UNIVALENT FUNCTIONS IN $H \cdot \overline{H}(D)$

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Abstract. Functions in $H \cdot \overline{H}(D)$ are sense-preserving of the form $f = h \cdot \overline{g}$ where $h$ and $g$ are in $H(D)$. Such functions are solutions of an elliptic nonlinear P.D.E. that is studied in detail especially for its univalent solutions.

1. Introduction. Let $D$ be a domain of $\mathbb{C}$, and $H(D)$ the set of all analytic functions defined on $D$ endowed with the topology of normal (locally uniform) convergence. Denote by $H \cdot \overline{H}(D)$ the set of all complex-valued mappings $f$ defined on $D$ of the form

$$f = H \cdot \overline{G}; \quad H \text{ and } G \text{ are locally in } H(D)$$

which are open and preserve the orientation. Such a mapping satisfies the nonlinear elliptic differential equation

$$f_z = \left[ a \cdot \frac{f}{f'} \right] f$$

where

$$a \in H(D) \quad \text{and} \quad a(D) \subset U = \{ \zeta ; |\zeta| < 1 \}.$$

The motivation behind the study of such a class comes from the fact that for any sense-preserving harmonic function $u = Hx + Gx$, $Hx$ and $Gx$ in $H(D)$, $e^u$ is a nonvanishing function of $H \cdot \overline{H}(D)$. Thus, of particular interest are those functions of $H \cdot \overline{H}(D)$ that vanish in $D$, as their zeros correspond to some singularities of harmonic functions.

In §2 we study solutions of (1.2) with $a$ as in (1.3). By a solution we mean a complex-valued function in the Sobolev space $W^{1,2}_{\text{loc}}$ which satisfies (1.2) almost everywhere. For $a \equiv 0$ we are led to the set of nonconstant function in $H(D)$. However, for other functions, $a$, satisfying (1.3) we may have solutions which are not in $H \cdot \overline{H}(D)$. For instance $f(z) = z |z|^{2\alpha}$, $\text{Re}(\alpha) > -\frac{1}{2}$ and $f(1) = 1$ is a solution of (1.2) in $\mathbb{C}$ with $a \equiv \overline{\alpha}/(1 + \alpha)$. We then denote by $\mathcal{F}(a, D)$ the set of all nonconstant solutions of (1.2) in $D$, where the given function $a$ always satisfies (1.3). The relation between $\mathcal{F}(a, D)$ and $H \cdot \overline{H}(D)$ is finally established.

§3 is concerned with the univalent solutions of (1.2) with $a$ as in (1.3). It includes the characterization of the univalent functions of $\mathcal{F}(a, C)$.

§4 contains an example showing that in general, the Riemann Mapping Theorem fails in our case. Instead, we establish the Mapping Theorem from the unit disk into a bounded simply connected domain of $\mathbb{C}$, the boundary of which is locally connected.

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2. **Representation theorems.** Let $D$ be a domain of $\mathbb{C}$ and $f \in \mathcal{F}(a, D)$. Then $f$ is a nonconstant locally quasiregular mapping and therefore it is open and sense preserving. Denote by $Z(f)$ the zero set of $f$, i.e.,

$$Z(f) = \{z \in D; f(z) = 0\}.$$  

For $z_0 \in D \setminus Z(f)$, let $B(z_0, \rho) \subseteq D \setminus Z(f)$, where

$$B(z_0, \rho) = \{z; |z - z_0| < \rho\}.$$  

Since

$$\overline{(\log f)'_z} = a(\log f)'_z; \quad z \in D \setminus Z(F),$$

we may choose a branch of $\log f$ which is harmonic in $B(z_0, \rho)$ [4]. Observe that $f'/f$ and $\overline{f'/f}$ are in $H(B(z_0, \rho))$. Put

$$H(z) = f(z_0) \exp \left( \int_{z_0}^z \frac{f_1(s)}{f(s)} \, ds \right)$$

and

$$G(z) = \exp \left( \int_{z_0}^z \frac{f_\overline{s}(s)}{f(s)} \, ds \right),$$

for $z \in B(z_0, \rho)$. Then $f = H \cdot \overline{G} \in H \cdot \overline{H}(B(z_0, \rho)).$ Note that although $f'/f$ and $\overline{f'/f}$ are in $H((D \setminus Z(f)))$, yet $H$ and $G$ can be multivalued locally analytic functions.

Conversely, let $f$ be in $H \cdot \overline{H}(D)$ and $0 \notin f(D)$. Then $f = H \cdot \overline{G}$, $H$ and $G$ in $H(D)$, is open and preserves orientation. Therefore, $a = (f'/f)/(\overline{f'/f}) = (G'/G)/(H'/H)$ is in $H((D \setminus Z(H' \cdot G)))$ and $a(D \setminus Z(h')) \subset U$. Since $f$ is sense preserving, $H$ is not a constant, which implies that $Z(H'G)$ is a discrete set in $D$. Therefore, $a \in H(D)$ and $a(D) \subset U$.

Summarizing, we have the following lemma.

**Lemma 2.1.** Let $D$ be a simply connected domain of $\mathbb{C}$. A nonvanishing function $f$ is in $H \cdot \overline{H}(D)$ if and only if $f$ is in $\mathcal{F}(a, D)$ for some function $a$ satisfying (1.3).

Next, we shall investigate the behavior of a solution $f$ in $\mathcal{F}(a, D)$ at a point $z_0 \in D$ where $f$ vanishes. We start by noting that $Z(f)$ is discrete in $D$. Indeed, $f$ is a nonconstant locally quasiregular mapping and therefore it is continuous, open and light. By a theorem of Stoiloff it follows that $f$ can be represented as a composition of two functions

$$f = E \circ \chi$$

where $\chi$ is a locally quasiconformal homeomorphism on $D$ and $E \in H(\chi(D))$. The result follows.

**Lemma 2.2.** Let $f$ be in $\mathcal{F}(a, D)$. Suppose that $f(z_0) = 0$ and that $B(z_0, \rho) \setminus \{z_0\} \subset D \setminus Z(f)$. Then $f$ admits the representation

$$f(z) = (z - z_0)^n |z - z_n|^{2\beta} \cdot h(z) \cdot \overline{g(z)}; \quad z \in B(z_0, \rho),$$

where $n = \arg \left( \frac{(f/z)'_z}{(f/z)'_{\overline{z}}} \right)$.
where \( n \in \mathbb{N} \), \( \beta = \overline{na(z_0)}(1 + a(z_0))/(1 - |a(z_0)|^2) \) and therefore \( \text{Re}\{\beta\} > -n/2 \).

**Proof.** Since \( f_z/f \) and \( \overline{f_z/f} \) are in \( H(B(z_0, \rho)) \), \( h(z_0) \neq 0 \) and \( g(z_0) = 1 \).

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The main result of this section is

**Theorem 2.3.** Let \( D \) be a simply connected domain of \( \mathbb{C} \). If \( f \) is in \( H \cdot \overline{H}(D) \) then \( f \) is in \( \mathcal{F}(a, D) \) for some function \( a \) satisfying (1.3) such that \( a(z) \) is a rational number in \([0, 1)\) whenever \( z \in Z(f) \). Conversely, let \( f \) be in \( \mathcal{F}(a, D) \) and suppose that for each \( z_0 \in Z(f) \) we have \( a(z_0) = p(z_0)/q(z_0) \in [0, 1) \) where \( p(z_0) \in \mathbb{N} \cup \{0\} \), \( q(z_0) \in \mathbb{N} \), and \( q(z_0) - p(z_0) \) is a divisor of \( p(z_0) \). Then \( f \) is in \( H \cdot \overline{H}(D) \).

**Proof.** Let

\[
f \in H \cdot \overline{H}(D).
\]
Then \( f \) is not a constant and belongs to \( \mathcal{H}(a, D \setminus Z(f)) \) for some \( a \) as in (1.3). Since \( Z(f) \) is discrete in \( D \), the function \( a \) has an analytic continuation in \( D \) and \( a(D) \subset U \). Let \( z_0 \in Z(f) \). Then by (1.1) we have

\[
a(z_0) = \lim_{z \to z_0} \frac{(z - z_0)G'(z)/G(z)}{((z - z_0)H'(z)/H(z))}
\]

where \( p \) and \( q \) are the zero order of \( H \) and \( G \) at \( z_0 \), respectively. Conversely, let \( f \in \mathcal{H}(a, D) \) and suppose that for each \( z \in Z(f) \) we have \( a(z) = p(z)/q(z) \in [0, 1) \) where \( p(z) \in \mathbb{N} \cup \{0\} \), \( q(z) \in \mathbb{N} \), and \( q(z) - p(z) \) is a divisor of \( p(z) \). Fix \( \xi \in D \). If \( f(\xi) \neq 0 \), then by Lemma 2.1 \( f \in H \cdot \overline{H}(B(\xi, \rho)) \) whenever \( B(\xi, \rho) \subset D \). If \( f(\xi) = 0 \), then by Lemma 2.2, (2.2) holds with \( \beta = p/(q - p) \in \mathbb{N} \) and again: \( f \in H \cdot \overline{H}(B(\xi, \rho)) \) whenever \( B(\xi, \rho) \subset D \). Observe that if \( f = H_1 \cdot \overline{G}_1 = H_2 \cdot \overline{G}_2 \) on a disk \( B(\xi, \rho) \subset D \) and \( G_1(z_0) = G_2(z_0) \) then \( H_1 = H_2 \) and \( G_1 = G_2 \). \( D \) being simply connected, there are \( H \) and \( G \) in \( H(D) \) such that \( f = H \cdot G \in H \cdot \overline{H}(D) \).

3. Univalent functions in \( H \cdot \overline{H}(D) \). Let \( D \) be a simply connected domain of \( \mathbb{C} \), and \( z_0 \in D \). Then the following characterization follows from Theorem 2.3.

**Theorem 3.1.** Let \( f \) be a univalent mapping defined on \( D \) such that \( f(z_0) = 0 \). Then \( f \) is in \( H \cdot \overline{H}(D) \) if and only if \( f \in \mathcal{H}(a, D) \) for some \( a \) satisfying (1.3) such that \( a(z_0) = m/(1 + m); m \in \mathbb{N} \cup \{0\} \).

**Proof.** If \( f \in H \cdot \overline{H}(D) \) is univalent, then the exponent \( n \) in (2.2) is one and \( a(z_0) = m/(1 + m) \) where \( m \) is a nonnegative integer. The converse is covered by Theorem 2.3. \( \square \)

**Lemma 3.2.** Let \( D \) be a simply connected domain of \( \mathbb{C} \) and \( f \) a univalent function in \( \mathcal{H}(a, D) \). Then we have

(a) \( f_z(z) \neq 0 \) for all \( z \in D \) whenever \( f(z) \neq 0 \), and

(b) If \( f(z_0) = 0 \) then \( \lim_{z \to z_0} (z - z_0)f_z(z)/f(z) \) exists and is in \( \mathbb{C} \setminus \{0\} \).

Therefore \( (z - z_0)f_z/f \) is a nonvanishing function in \( H(D) \).

**Proof.** (a) Let \( f(z) \neq 0 \). Then \( \log f \) can be defined as a univalent harmonic mapping in a small disk around \( z \). It follows that \( (\log f)_z(z) = f_z(z)/f(z) \neq 0 \) and therefore \( f_z(z) \neq 0 \).

(b) Suppose that \( f(z_0) = 0 \). Then by Lemma 2.2 and the univalence of \( f \) we have

\[
f(z) = (z - z_0)[z - z_0]^{2\beta} h(z) \cdot \overline{g(z)}, \quad z \in B(z_0, \rho) \subset D,
\]

where \( h \) and \( g \) are as in Lemma 2.2 and \( \text{Re} \beta > -\frac{1}{2} \). Therefore we have

\[
\lim_{z \to z_0} (z - z_0)f_z(z)/f(z) = 1 + \beta \neq 0. \tag{3.1}
\]

**Lemma 3.3.** Let \( f_0 \in \mathcal{H}(a_0, D) \) be univalent and \( \alpha \in \{ \alpha \in \mathbb{C}; \text{Re} \{ \alpha \} > -\frac{1}{2} \} \). Then \( f = f_0 \cdot |f_0|^{2\alpha} \) is univalent and belongs to \( \mathcal{H}(a, D) \) where

\[
a = \frac{1 + \overline{\alpha}}{1 + \alpha} \left[ \frac{a_0 + \overline{\alpha}/(1 + \alpha)}{1 + a_0\alpha/(1 + \alpha)} \right]
\]

satisfies (1.3).
Proof. Direct calculations show that $\bar{f}_z = a(\bar{f}/f)f_z$ in $D$. Since $\text{Re}\{\alpha\} > -\frac{1}{2}$ we have $|\bar{a}/(1 + \alpha)| < 1$ and therefore $a$ satisfies (1.3). Next, $f$ is not a constant since $f_0$ is not a constant and therefore $f \in \mathcal{F}(a, D)$. The univalence of $f$ follows from the fact that $w|w|^2a$, $\text{Re}\alpha > -\frac{1}{2}$, is univalent in $C$. ■

In our next result we consider univalent solutions in $\mathcal{F}(a, C)$. By Liouville's Theorem we know that $a(z) \equiv a \in U$.

**Theorem 3.4.** A function $f \in \mathcal{F}(a, C)$ is univalent in $C$ if and only if

$$f(z) = \text{const}(z - z_0)|z - z_0|^2\beta; \quad \beta = \bar{a}(1 + a)/(1 - |a|^2)$$

and $z_0 \in C$.

**Proof.** Let $f$ be of the form (3.2). In Lemma 3.2 put $D = C$, $f_0(z) = (z - z_0)$, $a_0(z) = 0$, and $\alpha = \beta$. Then we get that $f \in \mathcal{F}(\beta/(1 + \beta), C)$ and is univalent in $C$. Conversely, let $f$ be univalent and in $\mathcal{F}(a, C)$, $a(z) \equiv a \in U$. Put $f = |f|^{-2\beta}/(1 + \beta)$. Then by Lemma 3.3 $\hat{f}$ is an entire univalent function and therefore $\hat{f}(z) = \text{const}(z - z_0)$. Solving for $f$ we get that $f = \text{const}(z - z_0)|z - z_0|^{2\beta}$, $\beta = \bar{a}(1 + a)/(1 - |a|^2)$. ■

Let now $D$ be a simply connected domain of $C$, $D \neq C$, and $f \in \mathcal{F}(a, D)$. If $0 \not\in f(D)$, then $\log f$ can be defined as a univalent harmonic mapping on $D$. Since such mappings have been extensively studied [1-4], we assume that $0 \in f(D)$. Denote by $\phi$ a conformal mapping from the unit disk $U$ onto $D$. If $f \in \mathcal{F}(a, D)$ then $f \circ \phi \in \mathcal{F}(\bar{a}, U)$ where $\bar{a} = a \circ \phi$. Therefore we may assume that $D = U$ and $f(0) = 0$. Furthermore, by applying the postmapping $cw|w|^2a$, $\alpha = -a(0)/(1 + a(0))$ and $c$ an appropriate constant, we may assume that $a(0) = 0$ and $f_z(0) = 1$. We then denote

$$S_M = \bigcup_{a \in A} \{ f \text{ univalent in } \mathcal{F}(a, U); f(0) = 0, f_z(0) = 1 \},$$

where $A$ denotes the set of all functions $a \in H(U)$ such that $a(U) \subset U$ and $a(0) = 0$. As a direct consequence of Theorem 2.3 we get that

$$S_M = \{ f = z \cdot h \cdot \bar{g} \in H \cdot \bar{H}(U); f \text{ univalent and } h(0) = g(0) = 1 \}.$$

Our first result concerning $S_M$ is

**Theorem 3.5.** $S_M$ is compact in the topology of normal convergence.

**Proof.** Let $f_n, n \in \mathbb{N}$, be in $S_M$. Then by considering an appropriate subsequence of $\{f_n\}_{n=-1}^{\infty}$ we may assume that the corresponding $\{a_n\}_{n=-1}^{\infty}$ converges to some function $a$ in $A$. By Schwarz' Lemma for $a_n$ we know that each $f_n$ is a $K_r$-quasiconformal mapping in $rU$ for all $r < 1$. By a well-known result on quasiconformal mappings we know that $f_n$ converges normally in $rU$ to a $K_r$-quasiconformal function $f \in \mathcal{F}(a, rU)$ for all $r < 1$. Therefore $f$ is in $S_M$. ■

The following lemmas are needed later on.

**Lemma 3.6.** For $f \in S_M$ we have

$$1/16 \leq \text{dist}(0, \partial f(U)) \leq 1.$$
PROOF. Since \( a(0) = 0 \) we have \( |f_z(z)| \leq |z||f_z(z)| \) for all \( z \in U \) and from (3.3) we deduce that \( f(z) = z + O(|z|^2) \) near zero. By Lemma 3.3 in [3] we conclude that

\[
|f(z)| \geq |z|/4(1 + |z|^2)
\]

for all \( z \in U \). In particular the disk \( \{ w; |w| < 1/16 \} \) is in \( f(U) \).

On the other hand

\[
\text{dist}(0, \partial f(U)) = \lim_{|z| \to 1} |f(z)| = \lim_{|z| \to 1} |h(z)g(z)| \leq |h(0)g(0)| = 1.
\]

**Lemma 3.7.** Let \( f = zhg \) be in \( SM \). Then \( s = zh/g \) is locally univalent in \( U \).

PROOF. By Lemma 3.2 we know that \( zf_z/f \) is a nonvanishing function in \( H(U) \). Since \( zs'/s = (1 - a)zf_z/f \) for some \( a \in A \), \( zs'/s \) does not vanish in \( U \). But \( s'(0) = 1 \); therefore \( O \notin s'(U) \) and the result follows.

**4. Mapping theorem.** In this section we look for an analogue of the Riemann Mapping Theorem. Let \( \Omega \neq C \) be a simply connected domain in \( C \) and let \( a \in H(U) \), \( a(U) \subset U \) be given. Fix a \( z_0 \in U \) and \( w_0 \in \Omega \). We are interested in the existence of a univalent function \( f \in \mathcal{F}(a, U) \), \( f(U) = \Omega \), normalized by \( f(z_0) = w_0 \) and \( f_z(z_0) > 0 \). Let us start with an example which will show that this problem is not solvable in general.

Suppose that we want to find a univalent mapping \( f \in \mathcal{F}(-z, U) \) normalized by \( f(0) = 0 \) and \( f_z(0) > 0 \) such that \( f \) maps \( U \) onto \( \Omega = \mathbb{C}\setminus(-\infty, -1] \). Assume that such a function exists. Then \( f = zhg \equiv s|g|^2 \in H\cdot\overline{H}(U) \), \( s'(0) > 0 \), and \( g(0) = 1 \). Furthermore, we have

(i) \( s \in H(U) \) and \( s \) is locally univalent in \( U \) (Lemma 3.6), and
(ii) \( \arg f/z = \arg s/z \) is a bounded harmonic function on \( U \).

We will show that \( s(z)/s'(0) = k(z) = z/(1 - z)^2 \). First, observe that \( s \) is univalent in \( U \). Indeed, \( s \circ f^{-1}(w) = w/|g \circ f^{-1}(w)|^2 = w \cdot p(w) \), where \( p(w) > 0 \) on \( f(U) \), is a continuous locally univalent function in \( f(U) \) and therefore maps each radial line segment \( \{ w = Re^{it}, 0 \leq R < R_0 \} \) in \( f(U) \) injectively onto \( \{ w = \rho e^{it}, 0 \leq \rho < \rho_0 \leq \infty \} \). Since \( f(U) \) is a starlike domain with respect to the origin, we conclude that \( s \circ f^{-1} \) is univalent to \( f(U) \). Hence \( s \) is univalent in \( U \). Now \( \lim_{r \to 1}s(re^{it}) = s(e^{it}) \) exists almost everywhere on \( \partial U \) and by (ii) we know that \( s(e^{it}) \) lies on the negative real axis almost everywhere. Therefore \( s(z)/s'(0) = k(z) \).

Next, we shall determine the function \( g \) such that \( f \in \mathcal{F}(-z, U) \). We need to solve

\[
(4.1) \quad f_z/f = g'/g = -zf_z/f = -zk'/k = zg'/g, \quad g(0) = 1.
\]

The unique solution of (4.1) is \( g(z) = (1 - z) \) and therefore we get that

\[
(4.2) \quad f = \text{const} z(1 - \bar{z})/(1 - z).
\]

Observe that \( f \) is univalent in \( U \), but maps \( U \) onto a disk and not \( \Omega \). In other words, there is no univalent mapping in \( \mathcal{F}(-z, U) \) such that \( f(0) = 0 \), \( f_z(0) > 0 \), and \( f(U) = \Omega \). However, we have the following Mapping Theorem.
Theorem 4.1. Let \( \Omega \) be a bounded simply connected domain of \( \mathbb{C} \) whose boundary is locally connected. Fix \( 0 \in \Omega \) and let \( a \in H(U) \) such that \( a(U) \subset U \) be given. Then there is a univalent function \( f \in \mathcal{F}(a, \Omega) \) having the following properties:

(i) \( f(U) \subset \Omega \), normalized at the origin by \( f(z) = cz|z|^{2\beta}(1 + o(1)), \) where \( \beta = a(0)(1 + a(0))/(1 - |a(0)|^2) \) and \( c > 0 \).

(ii) \( \lim_{z \to e^it} f(z) = \hat{f}(e^{it}) \) exists and is in \( \partial \Omega \) for all \( t \in \partial U \setminus E \), where \( E \) is a countable set.

(iii) For each \( e^{it_0} \in \partial U \), we have that

\[
\begin{align*}
\hat{f}_{\ast}(e^{it_0}) &= \text{ess lim}_{t \uparrow t_0} \hat{f}(e^{it}) \quad \text{and} \quad f_{\ast}(e^{it_0}) = \text{ess lim}_{t \downarrow t_0} \hat{f}(e^{it})
\end{align*}
\]

exist and are in \( \partial \Omega \).

(iv) For \( e^{it_0} \in E \), the cluster set of \( f \) at \( e^{it_0} \) lies on a helix joining the point \( f_{\ast}(e^{it_0}) \) to the point \( \hat{f}_{\ast}(e^{it_0}) \).

Remarks. (1) If \( a(0) = m/1 + m, m \in \mathbb{N} \cup \{0\} \), then \( f \) is in \( H \cdot \overline{H}(U) \).

(2) In the case where \( ||a|| = \sup_{z \in U} |a(z)| < 1 \), properties (i) and (ii) imply that \( f(U) = \Omega \).

(3) If \( e^{it_0} \in E \) and \( f_{\ast}(e^{it_0}) = f_{\ast}(e^{it_0}) = B \), then there are infinitely many helices joining \( A \) and \( B \). Our claim is that the cluster set of \( f \) at \( e^{it_0} \) lies on one of them. Thus, for example, the cluster set of

\[
f(z) = \frac{z(1 - \bar{z})}{(1 - z)} \exp\left(-2 \text{arg}\left(\frac{1 - iz}{1 - z}\right)\right)
\]

at \( z = 1 \) lies on the helix \( \gamma(\tau) = \exp[-\tau + i(\pi/2 + \tau)] \) joining the points \( f_{\ast}(1) = -e^{-\pi/2} \) and \( f_{\ast}(1) = -e^{3\pi/2} \), whereas the cluster set of \( f \) at \( z = -i \) is the straight line segment from \( f_{\ast}(-i) = -e^{-\pi/2} \) to \( f_{\ast}(-i) = -e^{3\pi/2} \).

Proof. Assume first that \( a(0) = 0 \). Let \( \phi \) be the conformal mapping from \( U \) onto \( \Omega \) normalized by \( \phi(0) = 0, \phi'(0) > 0 \). Denote by \( \Omega_n = \{ w = \phi(z); |z| < r_n \} \), \( r_n = n/(n + 1), n \in \mathbb{N} \). Then, there exists a univalent function \( f_n \in \mathcal{F}(a_n \equiv a(r_n z), U) \), mapping \( U \) onto \( \Omega_n \) such that \( f_n(0) = 0 \) and \( (f_n)_{\ast}(0) > 0 \). Indeed, consider \( F_n = (1/r_n)\phi^{-1} \circ f_n \). Then \( F_n \) has to satisfy the nonlinear elliptic equation

\[
(F_n)_{z} = a_n \int f \cdot \frac{\phi' \circ F_n}{\phi' \circ F_n} \cdot (F_n)_{z}; \quad F(0) = 0, \quad (F_n)_{z}(0) > 0
\]

and map \( U \) onto \( U \) univalently. This has a solution (see for example the proof of Theorem 5.1 in [4]) and therefore the existence of \( f_n \) follows. Next, we show the existence of a mapping \( f \) having the properties of the theorem.

Since \( \Omega \) is bounded, then by applying the diagonal procedure on the exhaustion of \( U \), we conclude that there is a subsequence \( f_{n_k} \) which converges normally to a function \( f \) satisfying (1.2) with the given \( a \in A \) and \( f(0) = 0 \). By Lemma 3.6, we have

\[
dist(0, \partial \Omega_1) \leq (f_n)_{z}(0) \leq 16 \dist(0, \partial \Omega).
\]
Therefore \( f_\ast(0) > 0 \) and \( f \) is univalent. Furthermore, by the argument principle for quasiconformal mappings we have that \( f(U) \subset \Omega \). Now, since each prime end of \( \partial \Omega \) is singleton, \( \phi \) has a uniformly continuous extension to \( \overline{U} \) and \( 0 \notin \phi(\partial U) \). Observe that the branch of \( \log(\phi/z) \), \( \log(\phi')(0) \in \mathbb{R} \), is harmonic in \( U \) and continuous on \( \overline{U} \). Therefore \( \log(\phi/z) \) is bounded in \( \overline{U} \). Likewise we claim that the branch of \( g = \log(f/z) \), \( \Im g(0) = 0 \) is bounded in \( \overline{U} \). To see this, let \( g_n = \log(f_n/z) \), \( \Im g_n(0) = 0 \) be defined as continuous harmonic functions in \( U \). We shall show that \( g_n \) are uniformly bounded. Indeed, each \( f_n \) is a \( K_{r_n} \)-quasiconformal mapping on \( U \) with \( K_{r_n} = (1 + r_n)/(1 - r_n) \) and \( f_n(U) = \Omega_n \) is bounded by an analytic Jordan curve. Hence \( f_n \) has a continuous univalent extension to \( \overline{U} \) and therefore \( g_n \) admits a continuous extension to \( \overline{U} \). Evidently \( \Re g_n = \log|f_n/z| \) are uniformly bounded since \( \Omega_n \) are uniformly bounded. As of \( \Im g_n \), there are nondecreasing continuous functions \( \tau_n(t) \) defined on \( \mathbb{R} \) by

\[
\arg\left[ f_n(e^{it})/e^{it} \right] = \arg\left[ \phi(r_ne^{it\tau_n(t)})/e^{it\tau_n(t)} \right] + \tau_n(t) - t
\]

which satisfy \( \tau_n(t + 2\pi) = \tau_n(t) + 2\pi \) for all \( t \in \mathbb{R} \). Therefore there are \( k_n \in \mathbb{Z} \) such that

\[
|\tau_n(t) - t - 2k_n\pi| \leq 2\pi
\]

or

\[
2(|k_n| - 1)\pi \leq |\tau_n(t) - t| \leq 2(|k_n| + 1)\pi.
\]

On the other hand, \( \int_0^{2\pi} \arg[f_n(e^{it})/e^{it}] \, dt = 0 \), which implies that there is a \( t_n \) such that \( f_n(e^{it_n})e^{-it_n} > 0 \) and therefore

\[
2(|k_n| - 1)\pi \leq |\tau_n(t_n) - t_n| = |\arg\left[ \phi(r_ne^{it\tau_n(t_n)})/e^{it\tau_n(t_n)} \right]|
\]

\[
\leq 2(|k_n| + 1)\pi.
\]

But \( \sup_{|z|=1} |\arg(\phi(z)/z)| = M < \infty \) implies that \( |k_n| \leq 1 + M/2\pi \). Finally from (4.3) and (4.4) we get that

\[
\Im g_n(z) = \arg\left[ f_n(z)/z \right] \leq 2M + 4\pi.
\]

This concludes the proof of our claim.

Now, \( \lim_{r \to 1} \log f(re^{it})/re^{it} \) and therefore \( \tilde{f}(e^{it}) = \lim_{r \to 1} f(re^{it}) \) exists almost everywhere. In fact \( \tilde{f}(e^{it}) \subset \partial \Omega \), since \( f_n \) is quasiconformal on \( U \) and therefore extends to a homeomorphism from \( \overline{U} \) onto \( \overline{\Omega}_n \). Fix \( \varepsilon, 0 < \varepsilon < 1 \), and consider a finite covering \( \bigcup_1 B(e^{it}, \varepsilon) \) of \( \partial U \). Let \( \gamma_j \) be a conformal mapping from \( U \) onto \( C_j = U \cap B(e^{it}, \varepsilon) \). Then \( 0 \notin f(B(e^{it}, \varepsilon)) \) and therefore \( F_j = \log f \circ \gamma_j \) can be defined as a univalent harmonic function from \( U \) onto \( K_j \subset \Omega \). By Theorem 3.5 in [4] we conclude that except for at most a countable set \( E_j \) the unrestricted limit \( \tilde{F}_j(e^{it}) = \lim_{z \to e^{it}} F_j(z) \) exists, is continuous and belongs to \( K_j \). Let \( E = \bigcup_j E_j \); then since each \( \gamma_j \) can be extended to a homeomorphism to \( \overline{U} \) we conclude that \( \tilde{f}(e^{it}) = \lim_{z \to e^{it}} f(z) \) exists, is continuous, and belongs to \( \partial \Omega \) for \( e^{it} \in \partial U \setminus E \). By the same theorem, at the points \( e^{i\theta} \) of \( E \), the one-sided essential limits of \( \log \tilde{f}(e^{i\theta}) \) exist, are different, and belong to \( \partial \Omega \); and finally, the cluster set at \( e^{i\theta} \) of \( E \) is a straight line.
segment joining \((\log \hat{f})^*(e^{i\theta})\) and \((\log \hat{f})^*(e^{i\theta})\). Therefore \(A_0 = \hat{f}(e^{i\theta})\) and \(B_0 = \hat{f}(e^{i\theta})\) exist and belong to \(\partial \Omega\) for \(e^{i\theta} \in E\). The cluster set of \(f\) at such a point lies on a single helix \(\exp(\lambda \log A_0 + (1 - \lambda)\log B_0), 0 < \lambda < 1\), joining \(A_0\) and \(B_0\) (depending on the corresponding values of \(\log A_0\) and \(\log B_0\)). If for some point \(e^{i\theta} \in E\), \(\hat{f}^*(e^{i\theta}) = \hat{f}(e^{i\theta})\), then \(\log A_0 = \log B_0 + 2\pi i\) and therefore the cluster set of \(f\) at \(e^{i\theta}\) is \(B_0\exp[(1 - \lambda)2\pi i], 0 < \lambda < 1\), i.e., a circle centered at the origin of radius \(|f^*(e^{i\theta})|\).

To remove the assumption \(a(0) = 0\), we apply what has been proved to the domain

\[
\tilde{D} = \{ w|w|^{2(\overline{a(0)}/(1 + a(0))}; w \in D \}
\]

with

\[
\tilde{a}(z) = (a(z) - a(0))/(1 - \overline{a(0)} a(z))
\]

to obtain a mapping \(\tilde{f}: U \rightarrow \tilde{D}\). Then

\[
f = \tilde{f} \cdot f \cdot 2a(0)(1 + a(0))/(1 - |a(0)|^2)
\]

will be the desired solution. ■

**References**


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