishes on $\Delta_j \times \Delta_k$ when $|j-k|$ is large, Lemma 3.12 yields $A_5 \in M(\mathbf{R}^d \times \mathbf{R}^d)$ and thus

$$\|T_b(A_5)\|_{S_\infty} \le \|A_5\|_M \|\hat{b}(\xi-\eta)\|_{S_\infty} = C\|b\|_{L^\infty}. \quad \square \tag{7.26}$$

8. S_p-direct results.

THEOREM 8.1. *Suppose that A satisfies* A1 *and* A3(γ). *Suppose further that* $1 \le p \le \infty$, $s+t+d/p < \gamma$ *and* $s,t > \max(-d/2, -d/p)$. *Then*

$$\|T_b^{st}\|_{S_p} \le C\|b\|_{B_p^{s+t+d/p}}. \tag{8.1}$$

The case $p=\infty$ is Theorem 7.1. We treat also the cases $p=1$ and $p=2$ separately.

LEMMA 8.1. *Suppose that A satisfies* A1 *and* A3(γ). *Suppose further that* $s,t > -d/2$ *and* $s+t+d < \gamma$. *If* $\operatorname{supp}\hat{b} \subset B(0,R)$, *then*

$$\|T_b^{st}\|_{S_1} \le CR^{s+t+d}\|b\|_{L^1}. \tag{8.2}$$

PROOF. By homogeneity, we may assume that $R=1$. Assume further that $\|b\|_{L^1} = 1$. Let Q_n denote the cube with center n and side 1, and let $\tilde{Q}_n$ denote the concentric cube with side 3 for $n \in \mathbf{Z}^d$. Thus, if $\operatorname{supp}\hat{f} \subset Q_n$, then $\operatorname{supp}(T_b^{st}f)^\wedge \subset \tilde{Q}_n$. If $|n| > 3d/\delta$, then $\|A\|_{M(\tilde{Q}_n \times \tilde{Q}_n)} \le C|n|^{-\gamma}$ by A3, and thus, using Lemmas 3.1 and 3.3,

$$\begin{aligned}
&\|\hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\chi_{Q_n}(\eta)\|_{S_1(\mathbf{R}^d\times\mathbf{R}^d)} \\
&\quad = \|\hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\|_{S_1(\tilde{Q}_n\times Q_n)} \\
&\quad \le \|\hat{b}(\xi-\eta)\|_{M(\mathbf{R}^d\times\mathbf{R}^d)}\|A\|_{M(\tilde{Q}_n\times Q_n)}\|\,|\xi|^s|\eta|^t\|_{S_1(\tilde{Q}_n\times Q_n)} \\
&\quad \le C|n|^{-\gamma}\|\,|\xi|^s|\eta|^t\|_{S_1(\tilde{Q}_n\times Q_n)} \\
&\quad = C|n|^{-\gamma}\|\,|\xi|^s\|_{L^2(\tilde{Q}_n)}\|\,|\eta|^t\|_{L^2(Q_n)} \le C_1|n|^{-\gamma+s+t}.
\end{aligned} \tag{8.3}$$

This is good for n large. When n is small, we use the standard dyadic decomposition. By Lemmas 3.1 and 3.3 and A1,

$$\begin{aligned}
&\|\hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\|_{S_1(\Delta_j\times\Delta_k)} \\
&\quad \le C\|\,|\xi|^s|\eta|^t\|_{S_1(\Delta_j\times\Delta_k)} \\
&\quad = C\|\,|\xi|^s\|_{L^2(\Delta_j)}\|\,|\eta|^t\|_{L^2(\Delta_k)} = C_1 2^{j(s+d/2)}2^{k(t+d/2)}.
\end{aligned} \tag{8.4}$$

Put $E = \bigcup_{|n|\le 3d/\delta} Q_n$ and $\tilde{E} = \bigcup_{|n|\le 3d/\delta} \tilde{Q}_n$. Since $E \subset \tilde{E} \subset \bigcup_{-\infty}^{N} \Delta_k$ for some $N < \infty$, it follows that

$$
\begin{aligned}
&\|\hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\chi_E(\eta)\|_{S_1(\mathbf{R}^d\times\mathbf{R}^d)}\\
&\quad = \|\hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\|_{S_1(\tilde{E}\times E)}\\
&\quad = \left\|\hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\sum_{-\infty}^{N}\chi_{\Delta_j}(\xi)\sum_{-\infty}^{N}\chi_{\Delta_k}(\eta)\right\|_{S_1(\tilde{E}\times E)} \qquad (8.5)\\
&\quad \le \sum_{-\infty}^{N}\sum_{-\infty}^{N}\|\hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\|_{S_1(\Delta_j\times\Delta_k)} \le C.
\end{aligned}
$$

By (8.5) and (8.3),

$$
\|\hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\|_{S_1} \le C + C\sum_{|n|>3d/\delta}|n|^{-\gamma+s+t} = C_1. \quad \square \tag{8.6}
$$

If $b \in B_1^{s+t+d}$, and $b = \sum b_k$ is a dyadic decomposition as in (2.5), then Lemma 8.1 yields

$$
\|T_b^{st}\|_{S_1} \le \sum_k \|T_{b_k}^{st}\|_{S_1} \le C\sum_k 2^{k(s+t+d)}\|b_k\|_{L^1} = C\|b\|_{B_1^{s+t+d}}. \tag{8.7}
$$

REMARK 8.1. The argument above is a standard method to prove that a linear mapping T into a Banach space E is bounded on B_1^σ; if homogeneity of the right order applies, it suffices to show $\|T_b\|_E \le C\|b\|_{L^1}$ when $\operatorname{supp}\hat{b}$ lies in a fixed ball (or annulus). In fact, if further translation invariance holds, as in this case, then it suffices that $Tb_0 \in E$ for a single suitable function b_0, because B_1^σ is the minimal translation and dilation invariant Banach space that contains b_0, and thus $B_1^\sigma \subset \{b\colon\ Tb \in E\}$ (see e.g. Peetre (**1985a**), (**1985b**) for further remarks on minimal spaces).

The computations for S_2 are, as usual, simple, and we can use simpler assumptions on A than A1 and A3. Note that condition (8.8) below is unsymmetric only in appearance in view of the boundedness of A.

LEMMA 8.2. *Suppose that A is bounded and*

$$
|A(\xi,\eta)| \le C\left(\frac{|\xi-\eta|}{|\xi|}\right)^\gamma. \tag{8.8}
$$

If $s,t > -d/2$ and $s+t+d/2 < \gamma$, then

$$
\|T_b^{st}\|_{S_2} \le C\|b\|_{B_2^{s+t+d/2}}. \tag{8.9}
$$

PROOF.

$$
\begin{aligned}
\|T_b^{st}\|_{S_2}^2 &= (2\pi)^{-2d}\iint |\hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t|^2\,d\eta\,d\xi\\
&= (2\pi)^{-2d}\int |\hat{b}(\xi)|^2\int |A(\xi+\eta,\eta)|^2|\xi+\eta|^{2s}|\eta|^{2t}\,d\eta\,d\xi\\
&\le C\int |\hat{b}(\xi)|^2\int \min\left(1,\left(\frac{|\xi|}{|\eta|}\right)^{2\gamma}\right)|\xi+\eta|^{2s}|\eta|^{2t}\,d\eta\,d\xi \qquad (8.10)\\
&= C_1\int |\hat{b}(\xi)|^2|\xi|^{2s+2t+d}\,d\xi = C_2\|b\|_{B_2^{s+t+d/2}}^2. \quad \square
\end{aligned}
$$

Completion of the proof of Theorem 8.1. We use interpolation between the already proved cases $p = 1, 2, \infty$. For $2 < p < \infty$ we may argue as follows. First, extend the definition (5.1) of T_b^{st} to arbitrary complex s and t. It is clear that

$$||T_b^{st}||_{S_p} = ||T_b^{\mathrm{Re}\, s, \mathrm{Re}\, t}||_{S_p}, \tag{8.11}$$

because the factors $|\xi|^{i\,\mathrm{Im}\, s}$ and $|\eta|^{i\,\mathrm{Im}\, t}$ correspond to unitary operators. Let s_0 and t_0 be real numbers such that $s_0 + t_0 + d/2 < \gamma$, $s_0 > -d/2$, $t_0 > -d/2$, and let s_1 and t_1 be two real numbers such that $s_1 + t_1 < \gamma$, $s_1 > 0$, $t_1 > 0$. For $0 \le \mathrm{Re}\, z \le 1$ put $s(z) = s_0(1-z) + s_1 z$, $t(z) = t_0(1-z) + t_1 z$, $1/p(z) = \mathrm{Re}(1-z)/2 + \mathrm{Re}\, z/\infty$, and

$$T(z)b = T_b^{s(z),t(z)}. \tag{8.12}$$

Then $\{T(z)\}$ is an analytic family of linear operators (mapping functions into operators), and by interpolation $T(z)$ maps $B^{p(z)\mathrm{Re}\, s(z) + \mathrm{Re}\, t(z) + d/p(z)}$ into $S_{p(z)}$ because this is so if $\mathrm{Re}\, z = 0$ (Lemma 8.2) and if $\mathrm{Re}\, z = 1$ (Theorem 7.1). Choosing z real, $0 < z < 1$, and rewriting $s = s(z)$, $t = t(z)$, $p = p(z)$, we have $s + t + d/p < \gamma$, $s > -d/p$, $t > -d/p$. Conversely, any $p \in (2, \infty)$ and any pair s, t subject to these inequalities can be recaptured in this way with appropriate choices of z, s_0, t_0, s_1, t_1.

The case $1 < p < 2$ is similar. □

9. Converse results: Besov spaces. We prove converses to Theorem 8.1, under appropriate conditions on A, by duality (cf. Peller (**1980**)). We therefore consider besides the kernel A, a second kernel A_1. The main step is contained in the following lemma.

LEMMA 9.1. *Suppose that* $1 \le p \le \infty$, s *and* t *real,* $1/p + 1/q = 1$, $\gamma_1 > s_1 + t_1 + d/q$ *and* $s_1, t_1 > \max(-d/2, -d/q)$. *Suppose further that* A_1 *satisfies* A1 *and* A3(γ_1) *and let*

$$w(\xi) = |\xi|^{-s-t-s_1-t_1-d} \int A(\xi+\eta, \eta)\overline{A_1(\xi+\eta,\eta)}|\xi+\eta|^{s+s_1}|\eta|^{t+t_1}\, d\eta. \tag{9.1}$$

Then

$$||\hat{b}w||_{\hat{B}_p^{s+t+d/p}} \le C||T_b^{st}||_{S_p}. \tag{9.2}$$

PROOF. Theorem 8.1 gives

$$||T_g^{s_1 t_1}(A_1)||_{S_q} \le C||g||_{B_q^{s_1+t_1+d/q}}. \tag{9.3}$$

Furthermore, with $\sigma = s + s_1 + t + t_1 + d$ and $\hat{\beta}(\xi) = \hat{b}(\xi)w(\xi)$,

$$\begin{aligned}
&(2\pi)^{2d}\mathrm{Tr}(T_b^{st}(A)T_g^{s_1t_1}(A_1)^*) \\
&= \iint \hat{b}(\xi-\eta)\overline{\hat{g}(\xi-\eta)}A(\xi,\eta)\overline{A_1(\xi,\eta)}|\xi|^{s+s_1}|\eta|^{t+t_1}\, d\xi\, d\eta \\
&= \iint \hat{b}(\xi)\overline{\hat{g}(\xi)}A(\xi+\eta,\eta)\overline{A_1(\xi+\eta,\eta)}|\xi+\eta|^{s+s_1}|\eta|^{t+t_1}\, d\xi\, d\eta \\
&= \int \hat{b}(\xi)\overline{\hat{g}(\xi)}|\xi|^\sigma w(\xi)\, d\xi = \int (I^{-\sigma}\beta)^\wedge(\xi)\overline{\hat{g}(\xi)}\, d\xi = (2\pi)^d\langle I^{-\sigma}\beta, g\rangle.
\end{aligned} \tag{9.4}$$

Consequently,

$$\begin{aligned}|\langle I^{-\sigma}\beta, g\rangle| &= C|\mathrm{Tr}(T_b^{st}(A)T_g^{s_1t_1}(A_1)^*)| \le C||T_b^{st}(A)||_{S_p}||T_g^{s_1t_1}(A_1)||_{S_q} \\ &\le C||T_b^{st}(A)||_{S_p}||g||_{B_q^{s_1+t_1+d/q}}.\end{aligned} \tag{9.5}$$

Since g is arbitrary and $(B_q^u)^* = B_p^{-u}$ (with a standard modification if $q = \infty$), this implies

$$||\beta||_{B_p^{s+t+d/p}} = ||I^{-\sigma}\beta||_{B_p^{-s_1-t_1-d/q}} \le C||T_b^{st}(A)||_{S_p}. \quad \square \tag{9.6}$$

Let $\mathcal{M}(B_p^s)$ denote the algebra of Fourier multipliers on B_p^s, i.e. $\{w\colon\ b \to (w\hat{b})^\wedge$ is a bounded operator in $B_p^s\}$.

REMARK 9.1. It is easy to see that $\mathcal{M}(B_p^s)$ is independent of s, and that $w \in \mathcal{M}(B_p^s)$ iff $w\hat{\psi}_k$ are Fourier multipliers on L^p, uniformly in k, where ψ_k is as in (2.2). In other words, $\mathcal{M}(B_p^s)$ may be viewed as a "Besov space" based on the space of multipliers on L^p (cf. Peetre (**1976**)).

In order to obtain the sought implication from Lemma 9.1, we need to know that $w^{-1} \in \mathcal{M}(B_p^{s+t+d/p})$. In the homogeneous case, we can use the following Tauberian result.

LEMMA 9.2. *Suppose that $w \in \mathcal{M}(B_1^s)$ for some s and that $w(r\xi) = w(\xi)$ for some $r > 1$. If $w(\xi) \neq 0$ for $\xi \neq 0$, then $w^{-1} \in \mathcal{M}(B_p^t)$ for all t, p, $1 \le p \le \infty$.*

PROOF. $\hat{L}^1 = \{f\colon\ f \in L^1(\mathbf{R}^d)\}$ is a Banach algebra (without unit) with maximal ideal space $\mathbf{R}^d$. Let $\Delta = \{\xi\colon\ 1 \le |\xi| \le 8r\}$ and let $I = \{\hat{f} \in \hat{L}^1\colon\ f(\xi) = 0 \text{ on } \Delta\}$. I is a closed ideal in $\hat{L}^1$ and $\hat{L}^1/I = \{\hat{f}|_\Delta\colon\ f \in L^1\}$ is a Banach algebra (with unit) whose maximal ideal space equals I.

Choose $\varphi \in C_0^\infty$ with support in a larger annulus such that $\varphi = 1$ on Δ. Then $\varphi \in \hat{B}_1^s$, whence $w\varphi \in \hat{B}_1^s$ and $w\varphi \in \hat{L}^1$. Thus $w|_\Delta = w\varphi|_\Delta \in \hat{L}^1/I$. Since $w \neq 0$ on the maximal ideal space Δ, $w^{-1}|_\Delta \in \hat{L}^1/I$, i.e. there exists $h \in L^1$ with $\hat{h} = w^{-1}$ on Δ.

Suppose that $\hat{\psi}$ is a test function supported in $\tilde{\Delta}_0$ that satisfies (2.4), with ψ_k defined by (2.2). Let $\beta \in B_p^t$ and $\hat{b} = w^{-1}\hat{\beta}$. If $\tilde{\Delta}_k \subset \Delta$, then

$$\widehat{\psi_k * b} = \hat{\psi}_k w^{-1}\hat{\beta} = \hat{h}\hat{\psi}_k\hat{\beta} = \widehat{h * \psi_k * \beta},$$

and thus

$$||\psi_k * b||_{L^p} \le ||h||_{L^1}||\psi_k * \beta||_{L^p}. \tag{9.7}$$

By homogeneity (each $\tilde{\Delta}_k \subset r^j\Delta$ for some j), (9.7) holds for every k. Hence $||b||_{B_p^t} \le C||\beta||_{B_p^t}$. $\square$

THEOREM 9.1. *Suppose that A satisfies* A0, A1, A3, A4. *Then, for all real s and t, and $1 \le p \le \infty$,*

$$||b||_{B_p^{s+t+d/p}} \le C||T_b^{st}||_{S_p}. \tag{9.8}$$

PROOF. Choose s_1 and t_1 such that $s_1, t_1 > 0$ and $s + s_1$, $t + t_1 > 0$. Choose $s' = t'$ such that $-d/2 < s' < \gamma/2 - d/2$ and $s' < 0$. Let $s_1' = s + s_1 - s'$ and $t_1' = t + t_1 - t'$. Find an integer k such that $(2k-1)\gamma > \max(s_1 + t_1 + d, s_1' + t_1')$.

Let $A_1(\xi,\eta) = A(\xi,\eta)^k\overline{A(\xi,\eta)}^{k-1}$. Then A_1 satisfies A1 and A3($(2k-1)\gamma$). Let $p'=1$, $q'=\infty$. Lemma 9.1, applied to $p', s', t', \dots$, and Theorem 8.1 yield

$$(9.9)\qquad ||\hat{b}w||_{\hat{B}_1^{s'+t'+d}} \le C||T_b^{s't'}||_{S_1} \le C||b||_{B_1^{s'+t'+d}}.$$

Thus $w \in \mathcal{M}(B_1^{s'+t'+d})$. w is homogeneous by A0 and $w > 0$ by A4. Hence Lemmas 9.2 and 9.1, this time applied to $p, s, t, \dots$ (which gives the same w because $s + s_1 = s' + s_1'$ and $t + t_1 = t' + t_1'$), yield

$$(9.10)\qquad ||b||_{B_p^{s+t+d/p}} \le C||\hat{b}w||_{\hat{B}_p^{s+t+d/p}} \le C_1||T_b^{st}||_{S_p}. \quad \square$$

In the nonhomogeneous case, the Banach algebra argument in the proof of Lemma 9.2 shows only that w^{-1} locally is in $\hat{L}^1$, but not the required uniform norm estimates. Such estimates can be obtained from smoothness conditions on w, and thus from smoothness conditions on A. We will, however, not pursue this, but prove instead the following result.

THEOREM 9.2. *Suppose that A satisfies* A7. *If s and t are real numbers and $1 \le p \le \infty$, then*

$$(9.11)\qquad ||b||_{B_p^{s+t+d/p}} \le C||T_b^{st}||_{S_p}.$$

PROOF. Let δ_1, δ_2, A_1 be as in A7 and let $\varphi \in C_0^\infty(0,\infty)$ with $\operatorname{supp}\varphi \subset (\delta_1,\delta_2)\backslash\{1\}$ and φ not identically 0. Then, using Lemma 3.4, $\varphi(|\eta|/|\xi|) \in M(\mathbf{R}^d \times \mathbf{R}^d)$ and satisfies A0, A1, A3(∞), A4 and we obtain by Theorem 9.1 and Lemma 3.1

$$(9.12)\qquad \begin{aligned} ||b||_{B_p^{s+t+d/p}} &\le C||T_b^{st}(\varphi(|\eta|/|\xi|))||_{S_p} = C||T_b^{st}(\varphi(|\eta|/|\xi|)A_1A)||_{S_p} \\ &\le C||\varphi(|\eta|/|\xi|)||_M||A_1||_M||T_b^{st}||_{S_p}. \quad \square \end{aligned}$$

10. Converse results: BMO; the lost theorem. * In the preceding section we proved in particular that (under appropriate conditions on A), T_b bounded $\Rightarrow b \in B_\infty^0$. While this sometimes is sharp by Theorem 7.2, it is in other cases possible to improve this to $b \in$ BMO and to obtain a converse to Theorem 7.3.

THEOREM 10.1. *Suppose that A satisfies* A0, A1, A3, A5. *Then*

$$(10.1)\qquad ||b||_{I^s(\mathrm{BMO})} = ||I^{-s_b}||_{\mathrm{BMO}} \le C||T_b^{s0}||.$$

PROOF. Note that A5 and A0 imply that if $\xi_0 \ne 0$, k is large enough and $\eta \in V$, then $(r^k\xi_0 + \eta, \eta) \in U \times V$ and thus $A(\xi_0 + r^{-k}\eta, r^{-k}\eta) = A(r^k\xi_0 + \eta, \eta) \ne 0$. Thus A4 holds, and Theorem 9.1 yields

$$(10.2)\qquad ||b||_{B_\infty^s} \le C||T_b^{s0}||.$$

For simplicity, we assume that $s = 0$ and indicate the modifications when $s \ne 0$ at the end of the proof. We further assume that $||T_b|| = 1$. We have to show that for every ball B in $\mathbf{R}^d$, there exists a constant a_B such that

$$(10.3)\qquad \int_B |b - a_B| \le C|B|.$$

*Lost, not last. The proof of this theorem was found by one of the authors but then the other author mislaid the manuscript. This delayed further progress for more than a year.

By homogeneity, it suffices to prove (10.3) for $B = B(0,1)$. Let $|\xi_0| = 1$ and let δ, η_0, U, V be as in A5. If $f \in L^2$ with $\operatorname{supp} \hat{f} \subset V$, then

$$
\begin{aligned}
||\hat{b} * \hat{f}||_{L^2(U)} &= \left|\left| \int A(\xi,\eta)^{-1} \hat{b}(\xi - \eta) A(\xi,\eta) \hat{f}(\eta)\, d\eta \right|\right|_{L^2(U)} \\
&\le (2\pi)^d ||A^{-1}||_{M(U\times V)} ||T_b||\, ||\hat{f}||_{L^2(V)} \le C||f||_{L^2}.
\end{aligned} \tag{10.4}
$$

Hence, if $g \in L^2$ with $\operatorname{supp} \hat{g} \subset B(0,\delta)$, we obtain by letting $\hat{f}(\eta) = \hat{g}(\eta - \eta_0)$,

$$
||\widehat{bg}||_{L^2(U-\eta_0)} = (2\pi)^{-d} ||\hat{b} * \hat{g}||_{L^2(U-\eta_0)} = (2\pi)^{-d} ||\hat{b} * \hat{f}||_{L^2(U)} \le C||g||_{L^2}. \tag{10.5}
$$

By compactness, we may choose a finite number of points $\xi_0^{(1)}, \dots, \xi_0^{(N)}$ on the unit sphere such that the corresponding sets $U^{(j)}$ cover $\{\xi\colon |\xi| > 1\}$. Then

$$
\bigcup_{j=1}^{N} (U^{(j)} - \eta_0^{(j)}) \supset \{\xi\colon |\xi| > R\}
$$

for some large R. We fix $g \in L^2$ with $|g(x)| \ge 1$ when $|x| < 1$ and $\operatorname{supp} \hat{g} \subset B(0,\epsilon)$, where $\epsilon = \min_{j \le N} \delta^{(j)}$. Then

$$
||\widehat{bg}||_{L^2(\mathbf{R}^d \setminus B(0,R))} \le \sum_{1}^{N} ||\widehat{bg}||_{L^2(U^{(j)} - \eta_0^{(j)})} \le C. \tag{10.6}
$$

We may assume that $\epsilon < 1$ and $R = 2^m$ for some $m \ge 1$. Since $||b||_{B^0_\infty} \le C$ by (10.2), $b = \sum_{-\infty}^{\infty} b_k$ where $\operatorname{supp} \hat{b}_k \subset \tilde{\Delta}_k$ and $||b_k||_{L^\infty} \le C$ (cf. (2.5)). Put $b_- = \sum_{-\infty}^{m-3} b_k$ and $b_+ = \sum_{m-2}^{\infty} b_k$. Then $\operatorname{supp}(\widehat{gb_-}) = \operatorname{supp}(\hat{g} * \hat{b}_-) \subset B(0,R)$, whence

$$
||\widehat{gb_+}||_{L^2(\mathbf{R}^d \setminus B(0,R))} = ||\widehat{gb}||_{L^2(\mathbf{R}^d \setminus B(0,R))} \le C. \tag{10.7}
$$

On the other hand, $\operatorname{supp} \widehat{gb_k} \cap B(0,R) = \varnothing$ for $k \ge m+2$. Thus

$$
\begin{aligned}
||\widehat{gb_+}||_{L^2(B(0,R))} &\le \sum_{m-2}^{\infty} ||\widehat{gb_k}||_{L^2(B(0,R))} \\
&\le \sum_{m-2}^{m+1} ||gb_k||_{L^2} \le C \sum_{m-2}^{m+1} ||b_k||_{L^\infty} \le C_1.
\end{aligned} \tag{10.8}
$$

Hence, combining (10.7) and (10.8),

$$
||b_+||_{L^2(B(0,1))} \le ||gb_+||_{L^2(\mathbf{R}^d)} = (2\pi)^{-d/2} ||\widehat{gb_+}||_{L^2} \le C. \tag{10.9}
$$

For b_- we use the standard estimates

$$
||\nabla b_k||_\infty \le C 2^k ||b_k||_\infty \le C_1 2^k, \tag{10.10}
$$

and thus, if $|x| \le 1$,

$$
|b_-(x) - b_-(0)| \le \sum_{-\infty}^{m-3} ||\nabla b_k||_\infty \le C. \tag{10.11}
$$

Consequently,

$$\int_B |b - b_-(0)| \le \int_B |b_- - b_-(0)| + ||b_+||_{L^1(B)} \le C. \tag{10.12}$$

This completes the proof when $s = 0$.

For $s \neq 0$, we define $\beta = I^{-s}(b) \in B^0_\infty$ and use Lemma 3.6 and

$$\hat{\beta}(\xi - \eta) = A(\xi,\eta)^{-1}(|\xi - \eta|/|\xi|)^s \hat{b}(\xi - \eta) A(\xi,\eta)|\xi|^s$$

on $U \times V$ to conclude as above that $\beta \in \mathrm{BMO}$. □

In the nonhomogeneous case we use the same argument and Theorem 9.2.

THEOREM 10.2. *Suppose that* A *satisfies* A6, A7. *Then*

$$||b||_{I^s(\mathrm{BMO})} \le C||T_b^{s0}||. \quad \square \tag{10.13}$$

REMARK 10.1. The second half of the proof above contains a result on the space BMO which perhaps is of independent interest. It may be stated as follows:

PROPOSITION. *Let* $b \in B^{0\infty}_\infty$ *and assume that there exist constants* C *and* R *such that*

$$||\widehat{bg_{r,x_0}}||_{L^2(\mathbf{R}^d \setminus B(0,R/r))} \le Cr^{d/2} \tag{10.14}$$

for any $x_0 \in R^d$ *and* $r > 0$. *Here* $g_{r,x_0}(x) = g((x - x_0)/r)$ *and, as before,* g *is a fixed function in* L^2 *with* $|g(x)| \ge 1$ *for* $|x| \le 1$ *and* $\operatorname{supp}\hat{g} \subset B(0,\epsilon)$ *for some* $\epsilon > 0$. *Then* $b \in \mathrm{BMO}$.

PROOF. In view of translation and dilation invariance, it suffices to show that (10.3) holds for $B = B(0,1)$ with a constant that depends only on $||b||_{B^0_\infty}$ and C and R in (10.14). This was done in (10.7)–(10.12) above (solely utilizing (10.14) for $r = 1$, $x_0 = 0$). □

The point of this proposition is that it is often exceedingly simpler to verify that a function is in B^0_∞ than that it is in BMO.

11. The order of the zero at the diagonal. The conditions $s + t + d/p < \gamma$ in Theorem 5.1 and $s < \gamma$ in Theorems 5.2 and 5.3 can in general not be relaxed. In fact, the typical case is that (choosing γ in A3 as large as possible), T_b^{st} never belongs to S_p when $s + t + d/p \ge \gamma$ and $p < \infty$, or when $s + t > \gamma$ and $p = \infty$ (except when $T_b^{st} = 0$). More precisely we have the following result. For simplicity we only consider the case when A is homogeneous and C^∞ at the diagonal.

THEOREM 11.1. *Suppose that* A *satisfies* A0 *and is infinitely differentiable in a neighborhood of the diagonal* $\{(\xi,\xi): \xi \neq 0\}$. *Suppose further that* $k \ge 1$ *is an integer such that*

$$D^\alpha A(\xi,\xi) = 0 \quad \textit{for } |\alpha| \le k - 1 \tag{11.1}$$

but for each $\xi_0 \neq 0$ *there exists* $\xi_1 \neq 0$ *with*

$$D^k_{\xi_0} A(\xi_1,\xi_1) \neq 0. \tag{11.2}$$

Then, if $T_b^{st} \in S_p$ $(1 \le p \le \infty)$ *one of the following three conditions holds:*

(i) $s + t + d/p < k$ *and* $p < \infty$,

(ii) $s + t \le k$ *and* $p = \infty$,

(iii) b *is a polynomial.*

Before giving the proof we make some remarks. Note first that (11.1), Lemma 3.10 and homogeneity imply that A3(k) holds, while (11.2) implies that A3(γ) does not hold for any $\gamma > k$. Hence $\gamma = k$ is the best possible value.

An obvious technical defect in the theorem is that (iii) does not read "$T_b^{st} = 0$" as we have led the reader to expect. In fact, in most applications, if b is a polynomial then either $T_b^{st} = 0$ or T_b^{st} is not bounded, but that depends on the precise definition of T_b^{st}. Recall that if b is a polynomial, then $\hat{b}$ is supported by $\{0\}$, whence $\hat{b}(\xi - \eta)$ is supported on the diagonal where $A(\xi, \eta)$ vanishes. Hence some care is required in interpreting (0.1) or (5.1). In any case, the operator can be written $(T_b^{st} f)^\wedge = \sum m_\alpha D^\alpha \hat{f}$. Such an operator is bounded iff $m_\alpha = 0$ for $|\alpha| \geq 1$ and $m_0 \in L^\infty$, and it is never compact unless it vanishes. Hence no problem should arise when $p < \infty$, but in Example 11 below we will exhibit an operator of the type T_b^{st} with b a polynomial that is bounded without being zero.

Note further that for $p < \infty$ we have obtained a dichotomy (provided all other conditions hold): If $s + t + d/p \geq k$ then $T_b^{st} \in S_p$ is impossible by the present theorem, and if $s + t + d/p < k = \gamma$ then Theorem 5.1 applies. However, for $p = \infty$ there is a gap since Theorem 11.1 allows $s + t = k$. In fact, in general there are many bounded operators T_b^{st} with $s + t = k$ (see Example 8, §6 with $d = 1$ and $s = 1$ where $T_b^{10}(A_2) \in S_\infty$ iff $db/dx \in L^\infty$ (here $k = 1$)). This example also implies that neither Theorem 5.2 nor 5.3 extends to $s = \gamma$.

PROOF. For simplicity we assume that A is homogeneous, i.e. A0 holds for every $r > 0$. The general case is similar. By the homogeneity and a compactness argument there exists $\epsilon > 0$ such that if $\Gamma = \{(\xi, \eta)\colon |\xi - \eta| < \epsilon|\xi|\}$, then $A \in C^\infty(\Gamma)$ and

$$|D^\alpha A(\xi, \eta)| \leq C_\alpha |\xi|^{-\alpha}, \qquad (\xi, \eta) \in \Gamma. \tag{11.3}$$

Assume that (iii) does not hold. Let $\xi_0 \in \operatorname{supp} \hat{b}$ with $\xi_0 \neq 0$ and let ξ_1 be such that (11.2) holds. By continuity and homogeneity there exists a cone $W = \{\xi\colon |\xi/|\xi| - \xi_1/|\xi_1|| < \delta\}$ with

$$|D_{\xi_0}^k A(\xi, \xi)| > c|\xi|^{-k}, \quad \text{when } \xi \in W. \tag{11.4}$$

Thus, if $\xi \in W$ and $|\xi|$ is large enough, by Taylor's formula, (11.1), (11.4) and (11.3),

$$\begin{aligned} |A(\xi + \xi_0, \xi)| &\geq \frac{1}{k!}|D_{\xi_0}^k A(\xi, \xi)| - \frac{1}{(k+1)!} \sup_{0 \leq t \leq 1} |D_{\xi_0}^{k+1} A(\xi + t\xi_0, \xi)| \\ &\geq c_1 |\xi|^{-k} - c_2 |\xi|^{-k-1} \geq c_3 |\xi|^{-k}. \end{aligned} \tag{11.5}$$

Furthermore, if $0 < r < 1/2$ and $\xi_1 \in W$, define temporarily $f(\xi, \eta) = A(\xi + \xi_0, \eta) - A(\xi_1 + \xi_0, \xi_1)$ and conclude from Lemma 3.10 (using $f(\xi_1, \xi_1) = 0$), Taylor's formula, (11.1) and (11.3) that, provided $|\xi_1|$ is large enough,

(11.6)
$$\begin{aligned} ||f||_{M(B(\xi_1, r) \times B(\xi_1, r))} &\leq C \sup_{1 \leq |\alpha| \leq d+1} r^{|\alpha|} \sup_{\xi, \eta \in B(\xi_1, 2r)} |D^\alpha f(\xi, \eta)| \\ &\leq Cr \sup_{|\alpha| \leq d+1} \sup_{\xi, \eta \in B(\xi_1, 1 + |\xi_0|)} |D^\alpha A(\xi, \eta)| \leq C_1 r |\xi_1|^{-k}. \end{aligned}$$

We may by (11.5) choose r small enough (independent of ξ_1) such that the right-hand side of (11.6) is smaller than $\frac{1}{2}|A(\xi_1+\xi_0,\xi_1)|$. Consequently, $A(\xi+\xi_0,\eta) = A(\xi_1+\xi_0,\xi_1)+f(\xi,\eta)$ is invertible in $M(B(\xi_1,r)\times B(\xi_1,r))$ and

$$\begin{aligned}||A^{-1}||_{M(B(\xi_1+\xi_0,r)\times B(\xi_1,r))} &= ||A(\xi+\xi_0,\eta)^{-1}||_{M(B(\xi_1,r)\times B(\xi_1,r))}\\ &\le 2|A(\xi_1+\xi_0,\xi_1)|^{-1} \le C|\xi_1|^k,\end{aligned} \tag{11.7}$$

provided $\xi_1 \in W$ and $|\xi_1|$ is large enough.

Let $E = \{n \in \mathbf{Z}^d \cap W\colon |n| > N\}$, where N is a sufficiently large constant, and define $U_n = B(n+\xi_0, r)$ and $V_n = B(n,r)$ for $n \in E$. Let

(11.8)
$$A_1(\xi,\eta) = \begin{cases} |n|^{s+t-k}|\xi|^{-s}|\eta|^{-t}A(\xi,\eta)^{-1} & \text{if } \xi \in U_n,\ \eta \in V_n \text{ for some } n \in E,\\ 0 \quad \text{otherwise.} \end{cases}$$

Since, by (11.7),

$$||A_1||_{M(U_n\times V_n)} \le C|n|^{s+t-k}|n|^{-s}|n|^{-t}|n|^k = C, \tag{11.9}$$

Lemma 3.11 yields $A_1 \in M(\mathbf{R}^d\times\mathbf{R}^d)$. Consequently,

$$\begin{aligned}||A_1||_M||T_b^{st}||_{S_p} &\ge ||\hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t A_1(\xi,\eta)||_{S_p}\\ &= \left|\left|\hat{b}(\xi-\eta)\sum_E |n|^{s+t-k}\chi_{U_n}(\xi)\chi_{V_n}(\eta)\right|\right|_{S_p}\\ &= ||\,||\hat{b}(\xi-\eta)|n|^{s+t-k}||_{S_p(U_n\times V_n)}||_{l^p(E)}\\ &= ||\,|n|^{s+t-k}||_{l^p(E)}||\hat{b}(\xi-\eta)||_{S_p(B(\xi_0,r)\times B(0,r))}.\end{aligned} \tag{11.10}$$

Since $\xi_0 \in \operatorname{supp}\hat{b}$, $\hat{b}(\xi-\eta)$ does not vanish on $B(\xi_0,r)\times B(0,r)$, and (11.10) implies that $\{|n|^{s+t-k}\} \in l^p(E)$. Thus $p<\infty$ and $s+t-k < -d/p$ or $p=\infty$ and $s+t-k\le 0$, i.e. (i) or (ii) holds. □

EXAMPLE 11. Let $d=3$. Then $T_bf = \Delta b\cdot I^2f + d^3b/dx_1dx_2dx_3\cdot I^3f$ is a paracommutator with

$$A(\xi,\eta) = -|\xi-\eta|^2|\eta|^{-2} - i(\xi_1-\eta_1)(\xi_2-\eta_2)(\xi_3-\eta_3)|\eta|^{-3}. \tag{11.11}$$

This A satisfies the conditions of Theorem 11.1 with $k=2$. However, if b is the polynomial $x_1x_2x_3$, then $T_bf = I^3f$ and thus $T_b^{30} = T_b^{03}$ equals the identity mapping on L^2. Thus $T_b^{30} \in S_\infty$ although $3+0>k$.

ACKNOWLEDGMENT. The proof above is partly based on ideas by Peng (personal communication).

12. Toeplitz-like operators. We consider again the case when A does not vanish on the diagonal.

THEOREM 12.1. *Suppose that A satisfies* A8. *Then* $||b||_{L^\infty} \le C||T_b||$.

PROOF. With B_n as in A8 and $B_n' = B(0,n)$,

$$\begin{aligned}||\hat{b}(\xi-\eta)||_{S_\infty(B_n'\times B_n')} &= ||\hat{b}(\xi-\eta)||_{S_\infty(B_n\times B_n)}\\ &\le ||\hat{b}(\xi-\eta)A(\xi,\eta)||_{S_\infty}||A^{-1}||_{M(B_n\times B_n)} \le C||T_b||.\end{aligned} \tag{12.1}$$

Thus $|\iint \hat{b}(\xi-\eta)f(\eta)\overline{g(\xi)}| \le C||T_b||\,||f||_{L^2}||g||_{L^2}$ when f and g have support in B_n'. Since n is arbitrary, this holds for all f and g with compact support. Since such functions are dense in L^2,

$$||\hat{b}(\xi-\eta)||_{S_\infty} \le C||T_b||. \tag{12.2}$$

The proof is completed by Lemma 2.1. □

THEOREM 12.2. *Suppose that A satisfies* A8. *If T_b is compact, then $b = 0$.*

PROOF. Fix $r > 0$. It is possible to find a sequence $\{\xi'_n\}_1^\infty$ such that the balls $B'_n = B(\xi'_n, r)$ are disjoint and $\|A^{-1}\|_{M(B'_n \times B'_n)} \leq C$, $n = 1, 2, \ldots$. (We can achieve this by induction, taking ξ'_n as any point in $B(\xi_N, N - r) \setminus \bigcup_1^{n-1} B(\xi'_k, 2r)$, where N is such that this set is nonempty (any $N > 2rn$ will do).) Let

$$A_1(\xi, \eta) = \begin{cases} A(\xi, \eta)^{-1} & \text{if } (\xi, \eta) \in \bigcup_n B'_n \times B'_n, \\ 0 & \text{otherwise.} \end{cases} \tag{12.3}$$

Then $T_b(AA_1)$ is compact by Lemma 3.2. Let $\hat{f}$ and $\hat{g}$ belong to L^2 and have support in $B(0, r)$ and define $\hat{f}_n(\xi) = \hat{f}(\xi - \xi'_n)$, $\hat{g}_n(\xi) = \hat{g}(\xi - \xi'_n)$. Thus $\hat{f}_n$ and $\hat{g}_n$ have support in B'_n, whence

$$\begin{aligned} (2\pi)^d \langle T_b(AA_1) f_n, g_n \rangle &= \iint \hat{b}(\xi - \eta) A(\xi, \eta) A_1(\xi, \eta) \hat{f}_n(\eta) \overline{\hat{g}_n(\xi)} \\ &= \iint \hat{b}(\xi - \eta) \hat{f}_n(\eta) \overline{\hat{g}_n(\xi)} \\ &= \iint \hat{b}(\xi - \eta) \hat{f}(\eta) \overline{\hat{g}(\xi)} = \langle \hat{b} * \hat{f}, \hat{g} \rangle. \end{aligned} \tag{12.4}$$

On the other hand, $\{f_n\}$ is a bounded sequence in L^2, whence $\{T_b(AA_1) f_n\}$ is relatively compact. Furthermore, $\{g_n\}$ is bounded and converges weakly to zero. Hence $\langle T_b(AA_1) f_n, g_n \rangle \to 0$ as $n \to \infty$, and (12.4) yields

$$\langle \hat{b} * \hat{f}, \hat{g} \rangle = 0. \tag{12.5}$$

Since r is arbitrary, (12.5) holds for all $\hat{f}, \hat{g} \in L^2$ with compact support. Consequently $\hat{b} = 0$. □

13. Compactness. The theorems in §§7 and 8 yield as corollaries sufficient conditions for compactness of paracommutators. Peng (**1985**) has shown that, under suitable assumptions on A, these sufficient conditions also are necessary (i.e. that converses to the theorems below hold). See Uchiyama (**1978**) for the case of commutators (our Example 4).

Let b^s_∞ denote the closure of the test functions $\mathcal{S}$ in B^s_∞ and let CMO denote the closure of $\mathcal{S}$ in BMO.

THEOREM 13.1. *Suppose that A satisfies* A1 *and* A3(γ) *and that $s + t < \gamma$ and $s, t > 0$. If $b \in b^{s+t}_\infty$, then T^{st}_b is compact.*

PROOF. Let $\mathcal{C}$ denote the set of compact operators. Choose p so large that $s + t + d/p < \gamma$. By Theorem 8.1, the mapping $b \to T^{st}_b$ maps $B^{s+t}_\infty \to S_\infty$ and $\mathcal{S} \subset B^{s+t+d/p}_p \to S_p \subset \mathcal{C}$. The assertion follows because $\mathcal{C}$ is closed in S_∞. □

Results for the case $s = t = 0$ are obtained similarly using Theorems 8.1 and 7.2 or 7.3. (Similar results for T^{s0}_b are left as an exercise.)

THEOREM 13.2. *Suppose that A satisfies* A3 *and* (7.10)–(7.11). *If $b \in b^0_\infty$ then T_b is compact.* □

THEOREM 13.3. *Suppose that A satisfies* A1, A2, A3. *If $b \in$* CMO *then T_b is compact.* □

14. Supplement. We have defined paracommutators as integral operators on the Fourier side (in the "phase" variables ξ, η). But in practice one often encounters kernels in the "configuration" space (variables x, y) of the type

$$k(x,y) = a(x-y)b(ux+vy), \qquad (u+v=1) \tag{14.1}$$

or linear combinations of such, e.g.

$$k(x,y) = \int_{u+v=1} a(x-y)b(ux+vy)\, d\mu(u,v) \tag{14.2}$$

with some measure μ. Let us check that such kernels indeed define paracommutators. If, quite generally, $g(x) = \int_{\mathbf{R}^d} k(x,y)f(y)\,dy$, then

$$\hat{g}(\xi) = (2\pi)^{-d} \int_{\mathbf{R}^d} \hat{k}(\xi, -\eta)\hat{f}(\eta)\,d\eta.$$

In the case of the kernel (14.1) we get

$$\begin{aligned}\hat{k}(\xi,-\eta) &= \iint e^{-ix\xi+iy\eta} a(x-y)b(ux+vy)\,dx\,dy \\ &= \iint e^{-i(y+z)\xi+iy\eta} a(z)b(uz+y)\,dz\,dy \\ &= \iint e^{-i(w-uz)(\xi-\eta)-iz\xi} a(z)b(w)\,dz\,dw = \hat{b}(\xi-\eta)\hat{a}(v\xi+u\eta).\end{aligned}$$

With (14.2) this gives

$$\hat{k}(\xi,-\eta) = \hat{b}(\xi-\eta)\int_{u+v=1} \hat{a}(v\xi+u\eta)\,d\mu(u,v)$$

corresponding to a paracommutator with

$$A(\xi,\eta) = \int_{u+v=1} \hat{a}(v\xi+u\eta)\,d\mu(u,v). \tag{14.3}$$

EXAMPLE 12. In the case of a Calderón-Zygmund commutator $[b, K]$ (Example 4, §1) we have $\mu = \delta(1,0) - \delta(0,1)$ so (14.3) gives $A(\xi,\eta) = m(\eta) - m(\xi)$ $(m = \hat{K})$ in complete agreement with (1.3). Examples 7 and 8 are covered similarly. (The higher commutators (Example 5) though require a more evolved apparatus.)

EXAMPLE 13. Coifman and Meyer (**1978**, Proposition 4, pp. 160–161) considered the operator

$$\int_{-\infty}^{\infty} \frac{b(x+t)+b(x-t)-2b(x)}{t^2} f(x-t)\,dt = \int_{-\infty}^{\infty} \frac{b(2x-y)+b(y)-2b(x)}{(x-y)^2} f(y)\,dy \tag{14.4}$$

and proved that it is bounded on $L^2(\mathbf{R})$ when $db/dx \in L^\infty$. This is an operator of the above type with $a(x) = x^{-2}$ (in the principal value sense) and $\mu = \delta(2,-1) + \delta(0,1) - 2\delta(1,0)$. Since $\hat{a}(\xi) = -\pi|\xi|$, (14.3) shows that the operator (14.4) is the paracommutator $T_b f$ with

$$A(\xi,\eta) = -\pi(|2\eta-\xi| + |\xi| - 2|\eta|). \tag{14.5}$$

We rewrite T_b as $-2\pi T_b^{10}(A_1)$, with

$$(14.6)\qquad A_1(\xi,\eta) = -\frac{1}{2\pi}|\xi|^{-1}A(\xi,\eta) = \begin{cases} 1, & \eta/\xi \le 0, \\ 1-2\eta/\xi, & 0 \le \eta/\xi \le 1/2, \\ 0, & 1/2 \le \eta/\xi. \end{cases}$$

It is easy to prove (using Lemma 3.5 and a modification of the proof of Lemma 3.4) that A_1 satisfies A0, A1, A2, A3(∞), A4, A5. Hence Theorem 5.3 shows that the operator (14.4) is bounded iff $I^{-1}b \in \mathrm{BMO}$, i.e. iff $db/dx \in \mathrm{BMO}$. Theorem 5.1 shows that it belongs to S_p iff $b \in B_p^{1+1/p}$ $(1 \le p < \infty)$.

EXAMPLE 14. Jean Bourgain (personal communication) has considered operators of the type

$$(14.7)\qquad \int a(x-y)b\left(\frac{x+y}{2}\right) f(y)\,dy,$$

in fact, in the special case $d = 1$, $a(x) = 1/x$. Thus his case, by (14.3), is the paracommutator $T_b f$ with

$$(14.8)\qquad A(\xi,\eta) = \hat{a}\left(\frac{\xi+\eta}{2}\right) = -i\pi\,\mathrm{sign}(\xi+\eta).$$

A8 is satisfied and Theorems 12.1 and 12.2 apply. However, A1 is not satisfied (because of the jump at $\{(\xi,\eta)\colon\ \xi = -\eta\}$) and we do not know whether $b \in L^\infty$ is sufficient to make the operator bounded.

Returning to the general operator (14.7), we may subtract an expression $(bk + Kb)/2$, where K denotes convolution with a. We are led to consider kernels of the form ($-\frac{1}{2}$ times)

$$(14.9)\qquad a(x-y)\left(b(x) + b(y) - 2b\left(\frac{x+y}{2}\right)\right)$$

which we, with the formal parallel with the Weyl calculus (see e.g. Hörmander (**1979**), (**1985**, Chapter 18)) in view, call *Weyl commutators*. For them (14.3) yields ($m = \hat{a}$)

$$(14.10)\qquad A(\xi,\eta) = m(\xi) + m(\eta) - 2m\left(\frac{\xi+\eta}{2}\right).$$

Again, the present theory is not directly applicable, but results for a range of p may be obtained by a different method.

Let us return to the general formula (14.2) and define

$$b_\mu(x,y) = \int_{u+v=1} b(ux+vy)\,d\mu(u,v).$$

For the Hilbert-Schmidt norm of the kernel $k = a(x-y)b_\mu(x,y)$ and the corresponding operator we then have

$$(14.11)\qquad ||k||_{S_2}^2 = \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} |a(x-y)|^2 |b_\mu(x,y)|^2\,dx\,dy.$$

Under suitable conditions on a and μ (of the type used in the theorem below) one can show directly that this norm is equivalent to the norm of b in $B_2^{d/2}$, as it should. More generally, under similar assumptions one can estimate L^p-norms and mixed L^p-norms of the kernel in terms of the norm in $B_p^{d/p}$. This means that the machinery in Janson and Wolff (**1982**) (cf. Janson and Peetre (**1984**)) is applicable, and we obtain S_p results for $2 < p < \infty$. For example:

THEOREM 14.1. *Suppose that $a(x) = O(|x|^{-d})$ and that μ is a finite measure with compact support such that $\int u^j \, d\mu(u,v) = 0$ for $j = 0, \dots, N-1$. If $2 \le p < \infty$ and $d/N < p$, then with k defined by* (14.2),

$$\|k\|_{S_p} \le C \|b\|_{B_p^{d/p}}. \tag{14.12}$$

PROOF (SKETCH). If $b \in B_p^s$ with $0 < s < N$, then $|x-y|^{-s} b_\mu(x,y) \in L^p(|x-y|^{-d}\,dx\,dy)$. (This is easily proved using interpolation between $p = 1$ and $p = \infty$; cf. also Peetre (**1976**, Chapters 1 and 8).) Consequently, if $b \in B_p^{d/p}$, then $|x-y|^{-2d/p} b_\mu(x,y) \in L^p(dx\,dy)$ and thus

$$|a(x-y)|^{2/p} b_\mu(x,y) \in L^p(dx\,dy) = L^p(L^p). \tag{14.13}$$

(The last expression is an abbreviation of $L^p(L^p(dy), dx)$.)

This and (14.11) complete the proof when $p = 2$. For $2 < p < \infty$ we continue as follows. Clearly $|x-y|^{-d} \in L^\infty(L^{1\infty})$ and thus $a(x-y) \in L^\infty(L^{1\infty})$ and

$$|a(x \dot{-} y)|^{1-2/p} \in L^\infty(L^{q\infty}), \quad \text{with } 1/q = 1 - 2/p. \tag{14.14}$$

By a Lorentz version of Hölder's inequality, (14.13) and (14.14) yield, with $1/p' = 1 - 1/p = 1/p + 1/q$,

$$k(x,y) = a(x-y) b_\mu(x,y) \in L^p(L^{p'\infty}). \tag{14.15}$$

By symmetry, also $k(y,x) \in L^p(L^{p'\infty})$, and Janson and Wolff (**1982**, Lemma 2, p. 304) yields $k \in S_{p\infty}$. We have proved that the linear mapping $b \to k$ maps $B_p^{d/p}$ into $S_{p\infty}$, $\max(2, d/N) < p < \infty$, and an auxiliary interpolation yields (14.12). $\square$

Converse results can often be obtained by the results of §9, if necessary by first modifying A, e.g. as in the next example.

EXAMPLE 15. Let T_b be the Weyl commutator (14.9) with a Calderón-Zygmund kernel in $\mathbf{R}^d$. (For $d = 1$, this is essentially the case studied by Bourgain.) Theorem 14.1 with $N = 2$, and Theorem 9.1 applied to $\varphi(|\eta|/|\xi|)A(\xi,\eta)$ with $\varphi \in C_0^\infty(0,1)$ (cf. Lemma 3.3) yield $T_b \in S_p$ iff $b \in B_p^{d/p}$ provided $d/2 < p < \infty$ and $p \ge 2$. Theorem 11.1 applies (with $k = 2$) and shows that $p > d/2$ is a necessary requirement. However, we expect the result to be true also when $d/2 < p < 2$, but our methods fail in that case. Similarly, we do not know when T_b is bounded. ($b \in \mathrm{BMO}$ is necessary by Theorem 10.1 applied to a suitable modification of A.)

Other instances of operators generalizing the Calderón-Zygmund commutator $[K, b]$, but not fitting in our framework, are the Edmonton commutators, Peetre (**1983**, p. 321). The idea is to expand the product $K \cdot b$ according to the formal rules of Ψ.D.O. calculus (see e.g. Hörmander (**1985**, Chapter 18)). The first term is then bK and the remainder precisely $[K, b]$. In general

$$Kb = bK + \sum_j \frac{\partial b}{\partial x_j} K^{(j)} + \frac{1}{2} \sum_{jk} \frac{\partial^2 b}{\partial x_j \partial x_k} K^{(jk)} + \cdots$$

where $K^{(j)}$ has the "symbol" $m^{(j)}(\xi) = -i\partial m(\xi)/\partial\xi_j$ etc. This leaves us with a remainder of the type (with a Calderón-Zygmund kernel)

$$T_b f(x) = \int_{\mathbf{R}^d} a(x-y)\left(b(y) - b(x) - \sum(y_j - x_j)\frac{\partial b}{\partial x_j} - \cdots\right) f(y)\,dy, \tag{14.16}$$

where the parenthesis is the remainder in a Taylor expansion (of a preassigned order) of b at x. This is easily seen to be a paracommutator with

$$A(\xi,\eta) = m(\xi) - m(\eta) - \sum(\xi_j - \eta_j)\frac{\partial m}{\partial \eta_j} - \cdots, \tag{14.17}$$

a remainder term in the Taylor expansion of m at η. It would be nice to extend the results of this paper to such a case too; our results are not directly applicable since A is unbounded. (However, we obtain by Theorem 9.1, applied to $\varphi(|\eta|/|\xi|)A(\xi,\eta)$ for a suitable φ, that if $T_b \in S_p$, then $b \in B_p^{d/p}$. Furthermore, if the Taylor expansions contain all terms of order $\le N-1$, then Theorem 11.1 applies with $k = N$ and shows that $p > d/N$ is necessary for nontrivial results. Finally, the Hilbert-Schmidt case is, as usual, simple: $T_b \in S_2$ iff $b \in B_2^{d/2}$, provided $N - 1 < d/2 < N$.) See also Cohen (**1980**) and Cohen and Gosselin (**1982**) where operators of the type (14.16) (with a different homogeneity of the kernel a) are studied (together with multilinear generalizations).

Finally, we make one more point about Hankel operators (Example 3, §1). As is well known, Hankel operator theory is intimately connected with group invariance considerations (invariance under the Möbius group $SU(1,1)$ if we consider Hankel operators on $\mathbf{T}$, and invariance under $SL(2,\mathbf{R})$ if we consider them on $\mathbf{R}$). This is seen e.g. by the fact that in many theorems the relevant symbol classes turn out to be (Möbius) invariant. Indeed, a very simple purely "group theoretic" proof of Peller's trace ideal theorem in the case $1 < p < \infty$ (see Example 3, §1 and Peller (**1980**)) is available (see Peetre (**1984**), (**1985a**), (**1985b**), Ahlmann (**1984**) for various versions). This becomes even more pregnant if we instead consider Hankel forms. Then the Hankel forms appear as just one irreducible component under the natural action of the group on the space of all bilinear forms. There arises the question to identify the other irreducible components. This can be done, and in fact the corresponding forms can be realized as paracommutators so our results are applicable to them. One has thus especially a separate "Peller's theorem" for each of these irreducible components. The details of these computations have appeared elsewhere, Janson and Peetre (**1987**).

One can in principle also conceive paracommutators in the periodic case (on $\mathbf{T}^d$). Thus, still speaking of bilinear forms, we are led to consider the expression $(2\pi)^{-d}\sum\sum \hat{b}(m-n)A(m,n)\hat{f}(m)\hat{g}(n)$.

The question when a form simultaneously can be realized as a paracommutator on $\mathbf{R}^d$ and on $\mathbf{T}^d$ is somehow related to the above considerations concerning Hankel forms of "higher weights."

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