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SOME APPLICATIONS OF TREE-LIMITS TO GROUPS. PART 1

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ABSTRACT. Sharper applications to group theory are given of an elegant construction —the "tree-limit"—which S. Shelah circulated as a preprint in 1977 and used to obtain ∞-ω-enlargements to power 2^ω of certain countable homogeneous groups and skew fields. In this paper we enlarge the class of groups to which this construction can be interestingly applied and we obtain permutation representations of countable degree of the tree-limit groups; we obtain uncountable subgroup-incomparability for enlargements of countable existentially closed groups and even in nonhomogeneous cases we obtain the very strong "archetypal direct limit property" (which implies ∞-ω-equivalence (see (1.0)) of the permutation representations). We are able to control the permutation representations which get stretched by the tree-limit by varying the point-stabilizer subgroups (see (5.5)). In particular we can archetypally stretch in 2^ω subgroup-incomparable ways any homogeneous permutation representation of a countable locally finite group in which every finite subgroup has infinitely many regular orbits (Theorem 4). We discuss cases where tree-limits are subgroups of inverse limits.

1. Orientation and results. Here we give a number of new group-theoretical applications of an elegant construction of S. Shelah [**11**] which we call the "tree-limit" construction. This construction was circulated in preprint form in 1977 by Shelah (in fact, the present author had to decline an offer of coauthorship at that time). The version [**3**] is formally quite different from Shelah's original presentation which we have chosen to follow because of its naturalness. The progress we have made comes mainly from the special types of amalgamations we use which provide an easy way to verify Shelah's "crucial lemma" giving the tree limit group power 2^ω and which also provide a way of obtaining an embedding of the tree-limit group in $\mathrm{Sym}(\omega)$. We are also able to relax the requirement of homogeneity on the groups which get enlarged by the tree-limit by making use of special properties of wreath products. We also formulate and study the countable direct limit property which the tree-limit group often satisfies. This, we feel, is of such interest that it merits some new terminology.

(1.0) DEFINITION. Suppose G is a countable group and H is a group. We say that H is an *archetypal limit* of G iff $H = \bigcup\{F_s \mid s \in P(I)\}$ where $P(I)$ is the set of *finite subsets* of I, F_s is finitely generated for all $s \in P(I)$, $s \subseteq t \in P(I)$ implies $F_s \subset F_t$, and there exists $S \subseteq I$ with $|S| = \omega$ such that, for all $S \subseteq T \subseteq I$ with $|T| = \omega$, we have

$$F_T = \bigcup\{F_s \mid s \in P(T)\} \cong G.$$

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If ρ: $G \to \mathrm{Sym}(\omega)$ and τ: $H \to \mathrm{Sym}(\omega)$ are representations, we say that $(\omega, \tau(H))$ is an *archetypal limit* of $(\omega, \rho(G))$ provided every $(\omega, \tau(F_T))$ as above is isomorphic to $(\omega, \rho(G))$ as a permutation group.

Every archetypal limit of G is obviously ∞-ω-equivalent to G since it has a closed, unbounded set of countable subgroups isomorphic to G. All of the archetypal limits we produce from tree-limits have an even stronger property built into them, namely

(*) For every countable $T \subseteq I$ with $T \supseteq S$ there is a decomposition $T = \bigcup_{n \in \omega} t_n$ where $t_n \subset t_{n+1}$ are finite such that for all T, T' there is an isomorphism σ: $F_T \to F_{T'}$ such that $\sigma(F_{t_n}) = F_{t'_n}$ for all $n < \omega$.

This property is satisfied whenever the special wreath product amalgamations are used to construct the tree-limit and in some other cases as well (see (4.0) ff.). These special amalgamations provide conjugating elements (in the base groups) which construct the amalgams and define the direct limit isomorphisms σ of (*). This technique is given in §4. Of special interest is the fact that all isomorphisms σ in (*) are permutation group isomorphisms in our applications. Using the wreath amalgamations mentioned above we are also able to obtain sharper results on isomorphism-types. Our first two theorems illustrate all these points.

THEOREM 1. *Suppose G is a countable existentially closed group. Then there exists an isomorphism ρ: $G \to \mathrm{Sym}(\omega)$ and 2^ω groups G_α $(\alpha < 2^\omega)$ and isomorphism ρ_α: $G_\alpha \to \mathrm{Sym}(\omega)$ such that*

(i) $|G_\alpha| = 2^\omega$,

(ii) $(\omega, \rho_\alpha(G_\alpha))$ *is an archetypal limit of* $(\omega, \rho(G))$,

(iii) *If $\alpha \neq \beta < 2^\omega$, then G_α and G_β do not have isomorphic uncountable subgroups.*

The subgroup-incomparability property (iii) above has only been established previously for locally finite groups **[5, 3, 14]**. However, in **[13]** Shelah proved (among many other things) that if G is a countable e.c. group then no group of power ω_1 is embedded in every uncountable e.c. group having the same skeleton as G. Of course, this is an easy consequence of (iii). Because of a seeming technical hang-up, we are at present still unable to extend the subgroup-incomparability property to torsion-free groups (see the Remark after (5.5)).

In Theorem 2 we duplicate Theorem 1 for a large class of groups containing all of those dealt with by Thomas **[14]**. First we define the wreath products that we need. If A and B are groups then the unrestricted wreath product B WR $A = B^A A$ is the semidirect product of the Cartesian power B^A by A where A acts on functions $f \in B^A$ by right translations of their domain, that is, if $c, a \in A$, then $f^a(c) = f(ca^{-1})$. The restricted wreath product B wr A is the subgroup of B WR $A = B^A A$ generated by A and the direct product $B^{(A)}$ (the functions with finite support). Thus B wr $A = \langle B, A \rangle$ if we identify $B \subset B$ wr A with the 1-coordinate subgroup of $B^{(A)}$. Finally, if $A_0 \subset A$ we define B wr (A_0, A) to be the subgroup $\langle D, A \rangle \subset B$ WR $A = B^A A$ where $D \subset B^A$ is the diagonal of B over A_0, that is, $D = \{ f_b \mid b \in B \}$

where

$$f_b(a) = \begin{cases} 1 & \text{if } a \notin A_0 \\ b & \text{if } a \in A_0 \end{cases} \qquad (a \in A).$$

Thus A_0 centralizes D since $a \in A_0$ acts on $f_b \in D$ by permuting the A_0-coordinates among themselves.

THEOREM 2. *Suppose G is a countable group such that for all finitely generated $A_0 \subset A \subset G$ there are infinitely many $n < \omega$ such that $\mathbf{Z}_n \operatorname{wr} (A_0, A)$ is embeddable in G over A. Then the conclusions of Theorem* 1 *hold for G.*

If G is existentially closed in the class of locally finite-solvable π-groups, these tree-limit enlargements have properties similar to the ∞-ω-enlargements constructed by Thomas—they have chief series with countably many jumps. Thomas' enlargements are also complete (assuming $\diamondsuit$). Shelah once conjectured that the tree-limit group can be made complete by making suitably rigid amalgamations in the tree-limit system, but this remains a very speculative matter. The subgroup-incomparability property is of particular interest because it is stronger than can be obtained using logical stability theory. The previously known cases of subgroup-incomparability (which are all locally finite) satisfy an even stronger incomparability, namely, among uncountable subgroups of a given group [**5**, 0.3; **14**, Theorem 2]. For an existentially closed group such internal incomparability is impossible since many centralizers in e.c. groups contain themselves properly. Stability theory has been applied to e.c. groups by making use of Ziegler's "homogeneous limit" construction [**12**]; and an elementary construction of the maximum number of nonembeddable e.c. groups (with given skeleton) in power ω_1 is given in [**6**, Theorem 2]. Not every countable e.c. group satisfies the hypothesis of Theorem 2, but to prove this would require difficult recursion theory.

The simplest types of groups to which we can apply Theorem 2 are iterated wreath products of finite groups with top group embeddings:

$$G = \cdots A_3 \operatorname{wr} (A_2 \operatorname{wr} A_1)$$

where the A_i are finite groups of unbounded exponent. In this situation all of the tree-limit groups are contained in the inverse limit group of the system having the base groups as kernels and can be endowed with finite orbits on ω (see (5.11)). In the special case $A_i = \mathbf{Z}_{p_i}$ where p_i $(i < \omega)$ are distinct primes, G has the property that all of its maximal p-subgroups are finite. Let χ be the class of all such locally finite groups. Baer showed that every countable locally solvable χ-group which is embedded properly in itself can be enlarged to an ∞-ω-equivalent group of power ω_1 (in one way) [**1**]. We cannot even come close to applying tree-limits to every group which Baer's theorem enlarges, but at least Theorem 2 does give dramatic information in the case of the wreath products above. Thus we are left with many interesting spectrum problems in power 2^ω. Indeed 2^ω is the maximum possible power for a χ-group. In the locally solvable case this is easy because every composition factor of such a group is cyclic and there can be only finitely many for

a given prime—and so the result follows from [**2**, Theorem 1]. We have been unable to determine if some group of the form $G = \cdots A_3 \text{ wr } (A_2 \text{ wr } A_1)$ (A_k finite) can have an ∞-ω-enlargement beyond 2^ω, but this seems unlikely.

In *all* cases of Theorems 1 and 2 the tree-limit groups G_α are contained in $(\prod_{n=1}^\infty G_n)/(\sum_{n=1}^\infty G_n)$ (the cartesian product modulo the direct product) where $G = \bigcup_{n=1}^\infty G_n$, $G_n \subset G_{n+1}$, G_n finitely generated, is the decomposition of G used to construct the tree-limit system (see §2)). (In fact this is true for all tree-limits constructed from the skeleton of any countable model and is the sharp form that the varietal property of a direct limit takes in the case of a tree-limit.)

The tree-limits G_α of Theorems 1 and 2 have representations in $\mathrm{Sym}(\omega)$ which are constructed in a canonical way given in Lemma 3. These natural representations usually have infinitely many orbits on ω (see Lemma 3). However, the special situation in which we can obtain the archetypal limit property (see (4.0)) also permits us to obtain transitive representations ρ either by making the orbits of ρ faithful (see (4.0)) or by prescribing the stabilizer subgroup $H \subset G$ beforehand and constructing ρ to be the regular representation of G on the right cosets of H (see (5.5)). These considerations give

THEOREM 2*. *In Theorems* 1 *and* 2 *we can arrange for the representation* ρ: $G \to \mathrm{Sym}(\omega)$ *to be transitive.*

It is of considerable interest to see how Theorem 2* can be strengthened in the following special cases:

(a) $G = U =$ the countable universal locally finite group;

(b) $G = E_p =$ the unique countable existentially closed locally finite p-group (for any prime p) [**8–10**]; and

(c) G is a countable existentially closed group.

Two types of improvement in Theorem 2* are possible. We can obtain 2^ω inequivalent representations ρ for which the theorem holds, and we can try to make $\rho(G)$ act highly transitively or homogeneously on ω. In all of the above cases it is possible to obtain 2^ω inequivalent transitive representations ρ: $G \to \mathrm{Sym}(\omega)$ in which point-stabilizers are isomorphic to G and which satisfy Theorem 2*—that is, each of which gets archetypally stretched to power 2^ω in 2^ω drastically different ways by tree-limits. The proofs of this rely on the technique mentioned above of constructing many different "canonical" point-stabilizer subgroups H—since any canonical subgroup can serve as the point-stabilizer in transitive tree-limit representations (see (5.5)). For (b) we can use the result [**9**, 3.1, 3.2] to distinguish the H by the order-partition of the jumps which H creates in the unique chief series of E_p, while for (a) and (c) we can construct H so that the lattice of subgroups between H and G is totally ordered and has a specified order-type. In these proofs we cannot assert any more than transitivity of $\rho(G)$ on ω and we do not obtain many primitive representations with point-stabilizers $\cong G$ by these methods. Indeed, in the case $G = E_p$, there are no maximals (a maximal would have index p and hence contradict existential closure of E_p).

In the case $G = U$ we can give the representation ρ: $U \to \mathrm{Sym}(\omega)$ a very strong homogeneity property (implying high transitivity) introduced by Kegel [7] (see Proposition 1.5)—however, it is unknown if U has more than one such representation. In the case (c) above we can obtain 2^ω highly transtive representations ρ: $G \to \mathrm{Sym}(\omega)$ with nonembeddable point-stabilizers each of which can be used in Theorem 2*. We summarize these results in the following theorems.

THEOREM 3. *There exist 2^ω nonisomorphic transitive, faithful representations τ_β: $E_p \to \mathrm{Sym}(\omega)$ $(\beta < 2^\omega)$ in which point-stabilizers are isomorphic to E_p and such that the conclusions of Theorem* 1 *hold for each $\tau_\beta = \rho$ with $G = E_p$. Further an identical result holds with $G = U$ or $G =$ any countable existentially closed group (in place of E_p).*

THEOREM 1*. *Suppose G is a countable existentially closed group. Then there exist 2^ω isomorphisms τ_β: $G \to \mathrm{Sym}(\omega)$ $(\beta < 2^\omega)$ such that $\tau_\beta(G)$ is highly transitive on ω, distinct representations have nonembeddable point-stabilizer subgroups, and the conclusions of Theorem* 1 *hold for each $\tau_\beta = \rho$ $(\beta < 2^\omega)$.*

(For the homogeneous representation of U see Proposition 1.5.)

Finally, we note that in the situation $G = \cdots A_3 \text{ wr } (A_2 \text{ wr } A_1)$ (A_n finite and unbounded) we can obtain 2^ω natural representations τ_β: $G \to \mathrm{Sym}(\omega)$ having all their orbits finite and such that every $\tau_\beta = \rho$ satisfies Theorem 1. We leave the statement of this for later (see (5.11)). In this situation the tree-limit groups are (permutationally) subgroups of the Cartesian product of the top groups (in fact, as already mentioned, they lie in the inverse limit of them). One can also obtain many transitive representations of G in this case. Without the unboundedness assumption we can only obtain a single archetypal enlargement of G (see Corollary 4).

We mention here certain groups which cannot be enlarged by tree-limits or by any other means. One example of such a group is $G = (\mathbf{Z}_p \text{ wr } \mathbf{Z}_p) \text{ wr } \mathbf{Z}_p \cdots$. One can show for this and for similar groups of finitary permutations that, if every countable subgroup of a group H is contained in a subgroup isomorphic to G, then $H \cong G$. (The key to the proof is the observation that for all $1 \neq g \in G$, the action of G on the conjugacy class $\mathrm{Con}(g)$ of g consists of *finitary permutations* of $\mathrm{Con}(g)$; then, combining this with the fact that the centralizer $C_G(g^G)$ has the maximum condition on subgroups normal in G, allows one to prove that, for any $1 \neq g \in H$ as above, both $C_H(g^H)$ and $H/C_H(g^H)$ are countable.) In fact the above fact extends to groups G which are finite subdirect products of groups such as $(A_1 \text{ wr } A_2) \text{ wr} \cdots$. (These and other properties of finitary permutation groups are the subject of ongoing work with R. E. Phillips and Jon Hall.) A similar observation applies to the countable alternating group G; this follows from a recent result of Jon Hall [15].

Our remaining theorems give some specific applications of tree-limits to permutation representations. These applications were inspired by [7] in which Kegel constructed a homogeneous representation (see below) of the countable universal locally finite group. We have called the key concept involved in this "relative homogeneity" and initiated its study in [4] (where Theorem 4 below was announced). It involves a

strengthening of high transitivity to a condition of saturation on centralizers of f.g. subgroups (see below). This concept is usefully combined with that of sharpness (due to Peter Neumann) which is defined as follows: the permutation group (X, G) is *sharp* iff every $1 \neq g \in G$ moves all but finitely many elements of X.

Before stating Theorem 4 we must define and discuss relative homogeneity.

The permutational skeleton $\mathrm{skel}(X, G)$ of a permutation group (X, G) is the set of all permutation group isomorphism-types (Y, H) where H is a finitely generated subgroup of G and Y consists of finitely many orbits of H on X. (We expressly allow $Y = \varnothing$ in this definition because failure to do so would have an awkward consequence below.)

The centralizer of H in G is $C_G(H)$. We call $G \subset \mathrm{Sym}(X)$ *relatively homogeneous* if for every $(Y_1, H), (Y_2, H) \in \mathrm{skel}(X, G)$ and permutation isomorphism $f\colon (Y_1, H) \to (Y_2, H)$ with $f(h) = h$ $(h \in H)$, there exists $g \in C_G(H)$ such that $f(y) = g(y)$ for all $y \in Y_1$. This implies that G is highly transitive on X by taking $H = 1$.

If $G \subset \mathrm{Sym}(X)$ is relatively homogeneous and if $f\colon (Y_1, H_1) \to (Y_2, H_2)$ is any permutation group isomorphism of members of $\mathrm{skel}(X, G)$ such that there exists $x \in G$ such that $f(h) = xhx^{-1}$ for all $h \in H_1$ then we can find $g \in G$ which induces f (that is, $f(y) = g(y)$ $(y \in Y_1)$ and $f(h) = ghg^{-1}$ $(h \in H_1)$) by completing the diagram

$$\begin{tikzcd}
(Y_1, H_1) \arrow[rr, "f"] \arrow[dr, "z"'] & & (Y_2, H_2) \\
& (x^{-1}(Y_2), H_1) \arrow[ur, "x"'] &
\end{tikzcd}$$

using relative homogeneity to obtain $z \in G$ and defining $g = xz$. This is the reason for the terminology.

In Kegel [7] homogeneity of $G \subset \mathrm{Sym}(X)$ is defined by demanding that every isomorphism $f\colon (Y_1, H_1) \to (Y_2, H_2)$ between members of $\mathrm{skel}(X, G)$ be induced by some element of G. This means simply that G is relatively homogeneous as a permutation group and inner homogeneous as a group (that is, every isomorphism of f.g. subgroups of G is induced by conjugation in G). However, if we had defined $\mathrm{skel}(X, G)$ by disallowing (ϕ, H) (the group-theoretic skeleton) then Kegel's homogeneity would not obviously imply that G is inner-homogeneous since we might have isomorphic f.g. subgroups of G which need not be conjugate because none of their orbits are isomorphic. (I do not know if such representations exist.)

An important fact noted by Kegel is that any two countable permutation groups which are homogeneous (that is, relatively and inner homogeneous) and have the same permutational skeleton are isomorphic as permutation groups. This is proved with the usual back-and-forth construction with homogeneous models.

As an example let P be the group of finitary permutations of ω and suppose $P \subset G \subset \mathrm{Sym}(\omega)$ and that G is locally finite. Then G is relatively homogeneous because every permutation isomorphism $f\colon (Y_1, H) \to (Y_2, H)$ can be extended to a finitary permutation which centralizes H by completing the cycles of H-orbits that f has on $Y_1 \cup Y_2$ and fixing all other elements.

One interesting property of homogeneity is

(1.1) Suppose $G \subset \mathrm{Sym}(X)$ is relatively homogeneous and G is inner homogeneous. Then for all finite $S \subset X$ the stabilizer subgroup $G_S = \{g \in G \mid g(s) = s \text{ for all } s \in S\}$ is inner homogeneous also.

PROOF. Suppose φ: $A \to B$ is an isomorphism of two f.g. subgroups of G_S. Let $S = \{x_1, \ldots, x_n\}$. We first induce φ from A to B by an element $h \in G$ using inner homogeneity of G and then we follow this by an element $g \in C_G(B)$ which returns $h(x_1), \ldots, h(x_n)$ to $x_1, \ldots, x_n$ (using relative homogeneity) to obtain $gh \in G_S$ which induces φ. □

Thus, if $G \subset \mathrm{Sym}(X)$ is homogeneous and G_S has the same group skeleton (f.g. subgroups) as G then $G \cong G_S$. Notice that, regardless of what the group skeleton of G_S might be, the fact that $(X - S, G_S)$ is relatively homogeneous follows from relative homogeneity of (X, G) since S is finite.

Since Kegel's construction of a relatively homogeneous representation of U permits the inclusion of more orbits at each step, every stabilizer U_S is easily made to contain every finite group, implying $U \cong U_S$. (In fact there is only one known relatively homogeneous representation for U—namely, that having a universal skeleton.) In particular this gives

(1.2) The countable universal locally finite group U has a maximal subgroup M such that, for all $u_1, \ldots, u_n \in U$; $M^{u_1} \cap \cdots \cap M^{u_n} \cong U$.

To prove this we take M = a point-stabilizer U_x in the above representation. Such maximals (and representations) occur for many nonlocally finite homogeneous groups as well. This property seems interesting because I cannot in any situation see how to construct such a maximal internally using amalgamations because the subgroup cannot be kept proper even in situations where Britton's Lemma is available.

We next consider the concept of sharpness which was defined above.

(1.3) A locally finite $G \subset \mathrm{Sym}(X)$ is sharp if and only if every finite $A \subset G$ has only finitely many irregular orbits on X.

PROOF. ($\Leftarrow$) Clear.

($\Rightarrow$) Suppose $A \subset G$ has infinitely many irregular orbits X_i ($i \geqslant 1$). Then for each $i \geqslant 1$ we can find some $1 \neq a_i \in A$ and $x \in X_i$ such that $a_i(x_i) = x_i$. Since A is finite, infinitely many a_i's must be equal and this contradicts sharpness. □

We also note

(1.4) If $G \subset \mathrm{Sym}(\omega)$ is locally finite, sharp and highly transitive, then G has infinitely many conjugacy classes of elements of all orders > 1.

PROOF. Let $n > 1$ be given. Using high transitivity we can find elements $x_i \in G$ such that $|x_i| = n$ and x_i leaves fixed at least i elements of ω. If G had finitely many conjugacy classes of elements of order n, then infinitely many of the x_i would be conjugate in G and this would contradict sharpness. □

Thus U cannot have a sharp and highly transitive representation.

It turns out that Kegel's representation of U is ideally suited to the tree-limit construction—in fact, this was the way the permutation representation of "canonical" tree-limits was discovered.

The following result generalizes both the enlarging of Kegel's representation of U and the result announced in [**4**] on sharp, relatively homogeneous locally finite groups.

THEOREM 4. *Suppose (ω, G) is a countable, relatively homogeneous, locally finite permutation group such that every finite subgroup of G has infinitely many regular orbits on ω. Then there exist permutation groups (ω, G_α) $(\alpha < 2^\omega)$ with $|G_\alpha| = 2^\omega$ such that*

(i) *(ω, G_α) is an archetypal limit of (ω, G),*

(ii) *If $\alpha \neq \beta$, then G_α and G_β do not have isomorphic uncountable subgroups.*

In Theorem 4, if (ω, G) is sharp, then obviously every (ω, G_α) is also sharp. In [**4**] it was shown that 2^ω countable, sharp, relatively homogeneous locally finite groups exist and that all sharp representations of most countable locally finite groups are contained in sharp and relatively homogeneous ones (on the same set).

As mentioned above, we have as a corollary to Theorem 4

(1.5) PROPOSITION. *There exists a relatively homogeneous representation ρ: $U \to \mathrm{Sym}(\omega)$ of the countable universal locally finite group U and 2^ω subgroups $U_\alpha \subset \mathrm{Sym}(\omega)$ $(\alpha < 2^\omega)$ with $|U_\alpha| = 2^\omega$ such that*

(i) *(ω, U_α) is an archetypal limit of $(\omega, \rho(U))$,*

(ii) *If $\alpha \neq \beta$, then U_α and U_β do not have isomorphic uncountable subgroups.*

The representation of U involved in (1.5) is that with a universal skeleton. As mentioned before, it is unknown if this is the only relatively homogeneous representation of U.

In §§2 and 3 we present the tree-limit construction in the manner of Shelah [**11**] (but with more verbosity) and give our innovation in the amalgamations which yields permutation representations in $\mathrm{Sym}(\omega)$ of the tree-limit groups. In §4 we give our method of obtaining the archetypal limit property of the tree-limit group in the absence of homogeneity and prove Theorem 4. We prove Theorems 1, 2, 2*, and 3 in §5. In §6 we give the proof of Theorem 1*, which is distinctively combinatorial, and we state some further results to be proved in a sequel to this paper.

According to a suggestion of the referee we will outline the key components of the proofs. Special tree-limit systems $\mathscr{S}$ and their tree-limit groups $G_{\mathscr{S}}$ are defined and discussed in Lemmas 1 and 2. Special permutation representations of $G_{\mathscr{S}}$ in $\mathrm{Sym}(\omega)$ are constructed in Lemma 3. Some small technical lemmas follow (Descendence, Unique Ancestry, Ancestral Relations, and Branching) which are needed mainly for the proof of the Spectrum Lemma (3.11). The Spectrum Lemma gives algebraic conditions in the construction of 2^ω special tree-limit systems sufficient to guarantee that the resulting tree-limit groups do not possess isomorphic uncountable subgroups. (The condition we use here is implicit in Shelah [**11**].) In our applications the

Spectrum Lemma is coupled with two other results which give further properties to the 2^ω systems constructed by a single application of the Spectrum Lemma and their tree-limit groups. The Archetypal Limit Theorem (4.0) gives conditions in constructing the special permutation representations of Lemma 3 of $G_{\mathscr{S}} \subset \mathrm{Sym}(\omega)$ so that $G_{\mathscr{S}}$ has the archetypal limit property (with respect to a special countable subgroup of itself corresponding to the direct union of the system $\mathscr{S}$ itself). This is all the machinery needed to prove Theorem 4 since the countable representation to be archetypally stretched in many different ways is given beforehand. The Stabilizer Theorem (5.5) is needed for our other applications because it shows how a permutation representation suitable for use in the Archetypal Limit Theorem can be obtained by constructing its point-stabilizer subgroup. This stabilizer subgroup can be kept fixed during the construction of the 2^ω systems of the Spectrum Lemma, thus yielding many different archetypal blow-ups of the same countable representation of a great many countable groups. The representations can be controlled by varying or enriching the point-stabilizer subgroups. We meet the hypotheses of the Spectrum Lemma in the above context by using special wreath products. The necessary properties of the subgroups so constructed can then be transferred to e.c. groups. In e.c. groups it is further possible to enrich the point-stabilizer subgroups to obtain Theorem 1* as well as an exotic recursion-theoretic type of permutation group homogeneity (see §6).

We close this introduction with a simple example of a tree-limit (due to Shelah). Let T be the simple tree defined in the first paragraph of §2 below. Let F be the free abelian group with generators ζ_k (k odd). These are the vertices at which branchings occur in T. Let F_ω be the free abelian group with generators T_ω (the infinite paths through T). We obtain a torsion-free class 2 nilpotent group $G = \langle F_\omega \rangle$ such that $F = \zeta G$ (the center of G) by defining $[\alpha, \beta] = \alpha \wedge \beta$ for all α, $\beta \in T_\omega$ where $\alpha \wedge \beta = \zeta_k$ is the vertex where α and β branch away from each other. We have $|G| = 2^\omega$ and G has the property that *every abelian subgroup of G is countable.* It would be interesting to know if many nonisomorphic, nonembeddable, or even subgroup-incomparable such G exist, but we do not have the answers to any of these questions.

2. The tree-limit system. First we define the simple type of tree T involved in the construction.

We let $T = \bigcup\{T_k \mid k \geqslant 0\}$ where $T_k \subseteq \{0,1\}^k = \{\varepsilon_1 \cdots \varepsilon_k \mid \varepsilon_i = 0 \text{ or } 1\ (1 \leqslant i \leqslant k)\}$ subject to the conditions

(i) $T_0 = \varnothing$, $T_1 = \{0\}$,

(ii) If k is even, then $T_{k+1} = \{\sigma 0 \mid \sigma \in T_k\}$,

(iii) If k is odd, then there exists $\zeta_k \in T_k$ such that

$$T_{k+1} = \{\sigma 0 \mid \sigma \in T_k - \{\zeta_k\}\} \cup \{\zeta_k 0, \zeta_k 1\},$$

(iv) For all $\sigma \in T_k$ ($k \geqslant 0$) there exists $n \geqslant k$ such that n is odd and σ is an initial segment of ζ_n.

Thus T is a tree with ω finite levels having unique successors except at odd levels where a single binary branching occurs.

Condition (iv) guarantees that $T_\omega = \{(\varepsilon_i)_{i=1}^\infty \mid \varepsilon_1 \cdots \varepsilon_k \in T_k \ (k \geqslant 1)\}$ has power 2^ω.

If $\alpha, \beta \in T \cup T_\omega$ and α is an initial segment of β, we write $\alpha \leqslant \beta$.

If $X \subseteq \bigcup\{T_n \mid n \geqslant k\} \cup T_\omega$, we define $X|_k = \{\sigma \in T_k \mid \sigma \leqslant x$ for some $x \in X\}$.

If $k \leqslant n$, we define $Mp(k, n) = \{f\colon S \to T_n \mid S \subseteq T_k$ and $\alpha \leqslant f(\alpha)$ for all $\alpha \in S\}$. If k is even then $Mp(k, k+1)$ has a unique maximum member which we denote f_k, that is, $f_k(\alpha) = \infty 0$ for all $\alpha \in T_k$. If k is odd then $Mp(k, k+1)$ has two maximal members which we denote f_k^0 and f_k^1, that is, $f_k^0(\alpha) = \alpha 0$ for all $\alpha \in T_k$ and

$$f_k^1(\alpha) = \begin{cases} \alpha 0 & \text{if } \alpha \in T_k - \{\zeta_k\}, \\ \alpha 1 & \text{if } \alpha = \zeta_k. \end{cases}$$

We put $Mp = \bigcup\{Mp(k, n) \mid k \leqslant n < \omega\}$.

(2.0) Definition. A *T-limit system* $\mathscr{S} = \{G_S, \varphi_f\}$ is a collection of groups G_S $(S \subseteq T_k, k < \omega)$ and isomorphisms $\varphi_f\colon G_S \to G_{f(S)}$ $(S \subseteq T_k,\ f \in Mp(k, n)$ for some $n \geqslant k$ with $\operatorname{dom}(f) = S)$ such that T satisfies (i)–(iv) and

(v) $G_\varnothing = 1$ and $\varphi_{1_S} = 1_{G_S}$,

(vi) If $u \subseteq v \subseteq T_k$, then $G_u \subset G_v$,

(vii) $\varphi_f \circ \varphi_g = \varphi_{f \circ g}$ whenever $\operatorname{dom}(f) \subseteq \operatorname{range}(g)$ $(f, g \in Mp)$,

(viii) $f \subseteq g$ implies $\varphi_f \subseteq \varphi_g$ $(f, g \in Mp)$.

If k is even, put $\varphi_{f_k} = \varphi_k$; if k is odd, put $\varphi_{f_k^i} = \varphi_k^i$ $(i = 0, 1)$. We call φ_k^0 and φ_k^1 the *branching isomorphisms* at level k. We also put $G_k = G_{T_k}$.

A T-limit system $\mathscr{S}$ will be called *regular* provided

(ix) Every G_S is finitely generated,

(x) If k is odd and $\zeta_k 0, \zeta_k 1 \in S \subseteq T_{k+1}$, then

$$G_S = \langle \varphi_k^0(G_{S|_k}), \varphi_k^1(G_{S|_k}) \rangle,$$

(xi) For all $k < \omega$ and $f \in Mp(k, k+1)$ φ_f is surjective except possibly when k is even and $f = f_k$.

Conditions (x) and (xi) specify precisely how enlargements occur in the T-limit system. Namely, at each even level k, the embedding φ_k of G_k into G_{k+1} can be proper. This allows us to copy the skeleton of any countable group into the tree-limit in a game-theoretic way. While at odd levels k, condition (x) together with (viii) say that G_{k+1} is generated by a symmetrical amalgam, the two factors of which are the images of G_k under the branching isomorphisms. If k is odd and $\zeta_k \in S \subseteq T_k$ we put $S^* = S - \{\zeta_k\}$. By (viii) f_k^0 and f_k^1 agree on $G_k^* = G_{T_k^*}$; indeed the factors $\varphi_k^0(G_k)$ and $\varphi_k^1(G_k)$ are isomorphic over $\varphi_k^0(G_k^*) \equiv \varphi_k^1(G_k^*) \equiv \varphi_f(G_k^*)$ (where $f\colon T_k^* \to T_{k+1}$, $f \in Mp$) which may or may not be their intersection. Likewise, if k is odd and $\zeta_k \in S \subseteq T_k$, then $G_{S^+} = \langle \varphi_k^0(G_S), \varphi_k^1(G_S) \rangle \subset G_{k+1}$ where $S^+ = f_k^0(S) \cup f_k^1(S)$ is the join of a "symmetrical subamalgam" of $\varphi_k^0(G_k) \cup \varphi_k^1(G_k)$ since φ_k^0 and φ_k^1 agree on G_{S^*}.

In all of our applications the T-limit system will be regular although many properties of the tree-limit group (see §3) do not depend on this.

Finally we give the condition which will guarantee that the tree-limit group has power 2^ω. This is identical to the condition used by Shelah in **[11]**.

(xii) For all $k < \omega$ and $\alpha \in T_k$, there exists $x_\alpha \in G_{\{\alpha\}}$ such that $x_\alpha \notin G_{T_k - \{\alpha\}}$ and, for all $\alpha \leqslant \beta \in T$, $\varphi_f(x_\alpha) = x_\beta$ where $f = \{(\alpha, \beta)\}$.

The following condition, which all of our amalgamations will satisfy, will be helpful in verifying (xii).

(xiii) (Canonical Amalgamation) If k is odd there is a homomorphism h: $\langle \varphi_k^0(G_k), \varphi_k^1(G_k)\rangle \to G_k$ such that $h(\varphi_k^0(g)) = h(\varphi_k^1(g)) = g$ for all $g \in G_k$.

Actually, the tree-limit group will have power 2^ω under the weaker hypothesis that $x_\alpha \neq x_\beta$ for all $\alpha \neq \beta \in T_k$ (in place of $x_\alpha \notin G_{T_k - \{\alpha\}}$) in (xii), but our canonical amalgamations permit us to obtain Shelah's (xii).

LEMMA 1. *The following choices* (1) *and* (2) *inductively determine a unique regular T-limit system* $\mathscr{S} = \{G_S, \varphi_f\}$:

(1) *If k is even and* G_k *is given, we choose a finitely generated group* $G_{k+1} \supset G_k$,

(2) *If k is odd and* G_k *is given as well as* $G_k^* = A \subset G_k$, *we choose a group* $\langle G_k, \gamma(G_k)\rangle = G_{k+1}$ *where* γ *is an isomorphism such that* $\gamma(a) = a (a \in A)$.

If (2) *is replaced by* (2)*, *then the T-limit system* $\mathscr{S}$ *also satisfies conditions* (xii) *and* (xiii):

(2*) $G_1 \neq 1$ *and if k is odd and* G_k *is given as well as* $G_k^* = A \subset G_k$ *and some* $x_k \in G_k - A$ *we choose a group* $\langle G_k, \gamma_k(G_k)\rangle = G_{k+1}$ *as above such that* $x_k \neq \gamma_k(x_k)$ *and there exists a homomorphism* h_k: $G_{k+1} \to G_k$, *such that* $h_k(g) = h_k(\gamma_k(g)) = g$ $(g \in G_k)$.

(2.1) DEFINITION. We call a T-limit system constructed from choices (1) and (2*) as specified in Lemma 1 a *canonical T-limit system*. Note that if $\mathscr{S}$ is canonical, then $G_k \subset G_{k+1}$ and (xii) and (xiii) are satisfied by $\mathscr{S}$. In a canonical T-limit system we call the $\{h_k\}$ *canonical homomorphisms*.

Thus, Definition (2.1) declares any T-limit system constructed as described in the proof of Lemma 1 (using (2*)) to be a canonical T-limit system. To construct a canonical T-limit system we must satisfy (2*) for a particular choice of x_k at each odd level k determined inductively in the proof of Lemma 1.

PROOF OF LEMMA 1. Put $Mp(n) = \bigcup\{Mp(k, n) \mid k \leqslant n\}$. We need only verify the lemma for a canonical T-limit system using (2*). We assume the T-limit system is constructed up to level k. If k is even, $f \in Mp(k)$ and $f^+ \in Mp(k+1)$ with $\mathrm{dom}(f) = \mathrm{dom}(f^+)$ we define $\varphi_{f^+} = 1_{G_{k+1}} \circ \varphi_f$ and $G_{f_k(S)} = G_S$ for all proper $S \subseteq T_k$. If k is odd, then we define $G_{k+1} = \langle G_k, \gamma(G_k)\rangle$, $\varphi_k^0 = 1_{G_k}$, $\varphi_k^1 = \gamma$. If $\zeta_k \notin S \subseteq T_k$ and $f \in Mp(k, k+1)$ with $\mathrm{dom}(f) = S$, then $\varphi_f = 1_{G_S}$; if $\zeta_k \in S \subseteq T_k$, define $G_{S^+} = \langle G_S, \gamma(G_S)\rangle$. If $f \in Mp(k)$ we can "extend" f in one or two ways to $g \in Mp(k+1)$ with the same domain as f depending on whether $\zeta_k \notin \mathrm{range}(f)$ or $\zeta_k \in \mathrm{range}(f)$. In all cases the definition of φ_g is the obvious composition and the axioms for a regular T-limit system are easily checked up to level $k+1$. Obviously

condition (xiii) is met since it is part of the assumption (2*). So we need only verify (xii). We construct the elements x_α ($\alpha \in T$) inductively. Pick $1 \neq x_0 \in G_1$ recalling that $T_1 = \{0\}$. Assume that x_α ($\alpha \in T_1 \cup \cdots \cup T_k$) have been chosen for some $k \geqslant 1$ and that (xiii) is satisfied. If k is even and $\alpha \in T_k$ we define $x_{\alpha 0} = x_\alpha$. Since $\varphi_{f_k} = 1_{G_k}$ and $G_{f_k}(S) = G_S$ if $S \subseteq T_k$, $S \neq T_k$, (xii) is still satisfied. If k is odd and $\alpha \in T_k$, $\alpha \neq \zeta_k$, we define $x_{\alpha 0} = x_\alpha$; and we define $x_{\zeta_k 0} = x_{\zeta_k}$ $(= \varphi_k^0(x_{\zeta_k}))$ and $x_{\zeta_k 1} = \gamma(x_{\zeta_k})$ $(= \varphi_k^1(x_{\zeta_k}))$. Now we finish specifying the inductive use of (2*) by putting $x = x_k = x_{\zeta_k} \in G_k - G_k^*$. Thus $x_{\zeta_k 1} = \varphi_k^1(x_{\zeta_k}) = \gamma(x) \notin G_k = \varphi_k^0(G_k) = G_{T_{k+1} - \{\zeta_k 1\}}$ because if $\gamma(x) = g \in G_k$, then applying the canonical homomorphism h would give $x = g$, contradicting (2*). For the same reason we also have $x_{\zeta_k 0} = x \notin \gamma(G_k) = G_{T_{k+1} - \{\zeta_k 0\}}$. Now suppose $\alpha \in T_k - \{\zeta_k\}$. Thus $x_{\alpha 0} = x_\alpha$ and we must check that $x_\alpha \notin G_{T_{k+1} - \{\alpha 0\}} = G_{S^+}$ (where $S = T_k - \{\alpha\}$) $= \langle G_S, \gamma(G_S)\rangle$. But this also follows by applying h since $h(x_\alpha) = x_\alpha \notin G_S = h(\langle G_S, \gamma(G_S)\rangle)$. Thus (xii) is established at level $k + 1$.

3. The tree-limit group.

LEMMA 2. *Suppose $\mathscr{S} = \{G_S, \varphi_f\}$ is a T-limit system.*

(A) *A group $G_{\mathscr{S}}$ is obtained as a direct limit of groups $\{G_s \mid s \in P(T_\omega)\}$ with maps $\{\varphi(s,t) \mid s \subseteq t \in P(T_\omega)\}$ as follows:*

(a) *If $s \in P(T_\omega)$, then G_s is the direct limit of the groups $\{G_{s|_k} \mid j \leqslant k < \omega\}$ where j is minimal such that $|s| = |s|_k|$ with embeddings φ_f: $G_{s|_k} \to G_{s|_n}$ $(j \leqslant k \leqslant n)$ where f: $s|_k \to s|_n$, $f \in Mp(k,n)$ (this f is unique),*

(b) *If $s \subseteq t \in P(T_\omega)$, then the embedding $\varphi(s,t)$: $G_s \to G_t$ is defined by the commuting diagram*

$$\begin{array}{ccccccc}
G_{s|_j} & \to \cdots \to & G_{s|_k} & \xrightarrow{\varphi_{f(k)}} & G_{s|_{k+1}} & \to \cdots & G_s \\
\cap & & \cap & & \cap & & \varphi(s,t)\downarrow \\
G_{t|_j} & \to \cdots \to & G_{t|_k} & \xrightarrow{\varphi_{h(k)}} & G_{t|_{k+1}} & \to \cdots & G_t
\end{array}$$

where $f(k)$ and $h(k)$ are the maps of part (a) (*with $n = k + 1$*).

(B) *If $\mathscr{S}$ satisfies* (xi), *then the embedding $\varphi(s,t)(G_s) \subset G_t$ is isomorphic to each of the embeddings $G_{s|_k} \subset G_{t|_k}$ where $|t| = |t|_k|$ and either k is odd or $t|_k \neq T_k$,*

(C) *If G_k is countable $(k < \omega)$ and $\mathscr{S}$ satisfies* (xii), *then $|G_{\mathscr{S}}| = 2^\omega$,*

(D) *$G_{\mathscr{S}}$ is isomorphic to a subgroup of $(\prod_{k=1}^\infty G_k)/(\sum_{k=1}^\infty G_k)$ (the Cartesian product modulo the direct sum).*

PROOF. The groups G_s $(s \in P(T_\omega))$ are well defined because of (vii), while the diagram (b) commutes because of (v), (vi), and (viii) (note that $\varphi_{f(k)} \subseteq \varphi_{h(k)}$). Thus the direct limit system $\{G_s, \varphi(s,t)\}$ is well defined and yields $G_{\mathscr{S}}$. To prove (B) we note that if k is such that $|t| = |t|_k|$ and either k is odd or $t|_k \neq T_k$, then $\varphi_{f(k)}$ and $\varphi_{h(k)}$ are surjective isomorphisms and the vertical inclusions (which represent $\varphi(s,t)$) are isomorphic embeddings. To prove (C) we note that $|T_\omega| = 2^\omega$ by (iv). For all $\sigma \in T_\omega$ we obtain an element $x_\sigma \in G_{\mathscr{S}}$ as the image of the x_α $(\alpha \in T,\ \alpha < \sigma)$ under the direct limit maps into $G_{\{\sigma\}}$ and into $G_{\mathscr{S}}$. If $\sigma \neq \tau \in T_\omega$, then $x_\sigma \neq x_\tau$

because x_α and x_β are distinct in G_k for k large enough so that $\alpha \neq \beta \in T_k$ where $\alpha < \sigma$ and $\beta < \tau$. To prove (D), let $P = \prod_{k=1}^{\infty} G_k$ and $S = \sum_{k=1}^{\infty} G_k$. If $s \in P(T_\omega)$ and $x \in G_s$ we define $\sigma(x) \in P/S$ as follows. Let j be large enough so that x has a preimage $x_0 \in G_{s|_j}$ under the direct limit map into G_s (see (a) above). For all $k \geqslant j$ let $g(k) \in Mp(j,k)$ be such that $g(k)\colon s|_j \to s|_k$ and define

$$\sigma(x) = \left(\varphi_{g(k)}(x_0)\right)_{k \geqslant j} \pmod S.$$

Thus $\sigma(x)$ is the sequence of preimages of x in all sufficiently large G_k. (We need not specify the first j coordinates in P.) If $x \in G_s$ and $y \in G_t$ where $t \in P(T_\omega)$ then $x, y \in G_{s \cup t}$ can be multiplied by multiplying their (unique) preimages in sufficiently large G_k. Thus σ defines an isomorphism of $G_{\mathscr{S}}$ into P/S.

COROLLARY 1. *If $\mathscr{S}$ is a canonical T-limit system, then $|G_{\mathscr{S}}| = 2^\omega$.*

REMARK. If $s \subseteq t \in P(T_\omega)$ we will write $G_s \subset G_t \subset G_{\mathscr{S}}$. Thus, as in (B), this containment is isomorphic to $G_{s|_k} \subset G_{t|_k}$ for sufficiently large k.

(3.0) DEFINITION. If $S \subseteq T_\omega$ we put $G_S = \bigcup\{G_s \mid s \in P(S)\}$. If $\mathscr{S}$ is a T-limit system, the *archetype of* $\mathscr{S} = A_{\mathscr{S}} = G_S$ where $S \subseteq T_\omega$ consists of all sequences which are eventually 0.

REMARK. If every G_k $(k < \omega)$ is countable, then $A_{\mathscr{S}}$ is countable. $A_{\mathscr{S}}$ has a very natural interpretation with respect to $\mathscr{S}$ as a countable direct limit. Namely we regard the isomorphisms $\varphi_k\colon G_k \to G_{k+1}$ (k even) and $\varphi_k^0\colon G_k \to G_{k+1}$ (k odd) as inclusions and thus obtain an ascending union $\bigcup_{k<\omega} G_k$. We claim that this union is isomorphic to $A_{\mathscr{S}}$. To see this, let S_k consist of all the sequences in T_k followed by infinitely many zeros. Thus $S = \bigcup_{k<\omega} S_k$ (with S as in (3.0)) and the embedding $G_{S_k} \subset G_{S_{k+1}}$ is isomorphic to $\varphi_k(G_k) \subset G_{k+1}$ (k even) or $\varphi_k^0(G_k) \subset G_{k+1}$ (k odd) since $f_k^0(S_k|_k) = S_k|_{k+1}$, and our conclusion follows. Thus,

COROLLARY 2. *If $\mathscr{S}$ is a T-limit system obtained as in Lemma* 1 (*in particular, if $\mathscr{S}$ is canonical*), *then $A_{\mathscr{S}} \cong \bigcup_{k<\omega} G_k$.*

A central theme for us is to what extent the tree-limit group $G_{\mathscr{S}}$ can resemble its archetype $A_{\mathscr{S}}$. Obviously they have the same f.g. subgroups. However, even if $\mathscr{S}$ is obtained as in Lemma 1 and we assume that $A_{\mathscr{S}}$ is inner homogeneous, we cannot prove that $G_{\mathscr{S}}$ is inner homogeneous. To do this we must begin with the group we wish to be $A_{\mathscr{S}}$ and then construct $\mathscr{S}$ appropriately. Before entering into this we will show a major benefit of canonical amalgamations.

LEMMA 3 (CANONICAL REPRESENTATIONS). *If $\mathscr{S}$ is a canonical T-limit system then there exists an isomorphism $\rho\colon G_{\mathscr{S}} \to \mathrm{Sym}(\omega)$.*

PROOF OF LEMMA 3. Given a canonical T-limit system $\mathscr{S} = \{G_S, \varphi_f\}$ we claim that the following set of choices inductively define (via the direct limit) an embedding ρ: $G_{\mathscr{S}} \to \mathrm{Sym}(X)$ where X is the disjoint union of countable sets X_n $(n < \omega)$:

(a) Choose a representation $\rho_1\colon G_1 \to \mathrm{Sym}(X_1)$; and putting $W_n = X_1 \cup \cdots \cup X_n$,

(b) If k is even and ρ_k: $G_k \to \mathrm{Sym}(W_k)$ is given choose a representation ρ_{k+1}: $G_{k+1} \to \mathrm{Sym}(W_{k+1})$ such that $(W_k, \rho_k(G_k))$ is a subaction of $(W_{k+1}, \rho_{k+1}(G_{k+1}))$, that is, for all $x \in W_k$ and $g \in G_k$ we have (recalling that $G_k \subset G_{k+1}$ since $\mathscr{S}$ is canonical) $(x)\rho_k(g) = (x)\rho_{k+1}(g)$,

(c) If k is odd and ρ_k: $G_k \to \mathrm{Sym}(W_k)$ is given we first define, for all $z \in \langle G_k, \gamma(G_k)\rangle = G_{k+1}$ and $x \in W_k$, $(x)\rho_{k+1}(z) = (x)\rho_k(h(z))$ where h: $G_{k+1} \to G_k$ is the canonical homomorphism. Since $h(g) = h(\gamma(g)) = g$ $(g \in G_k)$, this means that, under ρ_{k+1}, $G_k = \varphi_k^0(G_k)$ and $\gamma(G_k) = \varphi_k^1(G_k)$ have identical actions on W_k (via the branching isomorphisms) which are identical to the way G_k acts on W_k under ρ_k. Next we choose any representation π: $G_{k+1} \to \mathrm{Sym}(X_{k+1})$ and define, for all $x \in X_{k+1}$ and $z \in G_{k+1}$, $(x)\rho_{k+1}(z) = (x)\pi(z)$. This defines the action of $\rho_{k+1}(G_{k+1})$ on $W_{k+1} = W_k \cup X_{k+1}$,

(d) For all $1 \neq g \in G_k$ $(k < \omega)$ we arrange that, for some $n \geqslant k$, $1 \neq \rho_n(\varphi_f(g)) \in \mathrm{Sym}(W_n)$ for all $f \in Mp(k, n)$ with $\mathrm{dom}(f) = T_k$.

We can summarize this construction of the ρ_k by saying that in the T-limit system the embeddings φ_k, φ_k^0, φ_k^1 correspond to permutation extensions—thus inducing a representation of the tree-limit group $G_{\mathscr{S}}$ on the union of the permuted sets at each level. However, to display this formally we must examine several diagrams as follows.

For all $s \in P(T_\omega)$ and $k < \omega$ with $|s| = |s|_k|$, the induced map γ_k in the diagram

$$\begin{array}{ccc} G_{s|_k} & \xrightarrow{\varphi_f} & G_{s|_{k+1}} \\ \rho_k \downarrow & & \downarrow \rho_{k+1} \\ R_k = \rho_k(G_{s|_k}) & \xrightarrow{\gamma_k} & R_{k+1} \\ \bigcap & & \bigcap \\ \mathrm{Sym}(W_k) & & \mathrm{Sym}(W_{k+1}) \end{array}$$

where $f \in Mp(k, k+1)$ takes every $\sigma \in R_k$ to an extension of σ in R_{k+1} since φ_f is either a restriction of φ_k if k is even or of φ_k^0 or φ_k^1 if k is odd. It follows (see Lemma 2(b)) that the representations $\{\rho_k\}$ define a representation ρ_s: $G_s \to \mathrm{Sym}(X)$ where $X = \bigcup_{k<\omega} X_k$. If $s \subseteq t \subseteq T_\omega$ are finite, then

$$\begin{array}{ccc} G_s & \xrightarrow{\varphi(s,t)} & G_t \\ \rho_s \downarrow & & \downarrow \rho_t \\ \rho_s(G_s) & \subset & \rho_t(G_t) \end{array}$$

commutes because $\varphi(s, t)$ is induced by the inclusion maps $G_{s|_k} \subset G_{t|_k}$ where $|t| = |t|_k|$, and because the diagrams

$$\begin{array}{ccc} G_{s|_k} & \subset & G_{t|_k} \\ \rho_k \downarrow & & \downarrow \rho_k \\ \rho_k(G_{s|_k}) & \subset & \rho_k(G_{t|_k}) \end{array}$$

clearly commute, implying $\rho_s \subseteq \rho_t$.

Thus the ρ_k $(k < \omega)$ define a reprentation ρ: $G_{\mathscr{S}} \to \mathrm{Sym}(X)$ as described at the outset—indeed we have $\rho = \bigcup\{\rho_s \mid s \in P(T_\omega)\}$—provided that we have a way to satisfy (b) at each even step k, that is, a way to extend the representation $(W_k, \rho_k(G_k))$ to $(W_{k+1}, \rho_{k+1}(G_{k+1}))$. Perhaps the easiest way to do this is Kegel's method of [7] whereby, if $A \subset B$ and ρ: $A \to \mathrm{Sym}(Y)$ is a representation, we extend the action of $\rho(A)$ on Y to the representation $\bar{\rho}$: $B \to \mathrm{Sym}(Y \times T)$ where $1 \in T$ is a right transversal of A in B such that $(y, t)\bar{\rho}(b) = (y', t')$ where $tb = at'$ $(t' \in T$, $a \in A)$ and $y' = (y)\rho(a)$ for all $y \in Y$, $t \in T$. (Thus if $a \in A$, $\bar{\rho}(a)$ extends $\rho(a)$ if we identify Y with $Y \times \{1\}$.)

Finally, we check easily that ρ is an isomorphism using (d) above. Indeed if $s \in P(T_\omega)$ is finite then λ: $G_{s|_k} \cong G_s$ for some $k < \omega$ where λ is the direct limit embedding. Thus if $1 \neq \lambda(g) \in G_s$, the property (d) above implies that $1 \neq \rho_n(\varphi_f(g)) \in \rho_n(G_{s|_n})$ where f: $s|_k \to s|_n$, $f \in Mp(k, n)$. Since $\varphi_f(g) \in G_{s|_n}$ is also mapped to $\lambda(g)$ by the direct limit embedding, we have $1 \neq \rho(\lambda(g))$.

Notice that (d) is easily satisfied if in (c) we let π be the regular representation of G_{k+1} on $X_{k+1} = G_{k+1}$.

Of course we can assume $X = \omega$ since X is countable.

COROLLARY 3. *In Lemma* 3 *all orbits of* $(\omega, \rho(G_{\mathscr{S}}))$ *will be finite provided every* G_k $(k < \omega)$ *is finite and either*

(i) $G_k = G_{k+1}$ *when* $k \geqslant 2$ *is even; or*

(ii) *the choices* (b) *are made in such a way that, for all even* k, *every orbit of* $\rho_k(G_k)$ *on* W_k *is an orbit of* $\rho_{k+1}(G_{k+1})$.

PROOF. In case (i) there are no choices (b). In (a) we let ρ_1 be a regular representation of $G_1 = X_1$ and in (c) we take $X_{k+1} = G_{k+1}$ and let π be regular also. Thus (d) is satisfied with $k = n$ and it is clear that every X_k is an orbit of $\rho(G_{\mathscr{S}})$. In case (ii) the conclusion is equally obvious that every X_k is an orbit of $\rho_n(G_n)$ for $n \geqslant k$ and hence of $\rho(G_{\mathscr{S}})$. □

(3.1) DEFINITION. An isomorphism ρ: $G_{\mathscr{S}} \to \mathrm{Sym}(\omega)$ obtained from choices (a)–(d) of Lemma 3 will be called a *canonical representation* of $G_{\mathscr{S}}$.

(3.2) DEFINITION. A countable group G has the *canonical amalgamation property* (c.a.p.) iff $G \neq 1$ and for all f.g. $A \subset B \subset G$ and $x \in B - A$ there exists an A-isomorphism γ: $B \to G$ (that is $\gamma(a) = a$ $(a \in A)$) such that $\gamma(x) \neq x$ and there is a homomorphism h: $\langle B, \gamma(B)\rangle \to B$ such that, for all $b \in B$, $h(b) = h(\gamma(b)) = b$.

LEMMA 4. *Suppose* G *is a countable group with the canonical amalgamation property. Then*

(a) *There is a canonical* T*-limit system* $\mathscr{S}$ *such that* $A_{\mathscr{S}} \cong G$,

(b) *If* G *is also* ω*-homogeneous, then* (*with* $\mathscr{S}$ *as above*) *we have* $G \cong G_X \subset G_{\mathscr{S}}$ *for every countable* $X \subseteq T_\omega$ *such that* $X|_k = T_k$ *for all* $k < \omega$. (*Hence* $G_{\mathscr{S}}$ *is an archetypal limit of* G (*see* (1.0)).)

PROOF OF LEMMA 4. The proof of part (a) is very easy. Making use of Corollary 2, we need only show that the choices (1) and (2*) of Lemma 1 can be made in such a way that $G = \bigcup_{k<\omega} G_k$, and this is immediate since the c.a.p. gives the canonical

amalgamations (2*) and we need only choose $G_{k+1} \supset G_k$ when k is even to exhaust G.

PROOF OF (b). This will follow if at even steps k we choose $G_k \subset G_{k+1} \subset G$ so that the following property holds.

(3.3) For all $i < \omega$, $A \subset G_i$, and $A \subset B \subset G$ with A, B f.g. there is an even $k \geqslant i$ such that, for all $f \in Mp(i,k)$ with $\mathrm{dom}(f) = T_i$, there is an isomorphism $g_f\colon B \to G_{k+1}$ such that the diagram below commutes:

$$\begin{tikzcd} B \arrow[rr, "g_f"] & & G_{k+1} \\ & A \arrow[ul, "1_A"] \arrow[ur, "\varphi_f"'] & \end{tikzcd}$$

Thus, to obtain $G_{k+1} \subset G$ we must "amalgamate" finitely many copies of B with G_k over $\varphi_f(A) \subset G_k$, and this is possible by ω-homogeneity of G.

Now suppose $X \subseteq T_\omega$ is countable and $X|_k = T_k$ for all $k < \omega$. We will show that (3.3) implies that $G_X = \bigcup\{G_s \mid s \in P(X)\} \cong G$, as required.

First we choose finite sets $F_k \subseteq X$ such that $F_k \subseteq F_{k+1}$, $X = \bigcup F_k$, $|F_k|_k| = |F_k|$, and $T_k = F_k|_k$ $(k < \omega)$. (This is possible by our hypothesis on X.)

Suppose $U \subset G_F \subset G_X$ where $F \in P(X)$ and U is f.g.. Choose i so that $F \subset F_i$ and let $A \subset G_i \subset G$ be the preimage of U under the direct limit embedding into $G_F \subset G_{F_i} \subset G_{\mathscr{S}}$. Let $A \subset B \subset G$ with B f.g. and $k < \omega$ be as given in (3.3). Since $k+1$ is odd, Lemma 2(B) says

(3.4) The embeddings $G_{F_i} \subset G_{F_k} \subset G_{F_{k+1}}$ $(\subset G_{\mathscr{S}})$ are isomorphic to $G_r \subset G_s \subset G_t = G_{k+1}$ $(\subset G)$ where $r = F_i|_{k+1}$, $s = F_k|_{k+1}$, $t = F_{k+1}|_{k+1} = T_{k+1}$ (under the direct limit embedding).

Let $\bar{A} \subset G_{k+1}$ be the preimage of U under the direct limit embedding into $G_F \subset G_{F_{k+1}} \subset G_{\mathscr{S}}$. Thus $\varphi_f(A) = \bar{A}$ where $f\colon F_i|_i \to F_i|_k$ $(f \in Mp(i,k))$ and (3.3) implies that G_{k+1} contains an extension $g_f(B) = \bar{B} \supset \bar{A} = \varphi_f(A)$ isomorphic to $B \supset A$, and hence, by (3.4), in $G_{\mathscr{S}}$ we have a copy $\tilde{B}$ of B such that $\tilde{B} \supset U$ is isomorphic to $\bar{B} \supset \bar{A}$. We leave these final diagrams to the reader. Hence $G_X \cong G$ by the usual back-and-forth argument. □

A more general method for obtaining the archetypal limit property of $G_{\mathscr{S}}$ will be found in (4.0). We conclude this section with a discussion of the isomorphism-type of $G_{\mathscr{S}}$.

(3.5) DEFINITION. Suppose $\mathscr{S}$ is a T-limit system and $z \in G_{\mathscr{S}}$. We say that z is a *descendant* of $g \in G_n$ iff there exists $s \in P(T_\omega)$ and $f \in Mp(n,m)$ where $f\colon s|_n \to s|_m$, $|s|_m| = |s|$, and $\varphi_f(g) = \bar{z}$ is the preimage of z under the direct limit embedding of $G_{s|_m}$ into G_s (see Lemma 2(a)). (We also say that g is an *ancestor* of z.)

It is crucial to specify the level n in a descendence relation since $g \in G_{n+1}$ might or might not embed the same way depending upon how s branches.

We note that if $z \in G_{\mathscr{S}}$ descends from $g \in G_n$ relative to $s \in P(T_\omega)$ (as in the above definition) and if $s \subseteq t \in P(T_\omega)$ then z also descends from $g \in G_n$ relative to t, that is, $\varphi_f(g) = \bar{z}$ for some $f \in Mp(n, m)$ such that $f\colon t|_n \to t|_m$ and $|t|_n| = |t|$ where $\bar{z}$ is the preimage in $G_{t|_m}$ of z. This is because the embedding $G_s \subset G_t$ is isomorphic to $G_{s|_m} \subset G_{t|_m}$ (for large m) and the φ_f we obtain for t will extend the φ_f for s. We also note that the descendence relation can misbehave in the following sense. Suppose for some odd k there exists $x \in \varphi_k^0(G_k) \cap \varphi_k^1(G_k) - \varphi_k^0(G_k^*)$ $(= \varphi_k^1(G_k^*))$, that is, the amalgamation of this step is not strong. Let $t \in P(T_\omega)$ consist of the members of T_{k+1} followed by infinitely many zeros. Then, if $z \in G_t$ is the element whose preimage in G_{k+1} is x, we see easily that z is a descendant of both $(\varphi_k^0)^{-1}(x)$ and $(\varphi_k^1)^{-1}(x) \in G_k$ and these elements need not be equal since the axioms say only that φ_k^0 and φ_k^1 are equivalent on G_k^*. We cannot even say that z has unique ancestors at each sufficiently large level because this mixing might recur for infinitely many values of k.

Despite this conceptive hazard, the relation of descendence is essential to the analysis of the structure of $G_{\mathscr{S}}$!

(3.6) (Descendence Lemma). *If every G_n is countable and X is an uncountable subset of $G_{\mathscr{S}}$ then there exists $Z \subseteq X$ with $|Z| = |X|$ as well as $g \in G_n$ for some $n < \omega$ such that every $z \in Z$ descends from $g \in G_n$.*

Proof. Immediate since $A = \bigcup_{n<\omega} G_n$ is countable and every $x \in G_{\mathscr{S}}$ descends from some element of A.

To make our conceptual task a bit easier we will use the fact that all the T-limit systems in our applications are canonical. In this case the misbehavior mentioned above does not occur and we have

(3.7) (Unique Ancestry). *Suppose $\mathscr{S}$ is a canonical T-limit system, $z \in G_{\mathscr{S}}$, and $n < \omega$. Then z is a descendant of at most one $g \in G_n$.*

Proof. Suppose $g_1 \neq g_2 \in G_n$. If n is even then obviously $\varphi_n(g_1) \neq \varphi_n(g_2)$; while if n is odd, $\varphi_n^i(g_1) \neq \varphi_n^i(g_2)$ $(i = 0, 1)$ and $\varphi_n^0(g_1) \neq \varphi_n^1(g_2)$ also: otherwise the canonical homomorphism $h_i\colon \langle \varphi_n^0(G_n), \varphi_n^1(G_n)\rangle \to G_n$ would give $g_1 = g_2$. It follows inductively from this that for all $m \geqslant n$ and $f_1, f_2 \in Mp(n, m)$ with $g_1, g_2 \in \operatorname{dom}\varphi_{f_i}$ $(i = 1, 2)$ we have $\varphi f_1(g_1) \neq \varphi f_2(g_2)$.

Now suppose both g_1 and $g_2 \in G_n$ are ancestors of $z \in G_{\mathscr{S}}$. Then there exist $s, t \in P(T_\omega)$, $q, r \geqslant n$, $f_1 \in Mp(n, q)$ and $f_2 \in Mp(n, r)$ such that $|s|_q| = |s|$, $|t|_r| = |t|$, $f_1\colon s|_n \to s|_q$, $f_2\colon t|_n \to t|_r$, and $\varphi_{f_1}(g_1) = w_1 \in G_{s|_q}$, $\varphi_{f_2}(g_2) = w_2 \in G_{t|_r}$ where w_1 is the preimage of $z \in G_s$ and w_2 is the preimage of $z \in G_t$. We can assume $r \geqslant q$ and let $f\colon s|_q \to s|_r$ with $f \in Mp(q, r)$. Thus $\varphi_f(\varphi_{f_1}(g_1))$ is also a preimage of $z \in G_s$ since φ_f is a map in the direct limit system defining G_s (Lemma 2(a)) and hence $\varphi_f(\varphi_{f_1}(g_1)) = \varphi_{f_2}(g_2)$ contrary to what we established at the outset.

(3.8) (Ancestral Relations). *Suppose $\mathscr{S}$ is a canonical T-limit system, $z \in G_{\mathscr{S}}$, and z is a descendant of both $g_1 \in G_k$ and $g_2 \in G_n$ where $n \geqslant k$. Then there exists $f \in Mp(k, n)$ such that $\varphi_f(g_1) = g_2$.*

PROOF. Note that (3.8) implies (3.7) by taking $n = k$ since every φ_f, $f \in Mp(n, n)$, is the identity by axiom (v) of (2.0). To prove (3.8) let $s \in P(T_\omega)$, $l < \omega$, and $f_1 \in Mp(k, l)$ such that $|s|_l| = |s|$, f_1: $s|_k \to s|_l$, and $\varphi_{f_1}(g_1)$ is a preimage of z. If $n \geqslant l$, then letting $h \in Mp(l, n)$ with h: $s|_l \to s|_m$, we see that $\varphi_f(g_1)$ is a preimage of z where $f = h \circ f_1$, and hence is an ancestor of z, and so $\varphi_f(g_1) = g_2$ by (3.7). If $n \leqslant l$, letting $f \in Mp(k, n)$ and $h \in Mp(n, l)$ such that $f_1 = h \circ f$ we find that f: $s|_k \to s|_n$, h: $s|_n \to s|_l$ and hence $\varphi_f(g_1)$ is an ancestor of z also and hence equals g_2 by (3.7).

(3.9) (BRANCHING LEMMA). *Suppose $\mathscr{S}$ is a canonical T-limit system and $z_0 \neq z_1 \in G_{\mathscr{S}}$ are descendants of $g \in G_n$. Choose $s \in P(T_\omega)$ such that $z_0, z_1 \in G_s$ and let $m \geqslant n$ be minimum such that $\langle z_0, z_1\rangle \cong \langle \bar{z}_0, \bar{z}_1\rangle$ by the generator map where $\bar{z}_0, \bar{z}_1 \in G_{s|_m}$ are ancestors of z_0 and z_1. Then $k = m - 1$ is odd and (after renumbering if necessary) we have $\bar{z}_0 \in \varphi_k^0(G_k) - \varphi_k^1(G_k)$ and $\bar{z}_1 \in \varphi_k^1(G_k) - \varphi_k^0(G_k)$.*

PROOF. First note that $m > n$ by (3.7) since z_0 and z_1 are descendants of $\bar{z}_0$ and $\bar{z}_1$. (Also note that m exists because G_s is the direct limit of groups of the form $G_{s|_m}$.) If $k = m - 1$ were even, then either $\bar{z}_0$ or $\bar{z}_1$ would not be in the range of φ_k since otherwise $\varphi_k^{-1}(\bar{z}_0)$, $\varphi_k^{-1}(\bar{z}_1)$ would also be ancestors of z_0 and z_1 contradicting the minimality of m; but if, say, $\bar{z}_0 \notin \varphi_k(G_k)$ then z_0 cannot descend from any $g \in G_n$ by (3.8). Hence k must be odd. Now if, say, $\bar{z}_0 \notin \varphi_k^0(G_k) \cup \varphi_k^1(G_k)$, then z_0 cannot descend from $g \in G_n$ again by (3.8), and similarly for $\bar{z}_1$. Hence $\bar{z}_0$, $\bar{z}_1 \in \varphi_k^0(G_k) \cup \varphi_k^1(G_k)$. If, say, $\bar{z}_0, \bar{z}_1 \in \varphi_k^0(G_k)$ then $(\varphi_k^0)^{-1}(\bar{z}_0)$ and $(\varphi_k^0)^{-1}(\bar{z}_1)$ are ancestors of z_0 and z_1 contradicting the minimality of m. □

We have chosen the above presentation because it highlights the nature of canonical systems. A slight variation of the Branching Lemma (3.9) is true for general T-limit systems. We give the statement of such a lemma for those interested.

(3.9)* Suppose $\mathscr{S}$ is a T-limit system, $s, t \in P(T_\omega)$ and $z_0 \in G_s$, $z_1 \in G_t$ both descend from $g \in G_n$ for some $n \geqslant 1$ and $z_0 \neq z_1$. Let $m \geqslant n$ be minimum such that $\langle z_0, z_1\rangle \cong \langle \bar{z}_0, \bar{z}_1\rangle$ by the generator map for some $\bar{z}_0, \bar{z}_1 \in G_{(s \cup t)|_m}$ which are ancestors of z_0 and z_1 such that $\bar{z}_0 = \varphi_f(g)$ and $\bar{z}_1 = \varphi_h(g)$ for some $f, h \in Mp(n, m)$. Then the conclusion of (3.9) holds.

This general formulation will not be needed in the sequel.

It is of interest to note that even in canonical systems if $x \in G_{\mathscr{S}}$, then there need not be a unique minimal $s \in P(T_\omega)$ such that $x \in G_s$. If k is minimal such that x has an ancestor $g \in G_{k+1}$ then either k is even and $g \notin \varphi_k(G_k)$ or k is odd and $g \notin \varphi_k^0(G_k) \cup \varphi_k^1(G_k)$. In the second case there might be numerous minimal $u \subseteq T_{k+1}$ with $g \in G_u \subset G_{k+1}$ depending upon the relations which hold in $G_{k+1} = \langle \varphi_k^0(G_k), \varphi_k^1(G_k)\rangle$ among the various symmetrical subamalgams. Thus the formulation (3.9) does skirt some conceptual problems which become apparent in (3.9)*.

The Branching and Descendence Lemmas provide the key to controlling the isomorphism-type of $G_{\mathscr{S}}$. First we need the following definition.

(3.10) DEFINITION. If $\mathscr{S}$ is a T-limit system and $x, y \in G_n$ we say that x and y are *similar* (in $\mathscr{S}$) iff for some $k \leqslant n$ there exists $g \in G_k$ and $f, h \in Mp(k, n)$ such that $x = \varphi_f(g)$ and $y = \varphi_h(g)$.

(3.11) (SPECTRUM LEMMA). *Suppose G is a countable group with the c.a.p. and that it is possible to make the choices* (2*) *of Lemma* 1 *to construct a canonical T-limit system $\mathscr{S}$ with $A_{\mathscr{S}} \cong G$ as in Lemma* 4(*a*) *in such a way that*

(3.12) *For each odd k, given finitely many groups $\langle a_i, b_i\rangle$ $(1 \leqslant i \leqslant p)$ and pairs $(x_j, y_j) \in G_k \times G_k$ $(1 \leqslant j \leqslant q)$ with $x_j, y_j \notin G_k^*$ and such that x_j and y_j are similar in $\mathscr{S}$, we can find $G_{k+1} = \langle G_k, \gamma(G_k)\rangle \subset G$ so that for all $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$, $\langle x_j, \gamma(y_j)\rangle \not\cong \langle a_i, b_i\rangle$ by the generator map.*

Then there exist 2^ω canonical T-limit systems $\mathscr{S}_\alpha$ $(\alpha < 2^\omega)$ with $A_{\mathscr{S}_\alpha} \cong G$ and such that for all $\alpha \neq \beta < 2^\omega$, $G_{\mathscr{S}_\alpha}$ and $G_{\mathscr{S}_\beta}$ do not have isomorphic uncountable subgroups. Further, any property of $G_{\mathscr{S}}$ which can be forced by the choices made for G_{k+1} at even steps k can be built into every $G_{\mathscr{S}_\alpha}$. Thus, if G is homogeneous, then every $G_{\mathscr{S}_\alpha}$ is an archetypal limit of G. (Note: The hypothesis means that the choices (2*) *can be made in accordance with* (3.12) *regardless of how the choices of G_{k+1} for even k are made; hence the final conclusion is almost tautological.)*

PROOF. To use the Descendence and Branching Lemmas we must ultilize (3.12) to build the following property into distinct T-limit systems $\mathscr{S}$ and $\tilde{\mathscr{S}}$.

(3.13) For all pairs (x, i), (y, j) such that $x \in G_i$ and $y \in \tilde{G}_j$, there exists $d \geqslant \max(i, j)$ such that for all odd $k, l \geqslant d$ and $f_1, f_2 \in Mp(i, k)$, $g_1, g_2 \in Mp(j, l)$ with $\operatorname{dom}(f_\delta) = T_i$, $\operatorname{dom}(g_\delta) = T_j$ $(\delta = 1, 2)$, if $\varphi_{f_\delta}(x) \notin G_k^*$ and $\tilde{\varphi}_{g_\delta}(y) \notin \tilde{G}_l^*$ $(\delta = 1, 2)$, then

(3.14) $$\langle \varphi_{f_1}(x), \varphi_k^1(\varphi_{f_2}(x))\rangle \not\cong \langle \tilde{\varphi}_{g_1}(y), \tilde{\varphi}_l^1(\tilde{\varphi}_{g_2}(y))\rangle \text{ by the generator map.}$$

(Note that $\varphi_{f_1}(x) = \varphi_k^0(\varphi_{f_1}(x))$, etc. since the systems are canonical).

The construction of $\mathscr{S}$ and $\tilde{\mathscr{S}}$ satisfying (3.13) makes direct use of (3.12) at odd steps n. At each such step we have finite subsets $F_i \subseteq G_i$, $\tilde{F}_j \subseteq \tilde{G}_j$ for $1 \leqslant i, j \leqslant n$ for which we will satisfy (3.13) for all $x \in F_i$, $y \in \tilde{F}_j$ and $k, l \leqslant n$. For fixed i, j we will increase the sets F_i, $\tilde{F}_j$ during the construction (so that they depend upon n) in such a way that eventually every $x \in G_i$ belongs to F_i and every $y \in \tilde{G}_j$ belongs to $\tilde{F}_j$. Thus (3.13) will be satisfied for new $x \in F_i$, $y \in \tilde{F}_j$ $(i, j \leqslant n)$ beginning at this stage with $d = n$; and for old pairs with $k, l \leqslant n$.

Since there are only finitely many possibilites for the four generators in (3.14) where $x \in F_i$, $y \in \tilde{F}_j$ $(i, j \leqslant n)$, (3.12) gives us the group $G_{n+1} = \langle G_n, \gamma(G_n)\rangle \subset G$ satisfying (3.14) for $k = n, l < n$ (recall that $\varphi_n^0 = 1_{G_n}$, $\varphi_n^1 = \gamma$). We next use (3.12) again to obtain $\tilde{G}_{n+1} = \langle \tilde{G}_n, \tilde{\gamma}(\tilde{G}_n)\rangle \subset G$ so that (3.14) is satisfied for the same $x \in F_i$, $y \in \tilde{F}_j$ and for $l = n$, $k \leqslant n$.

Now we will show that (3.13) implies subgroup-incomparability of the tree-limit groups $G_{\mathscr{S}}$ and $G_{\tilde{\mathscr{S}}}$. Suppose X and Y are uncountable subgroups of these groups respectively and that $\lambda\colon X \to Y$ is an isomorphism. By applying the Descendence

Lemma twice (once to each system) we obtain an uncountable $W \subseteq X$ such that every member of W is the descendant (in $\mathscr{S}$) of the same $x \in G_i$ for some $i < \omega$ and every member of $\lambda(W)$ is the descendant (in $\tilde{\mathscr{S}}$) of the same $y \in \tilde{G}_j$ for some $j < \omega$. According to (3.13) there exists $d < \omega$ such that for all $k, l \geqslant d$, (3.14) holds. By passing to a subset of W we can assume that every member of W has the same ancestor $\bar{x} \in G_d$ and that every member of $\lambda(W)$ has the same ancestor $\bar{y} \in \tilde{G}_d$, and so by (3.8) we have $\varphi_f(x) = \bar{x}$ and $\tilde{\varphi}_g(y) = \bar{y}$ for some $f \in Mp(i, d)$, $g \in Mp(j, d)$. We now choose $z_0 \neq z_1 \in W$ and apply the Branching Lemma to z_0, z_1 and also to $\lambda(z_0), \lambda(z_1)$. The first application gives us $k \geqslant d$ such that $\langle z_0, z_1 \rangle \cong \langle \bar{z}_0, \bar{z}_1 \rangle \subset G_{k+1}$ (by the generator map) where $a = (\varphi_k^0)^{-1}(\bar{z}_0)$ and $b = (\varphi_k^1)^{-1}(\bar{z}_1)$ satisfy $a, b \in G_k - G_k^*$ and such that $\bar{z}_0, \bar{z}_1$ and hence a and b are ancestors of z_0 and z_1, respectively. Hence by (3.8) we see that there exist $f_1, f_2 \in Mp(a, k)$ such that $f_1(\bar{x}) = a$ and $f_2(\bar{x}) = b$. Likewise, applying the Branching Lemma to $\lambda(z_0), \lambda(z_1)$ we obtain $l \geqslant d$ such that $\langle \lambda(z_0), \lambda(z_1) \rangle \cong \langle \tilde{\bar{z}}_0, \tilde{\bar{z}}_1 \rangle \subset \tilde{G}_{l+1}$ (by the generator map) and $\tilde{a}, \tilde{b}$ etc., and from (3.8) we obtain $g_1, g_2 \in Mp(d, l)$ such that $g_1(\bar{y}) = \tilde{a}$, $g_2(\bar{y}) = \tilde{b}$. This contradicts (3.14) which tells us that $\mathscr{S}$ and $\tilde{\mathscr{S}}$ were constructed so that $\langle \bar{z}_0, \bar{z}_1 \rangle \not\cong \langle \tilde{\bar{z}}_0, \tilde{\bar{z}}_1 \rangle$ by the generator map.

To construct 2^ω T-limit systems such that any two distinct ones satisfy (3.13) we proceed similarly. At some step we have already constructed finitely many systems $\mathscr{S}_1, \ldots, \mathscr{S}_m$ up to some odd level k and we can ensure that each pair of these will satisfy (3.13) in a manner identical to that above by choosing appropriate canonical amalgamations to obtain G_{k+1} in each system. We will also create branches in each system, that is, from this point on we construct more than one system which is the same as $\mathscr{S}_i$ $(1 \leqslant i \leqslant m)$ up to level k. This presents no difficulty and yields 2^ω systems as required. The details are obvious and so we omit them. Since for even n we are free to choose G_{n+1} as we please in each system being constructed independently of any of our other choices, the final conclusion does indeed follow. (The proof of Lemma 4(b) shows that if G is ω-homogeneous (in the algebraic sense) then the archetypal limit property can be built into $G_{\mathscr{S}}$ by the choices made for G_k when k is even.)

4. Proof of Theorem 4.

(4.0) (Archetypal Limit Theorem). *Suppose that in Lemma* 3 *the following conditions are satisfied for all odd* k:

(1) *There exists* $t \in G_{k+2}$ *such that* $t^{-1}gt = \gamma(g)$ *for all* $g \in G_k$ (*where* $G_{k+1} = \langle G_k, \gamma(G_k) \rangle$ *is the canonical amalgamation*);

(2) *The representation* ρ_{k+2}: $G_{k+2} \to \mathrm{Sym}(W_{k+2})$ *is defined so that* $\rho_{k+2}(t)$ *fixes every* $x \in W_k$.

Then, for all $S \subseteq T_\omega$ *such that* $|S| = \omega$ *and* $S|_k = T_k$ *for all* $k < \omega$, *there exists a permutation group isomorphism of* $(X, \rho(A_{\mathscr{S}}))$ *with* $(X, \rho(G_S))$ *which maps every orbit of* $(X, \rho(A_{\mathscr{S}}))$ *to itself. Hence* $(X, \rho(G_{\mathscr{S}}))$ *is an archetypal limit of* $(X, \rho(A_{\mathscr{S}}))$ *and, if* $W \subseteq X$ *is an orbit of* $\rho(A_{\mathscr{S}})$, *then* $(W, \rho(G_{\mathscr{S}}))$ *is an archetypal limit of* $(W, \rho(G_{\mathscr{S}}))$. *Further, the property* (*) *given after Definition* (1.0) *is satisfied and the isomorphism* σ *is a permutation isomorphism.*

PROOF. We have $A_{\mathscr{S}} = \bigcup_{k \geqslant 1} G_k$ since $\mathscr{S}$ is canonical by the remark after (3.0). Let $S = \bigcup_{k \geqslant 1} F_k$ with $F_k \subseteq F_{k+1}$ and

$$|F_k| = |F_k|_k| \quad \text{and} \quad F_k|_k = T_k. \tag{4.1}$$

(This is possible by our hypothesis on S.) Note that if k is even then $F_k = F_{k+1}$ since $|T_k| = |T_{k+1}|$. By Lemma 2 we have that $G_S = \bigcup_{k \geqslant 1} G_{F_k}$ is the direct limit of the groups $\{G_k | k \geqslant 1\}$ under the isomorphisms ψ_k: $G_k \to G_{k+1}$ where $\psi_k = \varphi_k$ if k is even and, if k is odd, then

$$\psi_k = \begin{cases} \varphi_k^0 & \text{if } \zeta_k 0 \in F_k|_{k+1}, \\ \varphi_k^1 & \text{if } \zeta_k 1 \in F_k|_{k+1}. \end{cases}$$

(Exactly one of the above cases holds; indeed, either $F_k|_{k+1} = T_{k+1} - \{\zeta_k 1\}$ or $T_{k+1} - \{\zeta_k 0\}$ by (4.1)). To obtain the isomorphism σ: $A_{\mathscr{S}} \to G_S$ of the conclusion we will construct inner automorphisms σ_k: $G_k \to G_k$ for odd values of k such that every diagram

$$\begin{array}{ccc} G_k & \xrightarrow{\sigma_k} & G_k \\ \cap & & \downarrow \varphi_{k+1} \circ \psi_k \\ G_{k+2} & \xrightarrow{\sigma_{k+2}} & G_{k+2} \end{array} \tag{4.2}$$

commutes. These $\{\sigma_k | k \text{ odd}\}$ will thus induce an isomorphism σ of $A_{\mathscr{S}}$ with G_S as desired. The reason we make k odd in (4.2) is that $G_{F_k} \cong G_k$ if k is odd but not necessarily if k is even (see property (xi) of regular T-limit systems and Lemma 2(B)).

In our construction we will also have for all odd k

(4.3) There exists $z_k \in G_k$ which induces σ_k by conjugation such that $\rho_{k+2}(z_{k+2})$ has the same action on W_k as does $\rho_k(z_k)$.

Since the representations $\{\rho_k\}$ define the action of $\rho(G_S)$ on X (recalling from Lemma 3 that $\rho_k(g)$ and $\rho_{k+2}(\varphi_{k+1} \circ \psi_k(g))$ have the same action on W_k for all $g \in G_k$) the $\{z_k\}$ will induce a permutation isomorphism of $(X, \rho(A_{\mathscr{S}}))$ with $(X, \rho(G_S))$ which takes every orbit of $(X, \rho(A_{\mathscr{S}}))$ to itself since it is obtained from the *inner* automorphisms $\{z_k\}$.

Thus to finish the proof it only remains to define the $\{z_k\}$ so that (4.2) and (4.3) hold where σ_k = conjugation by z_k. We let $z_1 = 1$ and, assume that $z_1, \ldots, z_k$ have been defined (for odd indices) and that (4.2) and (4.3) hold. Define

$$z_{k+2} = \begin{cases} z_k & \text{if } \zeta_k 0 \in F_k|_{k+1}, \\ z_k t & \text{if } \zeta_k 1 \in F_k|_{k+1}, \\ & \text{where } t \text{ is the element given in} \\ & (1) \text{ of the hypothesis.} \end{cases}$$

The verification of (4.2) and (4.3) is quite straightforward. In the first case above $\varphi_{k+1} \circ \psi_k$: $G_k \to G_{k+2}$ is the inclusion map since $\psi_k = \varphi_k^0$ and $\mathscr{S}$ is canonical, and (4.2) follows because in this case σ_{k+2} extends σ_k since $z_{k+2} = z_k$. Also $\rho_{k+2}(z_{k+2}) = \rho_{k+2}(z_k)$ has the same action on W_k as does $\rho_k(z_k)$ since the embeddings φ_{k+1}

and φ_k^0 correspond to permutation extensions in the construction of the $\{\rho_k\}$ in Lemma 3, and so (4.3) holds. In the second case above $\varphi_{k+1} \circ \psi_k = \varphi_{k+1} \circ \varphi_k^1$ is the branching isomorphism φ_k^1: $G_k \to \gamma(G_k)$ followed by the inclusion φ_{k+1}: $G_{k+1} \to G_{k+2}$ where $G_{k+1} = \langle G_k, \gamma(G_k)\rangle$. Hence for all $g \in G_k$ we have $\sigma_{k+2}(g) = t^{-1}(z_k^{-1}gz_k)t = \gamma(z_k^{-1}gz_k) = \varphi_{k+1} \circ \varphi_k^1 \circ \sigma_k(g)$ and (4.2) holds. Also, $\rho_{k+2}(z_{k+2}) = \rho_{k+2}(z_k t) = \rho_{k+2}(z_k)\rho_{k+2}(t)$ and $\rho_{k+2}(z_k)$ agrees with $\rho_k(z_k)$ on W_k (as before); thus our hypothesis (2) shows that $\rho_{k+2}(z_{k+2})$ agrees with $\rho_k(z_k)$ on W_k also.

PROOF OF THEOREM 4. Suppose (ω, G) is a countable, relatively homogeneous, locally finite permutation group such that every finite subgroup of G has infinitely many regular orbits on ω.

We will show how to construct a canonical T-limit system $\mathscr{S} = \{G_k, \varphi_f\}$ with a canonical representation ρ (see (3.1)) such that

(4.4) $A_{\mathscr{S}} = \bigcup_{k \geqslant 1} G_k = G$,

(4.5) ρ_k: $G_k \to \mathrm{Sym}(W_k)$ is the restriction of the permutations G_k to W_k where $\omega = \bigcup_{k \geqslant 1} W_k$, W_k finite (see Lemma 3),

(4.6) The hypothesis of the Archetypal Limit Theorem (4.0) are satisfied,

(4.7) The hypothesis of the Spectrum Lemma (3.11) is satisfied; indeed, the construction of $\mathscr{S}$ can be carried out so that for odd k the groups $G_{k+1} = \langle G_k, \gamma(G_k)\rangle \subset G$ can be chosen so that, for all $x, y \in G_k - G_k^*$, $|\langle x, \gamma(y)\rangle|$ is larger than any given natural number.

Note that (4.4) and (4.5) imply $(\omega, \rho(A_{\mathscr{S}})) \cong (\omega, G)$ and so the conclusion of Theorem 4 will follow from application of (3.11) and (4.0).

We assume that the T-limit system and the representations $\{\rho_k\}$ have been constructed up to some odd level k and we show how to continue the construction up to level $k + 2$.

Let $A = G_k^* = G_{T_k - \{\zeta_k\}}$ and let $F = G_k *_A \tilde{G}_k$ be the free product of two copies of G_k with A amalgamated (so $\tilde{a} = a$ for all $a \in A$). Let h: $F \to G_k$ be the natural homomorphism ($h(g) = h(\tilde{g}) = g$ for all $g \in G_k$) and let $K =$ the kernel of h. Since K is free and therefore residually finite we can obtain $N \subset K$, $N \lhd F$ such that $J = F/N = \langle G_k, \tilde{G}_k\rangle$ is finite and for all $x \in G_k - A$, $y \in \tilde{G}_k - A$, $|\langle x, y\rangle|$ is larger than any given natural number (for more details of this one can consult [**5**, Lemma 2.1]). We can also arrange that J has an automorphism τ such that $\tau(g) = \tilde{g}$ for all $g \in G_k$ since we can replace N by $N \cap N^\tau$ where τ is the involution of F with the stated property and then induce τ on $J = F/N \cap N^\tau$.

We cannot find a copy of J in G; however, we can use relative homogeneity and the availability of regular orbits to find an amalgamation in G which has J as an image. This is accomplished as follows.

Let c be the index of G_k in J. Choose distinct regular orbits $B_1, \ldots, B_c$ of G_k on ω which do not intersect W_k. Let λ: $J \to B$ be a permutation isomorphism of the right regular representation of G_k on J with the regular orbits $B_1, \ldots, B_c$ of G_k where $B = B_1 \cup \cdots \cup B_c$. (The group isomorphism involved here is 1_{G_k}.) Define r: $G_k \to \mathrm{Sym}(B)$ by $(b)r(g) = \lambda(\lambda^{-1}(b)g)$ and $\tilde{r}$: $G_k \to \mathrm{Sym}(B)$ by $(b)\tilde{r}(g) = \lambda(\lambda^{-1}(b)\tilde{g})$ for all $g \in G_k$, $b \in B$. Since $\tau \in \mathrm{Aut}\, J$ and $\tau(g) = \tilde{g}$ for all $g \in G_k$, τ is a permutation automorphism of the right regular representation of J (under the group automorphism τ) which takes the regular representation of G_k to that of $\tilde{G}_k$

—which are therefore conjugate in $\mathrm{Sym}(J)$ under τ. Since r and $\tilde{r}$ copy the regular representations of G_k and $\tilde{G}_k$ (on J) in B via λ, it follows that $\langle r(G_k), \tilde{r}(G_k)\rangle \subset \mathrm{Sym}(B)$ is isomorphic (as a group) to $J = \langle G_k, \tilde{G}_k\rangle$ by an extension of the maps $r(g) \mapsto g$, $\tilde{r}(g) \mapsto \tilde{g}$ $(g \in G)$. The permutation β: $B \to B$ which conjugates $r(G_k)$ to $\tilde{r}(G_k)$ is given by $\beta(b) = \lambda(\tau(\lambda^{-1}(b)))$ $(b \in B)$ and β centralizes $r(A) = \tilde{r}(A)$ since τ centralizes A.

Thus, we can apply relative homogeneity of (ω, G) to the subgroup $A = G_k^*$ and its orbits on B to obtain an element $t \in G \subseteq \mathrm{Sym}(\omega)$ such that t centralizes A, t is the identity on W_k, and t induces β on B.

Now we define $\gamma(g) = t^{-1}gt$ for all $g \in G_k$ and $G_{k+1} = \langle G_k, \gamma(G_k)\rangle$ to be the next group in $\mathcal{S}$. We define $X_{k+1} = B$ (see Lemma 3) and thus $W_{k+1} = W_k \cup B$ and ρ_{k+1} is the restriction of the permutations $G_{k+1} \subset G$ to W_{k+1}.

Let p: $G_{k+1} \to \mathrm{Sym}(B)$ be the restriction map. Thus $\langle p(G_k), p(\gamma(G_k))\rangle \cong \alpha J = \langle G_k, \tilde{G}_k\rangle$ where α extends the maps $p(g) \mapsto g$, $p \circ \gamma(g) \mapsto \tilde{g}$ $(g \in G_k)$. Thus, since J has a canonical homomorphism u onto G_k (which is induced by the canonical h: $F \to G_k$ since $J = F/N$ where $N \subset \ker(h)$), we see that $u \circ \alpha \circ p$: $G_{k+1} \to G_k$ is a canonical homomorphism. We also note here that condition (4.7) is satisfies by G_{k+1} because $|\langle \alpha(x), \alpha(\gamma(y))\rangle| \leqslant |\langle x, \gamma(y)\rangle|$.

We now choose G_{k+2} to be any finite subgroup of G such that $\langle G_{k+1}, t\rangle \subset G_{k+2}$, and we choose W_{k+2} to consist of finitely many orbits of G_{k+2} on ω such that $W_{k+1} \subseteq W_{k+2}$ (so $X_{k+2} = W_{k+2} - W_{k+1}$). Of course we define ρ_{k+2}: $G_{k+2} \to \mathrm{Sym}(W_{k+2})$ to be the permutation restriction. Thus ρ_{k+1} and ρ_{k+2} have been defined in accordance with properties (b) and (c) of canonical representations in the proof of Lemma 3 and we can satisfy (d) by choosing W_{k+2} so that G_{k+2} is faithful on it. We also choose G_{k+2} so that G is exhausted during the construction and W_{k+2} so that ω is exhausted, thus meeting the requirements (4.4) and (4.5). Since (4.7) has already been satisfied, we are left to check (4.6)—and to this end we observe that both (1) and (2) of (4.0) have been specifically built into our construction by the choice of t.

5. Applications using wreath products. In this section we prove Theorems 1, 2* and 3, except for the reference to e.c. groups in Theorem 3 which we defer until Part 2.

Our first task will be to give our method of making $\rho(A_{\mathcal{S}})$ transitive on ω by specifying what the point-stabilizer subgroup will be (see (5.5)). At the same time it is convenient to discuss the situation in which the tree-limit group $G_{\mathcal{S}}$ is contained in an inverse limit.

(5.0) Definition. A canonical T-limit system is *normal* iff for all odd k there exists $t \in G_{k+2}$ such that $t^{-1}gt = \gamma(g)$ for all $g \in G_k$ where $G_{k+1} = \langle G_k, \gamma(G_k)\rangle$ is the canonical amalgamation (2.1) and there is an extension $\bar{h}$: $D \to G_k$ of the canonical homomorphism h: $G_{k+1} \to G_k$ with $t \in D = \mathrm{dom}(\bar{h}) \subset G_{k+2}$ such that $\bar{h}(t) = 1$.

(5.1) Lemma. *Suppose $\mathcal{S}$ is a canonical normal T-limit system. Then*

(a) *There is a canonical representation ρ of $G_{\mathcal{S}}$ such that $(\omega, \rho(G_{\mathcal{S}}))$ is an archetypal limit of $(\omega, \rho(A_{\mathcal{S}}))$,*

(b) *If* $\ker(\bar{h})$ *is locally finite for all* k *(with* $\bar{h}$ *as in* (5.0)) *then* $x^{-1}y$ *has finite order whenever* $x, y \in G_k$ *are similar in* $\mathscr{S}$,

(c) *If* $\operatorname{dom}(\bar{h}) = G_{k+2}$ *for all odd* k, *then* $G_{\mathscr{S}}$ *is isomorphic to a subgroup of* $P = \Pi\{G_k \mid k \text{ odd}\}$ *contained in the inverse limit of these groups with respect to the homomorphisms* $\bar{h}$: $G_{k+2} \to G_k$; *under this embedding of* $G_{\mathscr{S}}$ *in* P *the canonical representation* $(\omega, \rho(G_{\mathscr{S}}))$ *is isomorphic to the subaction of the regular coordinate representation of* P *on the disjoint union* $\bigcup\{G_k \mid k \text{ odd}\}$. *(Thus, if the* G_k *are finite,* $(\omega, \rho(G_{\mathscr{S}}))$ *is sharp and has finite orbits since the inverse limit has these properties).*

Proof of (a). For odd k let $D_k = \operatorname{dom}(\bar{h}) \subset G_{k+2}$ and $N_k = \ker(\bar{h})$. Thus $G_{k+1} = \langle t^{-1}G_k t, G_k\rangle \subset \langle G_k, t\rangle \subset N_k G_k = D_k$ since $\bar{h}(t) = 1$ and N_k is normalized by G_k. To satisfy hypotheses (1) and (2) of the Archetypal Limit Theorem we put $X_{k+1} = \varnothing$ so that $W_{k+1} = W_k$ and define ρ_{k+1} in accordance with (c) of Lemma 3 on G_{k+1} and extend this definition to D_k so that for all $ug \in D_k$ $(u \in N_k, g \in G_k)$ we have

$$(w)\rho_{k+2}(ug) = (w)\rho_k(g) \qquad (w \in W_k).$$

Thus $\rho_{k+2}(t)$ fixes every $w \in W_k$ as required since $t \in N_k$ (and this is why the above definition extends the definition of ρ_{k+1} in (c) of Lemma 3 (using the canonical homomorphism h) since $\gamma(g) = t^{-1}gt = ug$ where $u = t^{-1}gtg^{-1} \in N_k$). Now we must finish the definition of ρ_{k+2} (as in (b) of Lemma 3) by extending the action of $\rho_{k+2}(D_k) \subset \operatorname{Sym}(W_k)$ in any way we want to $\rho_{k+2}(G_{k+2}) \subset \operatorname{Sym}(W_{k+2})$ taking care that (d) of Lemma 3 will be satisfied at these steps rather than at the steps (c) as was done in the proof of Lemma 3. (This is just a matter of bookkeeping since although $X_{k+1} = \varnothing$ we are free to define X_{k+2} to be whatever we want—thus we can include a regular orbit of G_{k+2} in X_{k+2} or do something else to satisfy (d).) (Actually (d) is *equivalent* to the condition that the representation $\rho(G_{\mathscr{S}})$ is faithful on W. The reader might enjoy verifying this using a finite projection-set argument.)

(5.2) Definition. A *normal canonial representation* ρ: $G_{\mathscr{S}} \to \operatorname{Sym}(\omega)$ is one constructed as in the proof of (5.1)(a) from a normal canonical T-limit system $\mathscr{S}$.

Proof of (b). We establish this by induction on k. Assume $x, y \in G_{k+2}$ are similar in $\mathscr{S}$. We can assume $x \neq y$ and hence $x, y \in \varphi_{k+1}(G_{k+1}) = G_{k+1}$ and there exist $a, b \in G_k$ which are similar in $\mathscr{S}$ and such that $x = \varphi_k^i(a)$, $y = \varphi_k^j(b)$ for some $i, j \in \{0, 1\}$. Since $\varphi_k^0 = 1$ and $\varphi_k^1 =$ conjugation by $t \in \ker(\bar{h}) = N_k$ we have $x = u_1 a$, $y = u_2 b$ for some $u_1, u_2 \in N_k$, and so $x^{-1}y = a^{-1}u_1^{-1}u_2 b = za^{-1}b$ where $z = a^{-1}(u_1^{-1}u_2)a \in N_k$. By the induction hypothesis $a^{-1}b$ has finite order and hence $x^{-1}y$ does also since $z \in N_k$ is locally finite.

Proof of (c). The present hypothesis gives $\operatorname{dom}(\bar{h}) = G_{k+2} = N_k G_k$ where $N_k = \ker(\bar{h})$ is normalized by G_k. Thus G_{k+2} is formed by adjoining a normal complement to G_k thus forming a semidirect product.

Let $J \subset P$ be the inverse limit of the $\{G_k \mid k \text{ odd}\}$ under the $\bar{h} = \bar{h}_k$: $G_{k+2} \to G_k$. Let $W =$ the disjoint union of the groups G_k, k odd. We will denote this as $W = \bigcup\{\tilde{G}_k \mid k \text{ odd}\}$ with $\tilde{G}_k \cap \tilde{G}_n = \varnothing$ if $k \neq n$.

First we define the representation r: $J \to \operatorname{Sym}(W)$ as follows. If $\pi = (g_j) \in J$, that is, $h_j(g_{j+2}) = g_j$ (j odd) and $\tilde{g} \in \tilde{G}_j$ then $(\tilde{g})r(\pi) = \tilde{g}\tilde{g}_j \in \tilde{G}_j$. Thus $r(\pi)$ is

the regular coordinate action. Obviously the $\tilde{G}_k$ (k odd) are the orbits of $r(J) \subset \mathrm{Sym}(W)$ and every $1 \neq \pi \in J$ acts without fixed points on the $\tilde{G}_k$ after some point —that is, whenever $g_k \neq 1$. (Hence, if the G_k are finite, $f(J)$ is sharp.)

We will check that there is an embedding σ: $G_{\mathscr{S}} \to J$ and that a normal canonical representation ρ: $G_{\mathscr{S}} \to \mathrm{Sym}(W)$ can be defined which agrees with $r \circ \sigma$. To embed $G_{\mathscr{S}}$ in J we use the embedding σ of $G_{\mathscr{S}}$ into P/S constructed in the proof of part (D) of Lemma 2; however, we only need to use odd values of k as coordinates. In that proof we defined

$$\sigma(x) = \left(\varphi_{g(k)}(x_0)\right)_{k \geqslant j} \pmod S$$

where $x \in G_s$ for some $s \in P(T_\omega)$ where $x_0 \in G_{s|_j}$ is a preimage of x under the direct limit embedding of $G_{s|_j}$ into $G_{\mathscr{S}}$ and $g(k)$: $s|_j \to s|_k$, $g(k) \in Mp(j,k)$. For the present proof we will adjoin the $j-2$, $j-4, \ldots, 1$-coordinates to $\sigma(x)$ by applying the inverse limit homomorphisms $h_{j-2}, h_{j-4}, \ldots, h_1$ successively to the jth-coordinate x_0 of $\sigma(x)$ to obtain $\sigma(x) \in P$. To see that $\sigma(x) \in J$ we need only check that all the remaining coordinates of $\sigma(x)$ are related via the inverse limit homomorphisms. To see this let $k \geqslant j$ be odd and let $y = \varphi_{g(k)}(x_0) \in G_k$ and $z = \varphi_{g(k+2)}(x_0) \in G_{k+2}$ be successive coordinates of $\sigma(x)$. Thus $z = \varphi_f(y)$ where f: $s|_k \to s|_{k+2}$ ($f \in Mp$), that is, $z = \varphi_k^0(y)$ or $\varphi_k^1(y) \in G_{k+1} \subset G_{k+2}$; that is, $z = y$ or $t^{-1}yt$ where $t \in \ker(\bar{h}_k)$; and so in either case we have $\bar{h}_k(z) = \bar{h}_k(y) = y$ since $y \in G_k$ and $\bar{h}_k$ extends the canonical homomorphism h: $\langle G_k, t^{-1}G_k t\rangle \to G_k$.

Note that we still have $\sigma(G_{\mathscr{S}}) \cong G_{\mathscr{S}}$ for the same reason as in the proof of Lemma 2—namely, the groups G_s ($s \in P(T_\omega)$) are recovered isomorphically by looking at any sufficiently large coordinates of members of $\sigma(G_s)$.

Now we obtain the desired normal canonical representation ρ: $G_{\mathscr{S}} \to \mathrm{Sym}(W)$ by putting $X_k = \tilde{G}_k$ and proceeding exactly as in part (a) of this proof. Since $D_k = G_{k+2}$ the action of $\rho_{k+2}(G_{k+2})$ on $W_k = \tilde{G}_1 \cup \cdots \cup \tilde{G}_k$ is prescribed for us (by applying $\bar{h}_k$, or equivalently, factoring out N_k from $G_{k+2} = N_k G_k$ and then using ρ_k) and we simply need to add the regular orbit $X_{k+2} = \tilde{G}_{k+2}$ of G_{k+2} to obtain $\rho_{k+2}(G_{k+2}) \subset \mathrm{Sym}(W_{k+2})$. One checks easily that this construction of ρ is the same as the direct product representation of $\sigma(G_{\mathscr{S}})$. □

We urge the reader to visualize the above embedding σ of the tree-limit group $G_{\mathscr{S}}$ in the inverse limit group J, noting that $G_{k+2} = N_k G_k$ and that the branching isomorphism φ_k^1 is obtained by conjugating G_k by some $t \in N_k$ which centralizes G_k^*.

Now we will define the type of group which possesses normal canonical systems and representations (of which it is the archetype) and the type of subgroup which can serve as the point-stabilizer in such representations.

Remark. We cannot assert that the representation ρ of (5.1)(a) is faithful on any of its orbits, although it is faithful. In general if G is a countable group with the c.a.p. and such that every f.g. subgroup of G is corefree in G then we could construct a canonical T-limit system $\mathscr{S}$ with $G \cong A_{\mathscr{S}}$ and a canonical representation ρ of $G_{\mathscr{S}}$ as in Lemma 3 such that $(\omega, \rho(A_{\mathscr{S}}))$ is faithful on all of its orbits using Kegel's coset extension procedure (as sketched in Lemma 3) and choosing the G_k (k even) to be sufficiently large. However we have no need of this because we will now develop a

much better way to obtain transitive archetypal limits which fits neatly into the context of the Archetypal Limit Theorem and which will allow us to employ the Spectrum Lemma while keeping the representation of the archetype fixed.

(5.3) DEFINITION. $G \neq 1$ has the *normal canonical amalgamation property* (n.c.a.p.) iff given f.g. subgroups $A \subset B \subset G$ and $x \in B - A$ there exists $t \in C_G(A)$ such that, defining $\gamma(b) = t^{-1}bt$ $(b \in B)$, the subgroup $\langle B, \gamma(B)\rangle$ is a canonical amalgamation, (that is, $\gamma(a) = a$ $(a \in A)$; $x \neq \gamma(x)$; and there exists a canonical homomorphism h: $\langle B, \gamma(B)\rangle \to B$ (see (2.1))) such that h has an extension to $\bar{h}$: $\langle B, t\rangle \to B$ with $\bar{h}(t) = 1$.

If G has the n.c.a.p. then $H \subset G$ is called a *canonical subgroup* of G iff H is corefree in G and for all f.g. subgroups $A \subset B \subset G$ and $x \in B - A$ there exist t and $\bar{h}$ as above such that $\ker(\bar{h}) \subset H$.

REMARK. The properties of the element t in the above definitions can be more conveniently formulated as follows: Given f.g. $A \subset B \subset G$ and $x \in B - A$ there exists $t = t(A, B, x) \in G$ such that $t \in C_G(A)$, $t^{-1}xt \neq x$, and $t^B \cap B = 1$ where $t^B = \langle b^{-1}tb \mid b \in B\rangle$. This is because we can obtain $\bar{h}$ by taking the quotient of $\langle B, t\rangle$ modulo t^B; and if $\bar{h}$: $\langle B, t\rangle \to B$ exists then $\langle B, t\rangle = \ker(\bar{h})B$ and we have $\ker(\bar{h}) = t^B$ since $t \in \ker(\bar{h})$. In defining a canonical subgroup H we require that $t^B \subset H$.

(5.4) LEMMA. *If G is a countable group with the n.c.a.p., then G has a canonical subgroup.*

PROOF. We construct a sequence of elements $t_0, t_1, \cdots$ and f.g. subgroups $J_0 \subset J_1 \subset \cdots$ of G such that $t_{n-1} \in J_n$ and $t_n^{J_n} \cap J_n = 1$.

Let $\{(A_i, B_i, x_i) \mid i < \omega\}$ be a list of all f.g. subgroups $A_i \subset B_i \subset G$ and elements $x_i \in B_i - A_i$.

We define $H_i = t_{2i}^{J_{2i}} \cdots t_2^{J_2} t_0^{J_0}$ (where only even indices occur) and $M_n = t_n^{J_n} t_{n-1}^{J_{n-1}} \cdots t_0^{J_0}$ and we note that these groups will be iterated semidirect products due to the assumptions in the first paragraph. (That is, M_{k+1} (resp., H_{k+1}) is obtained by adjoining a normal complement to M_k (resp., H_k).)

The construction proceeds as follows. If $n = 2i - 1$ is odd we will choose some $1 \neq y_n \in H_{i-1}$ and use the n.c.a.p. with $A = 1$ to find $t_n \in G$ so that $t_n^{J_n} \cap J_n = 1$ and $t_n^{-1}y_n t_n \neq y_n$. Then we define $J_{n+1} = J_{2i} = \langle J_n, t_n, B_i\rangle$ and use the n.c.a.p. to find $t = t_{2i} \in C_G(A_i)$ such that $t^{-1}x_i t \neq x_i$ and $t^{J_{2i}} \cap J_{2i} = 1$. Finally, we put $J_{2i+1} = \langle J_{2i}, t_{2i}\rangle$ for the next step.

Defining $H = \bigcup_{i=1}^{\infty} H_i$ we easily see that H will be a canonical subgroup of G provided that H is corefree in G by the Remark after (5.3) since $B_i \subset J_{2i}$. To ensure that H is corefree we arrange that every $1 \neq y \in H$ occurs as y_n for some odd n. We then have $1 \neq t_n^{-1}y_n t_n y_n^{-1} \in t_n^{H_{i-1}} \subset t^{J_n}$ where $n = 2i - 1$ and so because of the normal form of elements of M_k (for all $k \geqslant n$)—due to the semidirect product structure—we see that $t_n^{-1}y_n t_n y_n^{-1} \notin H$. So since $y_n = y \in H$ we conclude $t_n^{-1}yt_n \notin H$ and so H is corefree in G. □

The reason for considering canonical subgroups is as follows.

(5.5) (STABILIZER THEOREM). *Suppose G is a countable group satisfying the n.c.a.p. and $H \subset G$ is a canonical subgroup. Then there is a normal canonical T-limit system $\mathcal{S}$ with $G = A_{\mathcal{S}}$ and a normal canonical representation ρ: $G_{\mathcal{S}} \to \mathrm{Sym}(\omega)$ such that $\rho(A_{\mathcal{S}})$ is transitive on ω and $\rho(H)$ is a point-stabilizer subgroup of $\rho(A_{\mathcal{S}})$. Hence $(\omega, \rho(G_{\mathcal{S}}))$ is an archetypal limit of the regular representation of G on the right cosets of H (which is faithful since H is corefree in G).*

PROOF OF THE STABILIZER THEOREM. Suppose G has the n.c.a.p. and $H \subset G$ is a canonical subgroup. We assume that we have constructed the desired normal canonical T-limit system and representation up to some odd level $k < \omega$ and we will show how to construct the next two levels.

Specifically we assume that we have G_k and ρ_k: $G_k \to \mathrm{Sym}(W_k)$ and that $\rho_k(G_k)$ is transitive on W_k and that $\rho_k(G_k \cap H)$ is a point-stabilizer subgroup of $(W_k, \rho_k(G_k))$. Thus we can also assume inductively that ρ_k is the right regular representation of G_k on W_k = a right transversal of $G_k \cap H$ in G_k.

We have given $x \in G_k - G_k^*$ and we use the fact that H is canonical to obtain $t \in C_G(G_k^*)$ such that $t^{-1}xt \neq x$, $t^{G_k} \cap G_k = 1$ and $t^{G_k} \subset H$. We define ρ_{k+2}: $t^{G_k}G_k \to \mathrm{Sym}(W_k)$ by the natural homomorphism as prescribed in the construction of normal canonical representations given in Lemma (5.1)(a) (using $N_k = t^{G_k}$ as the kernel of $\bar{h}$). Thus $P = H \cap (N_kG_k) = N_k(H \cap G_k)$ is the point-stabilizer subgroup (under ρ_{k+2}) of $(W_k, \rho_{k+2}(N_kG_k))$ and this representation is again the right regular representation of N_kG_k on the cosets of P in N_kG_k since W_k is also a right transversal for P in N_kG_k.

Now to satisfy the induction hypothesis we must extend the representation ρ_{k+2}: $N_kG_k \to \mathrm{Sym}(W_k)$ to a right regular representation ρ_{k+2}: $G_{k+2} \to \mathrm{Sym}(W_{k+2})$ where W_{k+2} is a right transversal of $H \cap G_{k+2}$ in G_{k+2}. (Note that $W_{k+1} = W_k$ as in (5.1)(a).) We must be able to pick G_{k+2} to be any f.g. subgroup of $G = \bigcup G_k = A_{\mathcal{S}}$. This represents no difficulty since the elements of W_k belong to distinct right cosets of $G_{k+2} \cap H$ in G_{k+2} and so $W_k \subset W_{k+2}$ = a right transversal of $G_{k+2} \cap H$ in G_{k+2} and the regular action ρ_{k+2}: $G_{k+2} \to \mathrm{Sym}(W_{k+2})$ extends the regular action ρ_{k+2}: $N_kG_k \to \mathrm{Sym}(W_k)$ previously defined.

Thus in our construction we have $W = \bigcup W_k$ is a right transversal of H in G (since $G = \bigcup G_k = A_{\mathcal{S}}$) and $\rho(A_{\mathcal{S}})$ is the regular representation of G on the right cosets of H as desired. □

COROLLARY 4. *Every countable group with the n.c.a.p. has a faithful transitive permutational archetypal limit enlargement to power 2^ω. This includes every group of Theorem 2.*

PROOF. Every group of Theorem 2 has the n.c.a.p. since, given $A \subset B \subset G$, let $1 \neq f \in D \subset \mathbf{Z}_n \operatorname{wr} (A, B) = \langle D, B \rangle$, let φ: $\mathbf{Z}_n \operatorname{wr} (A, B) \to G$ be an embedding with $\varphi(b) = b$ $(b \in B)$, and put $t = \varphi(f)$. Then $\langle B, t^{-1}Bt \rangle$ is a normal canonical amalgamation since $\varphi(D) \cap B = 1$ and we have $t^{-1}xt = x[x, t] \neq x$ for all $x \in B - A$ since $[x, t] \neq 1$ since $x \notin A$ conjugates the A-diagonal element f to an Ax-diagonal element. □

REMARK. We could just as well use $\mathbf{Z}$ wr (A, B) in the above proof, thus obtaining many examples of countable torsion-free groups which have transitive archetypal limit enlargements to power 2^ω. The reason Theorem 2 does not encompass such groups is because of a technical defect in our method of satisfying the Spectrum Lemma using wreath products which relies on using elements of increasing finite orders at different levels. We had hoped to overcome this difficulty by using solvability lengths (of subgroups with two similar generators) instead of orders of elements, but we are at a loss to find the right technique for this.

The following lemma (as well as 5.1(b)) will be needed to apply the Spectrum Lemma.

(5.6) LEMMA. *Suppose $\mathscr{S}$ is a canonical T-limit system such that, for all odd $k < \omega$, we have $G_k \cap \gamma(G_k) = G_k^*$ (that is, the amalgamations are strong). Then if $x, y \in G_{k+1} - G_Z$ are similar in $\mathscr{S}$ where $Z = T_{k+1} - \{\eta\}$ for some $\eta \in T_{k+1}$, then for all j we have $y(x^{-1}y)^j \notin G_Z$. Without the hypothesis of strong amalgamation we still have $y(x^{-1}y)^j \neq 1$.*

PROOF. Note that the conclusion is true when $x = y$. Thus we can assume inductively that the property holds at level k and that $x \neq y$. If k is even then the conclusion is immediate since $G_{k+1} \supset G_Z = G_S \subset G_k$ where $S = \{\alpha \mid \alpha 0 \in Z\}$ and $x, y \in G_k$ because $x \neq y$ are similar. So assume k is odd.

Case 1. $\eta = \eta_k 0$. Thus $x, y \notin \varphi_k^1(G_k) = G_Z$. So evidently $x, y \in \varphi_k^0(G_k)$ since $x \neq y$ are similar in $\mathscr{S}$. If $z = y(x^{-1}y)^j \in \varphi_k^0(G_k)$ then $z \in G_k \cap \gamma(G_k) = G_k^*$. But this would contradict the inductive hypothesis since $x, y \in G_k$ are similar and $x, y \notin G_k^*$.

Case 2. $\eta = \eta_k 1$. This is obviously equivalent to the first case.

Case 3. $\eta = \tau 0$ where $\tau \in T_k - \{\eta_k\}$. Thus $G_Z = \langle G_S, \gamma(G_S)\rangle$ where $S = T_k - \{\tau\}$. Now $x, y \in G_k \cup \gamma(G_k)$ since $x \neq y$ are similar in $\mathscr{S}$, so we have $x, y \in (G_k - G_S) \cup \gamma(G_k - G_S)$. Thus, if $h\colon G_{k+1} \to G_k$ is the canonical homomorphism, we have $h(x), h(y) \in G_k - G_S$ are similar in $\mathscr{S}$ (since h inverts the branching isomorphism) and so $h(y(x^{-1}y)^j) \notin G_S$ whence $y(x^{-1}y)^j \notin G_Z$ as desired.

Strong amalgamations are used only in Cases 1 and 2 and in these cases we trivially have $y(x^{-1}y)^j \neq 1$. So the final conclusion follows. □

In order to prove Theorem 2* we must obtain canonical subgroups which are rich enough (in kernels) for us to satisfy the Spectrum Lemma within the proof of the Stabilizer Theorem. Specifically all we will need is

(5.7) LEMMA. *Suppose G is a group of Theorem 2. Then G has a canonical subgroup H such that for f.g. $A \subset B \subset G$, $x \in B - A$, and $m < \omega$ there exists $n > m$ and $f \in G$ such that $\langle f, B\rangle \cong \mathbf{Z}_n$ wr (A, B) where f is the diagonal over A of a generator of $\mathbf{Z}_n$ and $f^B \subset H$.*

PROOF. In the proof of Lemma (5.4) we can arrange for every triple (A_i, B_i, x_i) to occur infinitely often in our list and we choose $t_{2i} = f$ so that $\langle f, J_{2i}\rangle \cong \mathbf{Z}_n$ wr (A_i, J_{2i}) where $n > i$ and f is the diagonal of a generator of $\mathbf{Z}_n$ over A_i (using the hypothesis of Theorem 2). Thus $\langle f, B_i\rangle \cong \mathbf{Z}_n$ wr (A_i, B_i) and $f^{B_i} \subset f^{J_{2i}} \subset H$. □

PROOF OF THEOREM 2*. We will use the canonical subgroup $H \subset G$ given in (5.7) to satisfy the Spectrum Lemma while proceeding as in the proof of the Stabilizer Theorem. Thus we are given G_k (k odd), finitely many groups $\langle a_i, b_i \rangle$ ($1 \leqslant i \leqslant q$), and finitely many pairs of similar elements (x_j, y_j) ($1 \leqslant j \leqslant l$) with $x_j, y_j \notin G_k - G_k^*$, and we must construct $G_{k+1} = \langle G_k, t^{-1}G_k t \rangle$ so that $\langle x_j, t^{-1}y_j t \rangle \not\cong \langle a_i, b_i \rangle$ by the generator map (for all i, j) and so that $t^{G_k} \subset H$.

Referring to Lemma (5.1) we have $\ker(\bar{h}) = t^{G_k}$ since $\bar{h}$: $t^{G_k}G_k \to G_k$ is the natural homomorphism. We will always have $t = f$ where

$$\langle f, G_k \rangle \cong \mathbf{Z}_n \operatorname{wr} (G_k^*, G_k) \text{ for some } n \text{ with } f = \text{the diagonal of a generator of } \mathbf{Z}_n \text{ over } G_k^*. \tag{5.8}$$

So t^{G_k} is abelian with exponent n. Since this was true of previous steps (for various n), Lemma (5.1)(b) implies that every $x_j^{-1}y_j$ has finite order, say $|x_j^{-1}y_j| = p_j$. Now we choose m to be larger than every finite number in the set $\{|(a_i^{-1}b_i)^{p_j}|\}$ where $1 \leqslant i \leqslant q$, $1 \leqslant j \leqslant l$. Now choose $t = f$ so that (5.8) holds and $f^{G_k} \subset H$ where $n > m^2$ which is possible by Lemma (5.7).

Now all we need to do is to verify that $[x_j^{-1}(f^{-1}y_j f)]^{p_j}$ has finite order $> m$ (for all j) in order to conclude that $\langle x_j, f^{-1}y_j f \rangle \not\cong \langle a_i, b_i \rangle$ ($1 \leqslant i \leqslant q$) by the generator map and thus satisfy the Spectrum Lemma.

Omitting the index j we have $w = x^{-1}(f^{-1}yf) = x^{-1}y[y, f]$. Putting $u = x^{-1}y$ we easily compute

$$w^p = [y, f]^{u^{p-1}} \cdots [y, f]^u [y, f] \tag{5.9}$$

since $[y, f] = (y^{-1}f^{-1}y)f$ is in the base group f^{G_k} of $\mathbf{Z}_n \operatorname{wr} (G_k^*, G_k)$ and $u^p = 1$.

So $w^p \in f^{G_k}$ has finite order and we check that $|w^p| > m$ as follows. Put $A = G_k^*$. If $g \in f^{G_k} \subset \mathbf{Z}_n \operatorname{wr} (A, G_k)$, put $\operatorname{supp}(g) =$ the support of $g =$ the subset of G_k on which the function g is nontrivial. Thus $\operatorname{supp}(f) = A$ and $\operatorname{supp}(y^{-1}f^{-1}y) = Ay$. Considering the factors of w^p given in (5.9) we have

$$\operatorname{supp}\left([y, f]^{u^d}\right) = A\left(x^{-1}y\right)^d \cup Ay\left(x^{-1}y\right)^d. \tag{5.10}$$

Putting $\mathbf{Z}_n = \langle z \rangle$ where f is the diagonal of z over A we see that z occurs on the coordinates Au^d of (5.10) while z^{-1} occurs on the coordinates Ayu^d. If we show that none of the coordinates on which z^{-1} occurs overlaps A, then, in particular, $w^p(1) = z^v$ for some $1 \leqslant v \leqslant p$ in view of (5.9) and (5.10) and so $|w^p| > m$ since $n > m^2$ and $m > p$. But such an overlap is impossible since $y(x^{-1}y)^d \notin A$ by Lemma (5.6). □

PROOF OF THEOREM 3 WITH $G = E_p$. We need only construct 2^ω inequivalent canonical subgroups $H_\alpha \subset E_p$ ($\alpha < 2^\omega$) of the type used in the above proof of Theorem 2* and such that $H_\alpha \cong E_p$. That is, for $\alpha \neq \beta$, H_α must not be taken to H_β by any automorphism of E_p. Let $\{(U_\gamma, V_\gamma) \mid \gamma \in Q = \text{the rationals}\}$ be the jumps of the unique chief series of E_p. (The reader can consult **[8]** for basic properties of E_p.) If $H \subset E_p$ we put $j(H) = \{\gamma \in Q \mid H \cap (V_j - U_j) \neq \varnothing\}$.

Let $\Sigma = \{(\eta_x, \psi_x) \mid x \in I\}$ be a set of pairs of irrationals with $\eta_x < \psi_x$ such that I is an ordered set with no least element, $\{\eta_x \mid x \in I\}$ has no lower bound, and $x < y \in I$ implies $\psi_x < \eta_y$. We claim that we can construct $H = H(\Sigma)$ so that

$j(H) = \bigcup\Sigma$ (where the pairs in Σ are interpreted as rational intervals). This is an immediate consequence of the embedding theorem [**9**, 3.1] since at each step in the construction of H we are faced with embedding a *finite* wreath product $\mathbf{Z}_{p^n}$ wr G_k into E_p over $G_k \subset E_p$ and so we can embed the base group into the jumps (η_x, ψ_x) for some interval lower than any of the jumps occurring in $j(G_k)$. And by this same embedding theorem we can also enlarge H (in any manner we wish) so that $j(H)$ will include *all* the jumps in the intervals Σ and no others while simultaneously guaranteeing that $H \cong E_p$ by making appropriate enlargements of H at each step. We leave it to the reader to supply the (easy) details of this. Notice that H is automatically corefree in E_p since the only normal subgroups of E_p are those in its unique chief series and $j(H)$ omits arbitrarily small jumps.]

Since any automorphism of E_p induces an order-automorphism on the jumps of its unique chief series, if we use 2^ω nonisomorphic order-types I to construct the $H(\Sigma)$, then we will obtain 2^ω inequivalent H as desired.

"PROOF" OF THEOREM 3 FOR $G = U$. We will merely describe a rather natural construction which produces the canonical subgroups of U which we need here. This construction was presented in the author's dissertation in 1977 and was called the "injective chain" construction; however, it has not yet been published.

If $\{(U_\alpha, V_\alpha) \mid \alpha \in I\}$ are the jumps of a complete subgroup chain $\mathscr{C}$ of a locally finite group G, we call $\mathscr{C}$ an *injective chain* provided that whenever $A \subset G$ is finite and $A \subset B$ is a finite group and B is a subgroup series of B with jumps $\{(B_i, B_{i+1}) \mid 1 \leqslant i < m\}$ such that the restriction of B to A yields the same series as the restriction of $\mathscr{C}$ to A and φ: $m \to I$ is an injection which is order-preserving and takes every jump of B inducing a jump in A to the corresponding jump in $\mathscr{C}$, then there is an embedding f: $B \to G$ with $f(a) = a$ $(a \in A)$ such that f induces φ, that is, for all $1 \leqslant i < m$, $f(B_{i+1} - B_i) \subset V_{\varphi(i)} - U_{\varphi(i)}$.

It turns out that it is rather easy using permutational products to construct injective chains of locally finite groups whose jumps have any given order-type. It further turns out that every countable nontrivial member of such an injective chain is isomorphic to U and that if $\mathscr{C}$ is such an injective chain with $\bigcup\mathscr{C} = U$ and $H \in \mathscr{C}$, then $H \subset K \subset U$ implies $K \in \mathscr{C}$. If an injective chain $\mathscr{C}$ has a minimum jump $(1, H)$, then one easily checks that H is a canonical subgroup of $\bigcup\mathscr{C} = U$. Thus we need only vary the (countable) order-type of the jumps of $\mathscr{C}$—which is order-isomorphic to the subgroup lattice above H—to obtain 2^ω inequivalent canonical subgroups of U each of which is isomorphic to U.

The proof of Theorem 3 for e.c. groups will also rely on linearly ordering the subgroups of G which contain H.

PROOF OF THEOREM 1. Let G be a countable e.c. group. We will actually prove here the version of Theorem 2* for G. Namely, we will construct a canonical subgroup $H \subset G$ which permits the use of the Spectrum Lemma. This means, as it did in the above proof of Theorem 2*, that each of the 2^ω T-limit systems of the Spectrum Lemma is constructed as in the proof of the Stabilizer Theorem so that H is a point-stabilizer subgroup of the archetype G. To do this is a trifle more complex than Lemma (5.7) because we must reflect the properties of the wreath product construction by relations in G and there is no particularly natural way to do this.

(The proofs of Theorem 3 for e.c. groups and of Theorem 1* and its generalization to relative homogeneity will all involve using combinatorics to build further properties into the canonical subgroups constructed in this proof.)

We proceed as follows. We fix some $1 \neq g \in G$ which will lie outside of H. Let $\Delta = (A, B, x, P)$ be a quadruple such that $A \subset B \subset G$ are f.g., $x \in B - A$, and $P = \{(x_i, y_i) \mid 1 \leqslant i \leqslant l\}$ are finitely many pairs such that $x_i, y_i \in B - A$, x_i^{-1} has finite order, and such that $y_i(x_i^{-1}y_i)^j \neq 1$ for all $j < \omega$. Let $\{\Delta_k \mid k < \omega\}$ be a list of all such quadruples in which every quadruple occurs infinitely often. Put $H_1 = 1$ and suppose we have constructed H_{n-1} so that $g \notin H_{n-1}$ and H_{n-1} is a recursively enumerable subgroup of a f.g. subgroup of G. Put $J = \langle B_n, H_{n-1}, g\rangle$ and $K = \mathbf{Z}_n$ wr (A_n, J) where $\Delta_n = (A_n, B_n, x(n), P_n)$ and let $f \in K$ be the diagonal of a generator z of $\mathbf{Z}_n$ over A_n so that $K = \langle f, J\rangle = f^J J$.

We claim that we can find an element $t \in G$ such that

(a) t centralizes A_n,

(b) $t^{-1}x(n)t \neq x(n)$,

(c) $t^J \cap J = 1$,

(d) $g \notin t^J H_{n-1}$,

(e) For all $(x, y) \in P_n$ if $|x^{-1}y| = p$ and $w = x^{-1}t^{-1}yt$, then $[\![n/p]\!] \leqslant |w^p| < \infty$.

The proof of this claim uses the fact that (a)–(e) above are true in K *with* f *replacing* t which is evident from the proof of Theorem 2*. (Recall that (e) follows from (5.9) and (5.10) and the fact that $w^p(1) = z^v$ for some $1 \leqslant v \leqslant p$ which follows from the condition $y(x^{-1}y)^j \neq 1$ (which is satisfied in Lemma (5.6) without the hypothesis of strong amalgamation) and the fact that any consistent recursively enumerable set of conditions of the form "$w_1 = w_2$", "$w_1 \neq w_2$", and "$w_1 = 1 \Rightarrow w_2 = 1$" (where w_1, w_2 are words in elements of G and new variables) can be satisfied in the e.c. group G, which follows in a well-known fashion from the General Higman Embedding Theorem [**6**, §1].) (The reader can consult [**12**] for a discussion of implications.) Here we can state (a) with finitely many equations since A_n is f.g.; (d) with an r.e. set of inequations since J is f.g. and H_{n-1} is r.e. by the induction hypothesis; (e) can be satisfied by saying precisely what $|w|$ equals for every $(x, y) \in P_n$; and (c) can be stated with the r.e. set of implications "$uv = 1 \Rightarrow u = 1$" where $u \in t^J$ and $v \in J$.

We define $H_n = t^J H_{n-1}$ and $H = \bigcup H_n$. Since $g \notin H$ and G is simple, H is corefree in G. To complete the proof of Theorem 1 we need only to observe that we can duplicate the proof of Theorem 2*—using the Stabilizer Theorem and the Spectrum Lemma—with the canonical subgroup $H \subset G$ constructed above. This is obvious since to satisfy the Spectrum Lemma at some step we are given A, B, x, P as well as a finite set of numbers η which the orders of the $w_i = x_i^{-1}t^{-1}y_i t$ must avoid for all $(x_i, y_i) \in P$. So we need only choose n large enough so that $[\![n/p_i]\!]$ is larger than every member of η where $p_i = |x_i^{-1}y_i|$ and $(x_i, y_i) \in P$ and so that $\Delta_n = (A, B, x, P)$. □

To finish this section as promised we discuss tree-limits of the groups $G = \cdots A_3$ wr $(A_2$ wr $A_1)$ where the $A_n \neq 1$ are finite and have unbounded exponents. Let Σ be the class of all such wreath products.

(5.11) THEOREM. (i) *If* $G \in \Sigma$ *then* G *has* 2^ω *inequivalent representations in* $\mathrm{Sym}(\omega)$ *with finite orbits of the type dsecribed in Lemma* 5.1(c) *each of which satisfies Theorem* 2; (ii) *If* $G \in \Sigma$ *and* $A_n \cong \mathbf{Z}_{k_n}$ *where* $\lim_{n\to\infty}\{$*smallest prime power factor of* $k_n\} = \infty$ *then* G *has* 2^ω *inequivalent transitive representations in* $\mathrm{Sym}(\omega)$ *each of which satisfies Theorem* 2.

PROOF OF (i). We will give what amounts to a fairly complete sketch of a proof. We will show that one such representation can be constructed and that at every step we have the freedom to create a new orbit that is as large as we please, and from this it will be obvious that 2^ω such representations can be obtained by varying the sizes of orbits in the construction.

The construction is complicated by the fact that we are not using the Stabilizer Theorem and hence we must reconsider the proof of the Spectrum Lemma in order to satisfy it with a fixed representation of $G = A_{\mathscr{S}}$. Thus we are simultaneously constructing a binary tree of T-limit systems $\mathscr{S}_\alpha$ $(\alpha < 2^\omega)$ (as in the proof of the Spectrum Lemma). Each $\mathscr{S}_\alpha$ is to have archetype G and the same normal canonical representation of the archetype, which will be a product of regular actions of G on certain finite quotients of G as in Lemma (5.1)(c). We will satisfy the Archetypal Limit Theorem in exactly the same way we did in the proof of Theorem 2*, namely by taking $t = f =$ a partial diagonal function in the base group of a certain wreath product. Put $P_k = A_k \operatorname{wr}(\cdots \operatorname{wr}(A_2 \operatorname{wr} A_1)) =$ the kth top group of G.

Now assume inductively that we have defined finitely many T-limit systems, as required in proving the Spectrum Lemma, up to some odd level n and that G_n is the same for all of them and that the canonical representations ρ_n of G_n are also all the same. We further assume that $G_n = P_{f(n)}$ for some $f(n) < \omega$ and that ρ_n is built up by adding regular orbits isomorphic to G_n for every odd n in the manner of (5.1)(c). To continue we must choose $f(n+1)$ to be so large that $P_{f(n+1)} \cong B_n P_{f(n)}$ where $B_n = \langle A_{f(n)+1}, \ldots, A_{f(n+1)}\rangle^{P_{f(n)}}$ can be used to meet the requirements of the Spectrum Lemma at this stage, namely (in view of the method in the proof of Theorem 2*) B_n must contain a certain finite sequence of elements of increasing orders so that we can avoid finitely many isomorphism-types (in the various T-limit systems we have differentiated at this stage) by choosing the elements t (which perform the amalgamations to obtain G_{n+1} in these systems) to be partial diagonals in B_n having appropriately large orders. These elements t will be chosen from the base groups of the finite wreath products $A_k \operatorname{wr} P_{f(n)} = \langle A_k, P_{f(n)}\rangle \subset B_n P_{f(n)} = P_{f(n+1)}$ $(f(n) < k \leqslant f(n+1))$. Since the exponents of the A_k are unbounded we can pick new elements t of arbitrarily large orders. Thus at this stage of the construction we can choose $f(n+1)$ to be any number larger than a certain number we can compute from the partial T-limit systems already constructed and the exponents of the groups $\{A_k\}$. Having chosen $f(n+1)$ we define $G_{n+2} = P_{f(n+1)}$ to be the same in all of the T-limit systems under construction and we put $f(n+2) = f(n+1)$. (Of course the groups $G_{n+1} = \langle G_n, t^{-1}G_n t\rangle$ vary in the various T-limit systems because the elements $t = f$ are being varied in order to satisfy the Spectrum Lemma.) We then extend the canonical representation ρ_n in exactly the same way in all the partial T-limit systems to include a regular orbit isomorphic to G_{n+2} (see the proof of (5.1)(c)) to obtain

ρ_{n+2}. Thus we end up with a canonical representation ρ: $G \to \mathrm{Sym}(\omega)$ consisting of a product of regular orbits of G on its quotients $P_{f(n)}$ and 2^ω T-limit groups as in the Spectrum Lemma which are archetypal limits of this representation. As indicated at the outset there is no problem in creating branchings in this construction to obtain 2^ω such representations of G since we can avoid including any finite number of orbit sizes at each step by making $f(n+1)$ as large as we please.

PROOF OF (ii). The proof is considerably more canonical than the preceding one because we can use the Stabilizer Theorem. Obviously we obtain a canonical subgroup $H(f) \subset G$ for every function f: $\omega \to \omega$ such that $f(n+1) > f(n)$ by defining $H(f) = \langle A_{f(n)} \mid n < \omega \rangle$ provided that the range of f omits infinitely many natural numbers. (This is what makes $H(f)$ corefree, c.f. the proof of Lemma (5.4).) Since each $H(f)$ is also the type of canonical subgroup needed in the proof of Theorem 2* to satisfy the Spectrum Lemma, we need only show that different functions f yield inequivalent canonical subgroups of G.

As this ideal proof would run into considerable technical difficulty, we have made the illustrative hypothesis that $A_n = \mathbf{Z}_{k_n}$ and that the minimum prime power part of k_n tends to infinitely with n. (Thus the k_n can be different primes or different powers of the same prime.) Suppose k_n^* is another such sequence and let G^* be the corresponding wreath product of the $A_n^* = \mathbf{Z}_{k_n^*}$. Suppose further that

(*) There exists $N \geqslant 1$ such that for all $n, m \geqslant N$ k_n and k_m^* have no equal prime power parts.

In this case we can easily deduce that $G \not\cong G^*$ by considering the derived quotients Φ and Φ^* of these groups. Since $G' = \bigcup\{P_n' \mid n \geqslant 1\}$ where $P_n = A_n \cdots \mathrm{wr}\ (A_2\ \mathrm{wr}\ A_1) = B_n P_{n-1}$ (B_n = the base group) and $A_n = \mathbf{Z}_{k_n}$, we have $B_n/P_n' \cap B_n \cong \mathbf{Z}_{k_n}$ since it is quite well known that the intersection of the derived group with the base group in a restricted wreath product consists of all functions the sum of whose coordinates lies in the derived group of the base group. Hence $\Phi = G/G' \cong \oplus\{\mathbf{Z}_{k_n} \mid n \geqslant 1\}$ and $\Phi^* \cong \oplus\{\mathbf{Z}_{k_n^*} \mid n \geqslant 1\}$, whence $\Phi \not\cong \Phi^*$ under the assumption (*). Now we need only observe that we can choose 2^ω canonical subgroups $H(f_\alpha)$ ($\alpha < 2^\omega$) as defined above suitable for use in the Spectrum Lemma and such that if $\alpha \neq \beta < 2^\omega$ the numbers p_n and p_n^* defined by $p_n = k_i$ and $p_n^* = k_j$ where $i = f_\alpha(n)$ and $j = f_\beta(n)$ satisfy (*) (in place of k_n and k_n^*). Since in choosing a new $A_n = \mathbf{Z}_{k_n}$ we can make every prime power part of k_n as large as we please, there is no problem here. Thus no two $H(f_\alpha)$ and $H(f_\beta)$ are equivalent in G since they are not isomorphic.

6. Proof of Theorem 1* and related topics. To prove Theorem 1* we will give a simple combinatorial method to produce many highly transitive representations of an arbitrary countable e.c. group whose point-stabilizers are canonical subgroups of the type constructed in the proof of Theorem 1.

In an arbitrary e.c. group it is possible to strengthen high transitivity in two ways. We can replace finite maps by arbitrary relatively enumerable maps and we can build relative homogeneity with respect to certain nontrivial f.g. subgroups and their *regular* orbits, thus producing partial relative homogeneity. Before considering this let us consider relative homogeneity.

We apparently cannot construct a relatively homogeneous representation of an arbitrary countable e.c. group by the method of this section. The reasons are somewhat technical and involve our inability to guarantee the *properness* of the point-stabilizer subgroup if we try to homogenize over *irregular* orbits as it is constructed inductively. On the other hand we can homogenize the representation with respect to regular orbits while keeping the point-stabilizer free as we shall see in Lemma 5. Whereas, in view of (1.1), the point-stabilizer would itself be inner homogeneous in any relatively homogeneous representation of an e.c. group. This suggests that it would be difficult to control such a point-stabilizer in the construction. As already indicated, to homogenize irregular orbits, the cancellations which would occur (in certain HNN extensions) as we inductively enlarge the point-stabilizer do not even permit us to guarantee that it will remain a proper subgroup. (If we could guarantee this, we would have an "internal" construction of a maximal subgroup M satisfying (1.2) since the point-stabilizer could be further enlarged and made isomorphic to G). Kegel's "external" construction of relatively homogeneous representations is also inapplicable to e.c. groups because one cannot guarantee that the permutation extensions that one needs will remain in the skeleton of an arbitrary e.c. group.

Thus it is desirable to be able to say something about groups that possess relatively homogeneous representations which can be archetypally stretched before considering what can be done in e.c. groups. To this end we will state a result to be elaborated in a sequel to this paper. Kegel's method of coset-extensions and permutation-adjunctions suffices to build many countable homogeneous groups which possess at least one relatively homogeneous representation which can be archetypally stretched by tree-limits in 2^ω subgroup-incomparable ways. In fact one could even say recursion-theoretically how complicated the skeleton becomes. However, we can do better than this in the following theorem.

(6.0) THEOREM. *Suppose G is a countable group such that, for all f.g. $A \subset B \subset G$, the HNN extension $\langle B, t : t^{-1}at = a (a \in A)\rangle$ is embeddable in G over B. Then there are 2^ω isomorphisms τ_β: $G \to \mathrm{Sym}(\omega)$ $(\beta < 2^\omega)$ such that every $(\omega, \tau_\beta(G))$ is relatively homogeneous and if $\alpha \neq \beta < 2^\omega$ then* $\mathrm{skel}(\omega, \tau_\alpha(G))$ *is not contained in* $\mathrm{skel}(\omega, \tau_\beta(G)$. (*Thus distinct representations are mutually nonembeddable.*) *Further, every $(\omega, \tau_\alpha(G))$ has an archetypal tree-limit extension to power 2^ω within* $\mathrm{Sym}(\omega)$. *If G also satisfies the hypothesis of Theorem* 2, *then every $(\omega, \tau_\alpha(G))$ as above has 2^ω subgroup-incomparable archetypal tree-limit extensions to power 2^ω within* $\mathrm{Sym}(\omega)$ (*as in the conclusion of Theorem* 2).

Notice that many countable e.c. groups will satisfy the above hypotheses. The differentiation of the permutation skeletons in the above theorem is accomplished using recursion-theoretic invariants; the method of proof is similar to Kegel [7] and to [4] rather than the internal combinatorial method of this section.

We will prove a lemma which allows the construction of many "homogeneous" maximal subgroups in an arbitrary countable e.c. group. We are restricted, as delineated above, to homogenizing over *regular* orbits only, so we do not obtain

relatively homogeneous representations; however, we do obtain Theorem 1* and more.

(6.1) DEFINITION. A permutation group (X, G) is *regularly homogeneous* provided that it satisfies the definition of relative homogeneity for all finite mappings of *regular orbits* of f.g. subgroups.

Thus, a regularly homogeneous group is highly transitive since evey orbit of 1 is regular.

LEMMA 5. *If G is any countable e.c. group then by playing the following game we can construct a proper subgroup $H = \bigcup_{n=1}^{\infty} H_n \subset G$ such that the right regular representation of G on the right cosets of H is regularly homogeneous. At our play (the first player) of the game we are given a f.g. subgroup $H_n \subset G$ and finitely many regular orbits $H_n x_1 A_1, \ldots, H_n x_k A_k$ of f.g. subgroups with solvable word problems $A_1, \ldots, A_k \subset G$ on cosets of H_n. (This means that $H_n x_j A_j$ is a regular orbit of A_j under right translations, that is, $x_j A_j x_j^{-1} \cap H_n = 1$.) On this play of the game we are allowed to*

(1) *Enlarge the list of regular orbits by choosing finitely many more satisfying the above conditions, and*

(2) *Select any f.g. subgroup $K_n \supset H_n$ with $K_n \subset G$ such that for every orbit on our (enlarged) list we have $x_j A_j x_j^{-1} \cap K = 1$.*

(We can begin the game with $H_1 = 1$ and make our first selections.)

The second player (the group G) chooses a f.g. subgroup $H_{n+1} \supset K_n$ with $H_{n+1} \subset G$. This subgroup will always satisfy

(i) *For every orbit $H_n x_j A_j$ on our (enlarged) list we have $x_j A_j x_j^{-1} \cap H_{n+1} = 1$ (i.e., all orbits on our list remain regular orbits with respect to H_{n+1}),*

(ii) *If K_n has a solvable word problem then $H_{n+1} = K_n * F$ (free product) where F is a f.g. free group, and*

(iii) *If the primes dividing the orders of elements of K_n form a recursive set, then exactly the same primes are involved in H_{n+1}.*

We further have

Addendum A. *If K_n is always chosen to have a solvable word problem, then we can drop the requirement that the subgroups A_j on our list must have solvable word problems.*

Addendum B. *If H_n always has a solvable word problem and K_n is always chosen so that H_n is decidable in K_n then the subgroup $H = \bigcup\{H_n \mid n \geqslant 1\}$ will satisfy: for all f.g. $P \subset G$, $P \cap H = P \cap H_n$ for some $n \geqslant 1$.*

Addendum C. *If K_n satisfies a finite list of conditions of the form $y \notin K_n x_1 B \cup \cdots \cup K_n x_j B$ where $y, x_1, \ldots, x_j \in G$ and $B \subset G$ is f.g., then H_n will also satisfy each of these conditions (in place of K_n).*

We will use Lemma 5 to prove the following theorem.

THEOREM. 5. *Suppose G is a countable e.c. group and let K be any countable locally free group such that for all f.g. $A \subset K$ the free product $A * \mathbf{Z}$ is embeddable in G over A. Then there exists a regularly homogeneous representation $f\colon G \to \mathrm{Sym}(\omega)$ such that*

(i) *If $H \subset G$ is a point-stabilizer under f, then $H \cong K$;*

(ii) *The permutational skeleton of* $(\omega, f(G))$ *is always the same* (*as* K *varies*); *and*

(iii) *For all* $A \subset B \subset G$ *with* A, B *f.g. and* $A \cap H = 1$ *there exists an isomorphism* $g\colon B \to G$ *with* $g(a) = a$ $(a \in A)$ *and* $g(B) \cap H = 1$.

Notice that the representations of Theorem 5 satisfy the condition of partial relative homogeneity with respect to orbits of arbitrary f.g. *periodic* subgroups P because every orbit HxP is regular since $xPx^{-1} \cap H = 1$ since H is torsion-free. Condition (iii) guarantees that many other regular orbits exist besides these.

We cannot hope to strengthen regular homogeneity in Theorem 5 to relative homogeneity since relatively homogeneous groups with the same permutation skeleton are isomorphic as permutation groups if they are inner homogeneous.

The representations of Theorem 5 might not have archetypal tree-limit enlargements. To guarantee this we must further complicate the point-stabilizer H by making it a canonical subgroup. This can be done in the e.c. group G while keeping H torsion-free, but we only know how to obtain one archetypal tree-limit enlargement in the absence of elements of finite order.

To obtain *many* archetypal extensions our presently available technique (which in all likelihood can be improved and used in the torsion-free case as well) demands the introduction of a single prime p into H. However we want to obtain many such representations with nonembeddable point-stabilizers as Theorem 1* demands, and the only convenient way to do this is to vary the sets of primes involved in the point-stabilizers. Thus we will be content to prove the following result which contains Theorem 1*.

THEOREM 6. *Let* G *be a countable e.c. group and* π *a nonempty set of primes. Then there is a regularly homogeneous representation* $\tau\colon G \to \mathrm{Sym}(\omega)$ *such that* $(\omega, \tau(G))$ *has* 2^ω *archetypal tree-limit enlargements to power* 2^ω *in* $\mathrm{Sym}(\omega)$ *satisfying the uncountable subgroup-incomparability property* (iii) *of Theorem* 1. *The point-stabilizer* $H \subset G$ *under* τ *has an element of prime order* p *if and only if* $p \in \pi$, *and property* (iii) *of Theorem* 5 *is also satisfied by* H *provided* $A \subset B \subset G$ *have solvable word problems.*

Notice that in Theorem 5 we can obtain 2^ω nonembeddable locally free point-stabilizers K by putting $K(S) = *\{A_p \mid p \in S\}$ where S is a set of primes and $A_p = \{n/p^k \mid n, k \in \omega\}$ is the p-divisible subgroup of the additive rationals since no nontrivial element of $K(S)$ is infinitely divisible by any prime $q \notin S$.

We had hoped to use similar divisibility criteria in Theorem 6 to distinguish the point-stabilizers while only introducing a single prime p into them (that is, making them all p-groups (but not periodic ones) for the same prime p). However, there are combinatorial difficulties in doing this which we will address at a later date as well as the problem of keeping the permutation skeletons fixed (which could be done for special e.c. groups G if divisibilities could be used to distinguish the point-stabilizers).

The preceding theorems can be strengthened by allowing certain infinite maps f of orbits of f.g. subgroups $A \subset G$ to be induced by elements of $C_G(A)$ as in the definition of relative homogeneity. In an e.c. group this is particularly appropriate

because we can arrange for elements of G to induce arbitrary *relatively enumerable* maps of regular orbits (and the same idea is used to complicate the permutation skeletons in Theorem (6.0)). We will state this result for Theorem 1*. These matters will be pursued in a sequel to this paper where the Ziegler-Shelah construction **[12]** can also be brought into the picture.

THEOREM 1**. *In Theorem* 1* *all the representations* τ_β: $G \to \mathrm{Sym}(w)$ *can be made regularly homogeneous and endowed with the following "relative enumeration homogeneity" property. Suppose* $K \subset G$ *is f.g. and* $\{x_i\} \subseteq K$, $\{y_i\} \subseteq K$ *are sequences which are enumeration reducible to the set of all relations holding in some f.g. subgroup* $M \subset G$ *(with respect to some finite set of generators of* $M \supset K$*) and such that for all* $i, j \geqslant 1$ *we have* $Hx_i = Hx_j$ *iff* $Hy_i = Hy_j$ *where* $H \subset G$ *is a point stabilizer under* τ_β. *Letting* $\tau_\beta(H)$ *be the point-stabilizer of* $\mu \in \omega$, *there exists* $t \in G$ *such that for all* $j \geqslant 1$ *we have*

$$(\mu)\tau_\beta(x_j)\tau_\beta(t) = (\mu)\tau_\beta(y_j), \quad \textit{that is}, \quad x_j t y_j^{-1} \in H.$$

The point-stabilizers H *also satisfy property* (iii) *of Theorem* 5 *for subgroups* $A \subset B$ *having solvable word problems; in fact we can arrange that* $C_G(A) = \langle \varphi(B) \mid \varphi$ *as in* (iii) *of Theorem* 5$\rangle$.

(For a discussion of enumeration reducibility in e.c. groups see **[6]**.) Note that the "relative enumeration homogeneity" property implies high transitivity since every finite subset of G is ipso facto recursively enumerable.

The weakening of condition (iii) of Theorem 5 to subgroups with solvable word problem seems unavoidable; however, the final property given in Theorem 1** shows that the subgroups B of Theorem 5(iii) can be "deeply embedded" into G, thus giving considerably more force to the regular homogeneity property.

It also turns out that the method of proving Lemma 5 establishes homogeneity of *many* irregular orbits as well, but this is best left to be discussed after the proof.

PROOF OF LEMMA 5. We will use the

SUBLEMMA. *Suppose* $H \subset G$ *are groups and* $Hx_1A, \ldots, Hx_nA$ *and* $Hy_1A, \ldots, Hy_nA$ *are regular orbits of a subgroup* $A \subset G$ *acting on the right cosets of* H, *that is,* $zAz^{-1} \cap H = 1$ $(z = x_j, y_j,\ 1 \leqslant j \leqslant n)$. *Further assume that* $x_1, \ldots, x_n$ *are distinct* (H, A)*-double coset representatives and similarly for* $y_1, \ldots, y_n$. *Let* $K = (G, t : t^{-1}at = a\ (a \in A))$ *be the HNN extension and* $q_j = x_j t y_j^{-1}$. *Then* $F = \langle q_1 \rangle * \cdots * \langle q_n \rangle$ *is a free group,* $H * F \subset K$, *and* $(H * F) \cap G = H$.

PROOF. Let $w \in H * \langle q_1 \rangle * \cdots * \langle q_n \rangle$ (the abstract free product) be a reduced word. We will show that $w \neq 1$ in K. This will hold by Britton's Lemma provided that no portion of w has the form tct^{-1} or $t^{-1}ct$ with $c \in A$. To check this we examine sequential occurrences of t in segments of the form $\sigma = q_i^\varepsilon q_j^\delta$ and $q_i^\varepsilon z q_j^\delta$ where $\varepsilon\delta = -1$ and $1 \neq z \in H$. In the first type of segment we have $i \neq j$ since $\varepsilon = -\delta$ and w is freely reduced. So $\sigma = x_i t(y_i^{-1}y_j)t^{-1}x_j^{-1}$ or $y_i t^{-1}(x_i^{-1}x_j)ty_j^{-1}$ and these segments are K-reduced since the $\{x_i\}$ and $\{y_i\}$ belong to distinct left cosets of A. In the second type we have $\sigma = x_i t(y_i^{-1}zy_j)t^{-1}x_j^{-1}$ or $y_i t^{-1}(x_i^{-1}zx_j)ty_j^{-1}$ where

$1 \neq z \in H$ and possibly $i = j$. If $i \neq j$, then $y_i^{-1}zy_j \in A$ would imply $Hy_jA = Hy_iA$ contrary to y_i and y_j belonging to distinct double cosets by hypothesis. If $i = j$, then $y_i^{-1}zy_i \notin A$ because $y_iAy_i^{-1} \cap H = 1$ by our regularity hypothesis.

Since we have shown that no stable letter reductions can occur in the freely reduced word w, the final conclusion that $(H * F) \cap G = H$ also follows from Britton's Lemma. □

We need to show how the group G (player #2) can use the above Sublemma on its plays to build the regular homogeneity property into $H = \bigcup\{H_n \mid n \geqslant 1\}$. Suppose that $K_n \supset H_n$ has been chosen and let f be an arbitrary permutation isomorphism among finitely many *regular orbits* K_nx_jA, K_ny_jA $(1 \leqslant j \leqslant m)$ of the same f.g. subgroup $A \subset G$ on right cosets of K_n. Thus, let us suppose that $f(K_nx_ja) = K_ny_jf_j(a)$ where $a, f_j(a) \in A$. (We assume that the $\{K_nx_jA\}$ are distinct double cosets and similarly for the $\{K_ny_jA\}$ $(1 \leqslant j \leqslant m)$.) We first observe that every f_j: $A \to A$ must be equivalent to a left-translation since it centralizes the right regular action of A, that is, for all $a, c \in A$ we have $K_ny_jf_j(a)c = K_ny_jf_j(ac)$. So, putting $f_j(1) = b_j$ we have $f(K_nx_ja) = K_ny_jb_ja$ $(a, b_j \in A,\ 1 \leqslant j \leqslant m)$. Thus, without loss of generality we can redefine each y_j to be y_jb_j and we then have

$$f(K_nx_ja) = K_ny_ja \qquad (a \in A, 1 \leqslant j \leqslant m)$$

is an arbitrary permutation isomorphism of finitely many regular A-orbits on right cosets of K_n.

We will use the Sublemma to obtain a f.g. subgroup $H_{n+1} \supset K_n$ with $H_{n+1} \subset G$ such that there exists $\bar{t} \in C_G(A)$ such that for all $1 \leqslant j \leqslant m$ we have $H_{n+1}x_j\bar{t} = H_{n+1}y_j$, that is, the element $\bar{t}$ satisfies the relative homogeneity property with respect to the map f on the A-orbits corresponding to right cosets of H_{n+1}. We need also to check that $\bar{t}$ and H_{n+1} can be chosen to meet all the additional requirements of the game.

Put $Q = \langle K_n, A, B_i, x_j, y_j \mid 1 \leqslant j \leqslant m,\ 1 \leqslant i \leqslant k\rangle \subset G$ where the B_i are the conjugate subgroups corresponding to the regular orbits on our list (which intersect K_n trivially). According to the Sublemma we form the HNN extension $K = (Q, t\colon t^{-1}at = a\ (a \in A))$. We will pick an element $\bar{t} \in G$ which satisfies (over Q) the relations of K (since A is f.g.) as well as some recursively enumerable set of equations, inequations, and equational implications which t satisfies in K and which we are free to specify as we please. We will then define $\bar{q}_j = x_j\bar{t}y_j^{-1}$ and $H_{n+1} = \langle K_n, \bar{q}_1, \ldots, \bar{q}_m\rangle \subset G$. Then, clearly, the element $\bar{t}$ will induce the required permutation map f on the cosets $H_{n+1}x_jA$ $(1 \leqslant j \leqslant m)$.

The conditions $\bar{t}$ must satisfy are as follows.

(I) H_{n+1} must intersect trivally with finitely many f.g. subgroups of G having solvable word problems (the conjugate subgroups corresponding to the regular orbits on our list) given that each of these subgroups intersects K_n trivially.

(II) If K_n has a solvable word problem, then the subgroups in condition (I) need not have solvable word problems and $H_{n+1} = K_n * \langle\bar{q}_1\rangle * \cdots * \langle\bar{q}_n\rangle \subset G$ where the q_j have infinite order.

(III) Condition (iii) in Lemma 5 is satisfied.

For (I), since each subgroup B_i intersects $K_n * \langle q_1 \rangle * \cdots * \langle q_m \rangle = K_n * F \subset K$ trivially (by the Sublemma) we can form an HNN extension of K with stable letter τ which centralizes $K_n * F$ but such that $\tau^{-1} b \tau \neq b$ for all $1 \neq b \in B_i$ $(1 \leqslant i \leqslant k)$. Since each B_i has a solvable word problem we can enumerate all such inequations and so satisfy them with some $\bar{\tau} \in G$ (which also centralizes $H_{n+1} = \langle K_n, \bar{F} \rangle$), thus guaranteeing that $B_i \cap H_{n+1} = 1$. Condition (II) is satisfied similarly by observing that $K_n * F$ has a solvable word problem *if* K_n *does*. Thus we can arrange that $K_n * F \cong K_n * \langle \bar{q}_n \rangle * \cdots * \langle \bar{q}_m \rangle = H_{n+1}$. In the above argument we form a multiple HNN extension of K in which we have stable letters $\tau_1, \ldots, \tau_k$ where τ_i centralized B_i but $\tau_i^{-1} h \tau_i \neq h$ for all $1 \neq h \in H_{n+1} \cong K_n * F$ and we make $\bar{\tau}_i \in G$ satisfy these inequations also. To satisfy (III) in $H_{n+1} = \langle K_n, \bar{F} \rangle$ we use implications of the form $w^p = 1 \to w = 1$. By choosing $\bar{t}$ so that every such condition is satisfied for all $w \in H_{n+1}$ and every prime p *not* involved in K_n (since they are all satisfied in $K_n * F \subset K$) condition (iii) of Lemma 5 follows.

Now to prove Lemma 5 we observe that in choosing the mappings f at each play of the game we can arrange to consider *every* f.g. $A \subset G$ and *every* possible sequence $x_1, \ldots, x_m, y_1, \ldots, y_m \in G$ at some step. (Of course, many of these might not be regular orbits relative to K_n and so will be discarded.) However, defining $H = \bigcup \{ H_n \mid n \geqslant 1 \}$, this process obviously guarantees that G has the regular homogeneity property on the right cosets of H as required.

It remains to see how the Addendum cases B and C can be satisfied. In case B this is done by listing all f.g. subgroups $P_1, \ldots, P_n, \ldots$ of G and by choosing K_n and H_{n+1} so that we have for all $1 \leqslant j \leqslant n$, $P_j \cap K_n = P_j \cap H_{n+1} = P_j \cap H_n$. We can do this because H_n is decidable in K_n and since $H_{n+1} = K_n * \bar{F}$, K_n is also decidable in H_{n+1}. Hence we can arrange using a recursive set of inequations and equations that no element of $K_n - H_n$ or $H_{n+1} - K_n$ lies in any P_j $(1 \leqslant j \leqslant n)$ by obtaining an element of G which centralizes P_j but does not centralize any member of the above difference sets.

For the proof of Addendum C we need to consider the proof of the Sublemma and the use we have made of it. We can assume that for every relation $y \notin K_n x_1 B \cup \cdots \cup K_n x_j B$ on our (finite) list we have $y, x_1, \ldots, x_j \in Q$ and $B \subset Q$ (where $Q =$ the base group of the HNN extension K) since we can take $Q \subset G$ to be as large as we please. Suppose that $y \in (K_n * F) x_i B$ for some $1 \leqslant i \leqslant j$, say $y = w x_i b$ with $w \in K_n * F$, $b \in B$. If the stable letter t occurs in w, then we have a contradiction since $x_i b \in Q$ cannot cancel it. Thus $w \in K_n$ contrary to our hypothesis. Thus in choosing $\bar{t} \in G$ we can also satisfy the recursively enumerable set of inequations $y \notin \bigcup \{ \langle K_n, \bar{F} \rangle x_i B \mid 1 \leqslant i \leqslant j \}$ for each of the above relations on our list, as desired since $H_{n+1} = \langle K_n, \bar{F} \rangle$. (Notice that no recursiveness assumption on K_n was made here.) □

REMARKS. (1) If a certain orbit-map f is built into the action of some $\bar{t} \in G$ at some step (relative to H_{n+1} at that step) then $\bar{t}$ induces the same mapping on cosets of H, *even though the orbits* Hx_jA and Hy_jA *might not be regular* (although $K_n x_j A$ and $K_n y_j A$ were regular). Thus the proof exhibits the possibility of obtaining

relative homogeneity for many maps of irregular orbits, although it is not clear how systematically this could be done.

(2) The finite A-maps f built in at each step could be extended to any A-maps of regular orbits $f(K_n x_j) = K_n y_j$ of A such that the sequence (x_j, y_j) is enumeration reducible to the set of all relations holding in some f.g. subgroup of G (in particular $\{x_j, y_j \mid j \geqslant 1\}$ is contained in a f.g. subgroup of G) and such that $K_n x_i A = K_n x_j A$ iff $K_n y_i A = K_n y_j A$.

PROOF OF THEOREM 5. The representation $f: G \to \mathrm{Sym}(\omega)$ will be obtained by using Lemma 5 to construct an embedding $h: K \to G$ such that the regular representation of G on $h(K)$ has the properties required of $(\omega, f(G))$. We first ignore the requirement (ii) that the permutational skeletons be the same (regardless of K) and subsequently show how to fix the skeleton by making it universal.

At the nth step we have a f.g. $F_n \subset K$ and an embedding h_n: $F_n \to G$ such that $h_n(F_n) = H_n$. Thus H_n is a f.g. free group and so by Addendum A of Lemma 5 there is no restriction on the f.g. subgroups A_j which occur on the list of orbits $\{H_n x_j A_j\}$ (except that $x_j A_j x_j^{-1} \cap H_n = 1$). At this step we can extend this list as we please and choose a f.g. subgroup $G \supset K_n \supset H_n$ such that $K_n = H_n * L$ where L is f.g. and free and $x_j A_j x_j^{-1} \cap K_n = 1$ for every orbit on our list. (We will later use this freedom to enrich the permutational skeleton to make it universal.) By our hypothesis on K we can extend $h_n: F_n \to G$ to $h_n^*: F_n^* \to G$ where $F_n^* = F_n * M$ where $h_n^*(M) = L$, so that $h_n^*(F_n^*) = K_n$. Since K_n is free we have by Lemma 5 that $H_{n+1} = K_n * F$ where F is f.g. and free. So, as above, we obtain a f.g. $F_{n+1} \subset K$ and an embedding $h_{n+1} \to G$ which extends h_n^* and such that $h_{n+1}(F_{n+1}) = H_{n+1}$ and so the construction continues.

To guarantee that $K \cong H = \bigcup\{H_n \mid n \geqslant 1\}$ we must have $K = \bigcup\{F_n \mid n \geqslant 1\}$. This can be guaranteed by our choices of $K_n \supset H_n$ for odd values of n by choosing some f.g. $K \supset \overline{K}_n \supset F_n$. We can use $\overline{K}_n$ in place of F_n^* above (even though F_n might not be a free factor of $\overline{K}_n$) and obtain h_n^*: $\overline{K}_n \to G$ and $K_n = h_n^*(\overline{K}_n)$ as before. We can arrange that $x_j A_j x_j^{-1} \cap K_n = 1$ for every orbit on our list by the same argument used in the proof of Lemma 5 since $\overline{K}_n$ (the group we are embedding into G over H_n) has a solvable word problem since it is free. We choose the $\overline{K}_n$ so that $K = \bigcup\{\overline{K}_n \mid n \text{ odd}\}$.

To satisfy (iii) we merely need to enlarge our list of regular orbits in an appropriate way. This means that every pair $A \subset B \subset G$ of f.g. subgroups is considered at some play. If $A \cap H_n = 1$ we add $H_n A$ to our list as well as some orbit $H_n \overline{B}$ where $\overline{B}$ is an isomorphic copy of B over A (that is, $\bar{a} = a$ $(a \in A)$) such that $\overline{B} \cap H_n = 1$. (Such a $\overline{B}$ exists, as usual, because H_n has a solvable word problem.) Thus $\overline{B} \cap H = 1$ and (iii) will be satisfied.

To complete the proof of Theorem 5 we now only need to show how to use our choices for $K_n = H_n * L$ at even values of n to guarantee that $\mathrm{sk}(G:H) =$ the permutational skeleton of G acting on the right cosets of $H = \bigcup\{H_n \mid n \geqslant 1\} \cong K$ will always be the same. This is accomplished by observing that the A-orbit HxA is permutationally A-isomorphic to the regular action of xAx^{-1} on the right cosets of $xAx^{-1} \cap H$ since for all $a \in A$ we have $Hxa = Hx$ iff $xax^{-1} \in H$ iff

$$(xAx^{-1} \cap H)xax^{-1} = xAx^{-1} \cap H.$$

Now suppose $A \subset G$ is f.g.. By Addendum B to Lemma 5 (which applies to this proof since every K_n and H_n are f.g. free groups implying that H_n is decidable in K_n [**16**, 2.21]) we have $A \cap H = A \cap H_n$ for some $n \geqslant 1$. Thus every orbit HxA in $\mathrm{sk}(G:H)$ is permutationally A-isomorphic to the regular action of A on $A \cap H_n$ for some f.g. $A \subset G$ and some $n \geqslant 1$. In view of this we claim that we can always make $\mathrm{sk}(G:H)$ the same by satisfying

(*) If $F \subset G$ is f.g. and free and $A \subset G$ is f.g., then for all $m \geqslant 1$ there exist distinct (H, A)-double coset representatives $x_1, \ldots, x_m$ such that for all $1 \leqslant j \leqslant m$ we have $x_j A x_j^{-1} \cap H = x_j(A \cap F)x_j^{-1}$.

The proof that (*) guarantees the uniqueness of $\mathrm{sk}(G:H)$ is that every orbit in $\mathrm{sk}(G:H)$ is isomorphic to one of the forms given in (*), namely where $F = H_n$ for some n by the foregoing remarks. (Note that the A-orbit Hx_jA in (*) is A-isomorphic to the action of A on $A \cap F$.) But (*) guarantees that every such A-orbit occurs (A-isomorphically) infinitely many times in $\mathrm{sk}(G:H)$. Hence (*) implies that $\mathrm{sk}(G:H)$ depends only on the group-theoretic skeleton of G, as desired.

PROOF OF (*). Given $H_n, A, F \subset G$ and $m \geqslant 1$ as in (*) we will construct K_n so that the distinct orbits $K_n x_j A$ $(1 \leqslant j \leqslant m)$ are built into $\mathrm{sk}(G:H)$ at this step. We will arrange that $K_n \cap x_j A x_j^{-1} = H \cap x_j A x_j^{-1}$ for these orbits at subsequent steps as in Addendum B of Lemma 5—namely by adding these conjugate subgroups to our list $\{P_j\}$ of f.g. subgroups whose intersections with H are to be kept fixed at all subsequent steps.

We obtain $x_1, \ldots, x_m$ as follows. We first claim that there exist copies $A_1, \ldots, A_m \subset G$ of A and copies $F_1, \ldots, F_m \subset G$ of F and $t_1, \ldots, t_m \in G$ such that, for all $1 \leqslant j \leqslant m$, we have $t_j A t_j^{-1} = A_j$, $t_j F t_j^{-1} = F_j$, and $t_j(A \cap F)t_j^{-1} = A_j \cap F_j = A_j \cap K_n$ where $K_n = H_n * F_1 * \cdots * F_m \subset G$. This is done inductively. For example to obtain $\langle F_1, A_1, t_1 \rangle$ we need to guarantee that $H_n * t_1 F t_1^{-1} \subset G$ and $[(H_n * t_1 F t_1^{-1}) - t_1 F t_1^{-1}] \cap t_1 A t_1^{-1} = \varnothing$. This is possible in G since these relations are satisfied in $\langle H_n, F, A \rangle * \langle t_1 \rangle$ and they form a recursive set since H_n and F are both free. (To satisfy the intersection property we adjoin an element which centralizes $t_1 A t_1^{-1}$ but not any element in the above recursive difference set.) We then obtain t_2 by replacing H_n by $H_n * F_1$ in the above argument and so on.

To satisfy (*) we will define $x_j = t_j c_j$ where $c_j \in C_G(\langle F, A \rangle)$ is chosen appropriately. To guarantee that $x_1, \ldots, x_m$ lie in distinct (H, A)-double cosets we must have $x_{j+1} \notin Hx_1A \cup \cdots \cup Hx_jA$ for all $1 \leqslant j < m$. At the present step we can obviously arrange that $x_{j+1} \notin \langle K_n, A, x_1, \ldots, x_j \rangle \supset K_n x_1 A \cup \cdots \cup K_n x_j A$ by choosing $c_j \in C_G(\langle F, A \rangle)$ to lie outside of a suitable f.g. subgroup of G. However, at subsequent steps we cannot hope to show that $x_{j+1} \notin \langle K_n, A, x_1, \ldots, x_j \rangle$ implies $x_{j+1} \notin \langle H_{n+1}, A, x_1, \ldots, x_j \rangle$ because stable letter reductions might occur in the HNN extension used at this step. So at such steps we must use Addendum C of Lemma 5 to obtain $x_{j+1} \notin H_{n+1} x_1 A \cup \cdots \cup H_{n+1} x_j A$ from the inductive hypothesis that $x_{j+1} \notin K_n x_1 A \cup \cdots \cup K_n x_j A$. Thus at all subsequent steps in which H_{p+1} is obtained from K_p $(p \geqslant n)$ we will have accumulated finitely many such relations to satisfy in accordance with Addendum C.

Now we need to observe that we can also preserve finitely many such relations when we construct $K_p \supset H_p$. Thus, assuming inductively that the relation

$$(+) \qquad x_{j+1} \notin H_p x_1 A \cup \cdots \cup H_p x_j A$$

is satisfied, we must examine the requirements placed on our choice of K_p in this proof to observe that the condition $x_{j+1} \notin K_p x_1 A \cup \cdots \cup K_p x_j A$ (and finitely many others like it) can be satisfied. We first consider odd steps p. Here the construction of $K_p \subset G$ takes place by mirroring in G (via a homomorphism φ) relations which hold in some multiple HNN extension N of the group $P = \bar{K}_p *_{H_p} Q$ where $Q \subset G$ is a certain f.g. subgroup (which can be taken as large as we like). In this construction we have $\varphi(\bar{K}_p) = K_p$ and $\varphi(Q) = Q$ where $\varphi\colon N \to G$ is a homomorphism which we can take to preserve any recursively enumerable set of equations, inequations, and positive implications in P (and in N). So assuming that $x_1, \ldots, x_{j+1} \in Q$ and $A \subset Q$ (as we may) it follows from $(+)$ that in P we have $x_{j+1} \notin \bar{K}_p x_1 A \cup \cdots \cup \bar{K}_p x_j A$. Hence we can add this recursively enumerable set of inequations to those being satisfied in $\langle K_p, Q\rangle = \varphi(\langle \bar{K}_p, Q\rangle)$. At even steps p the argument is the same, except that we are mirroring relations in a multiple HNN extension of a group of the form $Q * \langle t\rangle$ and we have $K_p = \varphi(H_p * t^{-1}Ft)$ where $F \subset Q$. (Actually this procedure is done finitely many times to obtain K_p.) Since $Q * t^{-1}Ft = (H_p * t^{-1}Ft) *_{H_p} Q \subset Q * \langle t\rangle$ our conclusions in this case are the same. Thus each of our relations $x_{j+1} \notin H_p x_1 A \cup \cdots \cup H_p x_j A$ will be satisfied for all $p \geqslant n$ which establishes $(*)$ and Theorem 5. □

PROOF OF THEOREM 6. This proof is accomplished using Lemma 5 rather more easily than that of Theorem 5. In particular none of the Addenda are needed and we need not pay attention to recursiveness. Thus K_n and H_n will not have solvable word problems because of the kernels which must be adjoined to make H a canonical subgroup.

We will define $K_n = NH_n$ where N is a kernel of the type we need to adjoin to obtain a canonical subgroup suitable for use in the Spectrum Lemma as in the proof of Theorem 1. We recall that N is obtained from a group of the form $W = \mathbf{Z}_{p^k} \operatorname{wr} (A, J) = \langle f, J\rangle$ where $A \subset J \subset G$ are f.g. by choosing $t \in G$ to mirror certain relations satisfied by $f \in W$ and then putting $N = t^J$. (We will arrange to use every prime $p \in \pi$ at some step and no other primes.) These conditions guarantee that $NJ \supset NH_n = K_n$ will be a semidirect product. Here we can assume inductively that H_n is the point-stabilizer at this step in the construction of a canonical representation of a tree-limit system with archetype G. At this step we are free to make $N = t^J$ satisfy any other recursively enumerable set of relations satisfied in W and to embed NJ into G over J in any way we please. Here we need to do two things. We will make N a p-group by adding all relations $w^{p^k} = 1$ where $w \in t^J$ since W satisfies these and we will embed NJ into G over J in such a way that $WH_n = K_n$ will satisfy condition (2) of Lemma 5. This is possible since $N \cap J = 1$ and the subgroups involved in condition (2) have solvable word problems and intersect H_n trivially. Since only finitely many primes will be involved in K_n the group $K_n \supset H_n$ will be chosen to involve only these primes also and $H = \bigcup\{H_n \mid n \geqslant 1\}$ will be a canonical subgroup of the type desired. □

ADDED IN PROOF. The material in §6 and its extensions (some of which we have not mentioned here) is being supported by NSF Grant DMS-8701009. Many of these will be presented in the forthcoming paper *Highly transitive Jordan representations of groups* now in preparation.

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