REGULARITY OF SOLUTIONS OF TWO-DIMENSIONAL MONGE-AMPÈRE EQUATIONS

FRIEDMAR SCHULZ AND LIANG-YUAN LIAO

ABSTRACT. In the paper we investigate the regularity of solutions $z(x, y) \in C^{1,1}(\Omega)$, resp. $C^{1,1}(\overline{\Omega})$ of elliptic Monge-Ampère equations of the form

$$Ar + 2Bs + Ct + (rt - s^2) = E.$$  

It is shown that $z(x, y) \in C^{2,\alpha}(\Omega)$, resp. $C^{2,\alpha}(\overline{\Omega})$, with corresponding a priori estimates, if $A, B, C, E \in C^\alpha(\Omega \times \mathbb{R}^3)$. The results are deduced via the Campanato technique for equations of variational structure invoking a Legendre-like transformation.

I. Introduction and statement of the result. Let $\Omega$ be an open subset of the $x,y$-plane. We shall consider solutions $z(x,y)$ of class $C^{1,1}(\Omega)$ of elliptic Monge-Ampère equations of the general form

$$(1) \quad Ar + 2Bs + Ct + (rt - s^2) = E \quad (\Delta = AC - B^2 + E > 0).$$  

The coefficients $A, B, C, E$ are assumed to belong to the Hölder class $C^\alpha (0 < \alpha < 1)$ with respect to the variables $x, y, z, p, q$. Here $p, q, r, s, t$ represent the first and second derivatives of $z(x,y)$.

Suppose $a$ is a bound for the absolute values of $A, B, C, E$, and suppose $b$ is a bound for their Hölder constants. Furthermore let $1/c$ be a lower bound for $\Delta$ and $K$ be a bound for the $C^{1,1}$-norm of $z(x, y)$. The first result of the paper then reads as the following

THEOREM 1. The second derivatives of $z(x,y)$ are Hölder continuous in $\Omega$ with exponent $\alpha$. In every subset $\Omega'$, which is compactly contained in $\Omega$, they satisfy the Hölder conditions

$$(2) \quad |r(x',y') - r(x'',y'')|, \ldots, |t(x',y') - t(x'',y'')| \leq H((x' - x'')^2 + (y' - y'')^2)^{\alpha/2}$$

$((x',y'),(x'',y'') \in \Omega')$, where the constant $H$ only depends on $\alpha, a, b, c, K$ and the distance between $\Omega'$ and $\partial \Omega$.

Suppose now $\Omega$ is a bounded open set with boundary $\partial \Omega$ of class $C^{2,\alpha}$ and let $\varphi \in C^{2,\alpha}(\partial \Omega)$, with $k$ being a bound for its $C^{2,\alpha}(\partial \Omega)$-norm. The second part of the paper is concerned with the regularity near $\partial \Omega$ of solutions $z(x,y) \in C^{1,1}(\overline{\Omega})$ of the Dirichlet problem for the equation (1) subject to the boundary condition

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Theorem 2. The second derivatives of $z(x, y)$ are Hölder continuous in $\Omega$ with exponent $\alpha$. They satisfy the Hölder conditions (2) for $(x', y'), (x'', y'') \in \Omega$, where the constant $H$ depends only on $\alpha, a, b, c, k, K$ and $\Omega$.

The regularity parts of the theorems complete the results obtained by the author in [12–15]. The also stated interior a priori estimates have been established in [14], the boundary estimates in [15], if $u(x)$ is known to be regular, i.e., of class $C^{2,\alpha}(\Omega)$, resp. $C^{2,\alpha}(\bar{\Omega})$. The above-cited papers are subsequent to massive classical work, in particular by H. Lewy [7], A. D. Aleksandrov [1], Bakel’man [2], Pogorelov [9], Nirenberg [8], Heinz [6] and Sabitov [10]. We refer the reader to the account given in [14].

In addition, it should be noted that Trudinger [17] has recently shown a regularity result for fully nonlinear, concave equations in $n$ dimensions under certain differentiability assumptions. The class of equations under consideration includes general Monge-Ampère equations of the form (1). Such results can also be obtained via the Green’s function technique, as mentioned in [16], regularity results actually somewhat easier than the therein proven a priori estimates. We also wish to draw attention to Safonov’s approach [11], which yields the existence of an $\alpha$, $0 < \alpha < 1$, such that the estimates of Theorems 1 and 2 hold for $\alpha < \alpha$, if the coefficients belong to $C^\alpha$. The purpose of the present paper is to cover the case of merely $\alpha$-Hölder-continuous coefficients ($0 < \alpha < 1$).

Like [14], the proof of Theorem 1 is based on a Legendre-like transformation. But then the proof rests on a technique due to Campanato [3, 4], unlike [14], where the Schauder technique was employed, which unavoidably requires the regularity $z(x, y) \in C^{2,\alpha}(\Omega)$. At the beginning of the second section we shall review from [14], incorporating the necessary changes.

Similar remarks apply to the proof of Theorem 2. Like [15], we first straighten $\partial \Omega$ locally. It is then important to introduce the zero-boundary data not too early, i.e., only after performing the transformation.

It should also be noted, that we have achieved a few simplifications over [14, 15]. We adopt the notation used therein. The letter $C$ denotes various constants, which may change from line to line. If possible, we choose constants to be $\geq 1$.

II. Proof of Theorem 1. Suppose $D_R = D_R(x_0, y_0)$ is a circular disc of radius $R > 0$, centered at $(x_0, y_0) \in \Omega$. The very first thing to note is that the equation (1) can be written in the equivalent form

$$(r + C)(t + A) - (s - B)^2 = \Delta.$$ 

By putting

$$A_0 := A(x_0, y_0, \ldots, q(x_0, y_0)), \ldots, E_0 := E(\cdot, \cdot), \Delta_0 := \Delta(\cdot, \cdot),$$

the equation (1) can therefore be rewritten in the form

$$(r + C_0)(t + A_0) - (s - B_0)^2 = \Delta_0 + ((A_0 - A)r + 2(B_0 - B)s + (C_0 - C)t + (E - E_0))$$

$$= \hat{f}(x, y).$$
Hence the function
\[ \tilde{z}(x, y) := z(x, y) + \frac{1}{2}(C_0(x - x_0)^2 - 2B_0(x - x_0)(y - y_0) + A_0(y - y_0)^2) 
+ 2(K + a)(y - y_0) \]
solves the equation
\[ \tilde{\tau}^2 - \tilde{s}^2 = \tilde{f}(x, y) \geq 1/2c =: 1/\tilde{c} > 0 \]
in \( D := D_{R_0}(x_0, y_0) \), where
\[ R_0 := \min\{1/\sqrt{10bcK}, d/2\}. \]

Here \( d > 0 \) is a lower bound for the distance between \((x_0, y_0)\) and \( \partial \Omega \). We furthermore have the estimates \( \tilde{q} \geq 1, |\tilde{\tau}|, |\tilde{t}| \geq 1/\tilde{c}\tilde{K} \), where \( \tilde{K} \geq 1 \) is a bound for the \( C^{1,1} \)-norm of \( \tilde{z}(x, y) \).

Now we make the variable transformation
\[ T: \begin{cases} 
    u = x, \\
    v = \tilde{q}(x, y) 
\end{cases} \]
where \((x, y) \in D\). Compare also [15, Lemma 3], for the following list of its properties:

(i) \( T \) maps \( D \) homeomorphically onto the image \( T(D) \).

(ii) For \((x', y'), (x'', y'') \in D\), we have the dilation estimates
\begin{align*}
(u' - u'')^2 + (v' - v'')^2 &\leq \gamma_1^2((x' - x'')^2 + (y' - y'')^2), \\
(x' - x'')^2 + (y' - y'')^2 &\leq \gamma_2^2((u' - u'')^2 + (v' - v'')^2),
\end{align*}
with constants \( \gamma_1, \gamma_2 \geq 1 \), depending only on \( \tilde{c}, \tilde{K} \).

(iii) Hence the inclusions
\[ T(D_{R/\gamma_1}(x_0, y_0)) \subset D_{R}(u_0, v_0), \quad D_{R/\gamma_2}(u_0, v_0) \subset T(D_{R}(x_0, y_0)) \]
hold for all \( R, 0 < R \leq R_0 \).

(iv) The function \( y(u, v) \in C^{0,1}(T(D)) \) is a weak solution of the equation
\[ y_{uu} + (\tilde{f}y_v)_v = 0. \]

Only the proof of (4) needs to be modified slightly: On estimating
\[ |v' - v''| = \left| \int_0^1 \{s((1 - \tau)x' + \tau x'', (1 - \tau)y' + \tau y'')(x' - x'') + t(\cdots)(y' - y'')\} \, d\tau \right| \]
\[ \geq -\tilde{K}|x' - x''| + \frac{1}{\tilde{c}\tilde{K}}|y' - y''|, \]
we obtain
\[ (u' - u'')^2 + (v' - v'')^2 \geq (1 + \tilde{K}^2)(x' - x'')^2 - \frac{2}{\tilde{c}}|x' - x''||y' - y''| + \frac{1}{(\tilde{c}\tilde{K})^2}(y' - y'')^2 \]
\[ \geq \left(1 + \left(1 - \frac{1}{\delta}\right)\tilde{K}^2\right)(x' - x'')^2 + \frac{1 - \delta}{(\tilde{c}\tilde{K})^2}(y' - y'')^2. \]

Inequality (4) then follows by choosing \( \delta \) to be the mean of \( \tilde{K}^2/(1 + \tilde{K}^2) \) and 1.
Noting that
\[
\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{s}{t} & 1/t \end{pmatrix},
\]
we proceed to rewrite equation (5) in the form
\[
y_{uu} + (\Delta_0 y_v)_v = g_v,
\]
where
\[
g(u, v) := \frac{((A - A_0)r + 2(B - B_0)s - (C - C_0)t + (E_0 - E))/t.}
\]

We can now apply the Campanato technique [3, 4] as described in Chapter III of Giaquinta’s book [4]. It is unnecessary to carry out the first step, [5, Theorem 3.1], because of the boundedness of \(\nabla y(u, v)\). As in the proof of [5, Theorem 3.2, resp. Theorem 2.2], we obtain the inequalities
\[
\int \int_{D_\rho} |\nabla y - (\nabla y)_\rho|^2 \, du \, dv 
\leq C \left\{ \left( \frac{\rho}{R} \right)^4 \int \int_{D_R} |\nabla y - (\nabla y)_R|^2 + \int \int_{D_R} |g - g_R|^2 \right\}
\]
for all \(\rho \leq R \leq R_0/\gamma_2\). Here all discs are centered at \((u_0, v_0)\) and
\[
g_R := \frac{1}{|D_R|} \int \int_{D_R} g(u, v) \, du \, dv.
\]
Using the dilation inequality (3), we estimate
\[
\int \int_{D_R} |g - g_R|^2 \leq C\int \int_{D_R} |g|^2 \leq CR^{2+2\alpha},
\]
where the last constant also depends on \(b\). It therefore follows that
\[
\int \int_{D_\rho} |\nabla y - (\nabla y)_\rho|^2 \, du \, dv \leq C \left\{ \left( \frac{\rho}{R} \right)^4 \int \int_{D_R} |\nabla y - (\nabla y)_R|^2 + R^{2+2\alpha} \right\}.
\]
An iteration argument, [5, Lemma 2.1], yields for all \(\rho \leq R \leq R_0/\gamma_2\)
\[
\int \int_{D_\rho} |\nabla y - (\nabla y)_\rho|^2 \, du \, dv \leq C \left\{ \left( \frac{\rho}{R} \right)^{2+2\alpha} \int \int_{D_R} |\nabla y - (\nabla y)_R|^2 + \rho^{2+2\alpha} \right\}
\leq \frac{C}{R^{2\alpha}} \rho^{2+2\alpha},
\]
in incorporating the boundedness of \(\nabla \eta\).

We are not done, because the variables \(u, v\), and therefore also the function \(y(u, v)\), depend on the point \((u_0, v_0)\). Hence we reintroduce the \(x, y\)-variables, using some elementary geometric measure theory. By (6), we have a.e.
\[
y_v - (y_v)_\rho = \frac{1}{t} \frac{|T^{-1}(D_\rho)|}{|D_\rho|} (t : \rho - t),
\]
where
\[
g \, : \rho = g(x_0, y_0) : \rho := \frac{1}{|T^{-1}(D_\rho)|} \int \int_{T^{-1}(D_\rho)} g(x, y) \, dx \, dy.
\]
and consequently
\[ \int \int_{D_\rho} |y_v - (y_v)_\rho|^2 \, du \, dv = \frac{|T^{-1}(D_\rho)|}{|D_\rho|} \int \int_{T^{-1}(D_\rho)} \frac{1}{t} |t - t_\rho|^2 \, dx \, dy. \]

The inequalities
\[ \int \int_{D_{\rho/\gamma_1}(x_0, y_0)} |t - t_\rho|^2 \, dx \, dy \leq \frac{C}{R_0^2} \rho^{2+2\alpha} \]

hold therefore for all \((x_0, y_0) \in \Omega\) and all \(\rho, \ 0 < \rho \leq R_0/\gamma_2\). Here \(t_\rho\) depends on \((x_0, y_0)\) through the transformation \(T\).

Similarly
\[ y_u - (y_u)_\rho = \frac{1}{t} \frac{|T^{-1}(D_\rho)|}{|D_\rho|} (\tilde{s}, \tilde{t} - \tilde{s}, \tilde{t}_\rho) \]
a.e., hence
\[ \int \int_{D_\rho} |y_u - (y_u)_\rho|^2 \, du \, dv = \frac{|T^{-1}(D_\rho)|}{|D_\rho|} \int \int_{T^{-1}(D_\rho)} \frac{1}{t} |\tilde{s}, \tilde{t}, \rho - \tilde{s}, \tilde{t}_\rho|^2 \, dx \, dy. \]

On estimating
\[ |s - s_\rho| \leq \left| \tilde{s} - \tilde{s}_\rho \frac{1}{\tilde{t}, \rho} \right| + \left| \tilde{s}, \rho \frac{1}{t, \rho} - \tilde{s}, \rho \right| \leq C \left\{ \left| \tilde{s}, \rho - \tilde{s}, \rho \tilde{t}_\rho \right| + \left| \tilde{t} - \tilde{t}_\rho \right| \right\} \]
a.e. and using the differential equation (1), we therefore conclude that for all \((x_0, y_0) \in \Omega\) and all \(\rho, \ 0 < \rho \leq R_0/\gamma_2\),
\[(9) \quad \int \int_{D_{\rho/\gamma_1}(x_0, y_0)} \{|r - r_\rho|^2 + \cdots + |t - t_\rho|^2\} \, dx \, dy \leq \frac{C}{R_0^2} \rho^{2+2\alpha}. \]

We finally deduce the Hölder continuity of \(r, s, t\) by proceeding as in the proof of [5, Theorem 1.2], that the Campanato spaces \(L^{2,n+2\alpha}(\Omega)\) are contained in the Hölder spaces \(C^\alpha(\Omega)\) for \(0 < \alpha < 1\) (here \(n = 2\)). First we get the analogue of formula (1.8) of [5, p. 71], that there exist the limits
\[ \tilde{r}(x_0, y_0) := \lim_{\rho \to 0} r_\rho, \ldots, \tilde{t}(x_0, y_0) := \lim_{\rho \to 0} t_\rho, \]
and
\[ (10) \quad |r_\rho - \tilde{r}(x_0, y_0)|, \ldots, |t_\rho - \tilde{t}(x_0, y_0)| \leq C \rho^\alpha \]
for \(\rho \leq R_0/\gamma_2\). Here \(C\) depends also on \(R_0\).

Now suppose \(\Omega'\) is a disc in \(\Omega\) with sufficiently small radius, indeed \(\Omega' := D_{R_0/2\gamma_1}(x_0, y_0)\) would be fine. Let \((x', y'), (x'', y'') \in \Omega'\) and \((\rho/2\gamma_1)^2 := (x' - x'')^2 + (y' - y'')^2\). Then
\[ |\tilde{t}(x', y') - \tilde{t}(x'', y'')| \leq |\tilde{t}(x', y') - t(x', y')| + |t(x', y') - t(x'', y'')| + |t(x'', y'') - \tilde{t}(x'', y'')| \leq C \rho^\alpha + |t(x', y') - t(x, y)| + |t(x, y) - t(x'', y'')| \rho, \]

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and by integrating with respect to \((x, y)\) over \(D_{\rho/\gamma_1}(x', y') \cap D_{\rho/\gamma_1}(x'', y'')\), which includes \(D_{\rho/2\gamma_1}(x', y')\), we thus obtain

\[
|\ell(x', y') - \ell(x'', y'')| \leq C \rho^\alpha + \frac{1}{|D_{\rho/2\gamma_1}(x', y')|} \cdot \left\{ \int \int_{D_{\rho/\gamma_1}(x', y')} |t - t(x', y'); \rho| \, dx \, dy \right. \\
+ \left. \int \int_{D_{\rho/\gamma_1}(x'', y''), \rho} |t - t(x'', y''); \rho| \, dx \, dy \right\}.
\]

We use the Hölder inequality to get

\[
|\ell(x', y') - \ell(x'', y'')| \leq C \rho^\alpha,
\]

and, of course, similar inequalities for \(\bar{r}, \bar{s}\). But, by combining (9) and (10),

\[
\int \int_{D_{\rho/\gamma_1}(x_0, y_0)} \{|r - \bar{r}(x_0, y_0)|^2 + \cdots + |t - \bar{t}(x_0, y_0)|^2\} \, dx \, dy \leq C \rho^{2+2\alpha}
\]

for all \((x_0, y_0) \in \Omega\), and therefore \(r = \bar{r}, \ s = \bar{s}, \ t = \bar{t}\) on the respective Lebesgue sets, which proves the theorem.

**III. Proof of Theorem 2.** In a neighborhood of \((x_0, y_0) \in \Omega, \ \Omega\) is of the form

\[
\Omega_{R^*} := \Omega \cap D_{R^*} = \{(x, y) \in D_{R^*} | G(x, y) < 0\}
\]

for some \(R^* < \hat{R} \leq 1\), where \(G(x, y) \in C^{2,\alpha}(\overline{D}_{\hat{R}}), \ G^2_x \geq 1/\kappa > 0\). We apply the variable transformation \(\xi = G(x, y), \eta = y\), in order to flatten \(\partial \Omega \cap D_{\hat{R}}\). We calculate the derivatives of \(z(\xi, \eta) := z(x(\xi, \eta), y(\xi, \eta))\),

\[
\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} G_x \\ G_y \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\begin{pmatrix} r \\ s \\ t \end{pmatrix} = \begin{pmatrix} G_x \\ G_y \end{pmatrix} \begin{pmatrix} \hat{r} \\ \hat{s} \\ \hat{t} \end{pmatrix} + \hat{p} \begin{pmatrix} G_{xx} \\ G_{xy} \\ G_{yy} \end{pmatrix}.
\]

Hence

\[
\begin{pmatrix} r + C \\ s - B \\ s - B \quad t + A \end{pmatrix} = \begin{pmatrix} G_x \\ G_y \end{pmatrix} \begin{pmatrix} \hat{r} + \hat{C} \\ \hat{s} - \hat{B} \quad \hat{t} + \hat{A} \end{pmatrix} \begin{pmatrix} G_x \\ G_y \end{pmatrix},
\]

where

\[
\begin{pmatrix} \hat{C} \\ -\hat{B} \quad \hat{A} \end{pmatrix} := \frac{1}{G_x^2} \begin{pmatrix} G^2_x & 0 \\ -G_y & G_x \end{pmatrix} \begin{pmatrix} C \\ -B \quad A \end{pmatrix} + \hat{p} \begin{pmatrix} G_{xx} \\ G_{xy} \\ G_{yy} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \quad -G_y \end{pmatrix}.
\]

Let

\[
\hat{\Delta} := \Delta / G_x^2 \geq 1/\kappa c > 0;
\]

then \(\hat{z}\) solves the elliptic Monge-Ampère equation

\[
(\hat{r} + \hat{C})(\hat{t} + \hat{A}) - (\hat{s} - \hat{B})^2 = \hat{\Delta}.
\]

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When working near the boundary of $\Omega$, we can therefore assume w.l.o.g. that $\partial \Omega$ is flat.

From now on, we can follow the proof of Theorem 1, resp. [15], working in sets of the form $\Omega_R = \Omega \cap D_R$, resp. half-discs. We observe that the Legendre-like transformation preserves the flatness of $\partial \Omega \cap D_R$ and introduce zero-boundary data of $y(u, v)$, resp. $\eta(u, v)$ as in [15]. As far as the Campanato technique is concerned, we can refer to Lemma 12.II of [3, p. 355], for an inequality of the form (7) for half-discs, instead of referring to Lemma 8.II of [3, p. 338], or to Giaquinta’s book. Note that we have to use an inequality of the form (8) for the nontangential derivative, i.e., for $y_u$, resp. $\eta_u$. Theorem 2 is thus proved.

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DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF IOWA, IOWA CITY, IOWA 52242