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ARENS REGULARITY OF THE ALGEBRA $A \hat{\otimes} B$

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ABSTRACT. Let A and B be two Banach algebras. On the projective tensor product $A \hat{\otimes} B$ of A and B there exists a natural algebra structure. In this note we study Arens regularity of the Banach algebra $A \hat{\otimes} B$.

1. Introduction. As is pointed out by R. Arens in [**1**], given any Banach algebra A, on the second dual A'' of A there exist two, rather than one, algebra multiplications extending that of A. The details of the construction may be found in many places, including the book [**3**] and the survey article [**10**]. When these two multiplications coincide on A'', the algebra A is said to be "Arens regular". One natural and important problem in the Arens regularity theory is the construction of new regular algebras from the old ones. It is easily recognized that subalgebras and quotient algebras of regular algebras are regular. As is shown in [**2**], the direct sum algebra, defined by

$$\left(\bigoplus_{i\in I} A_i\right)_1 = \left\{x = (x_i)\colon x_i \in A_i \text{ for each } i \in I \text{ and } \|x\| = \sum_{i\in I} \|x_i\| < \infty\right\},$$

is Arens regular iff each component algebra A_i is Arens regular. On the other hand, the direct product algebra, defined by

$$\prod_{i\in I}(A_i) = \left\{x = (x_i)\colon x_i \in A_i \text{ for each } i \in I \text{ and } \|x\| = \sup_{i\in I} \|x_i\| < \infty\right\},$$

need not be Arens regular even if each component algebra A_i is finite dimensional [**25**].

Now let A and B be two Banach algebras. On the projective tensor product $A \hat{\otimes} B$ of A and B there exists a natural multiplication for which it is a Banach algebra (see below). The projective tensor product is an important construction and, as far as we know, the Arens regularity of the algebra $A \hat{\otimes} B$ has not been studied. It is the purpose of this note to study the Arens regularity of the algebra $A \hat{\otimes} B$. The first main result of the present note (Theorem 3.4) states a necessary and sufficient condition for regularity of the algebra $A \hat{\otimes} B$ involving a new concept, that of biregular bilinear form. The subsequent results demonstrate the utility of the condition in settling regularity of the algebra $A \hat{\otimes} B$ for several products involving classical Banach algebras. The second main result (Theorem 4.5) gives a useful sufficient condition for regularity of the algebra $A \hat{\otimes} B$. This theorem permits us to settle regularity of the algebras $l^p \hat{\otimes} A$ $(1 \le p < \infty)$

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and $L^p(G) \hat{\otimes} A$ $(1 < p \leq \infty)$. The main ingredient of the proofs here is the compactness of the multiplications in the algebra l^p and $L^p(G)$. The same theorem also settles the regularity of the algebra $C(K) \hat{\otimes} A$, where K is a dispersed compact set and A is a regular algebra whose dual does not contain c_0. Here the proof uses the compactness of the bilinear forms $m\colon C(K) \times A \to \mathbf{C}$. The third main result of the note (Theorem 4.20) settles regularity of the algebras $C(K) \hat{\otimes} C(S)$, $A(D) \hat{\otimes} A(D)$, $A(D) \hat{\otimes} C(S)$, $H^\infty(D) \hat{\otimes} H^\infty(D)$, $A(D) \hat{\otimes} H^\infty(D)$ and $H^\infty(D) \hat{\otimes} C(S)$, where K, S are two arbitrary compact sets, $A(D)$ is the disk algebra and $H^\infty(D)$ is the Hardy class on the unit disk D of the complex plane. Here the main ingredients are the Grothendieck inequality [**24**, Theorem 5.5] and some results due to J. Bourgain [**4**, **5**].

In a preliminary section (§2) we have collected the notation and basic facts we shall need. §§3 and 4 constitute the main body of the paper. In §3 we give a necessary and sufficient condition for the algebra $A \hat{\otimes} B$ to be Arens regular, and point out some immediate consequences of it. §4 consists of the applications of the necessary and sufficient conditions given in §3. In this section, the regularity of a certain number of algebras, including the algebras $l^p \hat{\otimes} A$, $L^p(G) \hat{\otimes} A$ and $C(K) \hat{\otimes} C(S)$, is settled. The note ends with some unsolved questions.

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2. Notation and preliminaries. Throughout this note, the letters X, Y, Z will stand for arbitrary (real or complex) Banach spaces and A, B for arbitrary (real or complex) Banach algebras. X' and X'' will be the first and second (topological) dual of X. The natural duality between X and X' will be denoted by $\langle x, x' \rangle$. We shall regard X as naturally embedded into X''. By X_1 we shall denote the closed unit ball of X. The letter K will stand for an arbitary (Hausdorff) compact space, and $C(K)$ for the space of scalar-valued continuous functions on K. The space $C(K)$ is equipped with the supremum norm. The spaces c_0, l^p $(1 \leq p \leq \infty)$ are the usual sequence spaces. When we consider $C(K), c_0$ or l^p as a Banach algebra, the multiplication we shall mean, unless the contrary is explicitly stated, is the pointwise multiplication. By a linear (bilinear) mapping we shall always mean a continuous linear (bilinear) mapping. The space of (continuous) linear mappings from X into Y will be denoted by $L(X, Y)$ and that of bilinear forms on $X \times Y$ by $B(X \times Y)$. Both spaces are equipped with their usual operator norms.

By $X \check{\otimes} Y$ and $X \check{\otimes} Y$ we shall denote, respectively, the projective and injective tensor products of X and Y. That is, $X \hat{\otimes} Y$ is the completion of $X \otimes Y$ for the norm

$$\|U\| = \inf \sum_{i=1}^{n} \|x_i\| \, \|y_i\|,$$

where the infimum is taken over all the representations of U as a finite sum of the form $U = \sum_{i=1}^{n} x_i \otimes y_i$, and $X \check{\otimes} Y$ is the completion of $X \otimes Y$ for the norm

$$\|U\| = \sup \left\{ \left| \sum_{i=1}^{n} \langle x_i, x' \rangle \langle y_i, y' \rangle \right| : \|x'\| \leq 1, \|y'\| \leq 1 \right\}.$$

The dual space of $X \hat{\otimes} Y$ is $B(X \times Y)$, and that of $X \check{\otimes} Y$ is (algebraically) a subspace of $B(X \times Y)$, which we shall denote by $I(X \times Y)$. An element of $I(X \times Y)$ is called an "integral bilinear form". For recent accounts of the tensor

product of Banach spaces, the reader may consult the books [**9** and **20**], and, for more information, the classics [**15** and **28**].

Although the injective tensor product of two Banach algebras A and B is not always a Banach algebra (see [**14**] for a counterexample) their projective tensor product is always a Banach algebra. The natural multiplication of $A \hat{\otimes} B$ is the linear extension of the following multiplication on decomposable tensors:

$$(a \otimes b) \cdot (\tilde{a} \otimes \tilde{b}) = a\tilde{a} \otimes b\tilde{b}.$$

For more information about the tensor product of Banach algebras, the reader may consult Chapter VI of the book [**3**] and the papers [**13**, **29**].

Finally, some words about Arens regularity. Let $m\colon X \times Y \to \mathbf{C}$ be a (continuous) bilinear form. Then there exists a linear operator $U\colon X \to Y'$ (resp. $V\colon Y \to X'$) such that

$$m(x,y) = \langle U(x), y\rangle \quad (\text{resp. } m(x,y) = \langle x, V(y)\rangle).$$

The operator V is nothing but the restriction of the adjoint U^* of U to Y. Consequently, U is (weakly) compact iff V is (weakly) compact [**11**, VI. 4.8 and VI. 5.2]. The bilinear form m will be called "(weakly) compact" if U is (weakly) compact. It is easy to see that with the notation of R. Arens [**1**], we have [**30**]

$$m^{***}(x'',y'') = \langle U^{**}(x''), y''\rangle \quad \text{and} \quad m^{t***t}(x'',y'') = \langle x'', V^{**}(y'')\rangle.$$

Moreover, m is Arens regular iff m is weakly compact. The weak compactness of m is equivalent to the following "double limit criterion" [**16**]: For any two bounded sequences (x_i) in X and (y_j) in Y,

$$\lim_i \lim_j m(x_i, y_j) = \lim_j \lim_i m(x_i, y_j),$$

provided that these limits exist. In the Banach algebra case, for each f in A' we define the bilinear form m_f on $A \times A$ by $m_f(x,y) = f(xy)$. The functional f is said to be "wap (weakly almost periodic) on A" if the bilinear form m_f is weakly compact. The collection of the wap functionals on A is denoted by $\mathrm{wap}(A)$. The set $\mathrm{wap}(A)$ is a closed linear subspace of A', and the equality $\mathrm{wap}(A) = A'$ is equivalent to the Arens regularity of A [**26**].

3. Biregular bilinear forms. As far as we know, the only general criterion to decide whether a given Banach algebra is Arens regular or not is the above-mentioned double limit criterion. However, when we want to apply this criterion to the algebra $A \hat{\otimes} B$ we are confronted with very complicated expressions, given that each element u of $A \hat{\otimes} B$ is of the form $u = \sum_{i=1}^{\infty} a_i \otimes b_i$ with $\sum_{i=1}^{\infty} \|a_i\| \cdot \|b_i\| < \infty$. Our aim in this section is to show that, instead of general elements u as above, one can work with decomposable tensors of the form $a \otimes b$. To this end, the following definition will be convenient.

DEFINITION 3.1. A bilinear form $m\colon A \times B \to \mathbf{C}$ is said to be "biregular" if, for any two pairs of sequences (a_i), $(\tilde{a}_j)$ in A_1 and (b_i), $(\tilde{b}_j)$ in B_1, we have

$$\lim_i \lim_j m(a_i\tilde{a}_j, b_i\tilde{b}_j) = \lim_j \lim_i m(a_i\tilde{a}_j, b_i\tilde{b}_j),$$

provided that these limits exist.

An immediate consequence of the definition is that if both algebras A and B are unital (i.e. they are with unit elements), then any biregular bilinear form is Arens regular. However, as the next example shows, if one of the algebras is not unital, a biregular bilinear form need not be Arens regular.

EXAMPLE 3.2. *A biregular nonregular bilinear form.* Let $A = l^\infty$, $B = l^1$ and $m(x,y) = \sum_{i=1}^\infty x_i y_i$. The bilinear form m is not Arens regular since the linear operator $U\colon A \to B'$ that corresponds to m is the identity operator of l^∞ and l^∞ is not reflexive. However, m is biregular. Indeed, let $(a_i), (\tilde{a}_j)$ in A_1 and (b_i), $(\tilde{b}_j)$ in B_1 be four sequences for which the limits

$$(*) \qquad \lim_i \lim_j m(a_i a_j, b_i b_j) \quad \text{and} \quad \lim_j \lim_i m(a_i a_j, b_i b_j)$$

exist. Now, introduce the bilinear form $\tilde{m}\colon l^1 \times l^1 \to \mathbf{C}$ defined by

$$\tilde{m}(x,y) = \sum_{i=1}^\infty x_i y_i.$$

Then $\tilde{m}(x,y) = f(x \cdot y)$, where $f = (1,1,1,\dots,1,\dots)$. As the algebra l^1 is Arens regular and f is in l^∞, the bilinear form $\tilde{m}$ is Arens regular. Moreover we have

$$m(a_i\tilde{a}_j, b_i\tilde{b}_j) = \tilde{m}(a_i b_i, \tilde{a}_j\tilde{b}_j).$$

Now put $x_i = a_i \cdot b_i$ and $y_j = \tilde{a}_j \cdot \tilde{b}_j$. Then, since l^1 is an ideal in l^∞, the sequences (x_i) and (y_j) are in l^1 and they are bounded there. So, $\tilde{m}$ being Arens regular, we have

$$\lim_i \lim_j \tilde{m}(a_i b_i, \tilde{a}_j\tilde{b}_j) = \lim_i \lim_j \tilde{m}(a_i b_i, \tilde{a}_j\tilde{b}_j).$$

Hence the limits in $(*)$ are equal, and m is biregular.

Another biregular nonregular bilinear form is the bilinear form $m = l^1 \times c_0 \to \mathbf{C}$, defined by

$$m(x,y) = \sum_{i=1}^\infty x_i y_i.$$

We shall see other examples below.

Now let $A_1 \otimes B_1$ be the set of elements of the form $a \otimes b$ with a in A_1 and b in B_1. The closed unit ball $(A \hat{\otimes} B)_1$ of $A \hat{\otimes} B$ is contained, by the very definition of the projective topology (see [**15**, p. 28 or **20** p. 176]), in the closed absolutely convex hull $\overline{\text{aco}}(A_1 \otimes B_1)$ of $A_1 \otimes B_1$. The next lemma, which is a particular case of [**16**, Corollary 2, p. 185], will permit us to work with $A_1 \otimes B_1$ instead of $(A \hat{\otimes} B)_1$, which is a much more complicated set.

LEMMA 3.3. *Let X be a Banach space, H a bounded subset of X' and D a bounded subset of X_1 such that $X_1 \subseteq \overline{\text{aco}}(D)$. Then the set H is relatively weakly compact iff, for any sequences (f_i) in H and (x_j) in D, we have*

$$\lim_i \lim_j f_i(x_j) = \lim_j \lim_i f_i(x_j),$$

provided that these limits exist.

Now we can characterize the Arens regularity of $A \hat{\otimes} B$ in terms of biregular bilinear forms.

THEOREM 3.4. *Let A and B be two arbitrary Banach algebras. Then the algebra $A \hat{\otimes} B$ is Arens regular iff every bilinear form $m\colon A \times B \to \mathbf{C}$ is biregular.*

PROOF. (1°) Assume that the algebra $A \hat{\otimes} B$ is Arens regular. Let $m\colon A \times B \to \mathbf{C}$ be a bilinear form and $\phi\colon A \times B \to A \hat{\otimes} B$ the canonical bilinear mapping, i.e. $\phi(a,b) = a \otimes b$. Then there exists a unique linear functional f on $A \hat{\otimes} B$ such that $f \circ \phi = m$. The algebra $A \hat{\otimes} B$ being Arens regular, the functional f is wap on $A \hat{\otimes} B$. Since we have

$$f(a \otimes b \cdot \tilde{a} \otimes \tilde{b}) = f(a\tilde{a} \otimes b\tilde{b}) = m(a\tilde{a}, b\tilde{b}),$$

the wap of f implies that m is biregular.

(2°) Conversely, assume that each bilinear form $m\colon A \times B \to \mathbf{C}$ is biregular. We have to show that any linear functional f on $A \hat{\otimes} B$ is wap. Let f be any linear functional on $A \hat{\otimes} B$ and $m = f \circ \phi$, where ϕ is the canonical bilinear mapping as above. For u in $A \hat{\otimes} B$, define the linear functional

$${}_u f\colon A \hat{\otimes} B \to \mathbf{C}$$

by ${}_u f(\tilde{u}) = f(u \cdot \tilde{u})$. Next, introduce the sets

$$E(f) = \{{}_u f\colon u \in (A \hat{\otimes} B)_1\}, \qquad H(f) = \{a \otimes b^f : a \in A_1 \text{ and } b \in B_1\}$$

and the mapping

$$\phi_f\colon A \hat{\otimes} B \to (A \hat{\otimes} B)', \quad \text{defined by } \Phi_f(u) = {}_u f.$$

The weak almost periodicity of f on $A \hat{\otimes} B$ is equivalent to the relative weak compactness of the set $E(f)$. Now we have

$$E(f) \subseteq \phi_f((A \hat{\otimes} B)_1) \subseteq \phi_f(\overline{\text{aco}}\,(A_1 \otimes B_1)) \subseteq \overline{\text{aco}}\,\phi_f(A_1 \otimes B_1) \subseteq \overline{\text{aco}}\,H(f).$$

Consequently, it is enough to show that the set $\overline{\text{aco}}\,H(f)$ is weakly compact. But, by the Krein-Smulyan theorem [**11**, p. 434], the set $\overline{\text{aco}}\,H(f)$ is weakly compact iff the set $H(f)$ is relatively weakly compact. Now, by the preceding lemma, the set $H(f)$ is relatively weakly compact iff, for any two sequences $(a_i \otimes b_i)$ and $(a_j \otimes b_j)$ in $A_1 \otimes B_1$, we have

$$\lim_i \lim_j \langle {}_{a_i \otimes b_i} f, \tilde{a}_j \otimes \tilde{b}_j \rangle = \lim_j \lim_i \langle {}_{a_i \otimes b_i} f, \tilde{a}_j \otimes \tilde{b}_j \rangle,$$

provided that these limits exist. Hence, since

$$\langle {}_{a_i \otimes b_i} f, \tilde{a}_j \otimes \tilde{b}_j \rangle = m(a_i\tilde{a}_j, b_i\tilde{b}_j)$$

and the bilinear form m is biregular, we conclude that the set $H(f)$ is relatively weakly compact. Consequently, f is wap on $A \hat{\otimes} B$, and the algebra $A \hat{\otimes} B$ is Arens regular.

We remark that, in the preceding theorem, no regularity hypothesis is made about the algebras A and B. However, as the next corollaries will make clear, the biregularity notion involves the Arens regularity of the algebras A and B, as well as that of the bilinear form itself.

We recall that an algebra A is said to be "trivial" if $a \cdot \tilde{a} = 0$ for each a and $\tilde{a}$ in A. The algebra $A \hat{\otimes} B$ is trivial iff at least one of the algebras A or B is trivial. The next corollary shows that, except in the trivial case, the algebra $A \hat{\otimes} B$ cannot be Arens regular unless A and B are.

COROLLARY 3.5. *Assume that the algebras A and B are not trivial. If the algebra $A \hat{\otimes} B$ is Arens regular then so are A and B.*

PROOF. Let us see, for example, that the algebra A is Arens regular. Let f be a functional on A and (a_i), $(\tilde{a}_j)$ two sequences in A_1 for which the limits

$$\lim_i \lim_j f(a_i \tilde{a}_j) \quad \text{and} \quad \lim_j \lim_i f(a_i \tilde{a}_j)$$

exist. The algebra B being nontrivial, there exist $b, \tilde{b}$ in B with $b \cdot \tilde{b} \neq 0$. Now, choose a linear functional g on B for which we have $g(b \cdot \tilde{b}) = 1$. Next, consider the bilinear form $m = f \otimes g$ on $A \times B$, i.e. $m(a, b) = f(a)g(b)$. Then, as we have

$$m(a_i \tilde{a}_j, b_i \tilde{b}_j) = f(a_i \cdot \tilde{a}_j)$$

the biregularity of m implies that the above-iterated limits are equal. Consequently, f is wap on A, and A is Arens regular.

As remarked after Definition 3.1, if the algebras A and B both are unital then any biregular bilinear form on $A \times B$ is Arens regular; next we record this fact. We do not know if the converse of this corollary is true.

COROLLARY 3.6. *If the algebras A and B are unital and the algebra $A \hat{\otimes} B$ is Arens regular, then any bilinear form $m\colon A \times B \to \mathbf{C}$ is weakly compact.*

REMARK 3.7 If an algebra A is not unital, we can consider its unitization $A \otimes \mathbf{C}1$. The multiplication and the norm of $A \oplus \mathbf{C}1$ are given, respectively, by

$$(a + \lambda)(\tilde{a} + \mu) = a\tilde{a} + \mu\tilde{a} + \lambda a + \lambda u \quad \text{and} \quad \|a + \lambda\| = \|a\| + |\lambda|.$$

It is easy to see that the algebra $A \oplus \mathbf{C}1$ is Arens regular iff A is. Now, although the spaces $(A \oplus \mathbf{C}1) \hat{\otimes} B$ and $A \hat{\otimes} B \oplus B$ are isometrically isomorphic as Banach spaces, they are not isometrically isomorphic as Banach algebras. Consequently, the Arens regularity of the algebras $A \hat{\otimes} B$ and $(A \oplus \mathbf{C}1) \hat{\otimes} B$ are not equivalent. Actually, as we shall see below, the algebra $l^1 \hat{\otimes} l^\infty$ is Arens regular but the algebra $(l^1 \oplus \mathbf{C}1) \hat{\otimes} l^\infty$ is *irregular* since, for example, the bilinear form

$$m\colon (l^1 \oplus \mathbf{C}1) \times l^\infty \to \mathbf{C}$$

defined by $m(x + \lambda, y) = \sum_{i=1}^\infty x_i y_i$ is not weakly compact.

Now, let $\tilde{A}$ (resp. $\tilde{B}$) be a subalgebra of A (resp. B). In general $\tilde{A} \hat{\otimes} \tilde{B}$ is not a subalgebra of $A \hat{\otimes} B$ and, when it is, the natural embedding need not be an isometry (see [**19** and **4**]). However, as we shall see below, there exist cases where each bilinear form $\tilde{m}\colon \tilde{A} \times \tilde{B} \to \mathbf{C}$ has a (continuous) bilinear extension $m\colon A \times B \to \mathbf{C}$. The biregularity of m then implies that of $\tilde{m}$, and we have the following result.

PROPOSITION 3.8. *Let $\tilde{A}$ (resp. $\tilde{B}$) be a subalgebra of A (resp. B). Assume that $A \hat{\otimes} B$ is Arens regular and each bilinear form $\tilde{m}\colon \tilde{A} \times \tilde{B} \to \mathbf{C}$ has a continuous bilinear extension $m\colon A \times B \to \mathbf{C}$. Then the algebra $\tilde{A} \hat{\otimes} \tilde{B}$ is also Arens regular.*

4. Applications. In this section we shall apply Theorem 3.4 to settle Arens regularity of the algebra $A \hat{\otimes} B$ for several pairs of algebras A and B. For the sake of clarity we shall divide this section into several subsections.

4.a. *Some classes of biregular forms.* Given that the Arens regularity of the algebra $A \hat{\otimes} B$ is characterized in terms of biregular bilinear forms, it is important to have as many biregular bilinear forms as one can. In view of Example 3.2 and Corollary 3.6 a characterization of the biregularity involving only the bilinear form itself does not seem to be possible. In this subsection we shall isolate some classes of biregular bilinear forms.

As a simple application of the double limit criterion we have the following propositions.

PROPOSITION 4.1. *Assume that $A \subseteq B$ and A is an ideal of B. Let $m\colon A \times B \to \mathbf{C}$ be a bilinear form such that there exists an Arens regular bilinear form $\tilde{m}\colon A \times A \to \mathbf{C}$ satisfying the equality $m(a\tilde{a}, b\tilde{b}) = \tilde{m}(ab, \tilde{a}\tilde{b})$. Then m is biregular.*

EXAMPLE 4.2 Let $A = l^p$ $(1 \leq 0 < \infty)$, $B = c_0$ and $m(x,y) = \sum_{i=1}^{\infty} x_i y_i$. Now, let $\tilde{m}\colon l^p \times l^p \to \mathbf{C}$ be given by $\tilde{m}(x,y) = \sum_{i=1}^{\infty} x_i y_i$. Then all the hypotheses of the preceding proposition are satisfied so that m is biregular.

PROPOSITION 4.3. *Assume A is commutative. For f in A' let $m_f\colon A \times A \to \mathbf{C}$ be the bilinear form defined by $m_f(a,b) = f(ab)$. Then, if f is wap on A, the bilinear form m_f is biregular.*

Although the injective tensor product $A \check{\otimes} B$ of two Banach algebras is not in general an algebra, there exist cases where not only $A \check{\otimes} B$ is a Banach algebra but its regularity also is known. For example, if B is a C^*-algebra, $C(K) \check{\otimes} B = C(K, B)$ is a C^*-algebra and so is Arens regular. For such algebras we have the following result.

PROPOSITION 4.4. *Let A and B be two Arens regular algebras such that $A \check{\otimes} B$ is an algebra and is Arens regular. Then any integral bilinear form $m\colon A \times B \to \mathbf{C}$ is biregular.*

PROOF. *Let $m\colon A \times B \to \mathbf{C}$ be an integral form. Then, interpreted as a functional on $A \check{\otimes} B$, m is* wap *on $A \check{\otimes} B$. So, applying the double limit criterion to m and to the elements of the form $a \otimes b$ of $A \check{\otimes} B$, we conclude that m is biregular.*

The next theorem is the first main result of this section. The hypotheses of this theorem involve in a combined way the multiplication of A, that of B, and the bilinear form itself. We shall need the following notations. For any a'' in A'', we define the left and right multiplications by a'' on A'' by

$$ {}_{a''}\tau\colon x'' \in A'' \to a''x'' \in A'' \quad \text{and} \quad \tau_{a''}\colon x'' \in A'' \to x''a'' \in A''. $$

The adjoint of ${}_{a''}\tau$ is given by

$$ {}_{a''}\tau^*\colon f \in A''' \to {}_{a''}f \in A''' $$

where ${}_{a''}f(x'') = f(a''x'')$. Similarly, $\tau^*_{a''}(f) = f_{a''}$, where $f_{a''}(x'') = f(x''a'')$. Remember also that we consider A as naturally embedded into A''.

THEOREM 4.5. *Assume that the algebras A and B are Arens regular. Let $U\colon A \to B'$ be a linear operator. Assume also that, for each a'' in A'' and each b'' in B'' the linear operators*

$$ {}_{b''}\tau^* \circ U^{**} \circ {}_{a''}\tau \quad \textit{and} \quad \tau^*_{b''} \circ U^{**} \circ \tau_{a''} $$

are compact. Then the bilinear form $m\colon A \times B \to \mathbf{C}$ *defined by* $m(a,b) = \langle U(a), b\rangle$ *is biregular.*

PROOF. Let (a_i), $(\tilde{a}_j)$ in A_1 and (b_i), $(\tilde{b}_j)$ in B_1 be four sequences for which the following iterated limits exist:

$$\lim_i \lim_j m(a_i\tilde{a}_j, b_i\tilde{b}_j) \quad \text{and} \quad \lim_j \lim_i m(a_i\tilde{a}_j, b_i\tilde{b}_j).$$

By the Alaoglu theorem [**11**, p. 424], from these sequences we can extract $\sigma(A'', A')$ (resp. $\sigma(B'', B')$) convergent subnets (a_α), $(\tilde{a}_\beta)$ and (b_α), $(\tilde{b}_\beta)$ such that

$$\begin{pmatrix} a_\alpha \to a'', \\ b_\alpha \to b'', \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{a}_\beta \to \tilde{a}'', \\ \tilde{b}_\beta \to \tilde{b}. \end{pmatrix}$$

The regularity of A and B implies that, in the weak* topologies of A'' and B'', we have the equalities

$$\lim_\alpha \lim_\beta a_\alpha\tilde{a}_\beta = \lim_\beta \lim_\alpha a_\alpha\tilde{a}_\beta = a''\tilde{a}''$$

and

$$\lim_\alpha \lim_\beta b_\alpha\tilde{b}_\beta = \lim_\beta \lim_\alpha b_\alpha\tilde{b}_\beta = b''\tilde{b}''.$$

Now,

$$\begin{aligned} m(a_\alpha\tilde{a}_\beta, b_\alpha\tilde{b}_\beta) &= \langle U(a_\alpha\tilde{a}_\beta), b_\alpha\tilde{b}_\beta\rangle = \langle U \circ {}_{a_\alpha}\tau(\tilde{a}_\beta), {}_{b_\alpha}\tau(\tilde{b}_\beta)\rangle \\ &= \langle {}_{b_\alpha}\tau^* \circ U^{**} \circ {}_{a_\alpha}\tau(\tilde{a}_\beta), \tilde{b}_\beta\rangle. \end{aligned}$$

Since the net $(\tilde{a}_\beta)$ converges weak* to $\tilde{a}''$ and U^{**} maps A'' into B''', the net $({}_{b_\alpha}\tau^* \circ U^{**} \circ {}_{a_\alpha}\tau(\tilde{a})\beta))_\beta$ converges weak* in B''' to ${}_{b_\alpha}\tau^* \circ U^{**} \circ {}_{a_\alpha}\tau(\tilde{a}'')$. This shows that the latter element is the unique norm cluster point of the net $({}_{b_\alpha}\tau^* \circ U^{**} \circ {}_{a_\alpha}\tau(\tilde{a}_\beta))_\beta$. By hypothesis, this net is contained in a compact subset of the space B'''. So, it converges in norm to its unique cluster point ${}_{b_\alpha}\tau^* \circ U^{**} \circ {}_{a_\alpha}\tau(\tilde{a}'')$. From this we conclude that

$$\lim_\beta m(a_\alpha\tilde{a}_\beta, b_\alpha\tilde{b}_\beta) = \langle {}_{b_\alpha}\tau^* \circ U^{**} \circ {}_{a_\alpha}\tau(\tilde{a}''), \tilde{b}''\rangle.$$

But, as we have

$$\langle {}_{b_\alpha}\tau^* \circ U^{**} \circ {}_{a_\alpha}\tau(\tilde{a}''), \tilde{b}''\rangle = \langle \tau^*_{\tilde{b}''} \circ U^{**} \circ \tau_{\tilde{a}''}(a_\alpha), b_\alpha\rangle,$$

for the same reason as above, taking into account the Arens regularity of A, we get

$$\lim_\alpha \langle \tau^*_{\tilde{b}''} \circ U^{**} \circ \tau_{\tilde{a}''}(a_\alpha), b_\alpha\rangle = \langle \tau^*_{\tilde{b}''} \circ U^{**} \circ \tau_{\tilde{a}''}(a''), b''\rangle$$

so that

$$\lim_\alpha \lim_\beta m(a_\alpha\tilde{a}_\beta, b_\alpha\tilde{b}_\beta) = \langle U^{**}(a''\tilde{a}''), b''\tilde{b}''\rangle.$$

Similarly

$$\lim_\beta \lim_\alpha m(a_\alpha\tilde{a}_\beta, b_\alpha\tilde{b}_\beta) = \langle U^{**}(a''\tilde{a}''), b''\tilde{b}''\rangle.$$

Hence

$$\lim_i \lim_j m(a_i\tilde{a}_j, b_i\tilde{b}_j) = \lim_j \lim_i m(a_i\tilde{a}_j, b_i\tilde{b}_j),$$

and m is biregular.

The mappings ${}_{b''}\tau \circ U^{**} \circ {}_{a''}\tau$ and $\tau^*_{b''} \circ U^{**} \circ \tau_{a''}$ may be compact for a variety of reasons. For example, they may be compact because ${}_{a''}\tau$ and $\tau_{a''}$ are compact or U^{**} is compact or again U^{**} is completely continuous and the mappings ${}_{a''}\tau, \tau_{a''}$ are weakly compact. In the next subsections we shall see several applications of this theorem.

4.b. *Arens regularity of $l^p \hat{\otimes} A$ and $L^p(G) \hat{\otimes} A$.* Let G be a compact topological group and μ its Haar measure. The space $L^p(G) = L^p_\mu(G)$, with the convolution as the multiplication, is a Banach algebra. As an application of Theorem 4.5, we shall see that the algebras $l^p \hat{\otimes} A$ and $L^p(G) \hat{\otimes} A$ $(1 < p < \infty)$ are Arens regular iff A is.

We shall omit the proof of the following lemma; a proof of it can easily be obtained from the characterization of the compact subsets of l^p spaces given in [**11**, V.13.3].

LEMMA 4.6. *Let $1 \leq p < \infty$, and let a be a fixed element in l^p. Then the linear operator ${}_a\tau \colon l^p \to l^p$ is compact.*

The following corollary is now an immediate consequence of Theorems 3.4 and 4.5 and Corollary 3.5.

COROLLARY 4.7. *Let $1 < p < \infty$. Then the algebra $l^p \hat{\otimes} A$ is Arens regular iff the algebra A is Arens regular.*

The next lemma is also well known and the proof of it can easily be deduced from any version of the proof of the Peter-Weyl Theorem.

LEMMA 4.8. *Let $1 < p < \infty$, and let a be a fixed element in $L^p(G)$. Then the linear operators ${}_a\tau$ and τ_a are compact as mappings from $L^p(G)$ into itself.*

Now, by Theorems 3.4 and 4.5 and Corollary 3.5 again we have the next corollary.

COROLLARY 4.9. *Let $1 < p < \infty$. Then the algebra $L^p(G) \hat{\otimes} A$ is Arens regular iff the algebra A is Arens regular.*

4.c. *Arens regularity of $l^1 \hat{\otimes} A, L^\infty(G) \hat{\otimes} A$ and $C(G) \hat{\otimes} A$.* In this subsection, too, G is a compact topological group and on $C(G)$ and $L^\infty(G)$ the multiplication is convolution. Let B stand for one of the algebras l^1, $L^\infty(G)$ of $C(G)$. In the preceding subsections we have seen that, for b in B, the mappings ${}_b\tau$ and τ_b are compact as mappings from B into itself. Here we shall first show that, for b'' in B'', the mappings ${}_{b''}\tau$ and $\tau_{b''}$ are compact as mappings from B'' into itself. We recall that, for any Banach space X, we have the canonical decomposition

$$X''' = X' \oplus X^\perp,$$

where $X^\perp$ is the annihilator of X in X'''. We shall need the following result which is of independent interest.

LEMMA 4.10. *Let $m \colon X' \times X' \to X$ be an Arens regular, separately (weak*) $\times$ (weak*) to (weak) continuous bilinear form. Let $m^{***} \colon X''' \times X''' \to X''$ be the Arens extension of m. Then, for $x''' = x' + \tilde{x}$ and $y''' = y' + \tilde{y}$ in X''', we have $m^{***}(x''', y''') = m(x', y')$.*

PROOF. As

$$m^{***}(x''', y''') = m(x', y') + m^{***}(x', \tilde{y}) + m^{***}(x, y') + m^{***}(\tilde{x}, \tilde{y})$$

it is enough to show that the last three terms are null. First remark that the hypothesis that m is (weak*)×(weak*) to (weak) separately continuous is equivalent to saying that, for each z' in X', the mappings $z' \circ m(\cdot, y')$ and $z' \circ m(x', \cdot)$ are evaluation functions at points of X. Now, let z' be in X'. Then

$$m^{***}(x', \tilde{y})(z') = m^{**}(\tilde{y}, z')(x') = \langle \tilde{y}, m^*(z', x') \rangle.$$

But, as $m^*(z', x') = z' \circ m(x', \cdot)$ and $z' \circ m(x', \cdot)$ are evaluation functions at points of X, we have

$$\langle \tilde{y}, m^*(z', x') \rangle = \langle y, z' \circ m(x', \cdot) \rangle = 0.$$

Thus $m^{***}(x', \tilde{y})(z') = 0$ for all z' in X'. This means that $m^{***}(x', \tilde{y}) = 0$. Similarly, we can see that $m^{***}(\tilde{x}, y') = m^{***}(\tilde{x}, \tilde{y}) = 0$.

LEMMA 4.11. *Let B stand for one of the algebras l^1, $L^\infty(G)$ or $C(G)$. Then, for b'' in B, the mappings ${}_{b''}\tau$ and $\tau_{b''}$ are compact from B'' into itself.*

PROOF. First let B be l^1 or $L^\infty(G)$. Then $B = X'$, where $X = c_0$ or $L^1(G)$, accordingly. Then, for b_1, b_2 in B, their product $b_1 b_2$ is in X and the bilinear mapping $m\colon B \times B \to X$ defined by $m(b_1, b_2) = b_1 b_2$ is separately continuous from the topology (weak*)×(weak*) to (weak). So, by the preceding lemma, for b'' in B'', if $b'' = b + \tilde{b}$ is the decomposition of b'' in $B'' = B \oplus X'$, the mapping

$${}_{b''}\tau\colon x'' \in B'' \to b''x'' \in B''$$

is given by

$${}_{b''}\tau(x'') = (b + \tilde{b})(x + \tilde{x}) = bx = {}_b\tau(x),$$

where ${}_b\tau\colon B \to B$ is the left multiplication. Now, Lemmas 4.6 and 4.8 imply that ${}_{b''}\tau$ is a compact operator from B'' into itself. The same conclusion holds for $\tau_{b''}$ too. Now, let B be $C(G)$. In this case, we have the decomposition

$$B'' = L^\infty(G) \oplus N,$$

where N is the ideal of two-sided annihilators of $C(G)''$ [**21**, 7.2(a)]. Hence, if $b'' = b + b_0$ and $x'' = x + x_0$ are the decompositions of two elements of B'', we have

$${}_{b''}\tau(x'') = b''x'' = bx = {}_b\tau(x),$$

where bx is the product of b and x in $L^\infty(G)$, i.e. $bx = b^*x$. So, ${}_{b''}\tau$ is compact by Lemma 4.8. Similarly $\tau_{b''}$ is also compact.

From the preceding lemmas, Theorems 3.4 and 4.5 and Corollary 3.5 we deduce the following result.

COROLLARY 4.12. *Let B be one of the algebras $l^1, L^\infty(G)$ or $C(G)$. Then the algebra $A \hat{\otimes} B$ is Arens regular iff the algebra A is Arens regular.*

4.13. REMARK. As $l^1 \hat{\otimes} A = (\bigoplus_n A_n)_1$, where $A_n = A$ for each n in $\mathbf{N}$, from [**2**] we could deduce immediately that $l^1 \hat{\otimes} A$ is Arens regular iff A is. We prefer our proof, which is completely different from that of [**2**], since it turns out to be a particular case of a more general result.

4.d. *Arens regularity of $A \hat{\otimes} B$ for some very particular algebras.* In all the examples given so far, the Arens regularity of $A \hat{\otimes} B$ is due to the compactness

of the multiplication in one of the algebras A or B. Now, for the Arens regular algebras A and B, as an immediate corollary of Theorem 4.5, we have the inclusions

$$K(A, B') \subseteq \mathrm{wap}(A \hat{\otimes} B) \subseteq L(A, B'),$$

where $K(A, B')$ is the space of compact linear operators from A into B'. So, if it happens that $K(A, B') = L(A, B')$, i.e. any linear operator $u\colon A \to B'$ is compact, then the algebra $A \hat{\otimes} B$ is Arens regular. In this section we shall consider some pairs of algebras A and B for which the equality $K(A, B') = L(A, B')$ holds. We shall need the following notions.

(a) A Banach space X is said to have the "Schur Property" if any weakly convergent sequence in X is norm convergent.

(b) A Banach space X is said to have the DP (Dunford-Pettis) property if, for any sequences (x_n) in X and (x'_n) in X', $x_n \to 0$ weakly and $x'_n \to 0$ weakly imply that $\langle x_n, x'_n \rangle \to 0$.

(c) A Banach space X is said to have the "Dieudonné Property" if, for any Banach space Y, any linear operator $u\colon X \to Y$ which sends weakly Cauchy sequences into weakly convergent ones is weakly compact.

For the Dunford-Pettis property, the reader may consult [**17** and **8**], and for the Dieudonné Property [**17** or **12**, p. 644].

The following result is an easy consequence of Rosenthal's l^1-theorem [**27**] (see [**8**, p. 23] for a proof).

LEMMA 4.14. *If a Banach space X has the DP property and does not contain an isomorphic copy of l^1 then the dual X' of X has the Schur property.*

The simplest space satisfying the hypotheses of this lemma is c_0. Next comes the space $C(K)$, where K is a dispersed compact space, i.e. K contains no nonempty perfect subset. Indeed, the fact that $C(K)$ has the DP property is well known [**17**]; the fact that $C(K)$ does not contain an isomorphic copy of l^1 is proven in [**23**]. Now, let (X_n) be a sequence of Banach spaces such that, for each n, X'_n has both the DP property and the RNP (Randon-Nikodým property) [**6**]—for example, if each X_n is finite dimensional or each X_n is a $C(K)$ space for some dispersed compact K, the dual of X_n has both properties. Then the space

$$X = \left(\bigoplus X_n\right)_0 = \{x = (x_n)\colon x_n \in X_n \text{ for each } n \in N \text{ and } \|x_n\| \to 0 \text{ as } n \to \infty\},$$

equipped with the norm $\|x\| = \sup_n \|x_n\|$, has the DP property and does not contain a copy of l^1. Indeed, as $X' = (\bigoplus X'_n)_1$ has the DP property (see e.g.[**7**]), X also has the DP property [**17**]. Moreover, as X' has the RNP [**6**, p. 169], X cannot contain an isomorphic copy of l^1. We summarize these results.

LEMMA 4.15. *Let K be a dispersed compact set and (X_n) a sequence of Banach spaces such that each X'_n has both the DP and RN properties. Then the spaces $C(K)$ and $X = (\bigoplus X_n)_0$ have the DP property and do not contain an isomorphic copy of l^1.*

Finally, we mention the following obvious result.

LEMMA 4.16. *Let X, Y be two Banach spaces. Assume that Y' has the Schur property. Then any weakly compact linear oeprator $u\colon X \to Y'$ is compact.*

Now, we can assert the Arens regularity of the algebra $A \hat{\otimes} B$ for certain pairs of algebras A and B.

COROLLARY 4.17. *Let K be a dispersed compact space and $B = C(K)$. Then $A \hat{\otimes} B$ is Arens regular in the following cases.*

(a) *A is Arens regular and the dual of A does not contain an (isomorphic) copy of c_0.*

(b) *A is Arens regular and has the Dieudonné property.*

PROOF. Indeed, any bilinear form $m\colon A \times B \to \mathbf{C}$ is weakly compact (so compact by the preceding lemma), in the first case by [**22**, Theorem 5], and in the second case by [**17**, Proposition 8].

An immediate consequence of this corollary is that $C(K) \hat{\otimes} A$ is Arens regular for any C^*-algebra A. This follows from Theorem 9.1 of [**24**], as well as from the fact that the dual of A does not contain a copy of c_0 (see Corollary 1 of [**24**, p. 221] in connection with Theorem 9.1 of [**24**]).

COROLLARY 4.18. *Let (B_n) be a sequence of Banach algebras with the property that, for each n, B_n' has both the DP and RN properties. Let $B = (\bigoplus_n B_n)_0$. Then, if A is an Arens regular Banach algebra which has the Dieudonné property, the algebra $A \hat{\otimes} B$ is Arens regular.*

PROOF. First remark that since B does not contain an isomorphic copy of l^1 and B', being a Schur space, is weakly sequentially complete, the algebra B is Arens regular (see [**31**]). Now, by Proposition 8 of [**17**], any linear operator $u\colon A \to B'$ is weakly compact, so compact. Hence $K(A, B') = L(A, B')$, and $A \hat{\otimes} B$ is Arens regular.

We remark that the space $C(K)$ for any compact K, the disk algebra $A(D)$, and the space $C(K) \check{\otimes} X = C(K, X)$ for any reflexive space X all have the Dieudonné property. (See [**31**] for other examples and references.)

Now, we consider the algebra $c_0 \hat{\otimes} A$ in particular. In the case of the injective tensor product $c_0 \check{\otimes} A$, from the equality

$$(c_0 \check{\otimes} A)'' = \prod_n (A_n''),$$

where $A_n = A$ for each integer n, one deduces immediately that the algebra $c_0 \check{\otimes} A$ is Arens regular iff A is. As the next example shows, the corresponding result is not true for the projective tensor product $c_0 \hat{\otimes} A$. All we know about the Arens regularity of $c_0 \hat{\otimes} A$ is that if A' does not contain a copy of c_0 or A has the Dieudonné property and is Arens regular then $c_0 \hat{\otimes} A$ is Arens regular.

EXAMPLE 4.19. Let $A = l^1 \oplus \mathbf{C}1$ be the unitization of l^1. The algebra A is Arens regular. Now, let $m\colon c_0 \times A \to \mathbf{C}$ be the bilinear form given by

$$m(x, y + \lambda) = \sum_{n=1}^{\infty} x(n)y(n),$$

where $x = (x(n))_n$ and $y = (y(n)))_n$. Since $(y + \lambda)(\tilde{y} + \tilde{\lambda}) = y\tilde{y} + \lambda\tilde{y} + \tilde{\lambda}y + \lambda\tilde{\lambda}$ for $y + \lambda$ and $\tilde{y} + \tilde{\lambda}$ in A, we have

$$m(x\tilde{x}, (y + \lambda)(y + \lambda)) = \sum_{n=1}^{\infty} x(n)\tilde{x}(n)(y(n)\tilde{y}(n) + \lambda\tilde{y}(n) + \tilde{\lambda}y(n)).$$

Next, we choose four sequences $(x_i), (\tilde{x}_j)$ in $(c_0)_1$ and $(y_i + \lambda_i)$, $(\tilde{y}_j + \tilde{\lambda}_j)$ in A_1 as

$$x_i = (1, 1, \dots, 1, 0 \dots 0 \dots) \quad (1 \text{ occurs } i \text{ times}),$$
$$\tilde{x}_j = (0 \dots 0, 1, 0 \dots 0 \dots) \quad (1 \text{ occupies the } j\text{th place})$$

and we take, for all i and j,

$$\tilde{y}_j = \tilde{x}_j, \quad \tilde{\lambda}_j = 0, \qquad \text{and} \qquad y_i = 0, \quad \lambda_i = 1.$$

Then we have

$$\begin{aligned} m(x_i\tilde{x}_j, (y_i + \lambda_i)(\tilde{y}_j + \tilde{\lambda}_j)) &= \sum_{n=1}^{\infty} x_i(n)\tilde{x}_j(n)\tilde{y}_j(n) \\ &= \sum_{n=1}^{\infty} x_i(n)\tilde{x}_j(n) = \begin{cases} 1 & \text{if } j \le i, \\ 0 & \text{if } j > i. \end{cases} \end{aligned}$$

It follows that the limits

$$\lim_i \lim_j m(x_i\tilde{x}_j, (y_i + \lambda_i)(\tilde{y}_j + \tilde{\lambda}_j)) \quad \text{and} \quad \lim_j \lim_i m(x_i, x_j, (y_i + \lambda_i)(\tilde{y}_j + \tilde{\lambda}_j))$$

exist and are different. Hence, m is not biregular and the algebra $c_0 \hat{\otimes} A$ is not Arens regular.

Here we remark that, c_0 being an ideal in its bidual l^∞, the multiplication operator ${}_a\tau\colon c_0 \to c_0$ is weakly compact [**10**, p. 318], so compact (Lemma 4.16 above). The preceding example shows that, in Theorem 4.5, the hypothesis (${}_{b''}\tau^* \circ U^{**} \circ {}_{a''}\tau$ is compact for each a'' in A'' and each b'' in B'') cannot be replaced by (${}_b\tau * \circ U \circ {}_a\tau$ is compact for each a in A and each b in B).

4.e. *Arens regularity of* $C(K) \hat{\otimes} C(S)$, $A(D) \hat{\otimes} A(D)$ *and* $H^\infty(D) \hat{\otimes} H^\infty(D)$. Let K and S be two arbitrary compact sets. The main result of this subsection is in the next theorem, which asserts that the algebra $C(K) \hat{\otimes} C(S)$ is Arens regular. In all the examples given above we have applied Theorem 3.4 through Theorem 4.5. In the present case we have neither the compactness of the multiplication of one of the algebras nor the equality $K(A, B') = L(A, B')$. Here our starting point is Theorem 3.4 itself.

THEOREM 4.20. *Let K and S be two arbitrary compact sets. Then the algebra $C(K) \hat{\otimes} C(S)$ is Arens regular.*

PROOF. Let $m\colon C(K) \times C(S) \to \mathbf{C}$ be a bilinear form. We want to show that m is biregular. According to the Grothendieck Theorem [**15**, p. 46, Corollary 4 or **24**, p. 54, Theorem 5.5], there exist two probability measures λ and μ on K and S, respectively, such that, for x in $C(K)$ and y in $C(S)$, we have the inequality

$$|m(x, y)| \le C\|m\| \left(\int_K |x|^2\, d\lambda\right)^{1/2} \left(\int_S |y|^2\, d\mu\right)^{1/2},$$

where C is an absolute constant, i.e. the Grothendieck constant. This inequality implies immediately that m has a continuous bilinear extension

$$\tilde{m}\colon L^2_\lambda(K) \times L^2_\mu(S) \to \mathbf{C}.$$

Let $(e_i)_{i\in I}$ (resp. $(f_j)_{j\in J}$) be an orthonormal Hilbert basis for the space $L^2_\lambda(K)$ (resp. $L^2_\mu(K)$). For x in $L^2_\lambda(K)$ and y in $L^2_\mu(S)$, we have the representations

$$x = \sum_{i\in J} \langle x, e_i\rangle e_i \quad \text{with } \langle x, e_i\rangle = \int_K x(t)e_i(t)\,d\lambda(t)$$

and

$$y = \sum_{j\in J} \langle y, f_j\rangle f_j, \quad \text{with } \langle y, f_j\rangle = \int_s y(s)f_j(s)\,d\mu(s).$$

Now, applying the bilinear form $\tilde{m}$ to the pair (x, y), we get

$$\tilde{m}(x,y) = \sum_{i,j} \tilde{m}(e_i, f_j)\langle x, e_i\rangle\langle y, f_j\rangle. \tag{1}$$

By the Fubini theorem, we have

$$\langle x, e_i\rangle\langle y, f_j\rangle = \int_{K\times S} x\otimes y \cdot e_i \otimes f_j\,d\lambda \otimes d\mu,$$

where $x\otimes y(t,s) = x(t)y(s)$ and $d\lambda\otimes d\mu$ is the product measure. Since $x\otimes y$ is in the space $L^2_{\lambda\otimes\mu}(K\times S)$ and the family $(e_i\otimes f_j)_{i,j}$ is an orthonormal Hilbert basis of $L^2_{\lambda\otimes\mu}(K\times S)$, we have

$$\sum_{i,j} \left| \int_{K\times S} x\otimes y\cdot e_i\otimes f_j\,d\lambda\otimes d\mu \right|^2 < \infty. \tag{2}$$

On the other hand, the set $\operatorname{span}\{x\otimes y\colon x\in L^2_\lambda(K) \text{ and } y\in L^2_\mu(S)\}$ being dense in $L^2_{\lambda\otimes\mu}(K\times S)$, from the relations (1) and (2), we deduce that

$$\sum_{i,j} |\tilde{m}(e_i, f_j)|^2 < \infty. \tag{3}$$

Now define a new bilinear form

$$n\colon L^2_{\lambda\otimes\mu}(K\times S)\times C(K\times S) \to \mathbf{C}$$

by

$$n(\phi,\psi) = \sum_{i,j} \tilde{m}(e_i, f_j) \int_{K\times S} \phi(t,s)\psi(t,s) e_i\otimes f_j\,d\lambda\otimes d\mu.$$

We shall first show that the bilinear form n is well defined and continuous. Indeed, since the function $\phi\cdot\psi$ is in $L^2_{\lambda\otimes\mu}(K\times S)$ and $(e_i\otimes f_k)_{i,j}$ is an orthonormal Hilbert basis of the space $L^2_{\lambda\otimes\mu}(K\times S)$,

$$\sum_{i,j} \left| \int_{K\times S} \phi\cdot\psi\cdot e_i\otimes f_j\,d\lambda\otimes d\mu \right|^2 < \infty.$$

Consequently, by (3), n is well defined and we have

$$|n(\phi,\psi)|^2 \le \sum_{i,j} |\tilde{m}(e_i, f_j)|^2 \cdot \sum_{i,j} \left| \int_{K\times S} \phi\cdot\psi\cdot e_i\otimes f_j\,d\lambda\otimes d\mu \right|^2.$$

But, as

$$\sum_{i,j} \left| \int_{K\times S} \phi \cdot \psi \cdot e_i \otimes f_j \, d\lambda \otimes d\mu \right|^2 = \|\phi \cdot \psi\|_2^2$$

and

$$\|\phi \cdot \psi\|_2^2 = \int_{K\times S} |\phi\psi|^2 d\lambda \otimes d\mu \leq \|\psi\|_\infty^2 \|\phi\|_2^2$$

we obtain the inequality

$$|n(\phi, \psi)| \leq \left(\sum_{i,j} |\tilde{m}(e_i, f_j)|^2 \right)^{1/2} \|\phi\|_2 \|\psi\|_\infty,$$

which proves the continuity of n. Now, let $x, \tilde{x}$ be in $C(K)$ and $y, \tilde{y}$ be in $C(S)$. Then,

$$\begin{aligned} m(x\tilde{x}, y\tilde{y}) &= \sum_{i,j} \tilde{m}(e_i, f_j) \langle x\tilde{x}, e_i \rangle \langle y\tilde{y}, f_j \rangle \\ &= \sum_{i,j} \tilde{m}(e_i, f_j) \int_{K\times S} x \otimes y \cdot \tilde{x} \otimes \tilde{y} \cdot e_i \otimes f_j \, d\lambda \otimes d\mu \\ &= n(x \otimes y, \tilde{x} \otimes \tilde{y}). \end{aligned}$$

Thus $m(x\tilde{x}, y\tilde{y}) = n(x \otimes y, \tilde{x} \otimes \tilde{y})$. Since the space $L^2_{\lambda\otimes\mu}(K \times S)$ is reflexive, the bilinear form n is Arens regular. Now, the equality $m(x\tilde{x}, y\tilde{y}) = n(x \otimes y, \tilde{x} \otimes \tilde{y})$ implies that m is biregular. We conclude that the space $C(K) \hat{\otimes} C(S)$ is Arens regular.

We proceed with some corollaries.

COROLLARY 4.21. *The algebra $l^\infty \hat{\otimes} l^\infty$ is Arens regular.*

In the next corollary, λ and μ are two arbitrary measures and on the spaces $L^\infty(\lambda)$ and $L^\infty(\mu)$ the multiplication is pointwise multiplication.

COROLLARY 4.22. *The algebra $L^\infty(\lambda) \hat{\otimes} L^\infty(\mu)$ is Arens regular.*

Now, let D be the closed unit disk of the complex plane and π be its boundary. By $A(D)$ we denote the disk algebra on D. The algebra $A(D) \hat{\otimes} A(D)$ naturally embeds into $C(\pi) \hat{\otimes} C(\pi)$ but the natural embedding is not isometric [**19**]. However, as J. Bourgain has shown [**4**, Theorem 1] any bilinear form $\tilde{m}\colon A(D) \times A(D) \to \mathbf{C}$ has a continuous bilinear extension $m\colon C(\pi) \times C(\pi) \to \mathbf{C}$. So, from Theorem 4.20 and Proposition 3.8 we deduce the next corollary.

COROLLARY 4.23. *The algebra $A(D) \hat{\otimes} A(D)$ is Arens regular.*

Next, let $H^\infty(D)$ be the Hardy class on D and μ the normalized Haar measure of π. Another result due to J. Bourgain [**5**, Corollary 3.10] says that any bilinear form $\tilde{m}\colon H^\infty(D) \times H^\infty(D) \to \mathbf{C}$ has a continuous bilinear extension $m\colon L^\infty_\mu(\pi) \times L^\infty_\mu(\pi) \to \mathbf{C}$. As above, we deduce the following corollary.

COROLLARY 4.24. *The algebra $H^\infty(D) \hat{\otimes} H^\infty(D)$ is Arens regular.*

We recall that a Banach space X is said to satisfy "Grothendieck's Theorem" (G.T. for short) if any linear operator from X into l^2 is absolutely summing [**24**, Chapter 6]. The results of J. Bourgain which we have used above mean that the dual spaces of $A(D)$ and $H^\infty(D)$ satisfy G.T. (For other spaces satisfying G.T., see [**24**, Chapter 6].) The next proposition is a particular case of [**24**, Proposition 6.3].

PROPOSITION 4.25. *Let K, S be two arbitrary compact sets and A a closed subalgebra of $C(K)$. Assume that the dual of A satisfies G.T. Then, any bilinar form $m\colon A \times C(S) \to \mathbf{C}$ has a continuous bilinear extension $\tilde{m}\colon C(K) \times C(S) \to \mathbf{C}$, so that $A \hat{\otimes} C(S)$ is Arens regular.*

As an immediate consequence of this proposition we have the following corollary.

COROLLARY 4.26. *For any compact set S, the algebras $A(D) \hat{\otimes} C(S)$ and $H^\infty(D) \hat{\otimes} C(S)$ are Arens regular.*

For the next corollary we shall need the "cotype" notion. (For this notion and related results, see Chapter 3 of [**24**].) The only results we shall need here are that any L^1-space, as well as the dual space of $A(D)$, are of cotype 2 (see [**24**, p. 34 and **4**, Corollary 1], respectively).

COROLLARY 4.27. *The algebra $A(D) \hat{\otimes} H^\infty(D)$ is Arens regular.*

PROOF. Let $m\colon A(D) \times H^\infty(D) \to \mathbf{C}$ be a bilinear form and $U\colon H^\infty(D) \to A(D)'$ the linear operator corresponding to it. Since $A(D)'$ is of cotype 2, U has a continuous linear extension $\tilde{U}\colon L^\infty(\pi) \to A(D)'$ [**5**, Corollary 3.9]. This means that m has a continuous bilinear extension $\tilde{m}\colon A(D) \times L^\infty(D) \to \mathbf{C}$. Since the space $L^\infty(\pi)$ is a $C(S)$-space for some compact set S, Proposition 4.25 above implies that m has a continuous bilinear extension $\bar{m}\colon C(\pi) \times L^\infty(\pi) \to \mathbf{C}$. The biregularity of $\bar{m}$ then implies that of m so that $A(D) \hat{\otimes} H^\infty(D)$ is Arens regular.

We end this note with some questions we would like to have settled but were unable to. These questions are very closely connected with each other.

1. Let A, B be two Arens regular Banach algebras. Example 3.2 shows that there may exist nonregular biregular bilinear forms on $A \times B$. We do not know if there exist regular nonbiregular forms on $A \times B$.

2. Although the analogue of the Grothendieck theorem used in the proof of Theorem 4.20 holds for noncommutative C^*-algebras as well [**24**, Theorem 9.1.], the proof of Theorem 4.20 does not seem to carry over the noncommutative case, and we do not know if the projective tensor product of two C^*-algebras is Arens regular.

3. Let A and B be two unital Arens regular algebras. Corollary 3.6 shows that if $A \hat{\otimes} B$ is Arens regular then any bilinear form $m\colon A \times B \to \mathbf{C}$ is weakly compact. We do not know if the converse holds.

4. Let A and B be two Arens regular algebras. Theorem 4.5 shows that any compact bilinear form $m\colon A \times B \to \mathbf{C}$ is biregular. We do not know if any integral bilinear form $m\colon A \times B \to \mathbf{C}$ is biregular.

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