SOME INEQUALITIES FOR SINGULAR
CONVOLUTION OPERATORS IN L^p-SPACES

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ABSTRACT. Suppose that a bounded function \( m \) satisfies a localized multiplier condition \( \sup_{t > 0} \| \phi m(t \cdot) \|_{M_p} < \infty \), for some bump function \( \phi \). We show that under mild smoothness assumptions \( m \) is a Fourier multiplier in \( L^p \). The approach uses the sharp maximal operator and Littlewood-Paley-theory. The method gives new results for lacunary maximal functions and for multipliers in Triebel-Lizorkin-spaces.

Introduction. Given a bounded function \( m \) the associated multiplier transformation \( T_m \) is defined by \( [T_m f]^\wedge(\xi) = m(\xi) f^\wedge(\xi) \), \( f \in \mathcal{S}(\mathbb{R}^n) \). Here \( \mathcal{S} \) denotes the Schwartz space of rapidly decreasing \( C^\infty \)-functions and \( \mathcal{F} f = f^\wedge \) the Fourier transform. \( m \) is called a Fourier multiplier in \( L^p(\mathbb{R}^n) \) if \( T_m \) extends to a bounded operator in \( L^p(\mathbb{R}^n) \); the multiplier norm \( ||m||_{M_p} \) equals the operator norm of \( T_m \).

Suppose that \( \phi \) is a radial \( C_0^\infty \)-function with compact support in \( \mathbb{R}_0^n = \mathbb{R}^n \setminus \{0\} \) and suppose that

\[
\|m\|_{\dot{M}_p} = \sup_{t > 0} \|\phi m(t \cdot)\|_{M_p} < \infty.
\]

The purpose of this paper is to find easily verified conditions that (1) implies \( m \in M_p \). The condition \( \|m\|_{\dot{M}_p} < \infty \) is satisfied if and only if \( m \) is a Fourier multiplier on the class of homogeneous Besov-spaces \( \dot{B}^p_{\infty q} \) (see Peetre [14, p. 132]). In fact the space \( M_p \) can be characterized by \( \dot{M}_p \); a theorem of Johnson [10] states that \( m \in M_p \) if and only if \( m(\cdot + y) \in \dot{M}_p \) for every \( y \in \mathbb{R}^n \). However, a straightforward verification of this condition seems to be impossible for many singular convolution operators.

In some applications it is useful to replace the ordinary dilations \( x \mapsto tx \) by anisotropic ones: \( x \mapsto t^P x = \exp(P \log t) x \), where \( P \) is a real \( n \times n \)-matrix with trace \( \nu \), the real parts of the eigenvalues being contained in \((a_0, a_0^0)\), \( a_0 > 0 \). Then we ask, under which conditions \( \sup_{t > 0} \|\phi m(t^P \cdot)\|_{M_p} < \infty \) implies \( m \in M_p \). Throughout this paper \( \phi \) will always be chosen as in the following

DEFINITION. \( \phi \in C_0^\infty(\mathbb{R}_0^n) \) satisfies a Tauber condition with respect to the dilations \( (t^P) \) if for every \( x \neq 0 \) there is a \( t_x \) such that \( \phi(t_x^P x) \neq 0 \).

Sometimes we need special bump functions of the following kind: Let \( \rho \in C^\infty(\mathbb{R}_0^n) \) be a \( P \)-homogeneous distance function; this means that \( \rho(t^P x) = t \rho(x) \),
\( x \in \mathbb{R}^n, \ t > 0 \) and \( \rho(x) > 0, \ x \neq 0 \). Then we set \( \phi = \phi_0 \circ \rho \), where

(2) \( \phi_0 \in C_0^\infty(\mathbb{R}_+), \ \text{supp} \phi_0 \subset \left(\frac{1}{2}, 2\right), \ \sum_{k \in \mathbb{Z}} \phi_0(2^k s) = 1, \ \text{all } s > 0. \)

We note that every \( P \)-homogeneous distance function satisfies a triangle inequality
\( \rho(x + y) \leq b[\rho(x) + \rho(y)] \), for some \( b \geq 1 \).

It is easily seen that the condition \( \sup_{t > 0} \| \phi_m(t^{1/p}) \|_{M_p} < \infty \) is independent of the special choice of \( \phi \). In fact, assume that \( \phi, \bar{\phi} \) are chosen as in the definition. By a compactness argument, there are \( s_0, \ldots, s_N > 0 \) such that \( \sum_{i=1}^N \phi^2(s_i^p x) > 0 \) for all \( x \in \text{supp} \phi \). Since \( M_1 M_p \subset M_p \), we have

\[
\| \phi_m(t^{1/p}) \|_{M_p} \leq c \sum_{i=0}^N \| \phi^2(s_i^p x) m(t^{1/p}) \|_{M_p}
\]

\[
\leq c \sum_{i=0}^N \| \phi(\cdot) m \left( \left( \frac{t}{s_i} \right)^p \cdot \right) \|_{M_p} \leq c \sup_{s > 0} \| \phi_m(s^{1/p}) \|_{M_p}.
\]

We are most interested in the cases \( 1 < p < \infty \). For \( p = 1 \) a satisfactory result is the Hörmander multiplier criterion \([9]\). Here the condition \( \sup_{t > 0} \| \phi_m(t^{1/p}) \|_{M_1} < \infty \) is replaced by the somewhat stronger assumption

(3) \( \sup_{t > 0} \int_{|x| \geq \omega} |\mathcal{F}^{-1}[\phi_m(t^{1/p})]| \, dx \leq B(1 + \omega)^{-\varepsilon}, \ \text{all } \omega > 0. \)

(3) implies that \( T_m \) is of weak type \((1,1)\) and \( m \in M_p, \ 1 < p < \infty \). The usual assumption

\[
\sup_{t > 0} \| \phi m(t^{1/p}) \|_{\mathcal{L}_2^\alpha} < \infty, \ \alpha > n/2,
\]

(\( \mathcal{L}_2^\alpha \) denoting the Bessel-potential space as in Stein \([18]\)) implies (3) for some \( B \), if \( \varepsilon < \alpha - n/2 \).

We use the following notations: \( \mathcal{S}_0 \) denotes the subspace of Schwartz functions whose Fourier transforms are compactly supported in \( \mathbb{R}^n_0 \). \( \Delta_h \) is the difference operator, \( \Delta_h f = f(\cdot + h) - f(\cdot) \). The Lipschitz space \( \Lambda_\varepsilon \) is normed by

\[
\| f \|_{\Lambda_\varepsilon} = \| f \|_\infty + \sup_{h} |h|^{-\varepsilon} \| \Delta_h f \|_{\infty}, \ \text{if } 0 < \varepsilon < 1.
\]

By \( |S| \) we denote the Lebesgue measure of a set \( S \). The barred integral \( \int_S f \) denotes the mean value \( |S|^{-1} \int_S f(y) \, dy \). \( c \) will be a general constant with different values in different occurrences.

1. Main result.

THEOREM 1. Suppose that \( m \) is a bounded function which satisfies for some \( p \), \( 1 < p < \infty, \ \varepsilon > 0 \)

(i) \( \sup_{t > 0} \| \phi_m(t^{1/p}) \|_{M_p} \leq A, \)

(ii) \( \sup_{t > 0} \int_{|x| \geq \omega} |\mathcal{F}^{-1}[\phi_m(t^{1/p})]| \, dx \leq B(1 + \omega)^{-\varepsilon}. \)
Then

\[ \|m\|_{M_p} \leq cA[\log(2 + B/A)]^{1/p - 1/2}. \]

**Remark.** Of course, condition (ii) alone implies \( m \in M_p, 1 < p < \infty \), with multiplier norm \( \leq cB \), which may however be much larger than the constant in the theorem. This constant is actually sharp; it cannot be replaced by \( A[\log(2 + B/A)]^{\gamma} \) with \( \gamma < |1/p - 1/2| \). This can be seen by a well-known counterexample of Littman, McCarthy, Rivière [12], modified in Triebel’s monograph [19]. Choose \( \phi \) as in (2) and vectors \( \sigma_k \), satisfying \( \rho(\sigma_k) = (2b)^k \). Define

\[ m_N(\xi) = \sum_{k=N}^{2N} e^{i\sigma_k \xi} \phi(\xi - \sigma_k). \]

Since \( \|\phi m_N(Ae^\cdot)\|_{M_p} \leq c \) and \( \|D^\alpha[\phi m_N(Ae^\cdot)]\|_{M_p} \leq c2^{\alphaN^2} \) for all multi-indices \( \alpha \), Theorem 1 implies \( \|m_N\|_{M_p} \leq c_pN^\gamma(p) \), with \( \gamma(p) = |1/p - 1/2| \).

On the other hand, the discussion in [19, p. 125] shows that \( \|m_N\|_{M_p} \geq c_pN^\gamma(p) \).

The counterexample shows that the condition (1) alone does not imply \( m \in M_p \). In the following corollaries we shall see that this is valid under weak smoothness assumptions on \( m \). The proof of Theorem 1 is given in §2.

**Corollary 1.** Suppose that for some \( 1 < p < \infty \)

(i) \[ \sup_{t > 0} \|\phi m(t^P \cdot)\|_{M_p} \leq A_0, \]

(ii) \[ \sup_{t > 0} \int_{|h| \leq 2^{-i}} \|\Delta_h[\phi m(t^P \cdot)]\|_{M_p} dh \leq A_i. \]

Then

\[ \|m\|_{M_p} \leq A_0 + \sum_{l > 1} l|1/p - 1/2|A_l. \]

**Proof.** We may choose \( \phi \) as in (2). Let \( \psi \) be a \( C^\infty \)-function, supported in \( \{\rho(\xi) \leq (8b)^{-1}\} \), \( \int \psi(\xi) d\xi = 1 \). Further set \( \psi_l = 2^{ln}\psi(2^l \cdot), \chi_l = \psi_l - \psi_{l-1} (l \geq 1), \chi_0 = \psi_0 \).

We split

\[ m = \sum_{j \in \mathbb{Z}} \phi(2^{-j}t^P \cdot)m \]

\[ = \sum_{j \in \mathbb{Z}} \sum_{l \geq 0} [\chi_l * (\phi m(2^j t^P \cdot))](2^{-j}t^P \cdot) =: \sum_{l \geq 0} m_l. \]
Set \( g_j = \phi m(2^jP) \). Then \( \chi_l \ast g_j \) is supported in \( \{ \frac{1}{4} \leq \rho(\xi) \leq 4 \} \). If \( l \geq 1 \), we have for \( 2^k \leq s \leq 2^{k+1} \) \((\delta \) denoting Dirac measure)

\[
\|\phi m(s^P')\|_{M_p} \leq c \sum_{j=k-4}^{k+4} \|\chi_l \ast g_j\|_{M_p}
\]

\[
\leq c \sum_{j=k-4}^{k+4} \|(\delta - \psi_{l-1}) \ast g_j + (\psi_l - \delta) \ast g_j\|_{M_p}
\]

\[
\leq c \sum_{j=k-4}^{k+4} \int |\psi_l(\eta)|\|\Delta_\eta g_j\|_{M_p} \, d\eta + \int |\psi_{l-1}(\eta)|\|\Delta_\eta g_j\|_{M_p} \, d\eta
\]

\[
\leq c \sum_{j=k-4}^{k+4} \int |\psi_l(\eta)|\|\Delta_\eta g_j\|_{M_p} \, d\eta + \int |\psi_{l-1}(\eta)|\|\Delta_\eta g_j\|_{M_p} \, d\eta
\]

\[
\leq c(A_{l-1} + A_l).
\]

For all multi-indices \( \alpha \) it follows by a similar computation \((2^k \leq s \leq 2^{k+1})\)

\[
\|D^\alpha(\phi m_l(s^P'))\|_2 \leq c \sum_{j=k-4}^{k+4} \sum_{\beta \leq \alpha} \int |\psi_l(\eta)|\|\Delta_\eta g_j(\xi - 2^{-l}\eta) - g_j(\xi - 2^{-l+1}\eta)\|_{\infty} \, d\eta
\]

\[
\leq c 2^{|\alpha|}(A_{l-1} + A_l).
\]

Now we apply Theorem 1 and obtain

\[
\|m_l\|_{M_p} \leq c l^{1/p-1/2}(A_{l-1} + A_l), \quad l \geq 1.
\]

Analogously \( \|m_0\|_{M_p} \leq c A_0 \), and the assertion follows by summation.

**Corollary 2.** Suppose that \( \sup_{t>0} \|\phi m(t^P')\|_{M_p} < \infty \), for some \( p \in (1, \infty) \).

(i) If for some \( \varepsilon > 0 \)

\[
\sup_{t>0} \sup_{h \in \mathbb{R}^n} |h|^{-\varepsilon}\|\Delta_h [\phi m(t^P')]\|_{M_p} < \infty
\]

then \( m \in M_p \).

(ii) If \( \sup_{t>0} \|\phi m(t^P')\|_{\Lambda_2} < \infty \), then \( m \in M_{\dot{r}}, \ |1/r - 1/2| < |1/p - 1/2| \).

**Proof.** (i) is weaker than the assertion of Corollary 1. (ii) then follows by interpolating the inequalities

\[
\|\Delta_h [\phi m(t^P')]\|_{M_p} \leq c, \quad \|\Delta_h [\phi m(t^P')]\|_{M_2} \leq c |h|^\varepsilon.
\]

**2. Proof of Theorem 1.**

2.1. Some tools needed in the proof. Let \( r \) be a distance function, homoge-
neous with respect to the adjoint dilations \( t^{P^*} \), satisfying a triangle inequality with
constant \( b \). Let \( \mathcal{W} \) be the collection of all \( r \)-balls

\[
Q = \{ x; r(x_0 - x) \leq 2^k \}, \quad x_0 \in \mathbb{R}^n, \ k \in \mathbb{Z},
\]

\( x_0 \) is the "center" of \( Q \), \( 2^k = \text{rad } Q \) its "radius".

The Hardy-Littlewood maximal operator with respect to \( \mathcal{W} \) is defined for functions with values in a Banach-space \( B \) by

\[
\mathcal{M} f(x) = \sup_{x \in Q \in \mathcal{W}} \int_Q |f(y)|_B \, dy.
\]
By $f^\#$ we denote the Fefferman-Stein sharp maximal function, defined by

$$
 f^\#(x) = \sup_{x \in Q} \int_Q |f(y) - f|_B \, dy.
$$

The basic fact about $f^\#$ is

**Proposition.** Assume that $1 < p < \infty$, $1 < p_0 < P$ and $f \in L^{p_0}(\mathbb{R}^n, \mathcal{L})$. If $f^\# \in L^P(\mathbb{R}^n)$, then $\mathcal{M} f \in L^P(\mathbb{R}^n)$ and $\|\mathcal{M} f\|_p \leq c\|f^\#\|_p$.

The proof is an adaptation of the proof given by Fefferman and Stein [8] in the more general setting of homogeneous spaces (see [15]). Another tool needed in the proof is Littlewood-Paley theory [18, 13]. Let $\phi \in C^\infty_0(\mathbb{R}^n)$ and $\eta_k = \mathcal{F}^{-1}[\phi(2^{-kP} \cdot)]$, $g(f) = (\sum_{k \in \mathbb{Z}} |\eta_k * f|^2)^{1/2}$. Then $\|g(f)\|_p \leq c\|f\|_p$, $1 < p < \infty$. We will choose $\phi = \Phi_0 * \rho$ as in (1). Then we also have $\|f\|_p \leq c\|g(f)\|_p$, $1 < p < \infty$. Let $\tilde{\phi} \in C^\infty_0(\mathbb{R}^n)$ be equal to 1 on supp $\phi$. Then we associate to $\tilde{\phi}$ in the same way the functions $\tilde{\eta}_k$ and $\tilde{g}(f)$.

2.2. **Proof of the theorem.** By duality we may assume $2 \leq p < \infty$. We associate to $T = T_m$ a vector-valued operator $\tilde{T}$, defined by $[\tilde{T}f]_k = \eta_k * Tf$. We will show that

$$
 \|([\tilde{T}f]^*)_k\|_p \leq cAN^{1/2-1/2^p} \|f\|_p
$$

where

$$
 N = \max(\varepsilon^{-1}, a_0^{-1}) \log_2(2 + B/A).
$$

If $f \in \mathcal{S}$, $\tilde{T}$ is a priori in $L^p(\ell^2)$. By Littlewood-Paley theory and the Fefferman-Stein inequality we get

$$
 \|Tf\|_p \leq c_1 \|g(Tf)\|_p = c_1 \|\tilde{T}f\|_{L^p(\ell^2)} 
\leq c_1 \|\mathcal{M} (\tilde{T}f)\|_p = c_2 \|([\tilde{T}f]^*)_k\|_p \leq c_3 AN^{1/2-1/p} \|f\|_p.
$$

It remains to prove (4). In order to apply interpolation arguments it is useful to linearize the operator $f \mapsto (\tilde{T}f)^\#$. Fix $f \in L^p$. Following [8, p. 157] we may find for each $x \in \mathbb{R}^n$ a ball $Q_x \in \mathcal{W}$ containing $x$, the center and the radius being measurable functions of $x$, further functions $\chi_k(x, y)$, with $(\sum |\chi_k(x, y)|^2)^{1/2} \leq 1$, $x \in \mathbb{R}^n$, $y \in Q_x$, such that the following inequality holds:

$$
 ([\tilde{T}f]^*_k)(x) \leq 2Sf(x)
$$

where

$$
 Sf(x) = \int_{Q_x} \sum_{k \leq l(x)} \left[ \eta_k * Tf(y) - \int_{Q_x} \eta_k * Tf(z) \, dz \right] \chi_k(x, y) \, dy.
$$

Define $l(x)$ by $\text{rad } Q_x = 2^{l(x)}$. Instead of $S$ we consider the following operators $\sigma_1$, $\sigma_2$ acting on sequence-valued functions $F = \{f_k\}$, $H = \{h_k\}$.

$$
 \sigma_1(F, x) = \int_{Q_x} \sum_{|k| \leq l(x)} \left[ \tilde{\eta}_k * f_k(y) - \int_{Q_x} \tilde{\eta}_k * f_k \chi_k(x, y) \right] \chi_k(x, y) \, dy,
$$

$$
 \sigma_2(H, x) = \int_{Q_x} \sum_{|k| > l(x)} \left[ \eta_k * T_h_k(y) - \int \eta_k * T_h_k \chi_k(x, y) \right] \chi_k(x, y) \, dy.
$$
In 2.3 and 2.4 we will show that
\begin{equation}
\|\sigma_1(F)\|_p \leq cN^{1/2 - 1/p}\|F\|_{L^p(I^p)}
\end{equation}
and
\begin{equation}
\|\sigma_2(H)\|_p \leq cA\|H\|_{L^p(I^2)},
\end{equation}
the constant c being independent of A, N and the choice of Q_x, \chi_k(x,y). We proceed by observing
\[ Sf = \sigma_1(\{\eta_k * Tf\}) + \sigma_2(\{\tilde{\eta}_k * f\}). \]
By Littlewood-Paley theory (9) implies
\[ \|\sigma_2(\{\tilde{\eta}_k * f\})\|_p \leq cA\|f\|_p. \]
Using the hypothesis (i) we get
\[ \|\{\eta_k * Tf\}\|_{L^p(I^p)}^p = \sum \|\eta_k * T(\tilde{\eta}_k * f)\|_p^p \leq A^p \sum \|\tilde{\eta}_k * f\|_p^p \leq A^p \|\{\eta_k * f\}\|_{L^p(I^2)}^p \leq cA^p\|f\|_p^p \]
and from (8) we conclude
\[ \|\sigma_1(\{\eta_k * Tf\})\|_p \leq cAN^{1/2 - 1/p}\|f\|_p. \]
These estimates imply (4).

2.3. Estimation of $\sigma_1(F)$. Since $\sigma_1(F) \leq 2\mathcal{M}[\sum_k |\eta_k * F_k|^2]^{1/2}$, it follows by $L^2$-boundedness of $\mathcal{M}$
\[ \|\sigma_1(F)\|_2 \leq c \left( \sum_k \|\eta_k * F_k\|_2^2 \right)^{1/2} \leq c'\|F\|_{L^2(I^2)}. \]
If $p = \infty$, we have
\[ \|\sigma_1(F)\|_\infty \leq \left\| \mathcal{M} \left[ \sup_{l \in \mathbb{Z}} \left( \sum_{|k+l| \leq N} |\eta_k * F_k|^2 \right)^{1/2} \right] \right\|_\infty \leq cN^{1/2} \sup_{k \in \mathbb{Z}} \|\eta_k * F_k\|_\infty \leq c'N^{1/2}\|F\|_{L^\infty(I^2)}. \]
Now an application of the Riesz-Thorin interpolation theorem establishes (8).

2.4. Estimation of $\sigma_2(H)$. The operator $\sigma_2(H)$ represents the "remainder"-terms similarly treated as in the Calderón-Zygmund theory. By $L^2$-boundedness of $\mathcal{M}$ and the Plancherel theorem we get
\[ \|\sigma_2(H)\|_2^2 = \sum_k \|\eta_k * Th_k\|_2^2 \]
\[ = \|\phi(2^{-kP}. mh_k)^2 \|_2 \leq A^2 \sum_k \|h_k\|_2^2 = A^2\|H\|_{L^2(I^2)}^2. \]
We show
\begin{equation}
\|\sigma_2(H)\|_\infty \leq cA\|H\|_{L^\infty(I^2)}
\end{equation}
and (9) follows by interpolation.
We need a further splitting of $\sigma_2$. Denote by $R_x$ the ball with same center as $Q_x$ and ${\text{rad}} R_x = 2b {\text{rad}} Q_x$. For a function $H$ we denote by $R_x H$ multiplication with the indicator function of $R_x$; similarly define $R_x^c H$ for the complement $R_x^c$. We have the majorization $\sigma_2(H, x) \leq I(x) + II(x)$, where

$$I(x) = \int_{Q_x} \left( \sum_k |\eta_k * T(R_x h_k)(y)|^2 \right)^{1/2} dy$$

and

$$II(x) = \int_{Q_x} \left( \sum_{|k+l(x)| > N} |\{\eta_k * T(R_x^c h_k)(y) - \int \eta_k * T(R_x^c h_x)(z) dz\}|^2 \right)^{1/2} dy.$$

By Hölder’s inequality and Plancherel’s theorem we get

$$|I(x)| \leq |Q_x|^{-1/2} \left( \sum_k \|\eta_k * T(R_x h_k)\|_2^2 \right)^{1/2}$$

$$\leq |Q_x|^{-1/2} A \left( \sum_k \|R_x h_k\|_2^2 \right)^{1/2}$$

$$\leq c A \int_{R_x} \sum_k |h_k(y)|^2 dy$$

$$\leq c A \|H\|_{L^\infty(l^2)}.$$

To estimate $II(x)$ set $K_k(x) = \mathcal{F}^{-1}[\phi m(A_{2^k} \cdot)]$. Then with

$$E_k(x, y, z) = \int_{R_x^c} 2^{k\nu} |K_k(2^k P^* (y - w)) - K_k(2^k P^* (z - w))| dw,$$

it happens that

$$E_k(x, y, z) \leq c B \min\{2^{-\epsilon(k+l(x))}, 2^{-a_0(k+l(x))}\},$$

whenever $y, z \in Q_x$.

Summing a geometrical series we obtain

$$||I(x)|| \leq \sup_{y, z \in Q_x} \left( \sum_{|k+l(x)| > N} [E_k(x, y, z)]^2 \right)^{1/2} \|H\|_{L^\infty(l^\infty)}$$

$$\leq c B \max\{2^{-\epsilon N}, 2^{-a_0 N}\} \|H\|_{L^\infty(l^\infty)}$$

$$\leq c A \|H\|_{L^\infty(l^\infty)} \leq c A \|H\|_{L^\infty(l^2)}.$$

(11) follows by a standard calculation. Denote by $x_0$ the center of $R_x$. Then for $w \in R_x^c, y \in Q_x$

$$r(y - w) \geq r(x_0 - w)/b - r(x_0 - y) \geq 2^{l(x)};$$

hence

$$E_k(x, y, z) \leq 2 \int_{r(u) \geq 2^{l(x)}} 2^{k\nu} |K_k(2^k P^* u)| du \leq c B 2^{-(k+l(x))\epsilon}$$
by hypothesis (ii). If \( k + l(x) < -N \), we use the fact that \( \hat{\phi} * K_k = K_k \) and obtain by Taylor’s formula

\[
E_k(x, y, z) \leq 2 \int_0^1 \int_{R^d_x} |2^{k\nu}[(2^{kP^*}(y - z) \cdot \nabla)K_k](2^{kP^*}(z - w + sy - sz))| \, dz \, ds
\]

\[
\leq c \|2^{kP^*}(y - z) \cdot \nabla|\hat{\phi} * K_k\|_1
\]

\[
\leq c \|2^{kP^*}(y - z) \cdot \nabla|\hat{\phi}\|_1 \|K_k\|_1
\]

\[
\leq cB2^{(k+l(z))a_0},
\]

if \( y, z \in R_x \).

This completes the estimation of \( \sigma_2(H) \) and concludes the proof of the theorem.

3. Some variants and applications.

3.1. The case \( p = 1 \). There is a simpler counterpart of Theorem 1.1 for \( p = 1 \) which strengthens slightly the Hörmander multiplier criterion. It involves a weak-type \((1,1)\) and an \((P_{1,1})\)-estimate for the operator \( T_m \). Here \( H^1 \) is the parabolic Hardy-space, defined as in [1] with respect to the \((tP^*)\)-dilations.

THEOREM 2. Suppose that the hypotheses (i), (ii) of Theorem 1 are satisfied with \( p = 1 \). Then

(a) \( \|T_m f\|_1 \leq cA \log(2 + B/A)^{1/2} \|f\|_{H^1} \).

(b) \( \sup_{\alpha > 0} \alpha \{ |T_m f| > \alpha \} \leq cA \log \left( 2 + \frac{B}{A} \right) \|f\|_1 \).

For \( \alpha > 0 \) we use the atomic decomposition (see Latter and Uchiyama [11]). Let \( a \) be an atom, supported in \( \{r(x_0 - x) \leq 2^l \} \), \( \|a\|_\infty \leq c2^{-l\nu} \). Choose \( N \) as in the proof of Theorem 1 and split \( T_a = T_{1,1}a + T_{1,2}a \), where \( T_{1,1}a = \sum_{|k+l| \leq N} \eta_k * Ta \). Using the standard Calderón-Zygmund estimates it follows

\[
\|T_{1,2}a\|_1 \leq cB \max(2^{-\varepsilon N}, 2^{-a_0 N}) \|a\|_1 \leq cA.
\]

Further

\[
\|T_{1,1}a\| \leq A \left\| \sum_{|k+l| \leq N} \eta_k * a \right\|_1
\]

\[
\leq cAN^{1/2} \left( \sum_k \left| \eta_k * a \right|^2 \right)^{1/2} \leq c'AN^{1/2},
\]

by Littlewood-Paley theory in \( H^1 \).

The proof of (b) is similar and involves a Calderón-Zygmund decomposition. We can only achieve the larger constant \( cAN \), because Littlewood-Paley functions do not define bounded operators in \( L^1 \).

REMARK. The counterexample \( m_N \) mentioned in §1 shows that the constants in Theorem 2 are sharp. For the \((H^1, L^1)\)-estimate this follows from [19, p. 125]. The essential part of the kernel \( \mathcal{F}^{-1}m_N \) lies near the points \( \sigma_k, N \leq k \leq 2N \), and a straightforward computation shows that \( \|\mathcal{F}^{-1}m_N\|_{L^{1,\infty}} \geq cN \) (\( L^{1,\infty} \) denotes the Lorentz-space). Let \( \chi \in \mathcal{S}, \hat{\chi}(\xi) = 1 \) near 0, \( \chi_l = 2^{l\nu} \chi(2^{lP^*}) \). For large \( l \)
we have $T_{m_N} \chi_l = \mathcal{F}^{-1} [m_N]$; this implies $\| T_{m_N} \|_{(L^1, L^{1,\infty})} \geq cN$ for the weak-type operator-"norm" of $T_{m_N}$.

3.2. Application to quasiradial multipliers.

**Corollary 3.** Let $\rho \in C^\infty(\mathbb{R}^n_0)$ be a $P$-homogeneous-distance function and $m = m_0 \circ \rho$, where $m_0 \in L^\infty(\mathbb{R}^+)$ and $m_0 \in L^1(\mathbb{R}_+)$, $m_0 \circ \rho$. Suppose that for some $p$, $1 \leq p < 2n/(n + 1)$,

$$\sup_{t > 0} \| \phi m_0 \circ t \rho \|_{M_p} < \infty.$$  

Then $m_0 \circ \rho \in M_r$, $p < r \leq 2$.

**Proof.** The smoothness assumption of Corollary 2, (ii) is satisfied, since the necessary conditions for quasiradial multipliers [17] imply

$$\sup_{t > 0} \| \phi m_0 (t \cdot) \|_{B^p_{\alpha p}(\mathbb{R}^n)} \leq c \sup_{t > 0} \| \phi m_0 \circ t \rho \|_{M_p},$$  

$\alpha = (n - 1)(1/p - 1/2)$. Here $\phi_0 \in C^\infty_0(\mathbb{R}^n_0)$ and $B^p_{\alpha p}(\mathbb{R})$ is the standard Besov space defined in [19]. Now $B^p_{\alpha p} \subset \Lambda^p_\varepsilon$ if $0 < \varepsilon < \alpha - 1/p'$ and $\alpha - 1/p' > 0$ if $p < 2n/(n + 1)$. The assertion follows from Corollary 2 and the elementary inequality

$$\| \phi_0 \circ pm_0 \circ t \rho \|_{\Lambda^p_\varepsilon(\mathbb{R}^n)} \leq c \| \phi_0 m_0 (t \cdot) \|_{\Lambda^p(R)}.$$  

The following criterion for quasiradial multipliers is proved in [16].

**Corollary 4.** Let $\rho \in C^\infty(\mathbb{R}^n_0)$, the unit sphere $\{ \rho(\xi) = 1 \}$ being strictly convex. Then

$$\| m_0 \circ \rho \|_{M_p} \leq c \sup_{t > 0} \| \phi m_0 (t \cdot) \|_{L^\alpha_{\beta}(\mathbb{R})},$$  

$\alpha > n[1/p - 1/2]$, $1 < p \leq 2(n + 1)/(n + 3)$.

The condition $\sup_{t > 0} \| \phi m \circ t \rho \|_{M_p} < \infty$ can be verified following Stein’s treatment of the Bochner-Riesz multiplier [7, 16]. The approach via Corollary 2 considerably simplifies the proof of Corollary 4 in [16]. It avoids also the weighted norm inequality in Christ’s proof of (essentially) the same result (see [3]).

3.3. Lacunary maximal operators. Given a multiplier $m$, we define for $f \in \mathcal{S}$ the lacunary maximal operator $T^*_m$ by

$$T^*_m f = \sup_{k \in \mathbb{Z}} |\mathcal{F}^{-1} [m(2^k P \cdot) \hat{f}]|.$$

To prove boundedness results for $T_m$ we shall need information about a vector-valued singular integral operator $\tau$, defined for functions $F = \{F_{k,l}\}$ with values in $l^2(\mathbb{Z}^2)$ by

$$[\tau(F)]_k = \sum_l \eta_{k+l} * F_{k,l}.$$  

**Lemma.** $\| \tau(F) \|_{L^p(l^2(\mathbb{Z}))} \leq c \| F \|_{L^p(l^2(\mathbb{Z}^2))}$, $1 < p < \infty$.

**Proof.** For $p = 2$ the inequality follows by Plancherel’s theorem. Then for $p < 2$, by Calderón-Zygmund theory we are led to verify the following weak Hörmander condition

$$\int_{|x| \geq 2bt \sum_k \sum_l |\eta_{k+l}(x - y) - \eta_{k+l}(x)| \alpha_{k,l}}^{1/2} dx \leq c \left( \sum_k \sum_l \alpha^2_{k,l} \right)^{1/2},$$  

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whenever \( r(y) \leq t \), \((\alpha_{k,l}, t)^{e} \in l^{2}(Z^{2})\). The verification of (12) is a routine matter, so it is omitted. The case \( p > 2 \) follows by observing that the adjoint \( \tau^{*} \) is similarly defined as \( \tau \) \((k,l \, \text{are exchanged}) \). □

**Theorem 3.** Suppose that for some \( 1 < p < \infty \), \( r = \min(p, 2) \), \( \varepsilon > 0 \)

\[
\left( \int_{0}^{\infty} \| \phi m(t^{P})\|_{M_{p}}^{r} \frac{dt}{t} \right)^{1/r} < \infty,
\]

\[
\left( \int_{0}^{\infty} \left[ \sup_{h} |h|^{-\varepsilon} \| \Delta h \phi m(t^{P})\|_{M_{p}} \right] \frac{dt}{t} \right)^{1/r} < \infty.
\]

Then

\[
\left\| \left( \sum_{k} |\mathcal{F}^{-1}[m(2^{k}P^{k})f^{k}]|^{2} \right)^{1/2} \right\|_{p} \leq c\|f\|_{p}.
\]

**Proof.** Choose \( \phi \) as in (2),

\[
a_{t} = \| \phi m(2^{k}P^{k})\|_{M_{p}}, \quad b_{t} = \sup |h|^{-\varepsilon} \| \Delta h [\phi m(2^{l}P^{l})]\|_{M_{p}}.
\]

Then the hypotheses of the theorem are equivalent with \( \sum (a_{t}^{2} + b_{t}^{2}) < \infty \); this essentially requires the same argument as in the Introduction.

We apply the lemma with \( F_{k,t} = \mathcal{F}^{-1}[^{\phi} (2^{-(k+l)}P^{k})]m(2^{k}P^{k})f^{k} \) to deduce

\[
\left\| \left( \sum_{k} |\mathcal{F}^{-1}[m(2^{k}P^{k})f^{k}]|^{2} \right)^{1/2} \right\|_{p} \leq \left\| \left( \sum_{k,l} |F_{k,l}|^{2} \right)^{1/2} \right\|_{p}.
\]

If \( p > 2 \) we have by Minkowski’s inequality

\[
\left\| \left( \sum_{k,l} |F_{k,l}|^{2} \right)^{1/2} \right\|_{p} \leq \left( \sum_{l} \left\| \left( \sum_{k} |F_{k,l}|^{2} \right)^{1/2} \right\|_{p} \right)^{1/2},
\]

whereas if \( p < 2 \) we use \( l^{p} \subset l^{2} \) and interchange summation and integration to get

\[
\left\| \left( \sum_{k,l} |F_{k,l}|^{2} \right)^{1/2} \right\|_{p} \leq \left( \sum_{l} \left\| \left( \sum_{k} |F_{k,l}|^{2} \right)^{1/2} \right\|_{p} \right)^{1/p}.
\]

Denote by \( r_{k} \) the sequence of Rademacher functions (see [18, p. 276]) and let

\[
m_{l,s} = \sum_{k} r_{k}(s) \overline{\phi}(2^{-(k+l)}P^{k})m(2^{k}P^{k}), \quad s \in [0, 1].
\]

An application of Corollary 1.2 gives

\[
\|m_{l,s}\|_{M_{p}} \leq \sum_{j=-3}^{3} a_{t+j} + b_{t+j}, \quad \text{uniformly in } s \in [0, 1].
\]
By Chinchin's inequality and interchanging the order of integrals we see
\[ \left\| \left( \sum_k |F_{k,l}|^2 \right)^{1/2} \right\|_p \leq c \int_0^1 \| m_{l,\alpha} \|_{M_p}^p \, dx \leq c \sum_{j=-3}^{3} a_{l+j}^p + b_{l+j}^p. \]

Now summation over \( l \) proves the assertion. \( \square \)

Of course, Theorem 3 implies boundedness of \( T_m^* \) in \( L^p \). For \( p > 2 \) there is a simpler result which follows from Littlewood-Paley theory and does not rely on Theorem 1.

**Corollary 5.** Suppose \( 2 \leq p < \infty \) and
\[ \left( \int_0^\infty \| \phi_m(tP^\cdot) \|_{M_p}^2 \frac{dt}{t} \right)^{1/2} < \infty. \]
Then \( \| T_m^* f \|_p \leq c \| f \|_p \).

**Proof.** We use the inequality
\[ \left\| \sum_l \eta_l * g_l \right\|_p \leq \left( \sum_l |g_l|^2 \right)^{1/2}, \]
\( 1 < p < \infty \), which, by duality, is a consequence of Littlewood-Paley theory. Now
\[ \| T_m^* f \|_p \leq \left( \sum_k \| \mathcal{F}^{-1} [m(2^{-kP^\cdot}) f^\wedge] \|^p \right)^{1/p} \]
\[ \leq c \left( \sum_k \left( \sum_l |\eta_{k+l} * \mathcal{F}^{-1} [m(2^{-kP^\cdot}) f^\wedge]|^2 \right)^{1/2} \right)^{1/2} \]
\[ \leq c \left( \sum_l \left( \sum_k \| \mathcal{F}^{-1} [\phi(2^{-k+l}P^\cdot) m(2^{-kP^\cdot}) \cdot \bar{\eta}_{k+l} * f \|_p]^2 \right)^{2/p} \right)^{1/2} \]
\[ \leq c \left( \sum_l \| \phi m(tP^\cdot) \|_{M_p}^2 \right)^{1/2} \left( \sum_k \| \bar{\eta}_k * f \|_p \right)^{1/p}, \]
and a second application of Littlewood-Paley theory implies the assertion.

**Corollary 6.** Suppose that \( m \in M_p \) satisfies for some \( \delta > 0 \) \( |m(\xi)| \leq c|\xi|^\delta \), if \( |\xi| \leq 1 \) and \( |m(\xi)| \leq c(|\xi|^{-\delta}, \text{ if } |\xi| \geq 1. \)
(i) If \( p > 2 \), then \( \| T_m^* f \|_r \leq c_r \| f \|_r \), \( 2 \leq r < p \).
(ii) If \( p < 2 \), and \( \sup_{t>0} \| \phi(tP^\cdot) \|_{M_r} < \infty \), then \( \| T_m^* f \|_r \leq c \| f \|_r \), \( p < r \leq 2 \).

The proof follows by interpolation. Note that (i) is already contained in [4].

**Remark.** In many cases, the decay condition at the origin is not valid, but \( m \) is smooth near \( \xi = 0 \). Then one may split \( m = m_0 + m_1 \), where \( m_0 \) is compactly supported and smooth and equals \( m \) near the origin. \( T_{m_0}^* f \) is majorized by the Hardy-Littlewood maximal function \( M^* f \), and \( T_{m_1}^* f \) can be handled by the above corollaries. For example we can deduce the following result of Duoandikoetxea and
Rubio de Francia [6] (which, however, does not require the full strength of Theorem 1.1):

Let \( \mu \) be a compactly supported measure satisfying \( \mu^\wedge(\xi) \leq c(1 + |\xi|)^{-\delta} \),

\[
\sup_{k \in \mathbb{Z}} \left| \int f(x - 2^k \xi) \, d\mu(y) \right|_p \leq c\|f\|_p, \quad 1 < p \leq \infty.
\]

Write \( \mu = m + \tilde{m} \), where \( m(\xi) = 0 \) near the origin and \( \text{supp} \tilde{m} \) is compact. Since \( \mu \) is compactly supported, \( m, \tilde{m} \) are smooth; further \( |D^\alpha \mu^\wedge(\xi)| \leq c(1 + |\xi|)^{-\delta} \) for every multi-index \( \alpha \). Then \( |T_m^\wedge f| \leq cMf \). If \( t \geq 1 \) we have

\[
\|\phi m(t^P \cdot)\|_\infty \leq c t^{-\delta a_0}, \quad \|D^\alpha (\phi m(t^P \cdot))\|_\infty \leq c t^{a_0}, \quad |\alpha| = 1
\]

which implies \( \sup_{t>0} \|\phi m(t^P \cdot)\|_{L_\infty} < \varepsilon \), for some \( \varepsilon > 0 \). \( \square \)

3.4. Multipliers on Triebel-Lizorkin spaces. Define for \( 1 < p, q < \infty \), \( \eta_q \) as in 2.1,

\[
g_q(f) = \left( \sum_{k = -\infty}^{\infty} |\eta_k \ast f|^q \right)^{1/q}
\]

and the homogeneous Triebel-Lizorkin space \( \dot{F}^{pq} = \dot{F}^{pq}(P) \) by \( \|f\|_{\dot{F}^{pq}} = \|g_q(f)\|_p \). \( \dot{F}^{pq} \) should be considered as a subspace of \( \mathcal{F}^\wedge(\mathbb{R}^n) \) modulo polynomials; the definition depends on the dilation group \( (t^P) \).

Let \( \mathcal{M}_{pq} = \mathcal{M}_{pq}(P) \) be the subspace of bounded functions whose norms

\[
\|m\|_{\dot{F}^{pq}} = \sup\{\|\mathcal{F}^{-1}[m^{\wedge}\cdot]\|_{\dot{F}^{pq}} ; f \in \mathcal{Z}_0, \|f\|_{\dot{F}^{pq}} \leq 1\}
\]

are finite. Note that \( \mathcal{Z}_0 \) is dense in \( \dot{F}^{pq} \). Multipliers in \( \dot{F}^{pq} \) are multipliers in the whole scale \( \dot{F}^{pq}_s \), \( -\infty < s < \infty \) (defined in [19] for isotropic dilations) since \( \dot{F}^{pq}_s = I_s \dot{F}^{pq} \), where \( I_s f = \mathcal{F}^{-1}[\rho^{-s} f^\wedge] \) for some \( P \)-homogeneous distance function \( \rho \in C^\infty(\mathbb{R}^n_0) \). For simple properties of \( \mathcal{M}_{pq} \) we refer to Triebel [19, p. 128], where the inhomogeneous case is discussed. Observe that \( \mathcal{M}_p = \mathcal{M}_{pq} \) equals the space of multipliers on anisotropic homogeneous Besov spaces \( B_{pq}^r \) as mentioned in the Introduction.

**THEOREM 4.** Suppose that \( m \) is a bounded function satisfying for some \( p, 1 < p < \infty, \varepsilon > 0 \)

(i) \( \sup_{t>0} \|\phi m(t^P \cdot)\|_{M_p} \leq A \),

(ii) \( \sup_{t>0} \int_{|x| \geq \omega} |\mathcal{F}^{-1}[\phi m(t^P \cdot)]| \, dx \leq c B(1 + \omega)^{-\varepsilon} \).

Then \( m \) is a Fourier multiplier in \( \dot{F}^{pq}(P) \), \( |1/q - 1/2| \leq |1/p - 1/2| \), and \( \|m\|_{\dot{F}^{pq}} \leq cA[\log(2 + B/A)]^{1/p-1/q} \).

**Proof.** By duality, we may assume \( p > q \). It suffices to consider the case \( q = p' \); the remaining cases follow by interpolation. The proof is a repetition of the arguments needed for Theorem 1, so we omit the details. The operators \( S, \sigma_1, \sigma_2 \) are defined as in (5), (6), (7) but now with \( (\sum |\chi_k(x, y)|^{p'})^{1/p'} \leq 1 \). Then

\[
\|\sigma_1(F)\|_p \leq cN^{1/p'-1/p} \|F\|_{L_p'(1^P)}, \quad \|\sigma_2(H)\|_p \leq c\|A\|_{L_p'(1^P)}.
\]
Instead of Plancherel’s theorem and Littlewood-Paley theory, we use the hypothesis
\[ \sup_{t>0} \| \phi m(t^p \cdot) \|_{\mathcal{M}_p} \leq A \text{ and the definition of } \hat{F}^\Phi. \]

As in §1, this theorem implies several corollaries, e.g.

**COROLLARY 7.** Suppose that for some \( \epsilon > 0 \)
\[ \sup_{t>0} \| \phi m(t^p \cdot) \|_{\Lambda_\epsilon} < \infty. \]

If \( m \) is a multiplier on the homogeneous Besov space \( \hat{F}^\Phi(P) \), then it is also a multiplier on \( \hat{F}^r(P) \), \( p < r, s < p' \).

It is an interesting problem whether the hypothesis of Corollary 7 implies \( m \in \mathcal{H}_2(P) \) for some \( |1/s - 1/2| > |1/p - 1/2| \).

During the preparation of this paper the author was informed by A. Carbery, that he also established some of the results of this paper (see [2]), using another approach. In particular he found Corollaries 2 and 6, as well as some weak-type estimates in the endpoint cases.

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