

REPRESENTATIONS OF ANISOTROPIC UNITARY GROUPS

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ABSTRACT. Let $SU(f)$ be the special unitary group of an anisotropic hermitian form f over a field k . Assume f represents only one norm class in k . The representations $\alpha: SU(f) \rightarrow SL(n, R)$ are characterized when R is a commutative ring with 2 not a zero divisor and $n = \dim f \geq 3$ with $n \neq 4, 6$. In particular, a partial classification of the normal subgroups of $SU(f)$ is given when k is the function field $C(X)$.

1. Introduction. We study representations $\alpha: U(f) \rightarrow GL(n, R)$ where $U(f)$ is the unitary or orthogonal group of an anisotropic form f over a field k with $\dim f = n \geq 3$ and R is a commutative ring with 2 not a zero divisor. Representations of the special unitary group $SU(f)$ will also be considered and, in particular, a partial classification of the normal subgroups of $SU(f)$ is obtained. However, the method only applies to a restricted class of fields k , including all real closed fields. The results are first established for R a local ring using a generalization of the fundamental theorem of projective geometry to construct a generalized place $\varphi: k \rightarrow R \cup \infty$ with the inverse image $A = \varphi^{-1}(R)$ a valuation ring of k . The kernel of the homomorphism $\varphi: A \rightarrow R$ is now an ideal \mathfrak{a} of A which need not be maximal. The kernel of the homomorphism α is a twisted congruence subgroup $U(\mathfrak{a}, \chi)$ of $U(f)$ defined with respect to \mathfrak{a} and a character $\chi: U(f) \rightarrow u(R)$, where $u(R)$ denotes the units of R . For general commutative rings R , the kernel of α is the intersection of the local kernels obtained after α is extended by localizing at the maximal ideals of R . These results generalize earlier work of Weisfeiler [16] and James [9, 10] where R is a field. Earlier, Borel and Tits [3] had studied abstract homomorphisms of isotropic algebraic groups and Tits [15] had considered representations of Lie groups. See [8] for a general survey of this area.

Let k be a field of characteristic not two with involution $*$, V a k -space of finite dimension $n \geq 3$ and $f: V \times V \rightarrow k$ an anisotropic hermitian form. Thus $f(x, x) = 0$ implies $x = 0$. Let $U(f)$ be the unitary group of f and $I(f)$ the subgroup of $U(f)$ generated by involutions. We allow the involution $*$ on k to be trivial, in which case f becomes a quadratic form and $U(f) = I(f)$ is an orthogonal group. The symmetry $\Psi(x): y \mapsto y - 2f(x, y)f(x, x)^{-1}x$ is an involution in $U(f)$ for each nonzero x in V . Henceforth we only consider forms f for which all symmetries in $U(f)$ are conjugate under $SI(f) = I(f) \cap SU(f)$. By Witt's Theorem this means that f can only represent one norm class (or square class when $*$ is trivial). Then necessarily f can be diagonalized as a sum of norms (or squares) and, since f is anisotropic, k will have characteristic zero. If f is a quadratic form then k must be

Received by the editors March 3, 1987.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 20G05, 51A10; Secondary 11E57.

This research was partially supported by the National Science Foundation.

pythagorean and formally real (see Lam [12]); this includes all real closed fields, as well as fields of Laurent series in several variables over \mathbf{R} or any other real closed field. When f is hermitian there are more possibilities. Now we can choose $k = K(a)$, with K formally real pythagorean and $a \notin K$ with $a^2 \in K$, or K any formally real field with pythagoras number two and $a^2 = -1$. In particular, our results now apply to the function field $\mathbf{C}(X)$ and this case will be considered in greater detail. Some of our results will also hold when f is an hermitian form over a quaternion division algebra of special type (see [10, p. 348]).

It is shown in [10, Proposition 3.1] that $U(f)$ is generated by $I(f)$ and any one-dimensional subgroup $U(1)$ of $U(f)$. Since $SI(f)$ is generated by products of two symmetries $\Psi(x)\Psi(y)$ where $\Psi(x)$ and $\Psi(y)$ are conjugate, it follows that $SI(f)$ is contained in the commutator subgroup $[U(f), U(f)]$. If $U(1)$ is the one-dimensional subgroup on the subspace kx of V then $\Psi(x) \in U(1)$. Hence the map $\det: U(f) \rightarrow k^*$ has kernel $SI(f)$. Thus $SI(f) = SU(f)$ is the commutator subgroup of $U(f)$ and

$$1 \rightarrow SU(f) \rightarrow U(f) \xrightarrow{\det} N_1(k) \rightarrow 1$$

is an exact sequence, where $N_1(k) = \{a \in k \mid a^*a = 1\}$ is the norm one subgroup of k^* . It is also easily seen that $SU(f)$ is perfect.

Our main result is the following.

THEOREM 1. *Let f be an anisotropic hermitian form which represents only one norm class and let G be a subgroup of $U(f)$ containing $SU(f)$. Let $\alpha: G \rightarrow GL(M)$ be a representation of G where M is a free module over a commutative local ring R with 2 a unit and $\alpha SU(f) \neq 1$. Assume $\dim f = \text{rank } M = n \geq 3$ and $n \neq 4, 6$. Then there exist*

- (i) a generalized place $\varphi: k \rightarrow R \cup \infty$ with $A = \varphi^{-1}(R) = A^*$ a valuation ring and $\varphi: A \rightarrow R$ a homomorphism with kernel \mathfrak{a} ,
- (ii) a free maximal A -lattice N in V of rank n with $U(N) = U(f)$,
- (iii) a φ -semilinear map $\beta: N \rightarrow M$ with kernel $\mathfrak{a}N$,
- (iv) a character $\chi: G \rightarrow \mathfrak{u}(R)$,
- (v) a twisted congruence subgroup $G(\mathfrak{a}, \chi)$ of G ,

such that

$$1 \rightarrow G(\mathfrak{a}, \chi) \rightarrow G \xrightarrow{\alpha} GL(M)$$

is an exact sequence. Moreover, for each $\sigma \in G$ and $x \in N$, $(\alpha\sigma)\beta(x) = \chi(\sigma)\beta\sigma(x)$. In particular, when $G = SU(f)$, the diagram

$$\begin{CD} N @>\sigma>> N \\ @V\beta VV @VV\beta V \\ M @>\alpha\sigma>> M \end{CD}$$

is commutative.

The theorem is false when $n = 4$ and $G = SU(f)$, see Artin [1, Theorem 5.23] since there are now nontrivial homomorphisms $\alpha: SO(f) \rightarrow SL(3, k)$ with kernel not a congruence subgroup. Probably the theorem is true when $n = 6$, but there are some technical difficulties at one point in our method of proof. However, if we

modify the assumptions on G to $I(f) \subseteq G$ and $\text{card } \alpha I(f) > 2$, then the theorem remains true for both $n = 4$ and 6 .

Theorem 1 can be used, with the help of a localization argument, to give the following partial classification of the normal subgroups of $SU(f)$ in terms of congruence subgroups.

THEOREM 2. *Let f be an anisotropic hermitian form which represents only one norm class. Assume $\dim f = n \geq 3$ and $n \neq 4, 6$. Let H be a nontrivial normal subgroup of $SU(f)$ such that $SU(f)/H$ is contained in $GL(n, R)$ for some commutative ring R with 2 not a zero divisor. Then H is the intersection of a family of congruence subgroups of the type $SU(\mathfrak{a}, 1)$ in Theorem 1.*

Special cases of Theorem 2 are known for orthogonal groups without the restriction $SO(f)/H \subseteq GL(n, R)$. When $k = \mathbf{R}$ the projective group $PSO(f)$ is simple, $n \neq 4$ (see [1]). Pollak [14] considered the field $k = \mathbf{R}((X))$ of formal Laurent series and Chang [5] generalized this to several variables. Archimedean ordered pythagorean fields are considered by Bröcker [4]. Probably their results can also be modified to unitary groups.

For specific fields k the results in Theorems 1 and 2 can be strengthened. In particular, if \mathbf{R} is contained in the fixed field of k , then the valuation on \mathbf{R} induced by the restricted place is trivial (Proposition 8).

We will consider unitary groups over the rational function field $k = \mathbf{C}(X)$, with involution induced by $i^* = -i$, in detail. For each complex number c the identity map $\varphi_c: \mathbf{C} \rightarrow \mathbf{C}$ extends to a unique place $\varphi_c: k \rightarrow \mathbf{C} \cup \infty$ by setting $\varphi_c(X) = c$. The associated valuation ring A_c is the set of rational functions $r(X) = g(X)/h(X)$ over \mathbf{C} with $g(X), h(X)$ relatively prime polynomials in $\mathbf{C}[X]$ and $h(c) \neq 0$. Fix an integer $m \geq 1$ and let $R(c, m)$ denote the local ring $\mathbf{C}[X]/(X - c)^m$. Then a generalized place $\varphi_{c,m}: k \rightarrow R(c, m) \cup \infty$ can be obtained as follows. If $r(X) \notin A_c$ set $\varphi_{c,m}(r(X)) = \infty$. Any $r(X) \in A_c$ can be expanded as a unique formal power series $r(X) = \sum_{j=0}^{\infty} a_j(X - c)^j$ with $a_j \in \mathbf{C}$. Set $\varphi_{c,m}(r(X)) = \sum_{j=0}^{m-1} a_j(X - c)^j$, viewed as an element in $R(c, m)$. The kernel of $\varphi_{c,m}: A_c \rightarrow R(c, m)$ is the ideal $(X - c)^m A_c$. Another place φ_{∞} on the field $\mathbf{C}(X)$ can be obtained by setting $\varphi_{\infty}(X) = \infty$; the associated valuation ring A_{∞} consists of all rational functions $r(X)$ with $\text{deg}(r(X)) \leq 0$. For each integer $m \geq 1$ there corresponds a generalized place $\varphi_{\infty,m}$. The kernel of the restriction of $\varphi_{\infty,m}$ to A_{∞} consists of all rational functions $r(X)$ with $\text{deg}(r(X)) \leq -m$.

Now let A be the intersection of a finite number of the valuation rings of $\mathbf{C}(X)$ just considered with $\varphi(X)$ real or infinite. Then A is a semilocal ring which can be described as follows. Choose a finite number of real numbers c_1, \dots, c_l (possibly none) and also possibly choose ∞ . Then A consists of all rational functions $r(X) = g(X)/h(X)$ with $h(c_i) \neq 0, 1 \leq i \leq l$, and also $\text{deg}(r(X)) \leq 0$ if ∞ is chosen. Let V be a vector space over $\mathbf{C}(X)$ with basis $u_1, \dots, u_n, n \geq 3$, and let f be the anisotropic hermitian form defined by $f(u_i, u_j) = \delta_{ij}, 1 \leq i, j \leq n$. Then $N = Au_1 \perp \dots \perp Au_n$ is a maximal lattice in V consisting of all $x \in V$ with $f(x, x) \in A$ (if any c_i is nonreal, N is not maximal). Hence $U(N) = U(f)$ where $U(N)$ consists of all $\sigma \in U(f)$ with $\sigma(N) = N$. Let \mathfrak{a} be an ideal of the principal ideal ring A and define $SU(\mathfrak{a})$ as the congruence subgroup of $SU(f)$ consisting of

all $\sigma \in SU(f)$ for which $(\sigma - 1_V)(N) \subseteq \mathfrak{a}N$. Then we have an exact sequence

$$1 \rightarrow SU(\mathfrak{a}) \rightarrow SU(f) \rightarrow SL(n, R)$$

where R is the commutative ring A/\mathfrak{a} . The ideal \mathfrak{a} is generated by a rational function $r(X)$ of the form

$$r(X) = (X - c_1)^{m_1} \dots (X - c_l)^{m_l} / (X - c)^{m+m_1+\dots+m_l}$$

with c_1, \dots, c_l, c all real and distinct; this gives the connection with the generalized places.

THEOREM 3. *Let H be a nontrivial normal subgroup of the special unitary group $SU(f)$ over the field $\mathbf{C}(X)$. Assume $n = \dim f \geq 3$, $n \neq 4, 6$, and $SU(f)/H \subseteq GL(n, R)$ for some commutative ring R with 2 not a zero divisor. Then H is a congruence subgroup $SU(\mathfrak{a})$.*

The above three theorems are a further contribution to the solutions of very general questions asked by Weisfeiler [11, XVII and 17, §7], although we have gone beyond his original framework in allowing the image group to be defined over a ring.

2. Collineations and projective geometry. Let R be a local ring with maximal ideal $\mathfrak{m} = \mathfrak{m}(R)$. In this section it is not necessary to assume that R is commutative or that 2 is a unit. Let M be a free R -module of finite rank $n \geq 3$. A *point* of M is a rank one direct summand of M and a *line* is a rank two direct summand of M . The *projective space* \mathbf{PM} of M is the set of all points of M . A *projective frame* of \mathbf{PM} is a set of n points P_1, \dots, P_n such that $M = P_1 + P_2 + \dots + P_n$. A special case of the above situation is a vector space V over a (skew) field k with associated projective space \mathbf{PV} .

DEFINITION. A *collineation* (or *projective homomorphism*) is a map $\pi: \mathbf{PV} \rightarrow \mathbf{PM}$ sending points of \mathbf{PV} to points of \mathbf{PM} such that

- (i) there exists a projective frame of \mathbf{PV} which is carried by π to a projective frame of \mathbf{PM} (and hence $\dim V = \text{rank } M$),
- (ii) if the images of the points P_1 and P_2 of \mathbf{PV} generate a line $\pi P_1 + \pi P_2$ in \mathbf{PM} , then any point P on the line $P_1 + P_2$ has image πP on the line $\pi P_1 + \pi P_2$.

If, moreover, in (ii) above, there always exists a point P_3 on the line $P_1 + P_2$ such that $\pi P_1 + \pi P_2 = \pi P_1 + \pi P_3 = \pi P_2 + \pi P_3$, then the collineation π is called *thick*.

If P_1 and P_2 are distinct points in \mathbf{PV} then $P_1 + P_2$ is always a line. However, this is false in \mathbf{PM} since the span of the points need not be a direct summand of M . This is the reason for the more careful phrasing in the definition of a collineation.

FUNDAMENTAL THEOREM. *Let $\pi: \mathbf{PV} \rightarrow \mathbf{PM}$ be a thick collineation. Then there exist*

- (i) a generalized place $\varphi: k \rightarrow R \cup \infty$ with $A = \varphi^{-1}(R)$ a valuation ring of k and the restriction $\varphi: A \rightarrow R$ a ring homomorphism,
 - (ii) a free A -module N in V with $\text{rank } N = \dim V$,
 - (iii) a φ -semilinear map $\beta: N \rightarrow M$,
- such that β induces π , namely, $\pi P = R\beta(P \cap N)$ for $P \in \mathbf{PV}$.

PROOF. Choose a basis u_1, \dots, u_n in V such that $\pi(ku_i) = Rv_i$, $1 \leq i \leq n$, is a projective frame for \mathbf{PM} . Thus $M = Rv_1 + \dots + Rv_n$. Fix j with $2 \leq j \leq n$.

Since Rv_1 and Rv_j span a line, it follows from the definition of a thick collineation that there exists a point $P_3 = k(u_1 + au_j)$ in PV , $a \neq 0$, with image $R(bv_1 + cv_j)$ satisfying

$$Rv_1 + Rv_j = Rv_1 + R(bv_1 + cv_j) = Rv_j + R(bv_1 + cv_j).$$

Hence b is a unit in R and we may assume $b = 1$. Also c is a unit in R . Replacing u_j by $a^{-1}u_j$ and v_j by $c^{-1}v_j$ we can normalize our choice of basis v_1, \dots, v_n for M such that

$$\pi(k(u_1 + u_j)) = R(v_1 + v_j), \quad 2 \leq j \leq n.$$

For each $a \in k$ the point $k(u_1 + au_j)$, $j \neq 1$, lies on the line $ku_1 + ku_j$ and hence its image is either $R(v_1 + bv_j)$ for some $b \in R$, or $R(pv_1 + v_j)$ for some $p \in \mathfrak{m}$. In the first case set $\varphi_j(a) = b$ and in the second case set $\varphi_j(a) = \infty$. Then for each $j \neq 1$ we have defined a map $\varphi_j: k \rightarrow R \cup \infty$ with $\varphi_j(0) = 0$ and $\varphi_j(1) = 1$. Let $A_j = \varphi_j^{-1}(R)$. We now prove that $\varphi_j = \varphi$ and $A_j = A$ are independent of j , that A is a valuation ring of k and $\varphi: A \rightarrow R$ is a ring homomorphism.

Fix $1 < i \neq j \leq n$ and let $a \in A_i$ and $b \in A_j$. Then by standard arguments it follows that the images of the points $k(u_1 + au_i + bu_j)$ and $k(au_i + u_j)$ are respectively, $R(v_1 + \varphi_i(a)v_i + \varphi_j(b)v_j)$ and $R(\varphi_i(a)v_i + v_j)$. Hence, for nonzero a, a' in A_i , the image of the point $k(u_1 + (a + a')u_i + u_j)$ lies on both of the lines $R(v_1 + \varphi_i(a)v_i) + R(\varphi_i(a')v_i + v_j)$ and $R(v_1 + \varphi_i(a)v_i + v_j) + Rv_i$. If $\varphi_i(a + a') = \infty$ then this image also lies on the line $R(pv_1 + v_i) + Rv_j$ where $p \in \mathfrak{m}$, which leads to a contradiction. Thus $\varphi_i(a + a') \neq \infty$ and A_i is closed under addition. It now easily follows that $\varphi_i(a + a') = \varphi_i(a) + \varphi_i(a')$.

Again take nonzero $a \in A_j$, $b \in A_i$. Since the images of ku_1 and $k(bu_i + u_j)$ span a line in PM , as do the images of $k(u_1 + au_j)$ and ku_i , it follows that the image of $k(u_1 + abu_i + au_j)$ is $R(v_1 + \varphi_j(a)\varphi_i(b)v_i + \varphi_j(a)v_j)$. Hence the image of $k(u_1 + abu_i)$ cannot be $R(pv_1 + v_i)$ and so necessarily $ab \in A_i$. Letting $b = 1$ it follows that $A_j \subseteq A_i$ and hence $A_i = A_j = A$ is closed under multiplication. Moreover, $\varphi_i(ab) = \varphi_j(a)\varphi_i(b)$ so that $\varphi_i = \varphi_j = \varphi$ and $\varphi(ab) = \varphi(a)\varphi(b)$. Note we still have not shown that A is a ring, namely $-1 \in A$.

Next consider $a \notin A$. Then $\pi(k(u_1 + au_2)) = R(pv_1 + v_2)$ where $p \in \mathfrak{m}$. Repeat all the previous arguments with u_1 and u_2 interchanged; there is no need to change the normalization of the basis, however, since it is already established that

$$\pi(k(u_2 + u_i)) = R(v_2 + v_i) \quad \text{for } i \neq 2.$$

We obtain a new set $B \subseteq k$, closed under addition and multiplication, and a map $\psi: B \rightarrow R$. For $c \in A$ the image of $k(u_1 + u_2 + cu_3)$ is $R(v_1 + v_2 + \varphi(c)v_3)$. If $c \notin B$, then $k(u_2 + cu_3)$ has image $R(pv_2 + v_3)$ where $p \in \mathfrak{m}$ and consequently the image of $k(u_1 + u_2 + cu_3)$ must be contained on the line $Rv_1 + R(pv_2 + v_3)$, which is a contradiction. Hence $A = B$ and $\varphi = \psi$. Again consider $a \notin A$. If $a^{-1} \notin A$, then $k(u_1 + au_2) = k(a^{-1}u_1 + u_2)$ forces $Rv_2 \equiv Rv_1 \pmod{\mathfrak{m}}$. Thus necessarily $a^{-1} \in A$; in particular $-1 = (-1)^{-1} \in A$. Hence A is a valuation ring and $\varphi: A \rightarrow R$ is a ring homomorphism.

Finally, define $N = Au_1 + \dots + Au_n$, a free A -module of rank n . Define $\beta: N \rightarrow M$ by $\beta(\sum a_i u_i) = \sum \varphi(a_i)v_i$. Then β is a φ -semilinear map. Let $x = \sum c_i u_i \in V$ with $c_i \in k$ not all zero. Since A is a valuation ring, there is a j with $c_j^{-1}c_i \in A$ for

all i . By standard arguments

$$\pi(kx) = R \left(\sum \varphi(c_j^{-1}c_i)v_i \right) = R\beta(kx \cap N)$$

completing the proof.

REMARK. Clearly N is not unique since it depends on the choice of the projective frames at the beginning of the proof. However, if k and R are commutative, a standard argument shows that the generalized place φ and its valuation ring A are unique; in general they are only determined up to conjugacy. Also, N and β are determined up to multiplication by a nonzero element of k and a unit of R , respectively. Note also that if $\varphi(a) = \infty$ then $\varphi(a^{-1}) \in \mathfrak{m}(R)$.

3. Involutions. Many of the results in this section are extensions of the work of Dieudonné [6, 7]. Let R be a local ring with 2 a unit, but not necessarily commutative, and M a free R -module of finite rank $m \geq 3$. Let σ be an involution in the general linear group $GL(M) = GL(m, R)$ and $P(\sigma) = \{x \in M | \sigma(x) = x\}$ and $N(\sigma) = \{x \in M | \sigma(x) = -x\}$ the positive and negative spaces of σ . Then $P(\sigma) \cap N(\sigma) = 0$ and $M = P(\sigma) + N(\sigma)$ is a direct sum so that both $P(\sigma)$ and $N(\sigma)$ are free modules. The involution σ is called *extremal*, or a 1-involution, if either $P(\sigma)$ or $N(\sigma)$ has rank one. In general, σ is an l -involution if either $P(\sigma)$ or $N(\sigma)$ has rank $l (\leq \frac{1}{2}m)$. Any set of mutually commuting involutions in $GL(M)$ contains at most 2^m elements and such a set can always be extended to a set of 2^m mutually commuting involutions in $GL(M)$ (see McDonald [13]). When R is commutative, a set of mutually commuting involutions in the special linear group $SL(M)$ can always be extended to a maximal set of 2^{m-1} involutions. All of the above holds for involutions in the unitary group $U(f)$ of an anisotropic hermitian form f , although now we also have $V = P(\sigma) \perp N(\sigma)$ is an orthogonal sum.

PROPOSITION 1. *Let f be an anisotropic hermitian form which represents only one norm class. Assume $n = \dim V$ is odd. Let $\alpha: SU(f) \rightarrow SL(m, R)$ be a nontrivial representation where $n \geq m \geq 3$ and R is a commutative local ring with 2 a unit. Then $n = m$ and α preserves l -involutions for $1 \leq l < \frac{1}{2}n$.*

PROOF. Observe first, since n is odd, that $-1_V \notin SU(f)$ and all extremal involutions in $SU(f)$ are of the form $-\Psi(x)$. Then $\alpha(-\Psi(x)) \neq 1_M$, for otherwise α is trivial. In fact, since any involution σ in $SU(f)$ is a product of mutually commuting extremal involutions, it is easily seen that $\alpha(\sigma) \neq 1_M$ provided $\sigma \neq 1_M$. Any involution in $SU(f)$ can be embedded into a maximal set S of 2^{n-1} mutually commuting involutions. Since $\text{card } \alpha(S) = \text{card } S = 2^{n-1}$, the image $\alpha(S)$ forms a set of 2^{n-1} mutually commuting involutions in $SL(m, R)$. Since $n \geq m$, this set is maximal and $n = m$. If $m = 3$ then α necessarily preserves extremal involutions. Hence we may assume $n \geq 5$. In S there are exactly n extremal involutions all conjugate under $SU(f)$. Any l -involution in S , where $1 < l < \frac{1}{2}n$, has at least $\frac{1}{2}n(n-1) > n$ conjugates in S . Since $\alpha: S \rightarrow SL(n, R)$ is injective and preserves conjugates, it follows that α preserves extremal involutions. Similarly, α preserves l -involutions.

REMARK. A slight modification of the above argument shows that a homomorphism $\alpha: U(f) \rightarrow GL(m, R)$ preserves extremal involutions for any $n \geq m \geq 3$ provided we assume $\text{card } \alpha I(f) > 2$; also, it is now not necessary to assume k and R are commutative (see James [9, Proposition 2.7]).

We next consider the case n even. Then $-1_V \in SU(f)$, but $SU(f)$ contains no extremal involutions.

PROPOSITION 2. *Let f be an anisotropic hermitian form which represents only one norm class. Assume $n = \dim V \geq 6$ is even. Let $\alpha: SU(f) \rightarrow SL(m, R)$ be a nontrivial representation where $n \geq m \geq 3$ and R is a commutative local ring with 2 a unit. Then $n = m$ and α preserves 2-involutions.*

PROOF. Any 2-involution in $SU(f)$ is of the form $\pm\Psi(x)\Psi(y)$ where $f(x, y) = 0$ and can be embedded into a maximal set S of 2^{n-1} mutually commuting involutions of $SU(f)$. Since $\alpha SU(f) \neq 1_M$ no noncentral involution can be killed by α . Hence the kernel of $\alpha: S \rightarrow SL(m, R)$ is contained in $\{\pm 1_V\}$. If $\alpha(-1_V) = -1_M$ then $\alpha(S)$ is a set of 2^{n-1} mutually commuting involutions in $SL(m, R)$ and hence $n = m$. If $\alpha(-1_V) = 1_M$ and m is even then $\alpha(S)$ and -1_M generate a set of 2^{n-1} mutually commuting involutions in $SL(m, R)$ and again $n = m$. If, however, m is odd we can only conclude $m \geq n - 1$ since now $-1_M \notin SL(m, R)$. Assume, if possible, $n = m + 1 \geq 6$. Now any noncentral element of S has at least $\frac{1}{2}n(n - 1)$ conjugates in S and hence its image in $\alpha(S)$ has at least $\frac{1}{4}n(n - 1) > n$ conjugates in $\alpha(S)$. Hence no involution in $\alpha(S)$ is extremal. Thus $\alpha(S)$ is not a maximal set of mutually commuting involutions in $SL(m, R)$ and, consequently, $m \geq n$. In all situations we now have $m = n$.

If $n = 6$, then all noncentral involutions in $SL(6, R)$ are 2-involutions and hence α preserves 2-involutions. Assume, therefore, $n = m \geq 8$. Then any 2-involution in S has exactly $\frac{1}{2}n(n - 1)$ conjugates in S while any other noncentral involution in S has at least $\binom{n}{4} > n(n - 1)$ conjugates in S . Since the kernel of $\alpha: S \rightarrow SL(n, R)$ has at most 2 elements, it again follows by counting conjugates that α preserves 2-involutions.

REMARK. Proposition 2 is false if $n = 4$ since there exist nontrivial representations of the type $\alpha: SO_4(f) \rightarrow SL(3, k)$ (see Artin [1, Theorem 5.23]). A slight modification of the above proof also shows that α preserves 4-involutions when $n \geq 8$.

The fact that $\alpha: SU(f) \rightarrow SL(n, R)$ preserves extremal involutions when n is odd sets up a correspondence between points in \mathbf{PV} and points in \mathbf{PM} and hence defines a map $\pi: \mathbf{PV} \rightarrow \mathbf{PM}$. However, when n is even the correspondence obtained from α is only between lines and we must get down to points by intersecting lines.

Assume $n \geq 8$ is even and let σ, τ be two commuting 2-involutions in $SL(n, R)$. Then (σ, τ) is called a *minimal pair* if $\sigma\tau$ is also a 2-involution. The same definition applies for involutions in $U(f)$. Denote by $L(\sigma)$ the rank two subspace of M (either the positive or negative space) associated with the 2-involution σ . Then $L(\sigma)$ can be viewed as a line in \mathbf{PM} . If (σ, τ) is a minimal pair then $P(\sigma, \tau) = L(\sigma) \cap L(\tau)$ is a rank one direct summand of M and hence a point in \mathbf{PM} . Note, if σ and τ are commuting 2-involutions, then $\sigma\tau$ is either $\pm 1_M$, a 2-involution or a 4-involution, according as the rank of $L(\sigma) \cap L(\tau)$ is 2, 1 or 0. When $n = 6$, the 2-involutions and 4-involutions cannot be distinguished; hence the assumption $n \geq 8$. A homomorphism $\alpha: SU(f) \rightarrow SL(n, R)$ with nontrivial image will now map a minimal pair (σ, τ) in $SU(f)$ to a minimal pair $(\alpha\sigma, \alpha\tau)$ in $SL(n, R)$. We can then set up a map $\pi: \mathbf{PV} \rightarrow \mathbf{PM}$ by putting $\pi P(\sigma, \tau) = P(\alpha\sigma, \alpha\tau)$. That π is well defined is an immediate consequence of the following result.

PROPOSITION 3. *Take the same assumptions as Proposition 2 except $n \geq 8$ and even. Let σ_1, σ_2 and σ_3 be three 2-involutions in $SU(f)$ whose associated lines in \mathbf{PV} have a unique common point, that is, $L(\sigma_1) \cap L(\sigma_2) \cap L(\sigma_3) = kx$. Then the lines $L(\alpha\sigma_1)$, $L(\alpha\sigma_2)$ and $L(\alpha\sigma_3)$ have a common point, which is unique if (σ_1, σ_2) is a minimal pair.*

PROOF. The subspace $U = L(\sigma_1) + L(\sigma_2) + L(\sigma_3)$ of V has dimension at most four. Hence there exist nonzero y_1, y_2 in V , orthogonal to U , with $f(y_1, y_2) = 0$. Then $\tau_1 = \Psi(x)\Psi(y_1)$, $\tau_2 = \Psi(x)\Psi(y_2)$ and $\tau_3 = \tau_1\tau_2$ are mutually commuting 2-involutions in $SU(f)$. Any pair of τ_1, τ_2, τ_3 form a minimal pair, in fact, $P(\tau_1, \tau_2) = kx$, $P(\tau_1, \tau_3) = ky_1$ and $P(\tau_2, \tau_3) = ky_2$. This constructs three points $P(\alpha\tau_1, \alpha\tau_2)$, $P(\alpha\tau_2, \alpha\tau_3)$ and $P(\alpha\tau_1, \alpha\tau_3)$ in \mathbf{PM} ; these three points must be distinct for otherwise $\alpha(\tau_1\tau_2\tau_3) = \alpha(1_V)$ is a 4-involution, which is absurd. By construction (σ_i, τ_j) , $1 \leq i \leq 3$, $1 \leq j \leq 2$, are minimal pairs; hence each $P(\alpha\sigma_i, \alpha\tau_j)$ is a point in \mathbf{PM} . Therefore, the line $L(\alpha\sigma_i)$ either contains the point $P(\alpha\tau_1, \alpha\tau_2) = L(\alpha\tau_1) \cap L(\alpha\tau_2)$, or lies in the plane (rank three direct summand) $L(\alpha\tau_1) + L(\alpha\tau_2)$ and hence $L(\alpha\sigma_i) \cap L(\alpha\tau_3) \neq 0$. But $\sigma_i\tau_3$ is a 4-involution. Hence $\alpha(\sigma_i\tau_3)$ is also a 4-involution and $L(\alpha\sigma_i) \cap L(\alpha\tau_3) = 0$. Thus necessarily the line $L(\alpha\sigma_i)$ goes through the point $P(\alpha\tau_1, \alpha\tau_2)$. This point need not be the unique point of intersection since the involutions $\alpha\sigma_i$ can all coincide. However, the additional assumption that (σ_1, σ_2) is a minimal pair ensures that the point is unique.

4. Thick collineations. Let f be an anisotropic hermitian form which represents only one norm class and $\alpha: SU(f) \rightarrow SL(n, R)$ a nontrivial homomorphism where R is a local ring with 2 a unit and $\dim f = n \geq 3$. We now use the results from the previous selection to construct a thick collineation $\pi: \mathbf{PV} \rightarrow \mathbf{PM}$ from which, with the help of the Fundamental Theorem, the main results will be obtained.

Consider first the case n odd. Then for each nonzero x in V we have $\alpha(-\Psi(x)) = E(X, y)$ where $E(X, y)$ is an extremal involution in $SL(n, R)$ with negative space X and positive space Ry . Define the map $\pi: \mathbf{PV} \rightarrow \mathbf{PM}$ by $\pi(kx) = Ry$.

PROPOSITION 4. *Let $\alpha: SU(f) \rightarrow SL(n, R)$ be a nontrivial homomorphism with $n \geq 3$ odd. Then the map $\pi: \mathbf{PV} \rightarrow \mathbf{PM}$ induced by α is a collineation.*

PROOF. Let u_1, \dots, u_n be any orthogonal basis of V and put $\alpha(-\Psi(u_i)) = E(U_i, v_i)$, $1 \leq i \leq n$. Since these involutions are mutually commutative it follows that $M = Rv_1 + \dots + Rv_n$ and each $U_i = \sum_{j \neq i} Rv_j$ (see McDonald [13]). Hence the map π sends the projective frame ku_1, \dots, ku_n of \mathbf{PV} to the projective frame Rv_1, \dots, Rv_n of \mathbf{PM} and consequently satisfies the first condition for a collineation.

Next let P_1 and P_2 be two points in \mathbf{PV} whose images πP_1 and πP_2 span a line in \mathbf{PM} . We must prove that any point P on the line $P_1 + P_2$ has image πP on the line $\pi P_1 + \pi P_2$. Assume first that $f(P_1, P_2) = 0$. Let $P_1 = ku_1$, $P_2 = ku_2$ and expand u_1, u_2 to an orthogonal basis u_1, u_2, \dots, u_n of V . Put $\alpha(-\Psi(u_i)) = E(U_i, v_i) = E_i$, $1 \leq i \leq n$, as above. Let $P = kx$ and $\alpha(-\Psi(x)) = E(X, y)$ so that $\pi P = Ry$. We must prove $y \in Rv_1 + Rv_2$. For each $i \geq 3$ the commutator $[\Psi(x), \Psi(u_i)] = 1_V$ and hence $E_i E(X, y) = E(X, y) E_i$. It follows that $E(X, y)(v_i)$ lies in the positive space Rv_i of E_i . If $E(X, y)(v_i) = v_i$ then $Ry = Rv_i$ and the positive space of the commuting involutions E_i and $E(X, y)$ coincide, which forces the contradiction

$\alpha(\Psi(x)\Psi(v_i)) = 1_M$. Hence, necessarily v_i lies in X , the negative space of $E(X, y)$, for $3 \leq i \leq n$. Let $y = \sum a_i v_i$ where $a_i \in R$. Then $E(X, y)E_i(y) = E_i E(X, y)(y) = E_i(y)$ and hence $E_i(y) \in Ry$, $3 \leq i \leq n$. Thus $cy = E_i(y) = 2a_i v_i - y$ for either $c = 1$ or $c = -1$. The first possibility again forces the contradiction $Ry = Rv_i$. Hence $c = -1$ and $a_i = 0$, $3 \leq i \leq n$. Thus $y \in Rv_1 + Rv_2$ and πP lies on $\pi P_1 + \pi P_2$ as required.

Finally assume $f(P_1, P_2) \neq 0$. Then $P_1 + P_2 = ku'_1 \perp ku'_2$ and the points $P'_1 = ku'_1$ and $P'_2 = ku'_2$ can be expanded to an orthogonal projective frame for \mathbf{PV} . As above $\pi P'_1 + \pi P'_2$ is a line in \mathbf{PM} and πP_1 and πP_2 must lie on this line. Hence $\pi P_1 + \pi P_2 = \pi P'_1 + \pi P'_2$ since both are rank two direct summands of M . Consequently, the collinearity property also holds when $f(P_1, P_2) \neq 0$.

We next consider the case $n \geq 8$ even. Then, as explained in the previous section, a nontrivial homomorphism $\alpha: SU(f) \rightarrow SL(n, R)$ induces a map $\pi: \mathbf{PV} \rightarrow \mathbf{PM}$ via its action on minimal pairs of 2-involutions in $SU(f)$.

PROPOSITION 5. *Let $\alpha: SU(f) \rightarrow SL(n, R)$ be a nontrivial homomorphism with $n \geq 8$ even. Then the map $\pi: \mathbf{PV} \rightarrow \mathbf{PM}$ induced by α is a collineation.*

PROOF. Let u_1, \dots, u_n be an orthogonal basis of V and put $\sigma_i = \Psi(u_i)\Psi(u_{i+1})$, $1 \leq i \leq n$, where we view the subscripts i modulo n . Then each (σ_{i-1}, σ_i) is a minimal pair in $SU(f)$ with associated point $P(\sigma_{i-1}, \sigma_i) = ku_i$. Let $L(\sigma_i)$ denote the 2-dimensional space of σ_i (now the negative space). Each $L(\sigma_i) = ku_i \perp ku_{i+1}$ can be viewed as a line in \mathbf{PV} . The image of the point ku_i under the map π is the point $P(\alpha\sigma_{i-1}, \alpha\sigma_i) = Rv_i$, say, of \mathbf{PM} . Then, from the definition of π , we have $Rv_i = L(\alpha\sigma_{i-1}) \cap L(\alpha\sigma_i)$. Hence $Rv_i + Rv_{i+1} \subseteq L(\alpha\sigma_i)$. Whenever $|i - j| \geq 2$ the product $\sigma_i\sigma_j$ is a 4-involution and since α preserves 4-involutions, it follows that $L(\alpha\sigma_i) \cap L(\alpha\sigma_j) = 0$. Hence $Rv_i \cap Rv_{i+1} = 0$ and, consequently, $L(\alpha\sigma_i) = Rv_i + Rv_{i+1}$. Since each $L(\alpha\sigma_i)$ is a direct summand of M , we now have $M = Rv_1 + \dots + Rv_n$ and π carries the projective frame ku_1, \dots, ku_n of \mathbf{PV} to the projective frame Rv_1, \dots, Rv_n of \mathbf{PM} .

If P_1 and P_2 are two orthogonal points in \mathbf{PV} whose images πP_1 and πP_2 generate a line $\pi P_1 + \pi P_2$ then we may take $P_1 = ku_1$ and $P_2 = ku_2$ as above. Let P be any point on the line $P_1 + P_2$ and choose a minimal pair (σ_1, τ) of 2-involutions, with σ_1 as before, and $P = P(\sigma_1, \tau)$. Then πP lies on $L(\alpha\sigma_1) = \pi P_1 + \pi P_2$ as required. The case $f(P_1, P_2) \neq 0$ can be handled as in Proposition 4.

PROPOSITION 6. *The collineations of Propositions 4 and 5 are thick.*

PROOF. Let $P_1 = ku_1$ and $P_2 = ku_2$ be points in \mathbf{PV} with images $\pi P_1 = Rv_1$ and $\pi P_2 = Rv_2$ spanning a line in \mathbf{PM} . We may assume $f(u_1, u_1) = f(u_2, u_2)$. Consider first $f(u_1, u_2) = 0$ and expand u_1, u_2 to an orthogonal basis u_1, \dots, u_n of V . By symmetry $R(v_1 + pv_2)$ is the image of $k(u_1 + u_2)$. If p is a unit then π is thick. Assume, therefore, $p \in \mathfrak{m}$. Let $k(u_1 - u_2)$ have image $R(a_1v_1 + a_2v_2)$. Since $u_1 + u_2, u_1 - u_2, u_3, \dots, u_n$ is an orthogonal basis of V it follows, as before, that $R(v_1 + pv_2)$ and $R(a_1v_1 + a_2v_2)$ must span $Rv_1 + Rv_2$. Hence a_2 is a unit which can be made 1. Again we are finished unless $a_1 = q \in \mathfrak{m}$. If n is odd, apply α to the identity $\Psi(u_2)\Psi(u_1 + u_2) = \Psi(u_1 - u_2)\Psi(u_2)$. Then, as in Proposition 4,

$E(U_2, v_2)E(X, v_1 + pv_2) = E(Y, qv_1 + v_2)E(U_2, v_2)$ where $U_2 = \sum_{i \neq 2} Rv_i$. Hence

$$\begin{aligned} E(Y, qv_1 + v_2)(-v_1 + pv_2) &= E(Y, qv_1 + v_2)E(U_2, v_2)(v_1 + pv_2) \\ &= E(U_2, v_2)E(X, v_1 + pv_2)(v_1 + pv_2) \\ &= E(U_2, v_2)(v_1 + pv_2) = -v_1 + pv_2 \end{aligned}$$

and, therefore, $-v_1 + pv_2 \in R(qv_1 + v_2)$. This is impossible since $q \in \mathfrak{m}$. When n is even a similar calculation can be made starting with the 2-involution identity $\sigma_2\Psi(u_3)\Psi(u_1 + u_2) = \Psi(u_1 - u_2)\Psi(u_3)\sigma_2$, where $\sigma_2 = \Psi(u_2)\Psi(u_3)$. The remaining case with $f(u_1, u_2) \neq 0$ is handled as in Proposition 4.

5. Representations over local rings. Let G be a subgroup of $U(f)$ containing $SU(f)$ and $\alpha: G \rightarrow GL(M)$ a representation, with $\alpha SU(f)$ nontrivial, where M is a free module over a commutative local ring R with 2 a unit and $n = \dim f = \text{rank } M \geq 3$. Since $SU(f)$ is perfect, $\alpha SU(f) \subseteq SL(n, R)$. Then except for $n = 4$ or 6 the homomorphism α induces a thick collineation $\pi: \mathbf{P}V \rightarrow \mathbf{P}M$. The cases $n = 4$ and 6 can be included if the hypotheses are modified to $I(f) \subseteq G$ and $\text{card } \alpha I(f) > 2$. By the Fundamental Theorem there now exists a generalized place $\varphi: k \rightarrow R \cup \infty$ with valuation ring $A = \varphi^{-1}(R)$ and homomorphism $\varphi: A \rightarrow R$, a free A -module N in V of rank n and a φ -semilinear map $\beta: N \rightarrow M$ inducing π . Thus $\pi P = R\beta(P \cap N)$ for any point P in $\mathbf{P}V$. We study these objects further and use them to describe the original representation α via a twisted congruence subgroup in G .

We first establish that $A^* = A$ and $\mathfrak{m}(A)^* = \mathfrak{m}(A)$. The A -module N in V is called *maximal* if $N = \{x \in V \mid f(x, x) \in \mathfrak{c}\}$ for some fractional ideal \mathfrak{c} of A . Denote by $U(N)$ the subgroup of $U(f)$ consisting of all $\sigma \in U(f)$ such that $\sigma(N) = N$. If N is maximal then $U(N) = U(f)$.

PROPOSITION 7. (i) $A^* = A$ and $\mathfrak{m}(A)^* = \mathfrak{m}(A)$.

(ii) The A -module N is maximal and has an orthogonal basis.

(iii) $U(N) = U(f)$.

PROOF. Let $x, y \in N$ be primitive (that is, x, y are not in $\mathfrak{m}(A)N$) with $\pi(kx) \neq \pi(ky)$. Assume, if possible, $c = 2f(x, x)^{-1}f(x, y)$ is not in A , or equivalently, that $f(x, y)^{-1}f(x, x) \in \mathfrak{m}(A)$. Let $\beta(x) = v$ and $\beta(y) = w$ so that $Rv \neq Rw$. Then

$$\Psi(x)\Psi(y) = \Psi(\Psi(x)(y))\Psi(x) = \Psi(y - cx)\Psi(x)$$

is an identity in $SU(f)$. If n is odd, applying the homomorphism α gives an identity $E(X, v)E(Y, w) = E(Z, \varphi(c^{-1})w - v)E(X, v)$ in $SL(M)$. Hence $E(X, v)(w) \in R(\varphi(c^{-1})w - v)$. But $E(X, v)(w) = -w + 2av$ for some $a \in R$ which forces the contradiction $Rv = Rw$. If $n \geq 8$ is even, choose z primitive in N and orthogonal to x and y . Then a similar contradiction can be achieved using the 2-involutions $\Psi(x)\Psi(z)$, $\Psi(y)\Psi(z)$ and $\Psi(y - cx)\Psi(z)$. Hence, in both cases, $f(x, y) \in f(x, x)A$.

Since A is a valuation ring the module N has an orthogonal decomposition into indecomposable components of rank one or two. If N has an indecomposable binary component $B = Ax + Ay$, then necessarily $f(x, x) \in f(x, y)\mathfrak{m}(A)$. This gives a contradiction with the previous paragraph since $\beta(x)$, $\beta(y)$ expands to a basis of M . Thus N has an orthogonal basis and we may assume $N = Au_1 \perp \cdots \perp Au_n$ and $M = Rv_1 + \cdots + Rv_n$ where $\beta(u_i) = v_i$, $1 \leq i \leq n$. Let $f(u_i, u_i) = a_i$,

$1 \leq i \leq n$, where each a_i is a nonzero element of k . Since A is a valuation ring there exists a_1 , say, such that $a_1^{-1}a_i \in A$, $1 \leq i \leq n$.

Let $e \in A$ be a unit. If $e^* \notin A$ replace e by e^{-1} . Hence $e^* \in A$. Assume e^* is not a unit, so $e^* \in \mathfrak{m}(A)$. Put $x = e^*u_1 + eu_2$ and $y = u_1$. Then $\pi(kx) \neq \pi(ky)$ and consequently $f(x, y) \in f(x, x)A$. Therefore, $ea_1 \in (ea_1e^* + e^*a_2e)A$ which gives the contradiction $1 \in \mathfrak{m}(A)$. Thus e^* must be a unit. Hence $(1 + \mathfrak{m}(A))^* \subseteq A$, so that $\mathfrak{m}(A)^* \subseteq A$. It follows that $A^* = A$ and $\mathfrak{m}(A)^* = \mathfrak{m}(A)$.

Next let $p \in \mathfrak{m}(A)$ and put $x = p^*u_1 + u_i$, $i \neq 1$, and $y = u_1$. Then $\pi(kx) \neq \pi(ky)$ so that $f(x, y) \in f(x, x)A$. Therefore, pa_1 lies in the fractional ideal $(pa_1p^* + a_i)A$ and, since A is a valuation ring, $pa_1 \in a_iA$. Thus $a_1\mathfrak{m}(A) \subseteq a_iA$. If $\mathfrak{m}(A)$ is not a principal ideal then $a_1^{-1}a_i$ must be a unit, since $a_1^{-1}a_i \in A$. In this case N is now a modular lattice. If, on the other hand, $\mathfrak{m}(A)$ is a principal ideal then either $a_1^{-1}a_i$ is a unit, or $a_iA = a_1\mathfrak{m}(A)$. The second possibility cannot occur since f represents only one norm class. Thus again N is modular.

Finally, let $x = \sum c_i u_i \in N$ with at least one coefficient, say c_1 , a unit in A . Assume $f(x, x) \in a_1\mathfrak{m}(A)$, so that necessarily a second coefficient, say c_2 , is also a unit. Put $y = u_2$. Then $\pi(kx) \neq \pi(ky)$ and $f(x, y) \in f(x, x)A$. This forces the contradiction $1 \in \mathfrak{m}(A)$. Hence $f(x, x)A = a_1A$ and N is now necessarily a maximal lattice.

DEFINITION. Let $\chi: G \rightarrow \mathfrak{u}(R)$ be a character, that is, a homomorphism from G into the group of units $\mathfrak{u}(R)$ of R , and \mathfrak{a} the kernel of $\varphi: A \rightarrow R$. Then $G(\mathfrak{a}, \chi)$ denotes the twisted congruence subgroup of the group G consisting of all $\sigma \in G$ such that

- (i) $\chi(\sigma) \in \varphi(A)$, so $\chi(\sigma) = \varphi(a_\sigma)$ for some $a_\sigma \in A$,
- (ii) $a_\sigma\sigma(x) \equiv x \pmod{\mathfrak{a}N}$ for all $x \in N$.

Note that the definition is independent of the choice of a_σ since \mathfrak{a} is the kernel of φ and condition (ii) is modulo $\mathfrak{a}N$.

PROOF OF THEOREM 1. Many parts of the theorem have already been established. It remains to construct a character $\chi: G \rightarrow \mathfrak{u}(R)$, show that $U(\mathfrak{a}, \chi)$ is the kernel of α , and describe the action of $\alpha\sigma$ for $\sigma \in G$.

Consider first $n \geq 3$ odd. Let $x \in N$ be primitive and $N = Ax \perp X$. Then the image of $-\Psi(x) \in SU(f)$ under α is $E(Y, \beta(x))$ where $Y = \beta(X)$. For if $w \in X$ is primitive, then $\Psi(x)$ and $\Psi(w)$ commute and hence $E(Y, \beta(x))$ and $E(Z, \beta(w))$ commute. Since $R\beta(x) \cap R\beta(w) = 0$, for otherwise $\alpha(\Psi(x)\Psi(w)) = 1_M$, it follows that $\beta(w) \in Y$. Hence $\beta(X) \subseteq Y$ and, since they have the same rank, $\beta(X) = Y$. For any $\sigma \in G$ and primitive $x \in N$ we have $\sigma(-\Psi(x))\sigma^{-1} = -\Psi(\sigma(x))$ in G . Applying the homomorphism α gives $(\alpha\sigma)E(\beta(X), \beta(x))(\alpha\sigma)^{-1} = E(\beta\sigma(X), \beta\sigma(x))$. Hence, from comparing the positive spaces,

$$(\alpha\sigma)(\beta(x)) = \chi(\sigma, x)\beta\sigma(x)$$

for some unit $\chi(\sigma, x)$ in R . We show $\chi(\sigma) = \chi(\sigma, x)$ is independent of the choice of x . Let $y \in N$ be primitive with $f(x, y) = 0$. Then $x + y \in N$ is also primitive. Moreover, if $\alpha(-\Psi(x)) = E(Y, \beta(x))$, then $\beta(y) \in Y$ and hence $R\beta(x) \cap R\beta(y) = 0$. Then, from $\chi(\sigma, x + y)\beta\sigma(x + y) = (\alpha\sigma)\beta(x + y) = \chi(\sigma, x)\beta\sigma(x) + \chi(\sigma, y)\beta\sigma(y)$ it follows that $\chi(\sigma, x + y) = \chi(\sigma, x) = \chi(\sigma, y)$. Thus $\chi(\sigma, x)$ is independent of x and we have constructed a map $\chi: G \rightarrow \mathfrak{u}(R)$. It is easily seen that χ is a group

homomorphism. Also, we have established that

$$(\alpha\sigma)(\beta(x)) = \chi(\sigma)\beta\sigma(x).$$

Now let $n \geq 8$ be even. Let $\tau_1 = \Psi(x)\Psi(y)$ and $\tau_2 = \Psi(x)\Psi(z)$ where x, y, z are primitive orthogonal elements of N . Then (τ_1, τ_2) is a minimal pair of 2-involutions with associated point $P(\tau_1, \tau_2) = L(\tau_1) \cap L(\tau_2) = kx$. From the construction of π in Proposition 5 and the definition of β we have $L(\alpha\tau_1) = \beta L(\tau_1)$, $L(\alpha\tau_2) = \beta L(\tau_2)$ and $P(\alpha\tau_1, \alpha\tau_2) = \beta P(\tau_1, \tau_2)$. For any $\sigma \in G$, $(\sigma\tau_1\sigma^{-1}, \sigma\tau_2\sigma^{-1})$ remains a minimal pair as also does its image under α . Since $L(\sigma\tau_1\sigma^{-1}) = \sigma L(\tau_1)$, $L(\sigma\tau_2\sigma^{-1}) = \sigma L(\tau_2)$ and $P(\sigma\tau_1\sigma^{-1}, \sigma\tau_2\sigma^{-1}) = \sigma P(\tau_1, \tau_2)$ it follows that $(\alpha\sigma)\beta(x) = \chi(\sigma, x)\beta\sigma(x)$ for some unit $\chi(\sigma, x)$ in R . Again it follows that $\chi(\sigma, x)$ is independent of x and that χ induces a group homomorphism from G to the units of R .

If \mathfrak{a} is the kernel of the homomorphism $\varphi: A \rightarrow R$ then clearly $\mathfrak{a}N$ is the kernel of the φ -semilinear map $\beta: N \rightarrow M$. Finally we must show that the twisted congruence subgroup $G(\mathfrak{a}, \chi)$ is the kernel of α . Let $\sigma \in \ker \alpha$. Then $\alpha(\sigma) = 1_M$ and consequently $\beta(x) = \chi(\sigma)\beta\sigma(x)$ for any $x \in N$. Hence $\chi(\sigma) \in \varphi(A)$. Therefore, $\beta(x - \mathfrak{a}_\sigma\sigma(x)) = 0$ where $\varphi(\mathfrak{a}_\sigma) = \chi(\sigma)$. Thus $\sigma \in G(\mathfrak{a}, \chi)$. The converse is similar. This completes the proof of Theorem 1.

REMARK. It is quite possible for $\mathfrak{a} = 0$ and φ and β to be injections. The kernel of α is then contained in the center of $U(f)$.

PROPOSITION 8. *Let $\varphi: k \rightarrow R \cup \infty$ be the generalized place in Theorem 1 and $A = \varphi^{-1}(R)$ the associated valuation ring. Let K be the fixed field of k under the involution $*$. Then*

- (i) $\mathbf{Q} \subseteq A$,
- (ii) if $\mathbf{R} \subseteq K$ then $\mathbf{R} \subseteq A$,
- (iii) $B/\mathfrak{m}(B)$ is formally real, where $B = K \cap A$ is the induced valuation ring of K .

PROOF. Since $\varphi(1) = 1$ we know $\mathbf{Z} \subseteq A$. Assume $\varphi(p) \in \mathfrak{m}(R)$ for some odd prime p in \mathbf{Z} . Then $p \in \mathfrak{m}(A)$. Let $N = Au_1 \perp \dots \perp Au_n$ be the maximal lattice in Theorem 1 where we may assume $f(u_i, u_i) = a \neq 0$, $1 \leq i \leq n$. Since the quadratic form $\langle 1, 1, \dots, 1 \rangle$ is isotropic over the finite field \mathbf{F}_p there exist integers c_1, \dots, c_n with $c_1 = 1$ such that $\sum c_i^2 \in p\mathbf{Z}$. Then $x = \sum c_i u_i \in N$ is primitive and $f(x, x) \in \mathfrak{m}(A)$, since the involution is trivial on \mathbf{Z} . This contradicts the fact that N is a maximal lattice (Proposition 7). Hence $\varphi(p)$ is a unit in R for all odd primes p . By hypothesis 2 is a unit in R and hence also in A . It follows that $\mathbf{Q} \subseteq A$ and the valuation induced by φ on \mathbf{Q} is trivial.

Now assume $\mathbf{R} \subseteq K$. Let $a \in \mathbf{R}$ with $0 < a < 1$. Assume $\varphi(a) = \infty$ so that $a^{-1} \in \mathfrak{m}(A)$. Let $a^{-1} - 1 = b^2$ where $b \in \mathbf{R}$. Then $x = bu_1 + u_2$ is primitive in N but $f(x, x) \in \mathfrak{m}(A)$, contradicting the fact that N is maximal. Therefore, $\mathbf{R} \subseteq A$ by the Archimedean axiom on \mathbf{R} , and the induced valuation on \mathbf{R} is now trivial. The proof of (iii) is similar.

REMARK. Proposition 8(ii) is more general. The same result holds for any subfield of \mathbf{R} all of whose positive elements are squares. If k is such a field then Theorem 1 suggests that the only nontrivial normal subgroup of the orthogonal group $SO(f)$, $n \neq 4$, is its center. This is true, see Bröcker [4]. A similar result should hold for unitary groups $SU(f)$ over $k(i)$, $i^2 = -1$.

6. Representations over commutative rings. We now extend the results of the previous section to representations $\alpha: G \rightarrow GL(n, R)$, where R is a commutative ring with 2 not a zero divisor. If 2 is not a unit in R , enlarge R by localizing with respect to the multiplicative set generated by 2. Assume, therefore, 2 is a unit in R . Let \mathfrak{p} be a prime ideal of R , $R_{\mathfrak{p}}$ the localization at \mathfrak{p} and $\varepsilon_{\mathfrak{p}}: R \rightarrow R_{\mathfrak{p}}$ the canonical homomorphism. Localize M at \mathfrak{p} and let $M \rightarrow M_{\mathfrak{p}}$ be the natural extension of $\varepsilon_{\mathfrak{p}}$. Then we have a homomorphism $\eta_{\mathfrak{p}}: GL(M) \rightarrow GL(M_{\mathfrak{p}})$ from the group $GL(M) = GL(n, R)$ to the group $GL(M_{\mathfrak{p}}) = GL(n, R_{\mathfrak{p}})$.

PROOF OF THEOREM 2. Let $\alpha: SU(f) \rightarrow GL(M)$ be a nontrivial representation of $SU(f)$. Then Theorem 1 can be applied to the composite map $\eta_{\mathfrak{p}} \circ \alpha: SU(f) \rightarrow GL(n, R_{\mathfrak{p}})$. If the image of $SU(f)$ is now trivial then obviously the kernel is $SU(f)$. Otherwise the kernel is a twisted congruence subgroup $SU(\mathfrak{a}_{\mathfrak{p}}, \chi_{\mathfrak{p}})$ of $SU(f)$. Let H be the intersection of all these twisted congruence subgroups for which $\eta_{\mathfrak{p}} \circ \alpha$ is nontrivial as \mathfrak{p} varies over the maximal ideals of R . Clearly H is a normal subgroup of $SU(f)$ containing the kernel of α . Conversely, let $\sigma \in H$. Then $\eta_{\mathfrak{p}}(\alpha\sigma) = 1$ for all maximal ideals \mathfrak{p} . Hence $(\alpha\sigma(x) - x)_{\mathfrak{p}} = 0$ in $M_{\mathfrak{p}}$ for all maximal ideals \mathfrak{p} and all $x \in M$. It follows (for example, Bass [2, p. 108]) that $(\alpha\sigma)(x) = x$ in M for all $x \in M$. Thus $\sigma \in \ker \alpha$ and

$$1 \rightarrow H \rightarrow SU(f) \xrightarrow{\alpha} GL(n, R)$$

is an exact sequence. Finally, since $SU(f)$ is perfect, the characters $\chi_{\mathfrak{p}}: SU(f) \rightarrow u(R_{\mathfrak{p}})$ are all trivial.

To prove Theorem 3 we need to study the generalized places $\varphi: \mathbf{C}(X) \rightarrow R \cup \infty$, with R a local ring, extending an injection $\varphi: \mathbf{C} \rightarrow R$. Recall the notation A_c and $\varphi_{c,m}$ from the introduction.

PROPOSITION 9. *Let $\varphi: \mathbf{C}(X) \rightarrow R \cup \infty$ be a generalized place, which induces the trivial valuation on \mathbf{C} , and $A = \varphi^{-1}(R)$ the associated valuation ring. Then one of the following occurs:*

- (i) $A = \mathbf{C}(X)$ and φ is an injection into R ,
- (ii) $A = A_c$ and $\varphi = \varphi_{c,m}$ for some $c \in \mathbf{C}$ and integer $m \geq 1$,
- (iii) $A = A_{\infty}$ and $\varphi = \varphi_{\infty,m}$ for some integer $m \geq 1$.

PROOF. We have $\varphi(\mathbf{C}) = \mathbf{C} \subseteq R$. Let \mathfrak{a} be the kernel of $\varphi: A \rightarrow R$. Assume first $\varphi(X) = x \in R$. Then $\mathbf{C}[x] \subseteq R$. If $g(x)$ is a unit in R for all nonzero polynomials $g(X) \in \mathbf{C}[X]$ then $A = \mathbf{C}(X)$, $\mathfrak{a} = 0$ and φ is an injection. Assume, therefore, there exists a monic nonconstant polynomial $g(X) \in \mathbf{C}[X]$ with $g(x) \in \mathfrak{m}(R)$. Let $g(X)$ have minimal degree and $c \in \mathbf{C}$ be a root of $g(X)$. Then $g(X) = X - c$. If $h(X) \in \mathbf{C}[X]$ is a polynomial relatively prime to $g(X)$, then $h(x)$ must be a unit of R , for otherwise $1 \in \mathfrak{m}(R)$. Hence $A = A_c$ and $\mathfrak{m}(A) = (X - c)A_c$. Since the valuation ring A_c is discrete, the kernel $\mathfrak{a} = (X - c)^m A_c$ for some integer $m \geq 1$. Then $\varphi = \varphi_{c,m}$. Finally assume $\varphi(X) = \infty$. Then $X^{-1} \in \mathfrak{m}(A)$ and $\varphi(X^{-1}) = x \in \mathfrak{m}(R)$. Hence $\varphi(X^{-1} - c)$ is a unit of R for all nonzero $c \in \mathbf{C}$. Therefore, $(X - a)/(X - b)$ is a unit in A for all $a, b \in \mathbf{C}$. Consequently $A = A_{\infty}$ consists of all rational functions with degree ≤ 0 . Hence $\mathfrak{m}(A) = X^{-1}A_{\infty}$ and the kernel $\mathfrak{a} = X^{-m}A_{\infty}$ for some $m \geq 1$. Thus $\varphi = \varphi_{\infty,m}$.

PROOF OF THEOREM 3. We apply Theorems 1 and 2 to the special case $k = \mathbf{C}(X)$. Let $\varphi: \mathbf{C}(X) \rightarrow R \cup \infty$ be a generalized place occurring in Theorem

1(i). By Proposition 8, $\varphi(\mathbf{R}) = \mathbf{R} \subseteq R$ (after identifying \mathbf{R} with a subfield of R). Also $\varphi(i)^2 = \varphi(-1) = -1$ and hence $\varphi(\mathbf{C}) = \mathbf{C} \subseteq R$. Therefore φ is one of the generalized places in Proposition 9. However, if $\varphi = \varphi_{c,m}$ with $c \in \mathbf{C}$, then $c \in \mathbf{R}$ since by Theorem 1(i) we must have $A_c^* = A_c$. Let H be a nontrivial normal subgroup of $SU(f)$ as in Theorem 3. By Theorem 2, $H = \bigcap SU(\mathfrak{a}_j, 1)$ where the \mathfrak{a}_j are kernels of generalized places $\varphi_j: \mathbf{C}(x) \rightarrow R_j \cup \infty$. We may assume $\mathfrak{a}_j \neq 0$ and hence any injective φ_j can be ignored. Only a finite number of \mathfrak{a}_j can be nonzero, for otherwise $H = 1$ is trivial. Let A be the intersection of the finite number of valuation rings A_c , $c \in \mathbf{R} \cup \infty$, associated with generalized places with nontrivial kernel, and \mathfrak{a} the intersection of these kernels. Then \mathfrak{a} is an ideal of A . By the weak approximation theorem for valuations we can choose a common basis u_1, \dots, u_n for all the associated maximal lattices in Theorem 1(ii). Then $N = Au_1 + \dots + Au_n$ is the intersection of these maximal lattices and hence $\sigma(N) = N$ for all $\sigma \in SU(f)$. Let $SU(\mathfrak{a})$ be the congruence subgroup of all $\sigma \in SU(f)$ with $(\sigma - 1)N \subseteq \mathfrak{a}N$. Then $H = \bigcap SU(\mathfrak{a}_j, 1) = SU(\mathfrak{a})$, completing the proof.

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