THE MORSE INDEX THEOREM
WHERE THE ENDS ARE SUBMANIFOLDS

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ABSTRACT. In this paper the Morse Index Theorem is proven in the case where submanifolds $P$ and $Q$ are at the endpoints of a geodesic, $\gamma$. At $\gamma$, the index of the Hessian of the energy function defined on paths joining $P$ and $Q$ is computed using $P$-focal points, and a calculation at the endpoint of $\gamma$, involving the second fundamental form of $Q$.

1. Introduction. Let $M$ be a complete Riemannian manifold with submanifolds $P$ and $Q$. The energy function $E$ is defined on the space $\Omega(M; P, Q)$ of piecewise $C^\infty$ paths joining $P$ and $Q$. A path $\gamma \in \Omega(M; P, Q)$ is a critical point for $E$ when $\gamma$ is a geodesic intersecting $P$ and $Q$ orthogonally. The tangent space, $T\Omega_\gamma$, consists of piecewise $C^\infty$ vector fields along $\gamma$ with initial and final vectors tangential to $P$ and $Q$, respectively. A symmetric bilinear map, $I$, is defined on $T\Omega_\gamma \times T\Omega_\gamma$ to $R$ and is called the Morse index form.

When $Q$ is a point, the Morse Index Theorem yields the index of $I$ as the sum of the $P$-focal points along $\gamma$ counted with multiplicities. Both Ambrose [1] and Bolton [2] have proven index theorems in the general case, where $Q$ is a submanifold. Ambrose defines a "$(P, Q)$ conjugate point," while Bolton uses the notion of a signed $(P, Q)$ focal point which is employed in the calculations of the index of $I$.

In this paper the index of a critical point $\gamma$ is found using $P$-focal points and a computation at the endpoint of $\gamma$ contained in $Q$, involving the second fundamental form of $Q$ with respect to $\gamma$. This method allows for a simpler proof of the theorem as well as an easy computation of the index of geodesics in many spaces. In a paper to follow this one, the homotopy type of some path spaces joining submanifolds on a Riemannian manifold have been computed. This result is obtained, with some minor modifications, by following the proof in Milnor [4, pp. 88-95].

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2. Definitions. $M$ is a complete Riemannian manifold of dimension $d$ with the Levi-Civita connection.

$\gamma(t)$, $t \in [0, T]$, is a geodesic in $M$.

$P$ and $Q$ are submanifolds of $M$ with $\gamma(0) \in P$; $\gamma'(0) \perp P_{\gamma(0)}$; $\gamma(T) \in Q$; $\gamma'(T) \perp Q_{\gamma(T)}$.

$r$ is the dimension of $Q_{\gamma(T)}$, $0 \leq r < d$. 

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$H$ is the linear space of continuous piecewise $C^\infty$ vector fields along $\gamma$ which are orthogonal to $\gamma$ and whose initial and final vectors are in $P_{\gamma(0)}$ and $Q_{\gamma(T)}$, respectively. Then,

$$H = \left\{ V(t) = \sum_{i=1}^{d-1} h_i(t) E_i(t) \mid V(0) \in P_{\gamma(0)}, V(T) \in Q_{\gamma(T)} \right\},$$

where $E_1, \ldots, E_{d-1}$ are orthonormal parallel vector fields along $\gamma$ and orthogonal to $\gamma$, and $h_1, \ldots, h_{d-1}$ are real valued continuous piecewise $C^\infty$ functions defined on $[0, T]$.

A **Jacobi field** $X$ is a vector field along $\gamma$ which satisfies the differential equation $X'' - RX = 0$, where $RX = R(\gamma', X)\gamma'$ is the curvature tensor of the Levi-Civita connection.

A **$P$-Jacobi field** is a Jacobi field which is orthogonal to $\gamma$ with $J(0) \in P_{\gamma(0)}$ and $J'(0) - S_0 J(0) \perp P_{\gamma(0)}$, where $S_0$ is the second fundamental form of $P$ at $\gamma(0)$ with respect to $\gamma'(0)$.

$J_1, \ldots, J_{d-1}$ are $d-1$ linearly independent $P$-Jacobi fields which span the space of $P$-Jacobi fields.

$B$ is the set of all $X \in H$ such that

$$X(t) = \sum_{i=1}^{d-1} f_i(t) J_i(t) \quad \text{with} \quad X(T) = 0,$$

and $f_1, \ldots, f_{d-1}$ are real valued continuous piecewise $C^\infty$ functions defined on $[0, T]$.

$I$ is a symmetric bilinear map from $H \times H$ to $R$ defined as follows.

$$I(X, Y) = \int_0^T (RX(t) - X''(t), Y(t)) \, dt + \sum_{i} (X'(p_i^-) - X'(p_i^+), Y(p_i)) + (X'(t) - S_t X(t), Y(t))_0^T$$

where $p_i$ is a point of discontinuity of $X'$ in $(0, T)$; $S_0$ is the second fundamental form of $P$ at $\gamma(0)$ with respect to $\gamma'(0)$; and $S_T$ is the second fundamental form of $Q$ at $\gamma(T)$ with respect to $\gamma'(T)$.

A **$P$-focal point** is a point $\gamma(t), \ t \in (0, T]$, for which there exists a nonzero $P$-Jacobi field which vanishes at $t$.

The **multiplicity**, $m$, of the $P$-focal point $\gamma(t)$ is the dimension of the space of $P$-Jacobi fields which vanish at $t$.

$A$ is a symmetric bilinear map defined on the space spanned by $J_1, \ldots, J_{d-1}$ whose value at $T$ is contained in $Q_{\gamma(T)}$ and is defined as follows:

$$A(V, W) = \langle V'(T) - S_T V(T), W(T) \rangle.$$

3. The Index Theorem.

**Theorem (Morse Index Theorem with Variable Endpoints).** The index of $I$ is equal to the number of points $\gamma(t), \ 0 < t < T$, such that $\gamma(t)$ is a $P$-focal point; each such $P$-focal point counted with its multiplicity, plus the index of $A$. (Assume $\gamma(T)$ is not a $P$-focal point.)
The above theorem states

$$i(I) = \sum_{i=1}^{k} m_i + i(A),$$

when $\gamma(T)$ is not a $P$-focal point, where $i(I)$ = index of $I$; $i(A)$ = index of $A$; $m_i$ is the multiplicity of $\gamma(t_i)$; and $\gamma(t_1), \ldots, \gamma(t_k)$ are the set of $P$-focal points along $\gamma$, $(0 < t_1 < \ldots < t_k < T)$.

Our aim will be to write $H = B \oplus B^C$, where $I$ is positive on $B$. (For definition of $B$ see §2.) We will show that $B^C$ is a finite dimensional space and construct a subspace of $B^C$ on which $I$ is negative definite and whose dimension is equal to or greater than any other subspace of $H$ on which $I$ is negative definite. This will yield $i(I)$.

**REMARK.** The subspace $B$ can be characterized as the set of all vector fields in $H$ whose values at the $P$-focal points are in the span of $J_1, \ldots, J_{d-1}$ and whose value at $T$ is zero.

This follows from the fact that any broken $C^\infty$ vector field which can be expressed as $\sum_{i=1}^{d-1} f_i(t) E_i(t)$ and is in the span of $J_1(t_i), \ldots, J_{d-1}(t_i)$ at all $P$-focal points $t_i$ can also be written as $\sum_{i=1}^{d-1} g_i(t) J_i(t)$ for broken $C^\infty$ functions $g_i$ [3, p. 231].

The next two definitions will yield $(\sum_{i=1}^{k} m_i + r)$ linearly independent elements of $H$ whose span will be denoted by $B^C$.

**DEFINITION OF $K_2$.** Since $T$ is not a $P$-focal point, we can choose $r$ linearly independent $P$-Jacobi fields $K_1, \ldots, K_r$ with the following properties: (1) $K_1(T), \ldots, K_r(T)$ span $Q_i(T)$; (2) $A$ is negative definite on the span of $K_1, \ldots, K_N$, where $N = \text{index } A$ and $N \leq r$; and (3) $A$ is positive ($\geq 0$) on the span of $K_{n+1}, \ldots, K_r$. (Recall that $A$ was defined as a symmetric bilinear map on those $P$-Jacobi fields which, when evaluated at $T$, lie in $Q_i(T)$. The dimension of this space is $r$.)

**DEFINITION OF $V_{ij}$.** Consider the $P$-focal point $\gamma(t_i)$ with multiplicity $m_i$. Let $Y_i^1, \ldots, Y_i^{m_i}$ be $m_i$ linearly independent $P$-Jacobi fields such that

1. $Y_i^{j_1}(t_i) = 0$ for $j_1 = 1, \ldots, m_i$,
2. $\{Y_i^{j_1}(t_i)\}$ for $j_1 = 1, \ldots, m_i$, form an orthonormal set.

Let $Z_{ij}^1$ be parallel vector fields along $\gamma$ such that

$$Z_{ij}^1(t_i) = -Y_i^{j_i}(t_i) \quad \text{for } j_1 = 1, \ldots, m_i.$$

Let $\phi_i: [0, T] \to R$ be a $C^\infty$ function such that (1) $\phi_i(t_i) = 1$; (2) $\phi_i(t_i)$ has small support about $t_i$; and (3) $0 \leq \phi_i(t) \leq 1$. Let

$$Z_{ij}^i(t) = \phi_i(t) Z_{ij}^1(t); \quad i = 1, \ldots, m_i.$$

Let

$$V_{ij}^i(t) = \begin{cases} Y_i^{j_i}(t) + \lambda Z_{ij}^i(t) & \text{for } 0 \leq t \leq t_i, \\ \lambda Z_{ij}^i(t) & \text{for } t_i \leq t \leq T, \end{cases}$$

where $\lambda > 0$.

**(Note:** It follows from (1) and (2) in the definition of $V_{ij}$ that

$$\langle \text{span}[Y_i^{j_1}(t_i), \ldots, Y_i^{m_i}(t_i)] \rangle = \text{span}(J_1(t_i), J_2(t_i), \ldots, J_{d-1}(t_i), \gamma'(t_i)).$$
DEFINITION OF $B^C$. Let $B^C$ denote the span of the vectors $V_i^{j_i}$ ($i = 1, \ldots, k; j_i = 1, \ldots, m_i$) and $K_l$ ($l = 1, \ldots, r$).

CLAIM. (1) The dimension of $B^C$ is $\sum_{i=1}^{k} m_i + r$,

(2) $B^C \cap B = 0$.

The claim follows from the definitions of $B$ and $B^c$, by making the support of each $\phi_i$ small enough, and by looking at Figure 1.

**Lemma 1.** $H = B \oplus B^c$.

**Proof.** Let $x \in H$. There is a $c \in B^c$ such that $x - c \in B$. This can be accomplished by choosing $c$ equal to a linear combination of elements in $B^c$, so that at $P$-focal point the value of $x - c$ lies in the span of the $P$-Jacobi fields and at $T$ is equal to zero. Thus $x - c \in B$. Since $B \cap B^c = 0$, we have Lemma 1.

Lemma 2 will show $I$ is positive on $B$ and Lemma 3 will exhibit a subspace of $B^C$ on which $I$ is negative definite.

**Lemma 2.** $I(V, V) \geq 0$ for $V \in B$.

**Proof.** Let $V \in B$. Then

$$V = \sum_{t=1}^{d-1} f_t J_t$$
and
\[
I(V, V) = \left\langle \sum_{i=1}^{d-1} f_i(T) J_i'(T), V(T) \right\rangle + \int_0^T \left\langle \sum f_i'(t) J_i(t), \sum f_i'(t) J_i(t) \right\rangle \, dt \\
= \int_0^T \left\langle \sum f_i'(t) J_i(t), \sum f_i'(t) J_i(t) \right\rangle \, dt \quad \text{(since } V(T) = 0) \\
\geq 0.
\]

For the first equality, see [3, p. 229]. This proves Lemma 2.

**Lemma 3.**

\[
\text{index}(I_{BC}) = \sum_{i=1}^k m_i + \text{index}(A).
\]

**Proof.** \(K_1, \ldots, K_r\) were chosen to be \(r\) linearly independent \(P\)-Jacobi fields such that \(A\) is negative definite \((< 0)\) on the span of \(K_1, \ldots, K_N\) and positive \((\geq 0)\) on \(K_{N+1}, \ldots, K_r\), \(N = \text{index } A\).

We wish to show that \(I\) is negative definite on the span of \(\{V_{i,j_i}\}_{i=1}^k, j_i=1, \ldots, m_k\) and \(K_1, \ldots, K_N\), and that \(I\) is positive on the span of \(K_{N+1}, \ldots, K_r\).

\[
I \left( \sum_{i,j_i} \alpha_i^{j_i} V_{i,j_i}^j + \sum_{l=1}^N \beta_l K_l \right) = I \left( \sum_{i,j_i} \alpha_i^{j_i} V_{i,j_i}^j \right) \\
+ 2I \left( \sum_{i,j_i} \alpha_i^{j_i} V_{i,j_i}^j, \sum_{l=1}^N \beta_l K_l \right) \\
+ I \left( \sum_{l=1}^N \beta_l K_l \right). 
\]

The computation of \((3)\) yields \(I(\sum \beta_l K_l) < 0\).

**Proof.** Let

\[
K = \sum_{i=1}^N \beta_i K_i.
\]

Then
\[
I(K) = \int_0^T (RK - K'', K) \, dt + \sum_{\text{jumps of } K'} \langle K'(p_i^-) - K'(p_i^+), V_{i,j_i}^j(p_i) \rangle \\
+ \langle K'(t) - S_i K(t), K(t) \rangle_{T_0}^T = \langle K'(T) - S_T K(T), K(T) \rangle.
\]

This follows from the fact that \(K\) is a \(P\)-Jacobi field which is smooth and satisfies \(RK - K'' = 0\) and \(K'(0) - S_0 K(0) \perp P_{\gamma(0)}\).

So \(I(K) = \langle K'(T) - S_T K(T), K(T) \rangle = A(K) < 0\), since \(A\) is negative definite on the span of \(K_1, \ldots, K_N\).
The computation of (2) yields
\[ I \left( \sum \alpha_i \psi_j, \sum \beta_i K_i \right) = 0. \]

**Proof.** When (2) is expanded we get linear combinations of terms of the form
\[ I(V_i^j, K_i). \]
\[ I(V_i^j, K_i) = \int_0^T \langle RK_i - K_i'', V_i^j \rangle dt + \sum_{\text{jumps}} \langle K'(p_i^-) - K'(p_i^+), K(p_i) \rangle + \langle K'_i(t) - S_i K_i(t), V_i^j(t) \rangle.T_0^T = 0, \]
since \( K_i \) is a P-Jacobi field, \( K'_i(0) - S_0 K(0) \perp P_{\gamma(0)} \) and \( V_i^j(T) = 0 \).

The computation of (1) yields \( I(\sum \alpha_i^j \psi_j) < 0 \).

**Proof.** Let
\[ \hat{Y}_i^j = \begin{cases} \hat{Y}_i^j(t), & 0 \leq t \leq t_i, \\ 0, & t_i \leq t \leq T. \end{cases} \]
Then \( V_i^{j}(t) = \hat{Y}_i^j(t) + \lambda Z_i^{j}(t); t \in [0, T] \).
\[ I(V_i^{j}, V_i^{h}) = I(\hat{Y}_i^j, \hat{Y}_i^h) + \lambda I(Z_i^{j}, Z_i^{h}) \]
\[ = \begin{cases} I(\hat{Y}_i^j, \hat{Y}_i^h) & \text{a)} \\ + \lambda^2 I(Z_i^{j}, Z_i^{h}) & \text{b)} \\ + \lambda I(Z_i^{j}, \hat{Y}_i^h) + \lambda I(\hat{Y}_i^j, Z_i^{h}) & \text{c)}. \end{cases} \]

(a) = \( I(\hat{Y}_i^j, \hat{Y}_i^h) = \int_0^T \langle R\hat{Y}_i^j - \hat{Y}_i^{m_j}, \hat{Y}_h^i \rangle dt + \langle \hat{Y}_i^{m_j}(t_i^-) - \hat{Y}_i^{m_j}(t_i^+), \hat{Y}_i^h(t_i) \rangle \)
\[ + \langle \hat{Y}_i^h(t) - S_i \hat{Y}_i^h(t), \hat{Y}_i^h(t) \rangle.T_0^T = 0, \]
when \( h \leq i \).

This follows from \( \hat{Y}_i^{m_j} - R\hat{Y}_i^{m_j} = 0, \hat{Y}_i^h(t_i) = 0, \hat{Y}_h^i(T) = 0 \), and \( \hat{Y}_i^{m_j}(0) - S_0 \hat{Y}_i^{m_j}(0) \perp P_{\gamma(0)} \).

(c) = \(-2\lambda\delta_{ih}\delta_{ji}\), which is shown to be true as follows. For \( i \neq h \), we get
\[ I(Z_i^{j}, \hat{Y}_h^i) = \int_0^T \langle R\hat{Y}_h^i - \hat{Y}_h^{m_l}, Z_i^{j} \rangle dt + \langle \hat{Y}_h^{m_l}(t_i^-) - \hat{Y}_h^{m_l}(t_i^+), Z_i^{j}(t_i) \rangle \]
\[ + \langle \hat{Y}_h^j(t) - S_i \hat{Y}_h^j(t), Z_i^{j}(t_i) \rangle.T_0^T = 0, \]
when the support of \( \phi_i \) is small enough.

For \( i = h \), we get the same as above except at \( t_i \).
\[ I(Z_i^{j}, \hat{Y}_h^i) = \langle \hat{Y}_h^{m_l}(t_i^-) - \hat{Y}_h^{m_l}(t_i^+), Z_i^{j}(t_i) \rangle \]
\[ = \langle Y_i^{m_l}(t_i^-), Z_i^{j}(t_i) \rangle \] (since \( \hat{Y}_h^i(t_i) = 0 \) for \( t \in [t_i, T] \))
\[ = \langle Y_i^{m_l}(t_i), -Y_i^{m_j}(t_i) \rangle \]
\[ = -\delta_{ji}\|Y_i^{m_j}(t_i)\|^2 \]
(since \( \{Y_i^{m_1}, Y_i^{m_2}, \ldots, Y_i^{m_r} \} \) is an orthonormal set evaluated at \( t_i \))
\[ = -\delta_{ji}. \]
Therefore, $\lambda I(Z^i_k, \dot{Y}^i_k) + \lambda I(\dot{Y}^i_k, Z^i_k) = -2\lambda \delta_{ih} \delta_{ij}$.

Putting together the results from (a), (b), and (c), we have

$$I(V_i^j, V_i^h) = \lambda^2 I(Z^i_k, Z^i_k) - 2\lambda \delta_{ih} \delta_{ij}.$$

**Notation.** Let $A$ be the $(\sum m_i) \times (\sum m_i)$ symmetric matrix $(I(Z^i_k, Z^i_k))$. If $X = \sum a^i_i V_i^j$, $Y = \sum b^i_i V_i^j$ let

$$\langle X, Y \rangle = \sum a^i_i b^i_j \quad \text{and} \quad \|X\|^2 = \langle X, X \rangle.$$

Then for $X = \sum_{i,j} a^i_j V_i^j$, we have

$$I(X, X) = \lambda^2 \langle AX, X \rangle - 2\lambda \langle X, X \rangle.$$

If $A = 0$, $I(X, X) < 0$ for $\lambda > 0$, $X \neq 0$. If $A \neq 0$, $\|A\| \neq 0$, let $0 < \lambda < 2/\|A\|$. Then

$$I(X, X) = \lambda^2 \langle AX, X \rangle - 2\lambda \langle X, X \rangle \leq \lambda^2 \|AX\| \|X\| - 2\lambda \|X\|^2 \leq \lambda \|A\| \|X\|^2 - 2\lambda \|X\|^2 = 0.$$

This gives

$$I \left( \sum_{i=1}^{r} \alpha_i^i V_i^i, \sum_{i=1}^{r} \alpha_i^i V_i^i \right) < 0 \quad \text{for} \quad \sum_{i=1}^{r} \alpha_i^i V_i^i \neq 0.$$

Thus we have the computation of (1).

The results from (1), (2), and (3) show that $I$ is negative definite on the span of $\{V_i^j, K_1, \ldots, K_N\}_{i=1}^{1}, \ldots, k; j_i=1, \ldots, m_i$.

In order to finish proving Lemma 3 we need to show $I$ is positive ($\geq 0$) on the span of $K_{N+1}, \ldots, K_r$.

Definition of $B^c_-, B^c_+$. Let

$$B^c_- = \text{Span}(K_1, \ldots, K_N) \quad \text{and} \quad B^c_+ = \text{Span}(K_{N+1}, \ldots, K_r).$$

From Lemma 3 we have that $I|_{B^c_-} < 0$ and $I|_{B^c_+} \geq 0$, while from Lemma 2 we have that $I|_{B} \geq 0$.

Write $H = B \oplus B^c_+ \oplus B^c_-$. In order to complete the Index Theorem we need to show that $I(B, B^c_+) = 0$, which will follow if

$$I \left( K_j, \sum_{i=1}^{r} f_i J_i \right) = 0 \quad \text{for} \quad j = N + 1, \ldots, r, \quad i = 1, \ldots, d - 1.$$

This is true since $K_j$ is a $P$-Jacobi field, $S_0 K_j(0) - K_j'(0) \perp (\sum f_i J_i)(0)$ and $(\sum f_i J_i)(T) = 0$.

We therefore have a subspace $B^c_-$ of $H$ on which $I$ is negative definite and whose dimension $\sum_{i=1}^{k} m_i + i(A)$ is the maximum value $I$ can attain on any subspace of $H$. Thus the Index Theorem is proven.
4. Remarks. 1. The proof of the Index Theorem is symmetric with respect to the $P$ and $Q$ submanifolds at the ends of the geodesic. That is, we can use $Q$-Jacobi fields and $Q$-focal points to prove the theorem.

2. If $\gamma(T)$ is a $P$-focal point, then the proof of the Index Theorem is still valid when
   
   (a) $Q\gamma(T)$ is contained in the span of the $P$-Jacobi fields or when
   
   (b) $\gamma(0)$ is not a $Q$-focal point, or if $P\gamma(0)$ is contained in the span of the $Q$-Jacobi fields.

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