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REGULARITY OF SOLUTIONS OF TWO-DIMENSIONAL MONGE-AMPÈRE EQUATIONS

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ABSTRACT. In the paper we investigate the regularity of solutions $z(x, y) \in C^{1,1}(\Omega)$, resp. $C^{1,1}(\overline{\Omega})$ of elliptic Monge-Ampère equations of the form

$$Ar + 2Bs + Ct + (rt - s^2) = E.$$

It is shown that $z(x, y) \in C^{2,\alpha}(\Omega)$, resp. $C^{2,\alpha}(\overline{\Omega})$, with corresponding a priori estimates, if $A, B, C, E \in C^{\alpha}(\Omega \times \mathbf{R}^3)$. The results are deduced via the Campanato technique for equations of variational structure invoking a Legendre-like transformation.

I. Introduction and statement of the result. Let Ω be an open subset of the x, y-plane. We shall consider solutions $z(x, y)$ of class $C^{1,1}(\Omega)$ of elliptic Monge-Ampère equations of the general form

$$Ar + 2Bs + Ct + (rt - s^2) = E \qquad (\Delta = AC - B^2 + E > 0). \tag{1}$$

The coefficients A, B, C, E are assumed to belong to the Hölder class C^{α} $(0 < \alpha < 1)$ with respect to the variables x, y, z, p, q. Here p, q, r, s, t represent the first and second derivatives of $z(x, y)$.

Suppose a is a bound for the absolute values of A, B, C, E, and suppose b is a bound for their Hölder constants. Furthermore let $1/c$ be a lower bound for Δ and K be a bound for the $C^{1,1}$-norm of $z(x, y)$. The first result of the paper then reads as the following

THEOREM 1. *The second derivatives of $z(x, y)$ are Hölder continuous in Ω with exponent α. In every subset Ω', which is compactly contained in Ω, they satisfy the Hölder conditions*

$$\begin{aligned} &|r(x', y') - r(x'', y'')|, \ldots, |t(x', y') - t(x'', y'')| \\ &\qquad \leq H((x' - x'')^2 + (y' - y'')^2)^{\alpha/2} \end{aligned} \tag{2}$$

$((x', y'), (x'', y'') \in \Omega')$, where the constant H only depends on α, a, b, c, K and the distance between Ω' and $\partial\Omega$.

Suppose now Ω is a bounded open set with boundary $\partial\Omega$ of class $C^{2,\alpha}$ and let $\varphi \in C^{2,\alpha}(\partial\Omega)$, with k being a bound for its $C^{2,\alpha}(\partial\Omega)$-norm. The second part of the paper is concerned with the regularity near $\partial\Omega$ of solutions $z(x, y) \in C^{1,1}(\overline{\Omega})$ of the Dirichlet problem for the equation (1) subject to the boundary condition

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$z\,|_{\partial\Omega} = \varphi$. The result can be stated as

THEOREM 2. *The second derivatives of $z(x,y)$ are Hölder continuous in $\overline{\Omega}$ with exponent α. They satisfy the Hölder conditions (2) for $(x',y'),(x'',y'') \in \Omega$, where the constant H depends only on α, a, b, c, k, K and Ω.*

The regularity parts of the theorems complete the results obtained by the author in [**12**–**15**]. The also stated interior a priori estimates have been established in [**14**], the boundary estimates in [**15**], if $u(x)$ is known to be regular, i.e., of class $C^{2,\alpha}(\Omega)$, resp. $C^{2,\alpha}(\overline{\Omega})$. The above-cited papers are subsequent to massive classical work, in particular by H. Lewy [**7**], A. D. Aleksandrov [**1**], Bakel′man [**2**], Pogorelov [**9**], Nirenberg [**8**], Heinz [**6**] and Sabitov [**10**]. We refer the reader to the account given in [**14**].

In addition, it should be noted that Trudinger [**17**] has recently shown a regularity result for fully nonlinear, concave equations in n dimensions under certain differentiabilty assumptions. The class of equations under consideration includes general Monge-Ampère equations of the form (1). Such results can also be obtained via the Green's function technique, as mentioned in [**16**], regularity results actually somewhat easier than the therein proven a priori estimates. We also wish to draw attention to Safonov's approach [**11**], which yields the existence of an $\bar{\alpha}$, $0 < \bar{\alpha} < 1$, such that the estimates of Theorems 1 and 2 hold for $\alpha < \bar{\alpha}$, if the coefficients belong to $C^{\bar{\alpha}}$. The purpose of the present paper is to cover the case of merely α-Hölder-continuous coefficients $(0 < \alpha < 1)$.

Like [**14**], the proof of Theorem 1 is based on a Legendre-like transformation. But then the proof rests on a technique due to Campanato [**3**, **4**], unlike [**14**], where the Schauder technique was employed, which unavoidably requires the regularity $z(x,y) \in C^{2,\alpha}(\Omega)$. At the beginning of the second section we shall review from [**14**], incorporating the necessary changes.

Similar remarks apply to the proof of Theorem 2. Like [**15**], we first straighten $\partial\Omega$ locally. It is then important to introduce the zero-boundary data not too early, i.e., only after performing the transformation.

It should also be noted, that we have achieved a few simplifications over [**14**, **15**]. We adopt the notation used therein. The letter C denotes various constants, which may change from line to line. If possible, we choose constants to be ≥ 1.

II. Proof of Theorem 1. Suppose $D_R = D_R(x_0, y_0)$ is a circular disc of radius $R > 0$, centered at $(x_0, y_0) \in \Omega$. The very first thing to note is that the equation (1) can be written in the equivalent form

$$(r + C)(t + A) - (s - B)^2 = \Delta.$$

By putting

$$A_0 := A(x_0, y_0, \ldots, q(x_0, y_0)), \quad \ldots, \quad E_0 := E(\cdots), \quad \Delta_0 := \Delta(\cdots),$$

the equation (1) can therefore be rewritten in the form

$$\begin{aligned}(r + C_0)(t + A_0) &- (s - B_0)^2 \\ &= \Delta_0 + ((A_0 - A)r + 2(B_0 - B)s + (C_0 - C)t + (E - E_0)) \\ &=: \tilde{f}(x,y).\end{aligned}$$

Hence the function

$$\begin{aligned}\tilde{z}(x,y) := z(x,y) + \tfrac{1}{2}(C_0(x-x_0)^2 - 2B_0(x-x_0)(y-y_0) + A_0(y-y_0)^2) \\ + 2(K+a)(y-y_0)\end{aligned}$$

solves the equation

$$\tilde{r}\tilde{t} - \tilde{s}^2 = \tilde{f}(x,y) \geq 1/2c =: 1/\tilde{c} > 0$$

in $D := D_{R_0}(x_0, y_0)$, where

$$R_0 := \min\{1/\sqrt[6]{10bcK}, d/2\}.$$

Here $d > 0$ is a lower bound for the distance between (x_0, y_0) and $\partial\Omega$. We furthermore have the estimates $\tilde{q} \geq 1$, $|\tilde{r}|, |\tilde{t}| \geq 1/\tilde{c}\tilde{K}$, where $\tilde{K} \geq 1$ is a bound for the $C^{1,1}$-norm of $\tilde{z}(x,y)$.

Now we make the variable transformation

$$T\colon \begin{cases} u = x, \\ v = \tilde{q}(x,y) \end{cases}$$

($(x,y) \in D$). Compare also [**15**, Lemma 3], for the following list of its properties:

(i) T maps D homeomorphically onto the image $T(D)$.

(ii) For $(x', y'), (x'', y'') \in D$, we have the dilation estimates

$$(u'-u'')^2 + (v'-v'')^2 \leq \gamma_1^2((x'-x'')^2 + (y'-y'')^2), \tag{3}$$

$$(x'-x'')^2 + (y'-y'')^2 \leq \gamma_2^2((u'-u'')^2 + (v'-v'')^2), \tag{4}$$

with constants $\gamma_1, \gamma_2 \geq 1$, depending only on $\tilde{c}, \tilde{K}$.

(iii) Hence the inclusions

$$T(D_{R/\gamma_1}(x_0,y_0)) \subset D_R(u_0,v_0), \quad D_{R/\gamma_2}(u_0,v_0) \subset T(D_R(x_0,y_0))$$

hold for all R, $0 < R \leq R_0$.

(iv) The function $y(u,v) \in C^{0,1}(T(D))$ is a weak solution of the equation

$$y_{uu} + (\tilde{f} y_v)_v = 0. \tag{5}$$

Only the proof of (4) needs to be modified slightly: On estimating

$$\begin{aligned}|v'-v''| &= \left|\int_0^1 \{s((1-\tau)x' + \tau x'', (1-\tau)y' + \tau y'')(x'-x'') + t(\cdots)(y'-y'')\}\, d\tau\right| \\ &\geq -\tilde{K}|x'-x''| + \frac{1}{\tilde{c}\tilde{K}}|y'-y''|,\end{aligned}$$

we obtain

$$\begin{aligned}&(u'-u'')^2 + (v'-v'')^2 \\ &\quad\geq (1+\tilde{K}^2)(x'-x'')^2 - \frac{2}{\tilde{c}}|x'-x''|\,|y'-y''| + \frac{1}{(\tilde{c}\tilde{K})^2}(y'-y'')^2 \\ &\quad\geq \left(1 + \left(1 - \frac{1}{\delta}\right)\tilde{K}^2\right)(x'-x'')^2 + \frac{1-\delta}{(\tilde{c}\tilde{K})^2}(y'-y'')^2.\end{aligned}$$

Inequality (4) then follows by choosing δ to be the mean of $\tilde{K}^2/(1+\tilde{K}^2)$ and 1.

Noting that

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\tilde{s}/\tilde{t} & 1/\tilde{t} \end{pmatrix}, \tag{6}$$

we proceed to rewrite equation (5) in the form

$$y_{uu} + (\Delta_0 y_v)_v = g_v,$$

where

$$g(u,v) := ((A - A_0)r + 2(B - B_0)s - (C - C_0)t + (E_0 - E))/\tilde{t}.$$

We can now apply the Campanato technique [**3**, **4**] as described in Chapter III of Giaquinta's book [**4**]. It is unnecessary to carry out the first step, [**5**, Theorem 3.1], because of the boundedness of $\nabla y(u,v)$. As in the proof of [**5**, Theorem 3.2, resp. Theorem 2.2], we obtain the inequalities

$$\begin{aligned} &\iint_{D_\rho} |\nabla y - (\nabla y)_\rho|^2 \, du \, dv \\ &\qquad \le C \left\{ \left(\frac{\rho}{R}\right)^4 \iint_{D_R} |\nabla y - (\nabla y)_R|^2 + \iint_{D_R} |g - g_R|^2 \right\} \end{aligned} \tag{7}$$

for all $\rho \le R \le R_0/\gamma_2$. Here all discs are centered at (u_0, v_0) and

$$g_R := \frac{1}{|D_R|} \iint_{D_R} g(u,v) \, du \, dv.$$

Using the dilation inequality (3), we estimate

$$\iint_{D_R} |g - g_R|^2 \le C \iint_{D_R} |g|^2 \le C R^{2+2\alpha}, \tag{8}$$

where the last constant also depends on b. It therefore follows that

$$\iint_{D_\rho} |\nabla y - (\nabla y)_\rho|^2 \, du \, dv \le C \left\{ \left(\frac{\rho}{R}\right)^4 \iint_{D_R} |\nabla y - (\nabla y)_R|^2 + R^{2+2\alpha} \right\}.$$

An iteration argument, [**5**, Lemma 2.1], yields for all $\rho \le R \le R_0/\gamma_2$

$$\begin{aligned} \iint_{D_\rho} |\nabla y - (\nabla y)_\rho|^2 \, du \, dv &\le C \left\{ \left(\frac{\rho}{R}\right)^{2+2\alpha} \iint_{D_R} |\nabla y - (\nabla y)_R|^2 + \rho^{2+2\alpha} \right\} \\ &\le \frac{C}{R^{2\alpha}} \rho^{2+2\alpha}, \end{aligned}$$

incorporating the boundedness of $\nabla \eta$.

We are not done, because the variables u, v, and therefore also the function $y(u,v)$, depend on the point (u_0, v_0). Hence we reintroduce the x, y-variables, using some elementary geometric measure theory. By (6), we have a.e.

$$y_v - (y_v)_\rho = \frac{1}{\tilde{t}} \frac{|T^{-1}(D_\rho)|}{|D_\rho|} (t_{\ ;\rho} - t),$$

where

$$g_{\ ;\rho} = g_{(x_0,y_0);\rho} := \frac{1}{|T^{-1}(D_\rho)|} \iint_{T^{-1}(D_\rho)} g(x,y) \, dx \, dy,$$

and consequently

$$\int\int_{D_\rho} |y_v - (y_v)_\rho|^2 \, du \, dv = \frac{|T^{-1}(D_\rho)|}{|D_\rho|} \int\int_{T^{-1}(D_\rho)} \frac{1}{\tilde{t}} |t - t_{;\rho}|^2 \, dx \, dy.$$

The inequalities

$$\int\int_{D_{\rho/\gamma_1}(x_0,y_0)} |t - t_{;\rho}|^2 \, dx \, dy \le \frac{C}{R_0^{2\alpha}} \rho^{2+2\alpha}$$

hold therefore for all $(x_0, y_0) \in \Omega$ and all ρ, $0 < \rho \le R_0/\gamma_2$. Here $t_{;\rho}$ depends on (x_0, y_0) through the transformation T.

Similarly

$$y_u - (y_u)_\rho = \frac{1}{\tilde{t}} \frac{|T^{-1}(D_\rho)|}{|D_\rho|} (\tilde{s}_{;\rho} \tilde{t} - \tilde{s} \tilde{t}_{;\rho})$$

a.e., hence

$$\int\int_{D_\rho} |y_u - (y_u)_\rho|^2 \, du \, dv = \frac{|T^{-1}(D_\rho)|}{|D_\rho|} \int\int_{T^{-1}(D_\rho)} \frac{1}{\tilde{t}} |\tilde{s} \tilde{t}_{;\rho} - \tilde{s}_{;\rho} \tilde{t}|^2 \, dx \, dy.$$

On estimating

$$|s - s_{;\rho}| \le \left| \tilde{s} - \tilde{s}_{;\rho} \frac{1}{\tilde{t}_{;\rho}} \right| + \left| \tilde{s}_{;\rho} \frac{1}{\tilde{t}_{;\rho}} - \tilde{s}_{;\rho} \right| \le C\{ |\tilde{s} \tilde{t}_{;\rho} - \tilde{s}_{;\rho} \tilde{t}| + |\tilde{t} - \tilde{t}_{;\rho}| \}$$

a.e. and using the differential equation (1), we therefore conclude that for all $(x_0, y_0) \in \Omega$ and all ρ, $0 < \rho \le R_0/\gamma_2$,

$$\int\int_{D_{\rho/\gamma_1}(x_0,y_0)} \{ |r - r_{;\rho}|^2 + \cdots + |t - t_{;\rho}|^2 \} \, dx \, dy \le \frac{C}{R_0^{2\alpha}} \rho^{2+2\alpha}. \tag{9}$$

We finally deduce the Hölder continuity of r, s, t by proceeding as in the proof of [**5**, Theorem 1.2], that the Campanato spaces $\mathcal{L}^{2,n+2\alpha}(\Omega)$ are contained in the Hölder spaces $C^\alpha(\Omega)$ for $0 < \alpha < 1$ (here $n = 2$). First we get the analogue of formula (1.8) of [**5**, p. 71], that there exist the limits

$$\bar{r}(x_0, y_0) := \lim_{\rho \to 0} r_{;\rho}, \ldots, \bar{t}(x_0, y_0) := \lim_{\rho \to 0} t_{;\rho},$$

and

$$|r_{;\rho} - \bar{r}(x_0, y_0)|, \ldots, |t_{;\rho} - \bar{t}(x_0, y_0)| \le C\rho^\alpha \tag{10}$$

for $\rho \le R_0/\gamma_2$. Here C depends also on R_0.

Now suppose Ω' is a disc in Ω with sufficiently small radius, indeed $\Omega' := D_{R_0/2\gamma_2}(x_0, y_0)$ would be fine. Let $(x', y'), (x'', y'') \in \Omega'$ and $(\rho/2\gamma_1)^2 := (x' - x'')^2 + (y' - y'')^2$. Then

$$\begin{aligned} |\bar{t}(x', y') - \bar{t}(x'', y'')| &\le |\bar{t}(x', y') - t_{(x',y');\rho}| \\ &\quad + |t_{(x',y');\rho} - t_{(x'',y'');\rho}| + |t_{(x'',y'');\rho} - \bar{t}(x'', y'')| \\ &\le C\rho^\alpha + |t_{(x',y');\rho} - t(x, y)| + |t(x, y) - t_{(x'',y'');\rho}|, \end{aligned}$$

and by integrating with respect to (x,y) over $D_{\rho/\gamma_1}(x',y') \cap D_{\rho/\gamma_1}(x'',y'')$, which includes $D_{\rho/2\gamma_1}(x',y')$, we thus obtain

$$|\bar{t}(x',y') - \bar{t}(x'',y'')| \le C\rho^\alpha + \frac{1}{|D_{\rho/2\gamma_1}(x',y')|} \times \left\{ \iint_{D_{\rho/\gamma_1}(x',y')} |t - t_{(x',y');\rho}|\,dx\,dy + \iint_{D_{\rho/\gamma_1}(x'',y'');\rho} |t - t_{(x'',y'');\rho}|\,dx\,dy \right\}.$$

We use the Hölder inequality to get

$$|\bar{t}(x',y') - \bar{t}(x'',y'')| \le C\rho^\alpha,$$

and, of course, similar inequalities for $\bar{r}, \bar{s}$. But, by combining (9) and (10),

$$\iint_{D_{\rho/\gamma_1}(x_0,y_0)} \left\{ |r - \bar{r}(x_0,y_0)|^2 + \cdots + |t - \bar{t}(x_0,y_0)|^2 \right\}\,dx\,dy \le C\rho^{2+2\alpha}$$

for all $(x_0,y_0) \in \Omega$, and therefore $\bar{r} = r$, $\bar{s} = s$, $\bar{t} = t$ on the respective Lebesgue sets, which proves the theorem.

III. Proof of Theorem 2. In a neighborhood of $(x_0,y_0) \in \overline{\Omega}$, Ω is of the form

$$\Omega_{\hat{R}} := \Omega \cap D_{\hat{R}} = \{(x,y) \in D_{\hat{R}} | G(x,y) < 0\}$$

for some $\hat{R}$, $0 < \hat{R} \le 1$, where $G(x,y) \in C^{2,\alpha}(\overline{D}_{\hat{R}})$, $G_x^2 \ge 1/\kappa > 0$. We apply the variable transformation $\xi = G(x,y), \eta = y$, in order to flatten $\partial\Omega \cap D_{\hat{R}}$. We calculate the derivatives of $\hat{z}(\xi,\eta) := z(x(\xi,\eta), y(\xi,\eta))$,

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} G_x & 0 \\ G_y & 1 \end{pmatrix} \begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix},$$

$$\begin{pmatrix} r & s \\ s & t \end{pmatrix} = \begin{pmatrix} G_x & 0 \\ G_y & 1 \end{pmatrix} \begin{pmatrix} \hat{r} & \hat{s} \\ \hat{s} & \hat{t} \end{pmatrix} \begin{pmatrix} G_x & G_y \\ 0 & 1 \end{pmatrix} + \hat{p} \begin{pmatrix} G_{xx} & G_{xy} \\ G_{xy} & G_{yy} \end{pmatrix}.$$

Hence

$$\begin{pmatrix} r+C & s-B \\ s-B & t+A \end{pmatrix} = \begin{pmatrix} G_x & 0 \\ G_y & 1 \end{pmatrix} \begin{pmatrix} \hat{r}+\hat{C} & \hat{s}-\hat{B} \\ \hat{s}-\hat{B} & \hat{t}+\hat{A} \end{pmatrix} \begin{pmatrix} G_x & G_y \\ 0 & 1 \end{pmatrix},$$

where

$$\begin{pmatrix} \hat{C} & -\hat{B} \\ -\hat{B} & \hat{A} \end{pmatrix} := \frac{1}{G_x^2} \begin{pmatrix} 1 & 0 \\ -G_y & G_x \end{pmatrix} \left\{ \begin{pmatrix} C & -B \\ -B & A \end{pmatrix} + \hat{p} \begin{pmatrix} G_{xx} & G_{xy} \\ G_{xy} & G_{yy} \end{pmatrix} \right\} \begin{pmatrix} 1 & -G_y \\ 0 & G_x \end{pmatrix}.$$

Let

$$\hat{\Delta} := \Delta / G_x^2 \ge 1/\kappa c > 0;$$

then $\hat{z}$ solves the elliptic Monge-Ampère equation

$$(\hat{r} + \hat{C})(\hat{t} + \hat{A}) - (\hat{s} - \hat{B})^2 = \hat{\Delta}.$$

When working near the boundary of Ω, we can therefore assume w.l.o.g. that $\partial\Omega$ is flat.

From now on, we can follow the proof of Theorem 1, resp. [**15**], working in sets of the form $\Omega_R = \Omega \cap D_R$, resp. half-discs. We observe that the Legendre-like transformation preserves the flatness of $\partial\Omega \cap D_{\hat{R}}$ and introduce zero-boundary data of $y(u, v)$, resp. $\eta(u, v)$ as in [**15**]. As far as the Campanato technique is concerned, we can refer to Lemma 12.II of [**3**, p. 355], for an inequality of the form (7) for half-discs, instead of referring to Lemma 8.II of [**3**, p. 338], or to Giaquinta's book. Note that we have to use an inequality of the form (8) for the nontangential derivative, i.e., for y_u, resp. η_u. Theorem 2 is thus proved.

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