NORMAL DERIVATIVE FOR BOUNDED DOMAINS WITH GENERAL BOUNDARY

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ABSTRACT. Let $D$ be a general bounded domain in the Euclidean space $R^n$. A Brownian motion which enters from and returns to the boundary symmetrically is used to define the normal derivative as a functional for $f$ with $f$, $\nabla f$ and $\Delta f$ all in $L^2$ on $D$. The corresponding Neumann condition (normal derivative = 0) is an honest boundary condition for the $L^2$ generator of reflected Brownian motion on $D$. A conditioning argument shows that for $D$ and $f$ sufficiently smooth this general definition of the normal derivative agrees with the usual one.

Introduction. $B_t$, $t \geq 0$, is standard Brownian motion on the Euclidean space $R^n$. The case when the dimension $n = 1$ is well understood and so we assume always that $n \geq 2$.

$D$ is a bounded open connected subset of $R^n$. An important role will be played by

$$\sigma = \inf\{t > 0: B_t \text{ is not in } D\},$$

the hitting time for the complement $R^n \setminus D$.

The absorbed process $X_t$, $t \geq 0$, is

$$X_t = B_t \quad \text{for } 0 \leq t < \sigma, \quad X_t = \partial \quad \text{for } t \geq \sigma$$

with $\partial$ a dead point adjoined to $D$ as an isolated point. All functions are understood to vanish at $\partial$.

The absorbed transition operators $P^0_t$, $t \geq 0$, with action

$$P^0_t f(x) = E_x f(X_t)$$

form a strongly continuous symmetric semigroup on $L^2$, the Hilbert space of real valued functions on $D$ that are square integrable with respect to Lebesgue measure $dx$.

The (strong) generator $A^0$ is selfadjoint and negative definite. By the spectral theorem as formulated and proved in Chapter IX of Segal and Kunze (1968), the operator $(-A^0)^{1/2}$ is a well-defined positive definite selfadjoint operator. The absorbed Dirichlet space is the pair $(F^0, E)$ where

$$(0.1) \quad F^0 = \text{domain}((-A^0)^{1/2}),$$

$$E(f, g) = \int dx (-A^0)^{1/2} f(x)(-A^0)^{1/2} g(x), \quad f, g \in F^0.$$
CONVENTION 1. All integrals are understood to be over $D$ unless otherwise specified.

We can explicitly identify $(F^0, E)$. The space $F^0$ is the set of functions $f$ in $L^2$ such that if $f$ is extended to all of $\mathbb{R}^n$ by setting $f(x) = 0$ for $x$ in the complement $\mathbb{R}^n \setminus D$ then $\nabla f$ is in $L^2$ on $\mathbb{R}^n$.

CONVENTION 2. The Laplacian $\Delta$, the gradient $\nabla$ and all partial derivatives are understood in the sense of distributions. If no set is specified then the distribution theory is on $D$.

CONVENTION 3. “$L^p$ on $K$” means the space of real valued $p$-integrable functions on $K$. If no $K$ is specified then $K = D$ is understood.

CONVENTION 4. $\nabla f$ in $L^2$ means that its norm $|\nabla f|$ is.

The form $E$ is

\begin{equation}
E(f, g) = \frac{1}{2} \int dx \nabla f(x) \cdot \nabla g(x)
\end{equation}

with “$\cdot$” denoting the usual inner product between two $n$-dimensional vectors. Also we will use the auxiliary form $E_a$ defined for $a > 0$ by

\begin{equation}
E_a(f, g) = E(f, g) + a \int f(x)g(x) \, dx.
\end{equation}

Reflecting Brownian motion on $D$ is easily specified in terms of its Dirichlet space $(F^r, E)$. The space $F^r$ is the set of functions $f$ in $L^2$ such that also $\nabla f$ is in $L^2$ (no condition on an extension of $f$). The form $E$ is still (0.2).

REMARK. If $D$ has a smooth boundary (Lipschitz is enough) then Theorem 7.3(iii) in SMP implies that $F^0$ is the set of $f$ in $F^r$ which vanish modulo polar sets on the boundary of $D$.

NOTATION. The abbreviation SMP is used in place of the reference Silverstein (1974).

CONVENTION 5. Functions $f$ in $F^r$ are represented by their quasi-continuous versions as specified in Theorem 3.13 in SMP. In particular $f$ is well defined modulo polar sets.

The strong $L^2$ generator $A^r$ of the reflecting transition semigroup $P_t^r, t \geq 0$, is also selfadjoint and nonpositive definite. From general principles $f$ in $L^2$ belongs to domain $(A^r)$ and $A^r f = g$ if and only if the following two conditions are satisfied.

GEN1. $f$ is in $F^r$.

GEN2. For all $h$ in $F^r$

\begin{equation}
E(f, h) + \int dx g(x)h(x) = 0.
\end{equation}

We will see in §4 that (0.3) can be checked separately for $h$ in $C_0^\infty(D)$ and for $h$ bounded and harmonic and in $F^r$.

CONVENTION 6. “$h$ is harmonic” means on $D$ unless another set is specified. “$h$ in $C_0^\infty(K)$” means $h$ is $C^\infty$ and has compact support contained in $K$.

For $h$ in $C_0^\infty(D)$, integration by parts shows that (0.3) is equivalent to

\begin{equation}
\frac{1}{2} \Delta f(x) = g(x).
\end{equation}

Suppose now that $h$ is harmonic and belongs to $F^r$. If both $f$ and $h$ are sufficiently smooth on the closure of $D$ and if $D$ itself has a sufficiently smooth boundary
then the classical “divergence theorem” can be applied to give

\[
\frac{1}{2} \int dx \nabla f(x) \cdot \nabla h(x) + \int dx g(x) h(x) \\
= \frac{1}{2} \int dx (\nabla f(x) \cdot \nabla h(x) + \Delta f(x) h(x)) \\
= \frac{1}{2} \int dx \text{div}(h \nabla f(x)) \\
= \frac{1}{2} \int_{\partial D} d\sigma(z) h(z) D_n f(z)
\]

with \( D_n f(z) \) denoting the “classical” exterior normal derivative. This suggests that the \( L^2 \) reflected generator is determined by the classical Neumann boundary condition

\[
D_n f(z) = 0.
\]

In fact the authors are not sure that (0.6) works as a boundary condition even for \( D \) smooth. Instead we use a functional version of the normal derivative introduced in Silverstein (1976) and prove that this works for general bounded \( D \) (no smoothness assumed).

To discuss the functional normal derivative \( \partial f/\partial n(\varphi) \) we must introduce some machinery. In §2 we describe an “approximate Brownian motion” which enters from and returns to the boundary with symmetric distribution. The process is denoted \( X_t, \sigma^* < t < \sigma, \) with \( \sigma^*, \sigma \) the times of entrance from and return to the boundary. The sample space and sample space measure are denoted \( \Omega_\infty \) and \( P_\infty, \) with expectation functional denoted \( E_\infty. \) An abstract boundary \( \partial^D D \) is introduced in §5 which represents all harmonic \( h \) in \( F^r \) and such that \( X_{\sigma^*} \) and \( X_\sigma \) are defined almost everywhere as points in \( \partial^D D \) by passage to the limit in \( t. \) A function \( \varphi \) defined on \( \partial^D D \) is a test function if it represents, or equivalently is the restriction to \( \partial^D D \) of, a harmonic function in \( F^r. \) The normal derivative \( \partial f/\partial n \) is the functional defined on test functions \( \varphi \) by

\[
\frac{\partial f}{\partial n}(\varphi) = \lim_{k \to \infty} E_\infty \varphi(X_{\sigma^*}) \{f(X_{\sigma^*}) - f(X_{\sigma(D_k)})\} I(\sigma(D_k) < \infty).
\]

with \( D_k, k \geq 1, \) an increasing sequence of open sets with closure contained in \( D \) and with union \( D. \) The symbol \( \sigma(D_k) \) represents the first hitting time for \( D_k, \) with the condition “\( \sigma(D_k) < \infty \)” meaning that \( D_k \) is hit before return to the boundary.

Our main result is the following theorem which will be proved in §6.

**Theorem.** Assume that \( f \) is in \( F^r \) and that \( \Delta f \) is in \( L^2. \) Then \( \partial f/\partial n(\varphi) \) exists for bounded test functions \( \varphi \) and if \( h \) is the harmonic function represented by \( \varphi \)

\[
\frac{\partial f}{\partial n}(\varphi) = \frac{1}{2} \int \{\nabla f(x) \cdot \nabla h(x) + \Delta f(x) h(x)\}.
\]

In particular \( f \) belongs to domain \( (A^r) \) if and only if

\[
\frac{\partial f}{\partial n}(\varphi) = 0
\]

for all bounded test functions \( \varphi. \)
We will see in §7 that for $D$ smooth we can arrange matters so that $\partial^* D$ is the Euclidean boundary $\partial D$. Then (0.8) and (0.5) together imply that for $D$ and $f$ smooth

$$\partial f/\partial n(\varphi) = \frac{1}{2} \int \varphi(z) D_n f(z) \, d\sigma(z)$$

with $d\sigma$ denoting the usual surface measure on $\partial D$. We believe that it is instructive to verify (0.10) directly from (0.7) and so in §8 we treat two special cases, the interval $(0, 1)$ and the half plane $R^2_+$, even though they do not fit the hypotheses of the main text. In both cases we can explicitly compute the joint distribution of $X_{\sigma^*}$ and $X_{\sigma(D_k)}$, assuming a particular choice for the $D_k$. In §7 we use a more robust technique based on a conditioning argument suggested to one of the authors by Richard F. Gundy. This works for general smooth and bounded $D$.

We emphasize that this functional normal derivative and its connection with boundary conditions is the new result. The reflected Brownian motion itself was constructed by Fukushima (1967). Fukushima constructs the process using the Martin-Kuramochi boundary. We introduce a different boundary $\partial^* D$ in §5 and do not construct the process—not needed for our result.

For $K$ a subset of $D$ its hitting time is

$$\sigma(K) = \inf\{t > 0: X_t \text{ is in } K\}$$

with the usual understanding that $\sigma(K) = +\infty$ when the set on the right is empty. The hitting operator $H^K$ is defined by

$$H^K f(x) = E_x I(\sigma(K) < \infty) f(X_{\sigma(K)}).$$

**REMARK.** In SMP a set $N$ is said to be polar if it has capacity 0 in the sense of Definition 3.3 there. By Proposition 4.5 there, this implies

$$P_x(\sigma(N) < \infty) = 0$$

for almost every $x$. In general this is weaker than the standard concept which requires (0.12) for all $x$. Fortunately, for Brownian motion, the two concepts are equivalent. This and much more follows from the very interesting results in Fukushima (1971). So we can use the term "polar" without worrying about which concept is intended.

1. **The absorbed space.** The absorbed transition operators can be represented

$$P_t^0 f(x) = \int P_t^0(x, y) f(y) \, dy$$

with $P_t^0(x, y)$ jointly continuous and symmetric:

$$P_t^0(x, y) = P_t^0(y, x), \quad x, y \text{ in } D.$$  

The potential density

$$G^0(x, y) = \int_0^\infty dt P_t^0(x, y)$$

and for $a > 0$ there resolvent density

$$G_a^0(x, y) = \int_0^\infty dt e^{-at} P_t^0(x, y)$$
are finite and jointly continuous for \( x \neq y \). Also \( G^0(x,y) \) is harmonic in each variable for \( x \neq y \) and \( G^0_a(x,y) \) is in the same way \( a \)-harmonic—that is, annihilated by \( \frac{1}{2}\Delta - a \). Of course (1.2) implies the same symmetry for \( G^0(x,y) \) and \( G^0_a(x,y) \). The operators \( G^0 \) and \( G^0_a \) are defined as \( P^0_t \) is in (1.1). All this can be found in §4 of Port and Stone (1978).

For \( K \) a subset of \( D \)

\[ h^K(x) = H^K 1(x). \]  

If \( K \) has closure contained in \( D \) then by Chapter 6, §5, ibid., there exists a Radon measure \( \mu_K \) on the boundary of \( K \) such that

\[ h_K(x) = \int G^0(x,y) \, d\mu_K(y). \]

In general \( \mu_K \) and \( h_K \) are referred to as the equilibrium measure and potential for \( K \).

The first two lemmas are known. We include our own proofs as a matter of convenience.

**Lemma 1.1.** The absorbed semigroup \( P^0_t, t \geq 0 \), is strongly continuous on \( L^2 \).

**Proof.** It is enough to show that for a dense set of \( f \) in \( L^2 \), in particular for bounded \( f \)

\[ P^0_t f \to f \text{ strongly in } L^2 \]

as \( t \downarrow 0 \). We take as known that the lemma is true for the transition operators

\[ P_t f(x) = E_x f(B_t) \]

for the original Brownian motion on \( \mathbb{R}^n \). Therefore it is enough to show that the difference \( E_x I(\sigma \leq t) f(B_t) \to 0 \) strongly in \( L^2 \). But this is clear since \( P_x(\sigma > 0) = 1 \) for \( x \) in \( D \) and since \( D \) has bounded measure. Q.E.D.

By §14 in Silverstein (1976) the Dirichlet space \( F \) for Brownian motion on \( \mathbb{R}^n \) is the set of functions \( f \) such that both \( f \) and \( \nabla f \) are in \( L^2 \) on \( \mathbb{R}^n \). The Dirichlet form is (0.2) with the integral over \( \mathbb{R}^n \) instead of \( D \). Then Theorem 7.3 in SMP justifies the description of the absorbed space \( (F^0, E) \) given in the Introduction.

By §1 in SMP every Dirichlet space \( (F, E) \) is contractive. This means that if \( g(t) \), a real valued function of real \( t \), satisfies

\[ g(0) = 0, \quad |g(s) - g(t)| \leq |s - t| \]

and if \( f \) is in \( F \), then also \( g(f) \) is in \( F \) and

\[ E(g(f), g(f)) \leq E(f, f). \]

Important examples for \( g(t) \) are \( t^+, t^-, |t|, a \wedge t, a \vee t \) and \( \frac{1}{2}(1 \wedge t^2) \).

The absorbed Dirichlet space \( (F^0, E) \) satisfies the following two conditions.

**REG1.** \( F^0 \cap C_{\text{com}}^\infty (D) = C_{\text{com}}^\infty (D) \).

**REG2.** \( F^0 \cap C_{\text{com}}^\infty (D) \) is \( E_1 \) dense in \( F^0 \).

To verify Reg2 it is enough to approximate \( f \) in \( F^0 \) by convolutions \( f \ast \varphi_k \) with \( \varphi_k(x) = k^n \varphi(kx) \) where \( \varphi \) is in \( C_{\text{com}}^\infty (k) \) and integrates to 1, after first approximating by functions in \( F^0 \) which are bounded and have compact support in \( D \).
For the former use bounded contractions. For the latter subtract projections onto complements of open sets with compact closure in $D$. (See Theorem 7.3 in SMP.)

Reg1 and Reg2 together with the elementary fact that $dx$ charges every nonempty open subset of $D$ imply regularity in the sense of §2 in SMP.

Nothing is gained by introducing the extended Dirichlet space $F_{(e)}^0$ of §1 in SMP since our assumption that $D$ is bounded implies that $F_{(e)}^0 = F^0$. This follows from the next lemma.

**Lemma 1.2.** The absorbed Dirichlet space $F^0$ is complete with respect to the Dirichlet norm $E$.

**Proof.** Consider first the case when $D$ is the cube

$$Q = \{x \in \mathbb{R}^n : 0 < x_j < R \text{ for } 1 \leq j \leq n\}.$$ 

Then an eigenbasis for the absorbed generator $A^0$ is the set of functions

$$\varphi_k(x) = \prod_{j=1}^n \sin\{\pi k_j x_j / R\}$$

with $k = (k_1, \ldots, k_n)$ an $n$-tuple of positive integers. The eigenvalue for $\varphi_k$ is $-\sum_{j=1}^n (k_j \pi / R)^2$ and this implies that for any $f$ in $F^0$

$$\int dx f^2(x) \leq (R^2 / n \pi^2) E(f, f)$$

and completeness with respect to $E$ follows from the known completeness of $F^0$ with respect to $E_1$. In general $D$ is contained in such a cube $Q$ and completeness with respect to $E$ follows since the absorbed space for $D$ is a subspace of the one for $Q$. Q.E.D.

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**2. An approximate Brownian motion.** Up to now we have been concerned only with Brownian motion starting in the interior of $D$ and eventually reaching the boundary. For this we have implicitly used the standard sample space $\Omega$. This is the set of trajectories $\omega$ with $\omega(t)$ defined for $t \geq 0$, taking values in $D \cup \{\partial\}$ and satisfying the following two conditions.

**TRAJ1.** There exists a time $\sigma > 0$ such that $\omega(t) = \partial$ for $t \geq \sigma$ and $\omega(t)$ is in $D$ for $0 \leq t < \sigma$.

**TRAJ2.** $\omega(t)$ is continuous for $t \neq \sigma$.

The coordinate variables $X_t$ are defined on $\omega$ in $\Omega$ by $X_t(\omega) = \omega(t)$. The $\sigma$-algebra $F_s$ on $\Omega$ is the one generated by the $X_t$. The sample space probabilities $P_x$, $x$ in $D$ are defined on $F_s$.

In this section we introduce a process which enters $D$ from the boundary and then acts like absorbed Brownian motion. An appropriate sample space is the extended sample space $\Omega_\infty$. This is the set of trajectories $\omega$ with $\omega(t)$ defined for all real $t$, taking values in $D \cup \{\partial\}$ and satisfying:

**TRAJ1'.** There exist times $\sigma^* < \sigma$ such that $\omega(t) = \partial$ for $t \leq \sigma^*$ and for $t \geq \sigma$ and such that $\omega(t)$ is in $D$ for $\sigma^* < t < \sigma$.

**TRAJ2'.** $\omega(t)$ is continuous for $t \neq \sigma$ or $\sigma^*$.

On $\Omega_\infty$ the coordinate variables $X_t$ are defined as on $\Omega$ and the $\sigma$-algebra generated by them is denoted $F_\infty$.  

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The hitting times $\sigma_K$ are defined on $\Omega_\infty$ as on $\Omega$:

$$\sigma_K = \inf\{t > \sigma^*: X_t \text{ is in } K\}$$

and $+\infty$ if the set on the right is empty. The shift $\theta_K$ maps $\Omega_\infty$ into $\Omega$ according to the formula

$$\theta_K \omega(t) = \omega(\sigma_K + t)$$

when $\sigma_K < \infty$.

The following theorem is proved in §5 in SMP.

**Theorem 2.1.** There exists a unique measure $P_\infty$ on $F_\infty$ with the following properties.

1. For Borel $K$ with closure contained in $D$ and for $\phi \geq 0$ and $F_\omega$ measurable on $\Omega$, the composition $\phi \cdot \theta_K$ is $F_\infty$ measurable on $\Omega_\infty$. If $\psi \geq 0$ is defined on $\Omega_\infty$ and measurable with respect to the $\sigma$-algebra generated by $X_{t \wedge \sigma(K)}$, $t$ real (the past) and if $\psi = 0$ when $\sigma_K = \infty$, then

$$E_\infty(\psi)(\phi \cdot \theta_K) = E_\infty \psi E_X(\sigma(K)) \phi.$$

Also

$$E_\infty I(\sigma(K) < \infty) \phi \cdot \theta_K = \int d\mu_K(x) E_x \phi$$

with $d\mu_K$ equilibrium measure for $K$.

2. For $f \geq 0$ and Borel measurable on $D$

$$E_\infty \int_{\sigma^*}^\sigma f(X_t) \, dt = \int f(x) \, dx.$$

3. Let the "time reversal operator" $\rho$ be defined on $\omega$ in $\Omega_\infty$ so that $\rho \omega(\sigma^* + t) = \omega(\sigma - t)$. Then $P_\infty$ is $\rho$ invariant (time reversal invariant). That is, for $\phi \geq 0$ and $F_\infty$ measurable on $\Omega_\infty$, also $\rho \cdot \phi$ is and

$$E_\infty \rho \cdot \phi = E_\infty \phi.$$

**Remark.** "$\phi \cdot \theta_K$ is measurable" in Conclusion 1 and "$\phi \cdot \rho$ is measurable" in Conclusion 3 mean measurable with respect to the $P_\infty$ completion of $F_\infty$.

The adjective "approximate" refers to $X_t$, $T \geq \sigma^*$, not being Brownian motion in general. The terminology as well as the idea go back to G. A. Hunt (1960) who worked in a discrete time context. A continuous time version was first constructed by M. Weil (1970).

It is not hard to see that $P_\infty$ is an unconditioned version of the excursion law as defined for example in Burdzy (1985).

3. **The reflected Dirichlet space.** Our candidate for the reflected Dirichlet space is the pair $(F^r, E)$ described in the Introduction. To show that $(F^r, E)$ really is a Dirichlet space on $L^2$ as in Fukushima (1969) and as in SMP, we must establish contractivity as described in §1 and we must show that for $a > 0$ the function space $F^r$ is complete relative to

$$E_a(f, g) = E(f, g) + a \int dx f(x) g(x), \quad f, g \in F^r,$$

already introduced in the introduction.
Lemma 3.1. \( F^r \) is \( E_a \) complete, any \( a > 0 \).

Proof. Let \( f_n, n \geq 1 \), in \( F^r \) be \( E_a \) Cauchy. Then there exists scalar \( f \) and vector valued \( g \) in \( L^2 \) such that \( \nabla f_n \to g \) and \( f_n \to f \) in \( L^2 \). The lemma will be proved when we show that \( \nabla f = g \). But this follows since for vector valued \( \varphi \) in \( C^{\infty}_{\text{com}}(D) \)

\[
\int dx \, g(x) \cdot \varphi(x) = \lim_{n \to \infty} \int dx \, \nabla f_n(x) \cdot \varphi(x)
\]

\[
= -\lim_{n \to \infty} \int dx \, f_n(x) \nabla \cdot \varphi(x)
\]

\[
= -\int dx \, f(x) \nabla \cdot \varphi(x)
\]

\[
= dx \, \nabla f(x) \cdot \varphi(x). \quad \text{Q.E.D.}
\]

Lemma 3.2. If \( f \) is in \( F^r \) and if \( g \) satisfies (1.6), then also \( g(f) \) is in \( F^r \) and

\[
E(g\{f\}, g\{f\}) \leq E(f, f).
\]

Proof. Let \( K \) be an open set with closure contained in \( D \). It suffices to show that for all such \( K \), the distribution \( \nabla g(f) \) is in \( L^2 \) on \( K \) and

\[
\int_K |\nabla g(f)|^2(x) \, dx \leq \int_K |\nabla f|^2(x) \, dx.
\]

If \( g \) and \( f \) are both in \( C^1 \), then by the classical chain rule \( \nabla g(f) = g'(f) \nabla f \) which implies for all \( x \) in \( K \)

\[
|\nabla g(f)(x)| \leq |\nabla f|(x)
\]

and this certainly implies (3.3). Suppose now that \( f \) is in \( C^1 \) but \( g \) is not. There exists an at most countably infinite set \( S \) of times \( t \) such that \( \int_K dx \, I(f(x) = t) > 0 \). Assume at first that 0 is not in \( S \). Then there exist

\[
t_i, j, \quad i \geq 1, \quad -\infty < j < +\infty,
\]

such that none of the \( t_i, j \) belong to \( S \) and such that

\[
t_i, 0 = 0, \quad t_i, j < t_i, j+1 < t_i, j + 1/i,
\]

\[
\inf_j t_i, j = -\infty, \quad \sup_j t_i, j = +\infty.
\]

For each \( i \) let \( g_i \) be the piecewise linear function which interpolates \( g \) at the points \( t_i, j, \quad -\infty < j < +\infty \). Then \( g_i \) satisfies (1.6), the chain rule \( \nabla g_i(f) = g'_i(f) \nabla f \) is valid almost everywhere on \( K \) and therefore so is (3.4). Also \( g_i(f) \to g(f) \) in \( L^2 \) and by the argument on p. 1.12 in SMP (Banach and Saks) Cesàro sums of a subsequence of \( \nabla g_i(f) \) converge in \( L^2 \) to a vector valued function \( h \). By the proof of Lemma 3.1, \( h = \nabla g(f) \) and we are done with this case.

If 0 is in \( S \) choose \( a_k, k \geq 1 \), not in \( S \) such that \( a_k \to 0 \) and let \( f_k = f - a_k \). The above argument is valid for \( f_k \) and so \( g(f_k) \) is in \( F^r \) with smaller \( E \) norm than \( f_k \). Certainly \( g(f_k) \to g(f) \) in \( L^2 \) and again we finish with the "argument of Banach and Saks."
For $f$ not in $C^1$ use convolutions to find $f_k$, $k \geq 1$, in $C^1$ on $A$ such that $f_k$ and $\nabla f_k$ converge in $L^2$ on $K$ to $f$ and $\nabla f$ respectively. For a subsequence $f_k \to f$ and therefore $g(f_k) \to g(f)$ almost everywhere on $K$. By the previous result
\[
\sup_k \int_K dx |\nabla g(f_k)|^2(x) \leq \sup_k \int_K dx |\nabla f_k|^2(x)
\]
which is finite since $\nabla f_k \to \nabla f$ in $L^2$ on $K$. One more application of the "argument of Banach and Saks" finishes the proof. Q.E.D.

Thus $(F^r, E)$ is indeed a Dirichlet space on $L^2$. By Fukushima (1969) or by SMP there exists a strongly continuous symmetric semigroup of submarkovian (actually markovian) transition operators $F_t^r$ with strong $L^2$ generator $A^r$, selfadjoint and nonpositive definite, such that (0.1) is true with $F^0, A^0$ replaced by $F^r, A^r$.

4. The Dirichlet principle. For $a > 0$ let $H_a$ denote the $E_a$ orthogonal complement in $F^r$ of $F^0$.

CONVENTION. We use the same symbol $H_a$ to denote the operator which implements $E_a$ orthogonal projection onto the space $H_a$.

**Lemma 4.1.** A function $h$ in $F^r$ belongs to the subspace $H_a$ if and only if $h$ (has a version which) is $C^\infty$ on $D$ and satisfies
\[\frac{1}{2} \Delta h = ah.\]

**Proof.** If $h \in F^r$ is $C^\infty$ and satisfies (4.1) then for any $\varphi \in C_c^\infty(D)$
\[E_a(h, \varphi) = \frac{1}{2} \int dx \nabla h(x) \nabla \varphi(x) + a \int dx h(x) \varphi(x) = \int dx \varphi(x) \left\{-\frac{1}{2} \Delta h(x) + ah(x)\right\} = 0.\]
Such $\varphi$ are dense in $F^0$ by Reg2 in §1 and so $h$ is in $H_a$.

Conversely, assume that $h$ is in $H_a$ and let $B$ be an open ball with closure contained in $D$. The function $h$ coincides on the closure of $B$ with a function $h_0$ in $F^0$. (Multiply $h$ by $\varphi$ in $C_c^\infty(D)$ with $\varphi = 1$ everywhere on $B$.) The absorbed space for $B$ is contained in $F^0$ and so $E_a(h_0, g) = E_a(h, g) = 0$ with $g$ in the absorbed space for $B$. (Replacement of $h_0$ by $h$ is allowed by the local nature of $E$.) By Theorem 7.3-(i) in SMP
\[h_0(x) = E_x e^{-a\tau} h_0(X_{\tau}), \quad x \text{ in } B,\]
with $\tau$ the first exit time from $B$. (That is $\tau = \sigma_{D\setminus B}$.) Of course $h_0$ can be replaced by $h$ in (4.3) and so
\[h(x) = E_x e^{-a\tau} h(X_{\tau}) = E_x \left\{1 - a \int_0^\tau dt e^{-at}\right\} h(X_{\tau}) = E_x h(X_{\tau}) - \int_B dy aG_a^B(x, y) E_y h(X_{\tau}) = E_x h(X_{\tau}) - a \int_B dy G^B(1 - aG_a^B)(x, y) E_y h(X_{\tau})\]
with $G^B(x,y)$, $G^B_a(x,y)$ the potential density and resolvent density for the corresponding operators $G^B$ and $G^B_a$ on $B$ and with $G^B G^B_a(x,y)$ the density for the operator $G^B G^B_a$. Formulae are known for the Poisson kernel and potential density on spheres which imply that the last line in (4.4) is $C^\infty$ on $B$. (See §§1.10 and 1.11 in Durrett (1984).) Thus $h$ is $C^\infty$ on $B$ and therefore, after varying $B$, on $D$. Reversing (4.2) gives (4.1). Q.E.D.

**Lemma 4.2.** For $f$ in $F^0$ and for every $x$ in $D$

$$
(4.5) \lim_{t \uparrow \sigma} f(X_t) = 0 \quad \text{a.e. } dP_x.
$$

**Proof.** Certainly (4.5) is true for $f$ in $C^\infty_{\text{com}}(D)$. For general $f$ in $F^0$ there exists by Reg2 in §1 a sequence $f_k$, $k \geq 1$, in $C^\infty_{\text{com}}(D)$ such that $f_k \to f$ in $F^0$. By the argument on the bottom half of p. 4.4 in SMP, and possibly after selecting a subsequence from the $f_k$, we conclude that for almost every $x$ in $D$ and a.e. $dP_x$, $f_k(X_t) \to f(X_t)$ uniformly in $t \leq \sigma$, implying (4.5). The “almost every $x$” restriction is removed by observing that if $\Gamma$ is the subset of $\Omega$ where (4.5) is true, then $h(x) = P_x(\Gamma)$ is a harmonic function which $= 1$ almost everywhere and therefore everywhere on $D$. Q.E.D.

**Lemma 4.3.** Let $a > 0$ and let $h$ be in $H_a$. Then for every $x$ in $D$

$$
(4.6) \ h(X_{\sigma-}) = \lim_{t \uparrow \sigma} h(X_t)
$$

exists a.e. $dP_x$ and

$$
(4.7) \ h(x) = E_x e^{-a\sigma} h(X_{\sigma-}).
$$

Moreover (4.6) is true with $h$ replaced by general $f$ in $F^\tau$ and also

$$
(4.8) \ H_a f(x) = E_x e^{-a\tau} f(X_{\tau}).
$$

**Proof.** As in the proof of Lemma 4.1

$$
(4.9) \ h(x) = E_x e^{-a\tau} h(X_{\tau})
$$

for $D'$ an open set with closure contained in $D$, for $x$ in $D'$ and with $\tau$ denoting the exit time from $D'$. Also

$$
E_x \{e^{-a\tau} h(X_{\tau})\}^2 = E_x \left\{ h(X_0) + \int_0^\tau \nabla h(X_t) e^{-at} \cdot dX_t \right\}^2
= h^2(x) + E_x \int_0^\tau |\nabla h(X_t)|^2 e^{-2at} dt
$$

by Ito’s formula (p. 64 in Durrett (1984)) which is applicable since $h$ is in $C^\infty$ by Lemma 4.1. The last expression is dominated independent of $D'$ by $G^0|\nabla h|^2(x)$ which is finite for almost every $x$ by Definition 5.1 in SMP. For such $x$ the process $e^{-at} h(X_t)$, $t \geq 0$, is an $L^2$ bounded martingale and passage to the limit in $D'$ gives (4.6) and (4.7) for almost every $x$. The “almost every” restriction is removed as in the proof of Lemma 4.2. The conclusions for general $f$ in $F^\tau$ follow from Lemma 4.2 since $f - H_a f$ belongs to $F^0$. Q.E.D.
THEOREM 4.4 (DIRICHLET PRINCIPLE). For \( f \) in \( F^r \) the function

\[
H_0 f(x) = E_x f(X_\sigma^-)
\]

belongs to \( F^r \) and satisfies

\[
E(H_0 f, H_0 f) \leq E(f, f).
\]

Also \( H_0 f \) is harmonic, \( f - H_0 f \) is in \( F^0 \), and

\[
E(H_0 f, g) = 0, \quad g \in F^0.
\]

PROOF. As above we can assume \( f \geq 0 \) without losing generality. For \( 0 < a < b \) define

\[
Q_{a,b}(f, f) = (b-a) \int dx \, H_a f(x) H_b f(x).
\]

The calculation

\[
H_a f(x) - H_b f(x) = E_x \{e^{-a\sigma} - e^{-b\sigma}\} f(X_\sigma^-) = (b-a) E_x \sigma \int_0^\sigma dt \ e^{-(b-a)t} e^{-a\sigma} f(X_\sigma^-) = (b-a) E_x \sigma \int_0^\sigma dt \ e^{-bt} H_a f(X_t)
\]

establishes the identity

\[
(4.13) \quad H_a f - H_b f = (b-a) G_b^0 H_a f
\]

which implies that \( H_a f - H_b f \) is in \( F^0 \). (This can also be deduced from the already used fact that both \( f - H_a f \) and \( f - H_b f \) belong to \( F^0 \).) Thus

\[
E_a(H_a f, H_a f - H_b f) = 0, \quad E_b(H_b f, H_a f - H_b f) = 0
\]

and subtraction gives

\[
E_a(H_a f - H_b f, H_a f - H_b f) = (b-a) \int dx \, H_b f(x) \{H_a f(x) - H_b f(x)\}
\]

which implies

\[
(4.14) \quad E_a(H_a f - H_b f, H_a f - H_b f) \leq Q_{a,b}(f, f).
\]

Also, again since \( H_a f - H_b f \) is in \( F^0 \),

\[
E_b(H_b f, H_b f) - E_a(H_a f, H_b f)
= E_b(H_b f, H_b f) - E_a(H_a f, H_a f) = (b-a) \int H_a f(x) H_b f(x) \, dx
\]

and therefore

\[
(4.15) \quad E_b(H_b f, H_b f) - E_a(H_a f, H_a f) = Q_{a,b}(f, f).
\]

the inequality (4.14) implies \( Q_{a,b}(f, f) \geq 0 \) and then equality (4.15) implies that \( E_a(H_a f, H_a f) \) decreases with \( a \) and therefore \( Q_{a,b}(f, f) \rightarrow 0 \) as \( b \downarrow 0 \). Finally (4.14) then implies that \( H_a f \), \( a > 0 \), is \( E \) Cauchy as \( a \downarrow 0 \). The same is true for \( (f - H_a f) \), \( a > 0 \), which belong to \( F^0 \) and by Lemma 1.2 has a limit in \( F^0 \). Equivalently,
the $H_\alpha f$ converge in $F^r$ as $a \downarrow 0$ to something which in fact must be $H_0 f$ since $f \geq 0$ implies that $H_\alpha f \to H_0 f$ pointwise by the monotone convergence theorem.

We have shown that $H_0 f$ is in $F^r$ and that $f - H_0 f$ is in $F^0$. The inequality (4.11) and the orthogonality relation (4.12) follow by passage to the limit $a \downarrow 0$. Finally harmonicity follows from (4.10). Q.E.D.

NOTATION. $H_0$ denotes also the collection of harmonic functions in $F^r$.

For $f$ in $C_\infty^\infty(D)$ and $h$ in $H_0$, integration by parts gives

$$E(f, h) = 0$$

valid for $f$ in $F^0$ and $h$ in $H_0$ by passage to the limit in $f$. We can view $H_0$ as the $E$ orthogonal complement of $F^0$ in $F^r$, keeping in mind that $E$ is not a true inner product on $F^r$ since $E(1, 1) = 0$.

By harmonicity of $H_0 f$ the integral on the right in (4.10) converges absolutely for all $f$ in $F^r$ and for all $x$ in $D$.

5. The boundary.

LEMMA 5.1. There exists a countable collection $Q_0$ of bounded functions in $H_0$ which are $E$ dense there.

PROOF. If we knew that $F^r$ were a separable Hilbert space then by contractivity there would be a dense sequence of bounded functions $f_m, m \geq 1$, in $F^r$. The images $H_0 f_m$ are $E$ dense in $H_0$ and bounded by (4.10). So we need only prove that $F^r$ is separable.

Let $E(\lambda), \lambda \geq 0$, be the spectral projections associated to the nonnegative definite selfadjoint operator $-A^r$. (See Kunze and Segal (1968).) For integer $k \geq 1$ let $M_k = E(k)L^2$. Each $M_k$ is a closed, possibly empty, subspace of both $L^2$ and $F^r$ on which the $L^2$ and $E_1$ norms are equivalent. Separability of the $M_k$ and therefore of $F^r$ then follow from separability of $L^2$. Q.E.D.

Represent $Q_0$ from Lemma 5.1 as a sequence $\{h_1, h_2, \ldots\}$ and for $m \geq 1$ discard $h_m$ if it is (0 for $m = 1$) linearly dependent on $h_1, \ldots, h_{m-1}$. Denote by $Q_1$ the set of functions surviving this depletion procedure.

For bounded $f, g$ in $H_0$ the product

$$fg = \frac{1}{4}(f + g)^2 - \frac{1}{4}(f - g)^2$$

belongs to $F^r$ by contractivity and therefore

$$f \# g = H_0(fg)$$

is a bounded function in $H_0$.

Denote by $Q$ the algebra over the rationals generated by $Q_1$, multiplication being the operator $\#$ of (5.2). Of course $Q$ also is a countable $E_1$ dense subset of $H_0$.

Let $W$ be the collection of functions $w$ defined on $f$ in $Q$ so that $|w(f)| \leq \|f\|_\infty$ (usual $L^\infty$ norm on $D$). Equip $W$ with the usual product topology so that it becomes a separable compact Hausdorff space. The boundary $\partial W$ is the subset of $w$ in $W$ satisfying

$$w(f \# g) = w(f)w(g), \quad w(af) = aw(f),$$

$$w(f + g) = w(f) + w(g), \quad f \geq 0 \implies w(f) \geq 0$$
with $f, g$ in $Q$ and with a rational. Then $\partial^#D$ is a closed subspace of $W$ and is itself a separable compact Hausdorff space.

All functions $f$ in $Q$ are extended to $\partial^#D$ by the formula $f(w) = w(f)$. Our state space then is $D^# = D \cup \partial^#D$. Equip $D^#$ with the coarsest topology so that $D$ is open and its subspace topology agrees with its original Euclidean topology and such that every $f$ in $Q$ is continuous on $D^#$. Then for $w$ in $\partial^#D$, sets $\{x \in D^# : |f_j(x) - f_j(w)| < \varepsilon_j \text{ for } 1 \leq j \leq m\}$ with the $f_j$ in $Q$ form a neighborhood basis at $w$. Thus $x_k$, $k \geq 1$, in $D$ converge to $w$ in $\partial^#D$ if and only if $f(x_k) \to f(w)$ for all $f$ in $Q$.

**Lemma 5.2.** For every $x$ in $D$ and a.e. $dP_x \ X_\sigma = \lim_{t \uparrow \sigma} X_t$ exists in the topology of $D^#$ and belongs to $\partial^#D$.

**Proof.** For $f$ in $Q$ the process $f(X_t)$, $0 \leq t < \sigma$, is a bounded martingale and so $L(f) = \lim_{t \uparrow \sigma} f(X_t)$ exists a.e. $dP_x$ for $x$ in $D$. Since $Q$ is countable, $L$ is a random functional on $Q$ defined a.e. $dP_x$, $x$ in $D$. Certainly $L$ satisfies the second, third and fourth identities in (5.3). For $f, g$ in $Q$ $fg = f\#g + h$ with $h$ in $F^0$. By Lemma 4.2 $h(X_t) \to 0$ as $t \uparrow \sigma$ a.e. $dP_x$ and the first identity in (5.3) is verified. Q.E.D.

**Notation.** From now on $X_\sigma$ and $X_\sigma^*$ denote the appropriate limit in $\partial^#D$ rather than the dead point $\partial$.

The next lemma follows from an isoperimetric result of Weinberger (1962). This and more is proved by Aizenman and Simon (1982), Appendix 3.

**Lemma 5.3.** There exists $C > 0$ such that

$$\int G^0(x, y) \, dy \leq C$$

for all $x$ in $D$.

**Proof.** Choose $R > 0$ such that $x$ in $D$ and $|x - y| \geq R$ implies that $y$ is not in $D$. Then the integral is

$$\leq E_x \inf\{t > 0 : |B_t - x| = R\} = E_0 \inf\{t > 0 : |B_t| = R\}.$$

For dimension $n \geq 3$ the last expression is $c \int_0^R dr r^{n-1} r^{2-n}$ and for $n = 2$, $c \int_0^R dr r \log(R/r)$. Q.E.D.

**Definition.** Harmonic measure $d\nu$ is defined on Borel subsets $V$ of $\partial^#D$ by

$$\nu(V) = \int \, dP_x (X_\sigma \text{ is in } V).$$

For $h$ in $Q$ and $x$ in $D$

$$E_x h^2(X_\sigma) = E_x \left\{ h(X_0) + \int_0^\sigma \nabla h(x_t) \, dX_t \right\}^2 = h^2(x) + E_x \int_0^\sigma |\nabla h|^2(X_t) \, dt$$
and integration in $x$ gives
\[ \int \Delta d\nu(z)h^2(z) = \int dx h^2(x) + \int dx \int dy G^0(x,y) |\nabla h|^2(y) \leq \int dx h^2(x) + C \int dy |\nabla h|^2(y) \]
by Lemma 5.3. This establishes the useful estimate
\[ (5.6) \int d\nu(z)h^2(z) \leq c E_1(h,h) \]
for $h$ in $Q$. Bounded $h$ in $H_0$ can be $E_1$ approximated by functions in $Q$ and then
general $h$ in $H_0$ can be $E_1$ approximated by its truncations. Thus $(5.6)$ allows us
to extend by continuity from $h$ in $Q$ to $h$ in $H_0$ the definition of $h$ on $\partial^#D$ as a
member of $L^2(\partial^#D, d\nu)$. General $f$ in $F^r$ is defined on $\partial^#D$ to agree with $H_0f$.
In particular $f$ in $F^0$ vanishes identically on $\partial^#D$. Of course $(5.6)$ continues to be
true for $h$ in $H_0$.

CONVENTION. From now on
\[ H_0(x,V) = P_x(X_\sigma \text{ is in } V), \]
\[ H_a(x,V) = E_x e^{-a\sigma} I(X_\sigma \text{ is in } V). \]

If a subset $V$ of $\partial^#D$ is $d\nu$ null—that is $\nu(V) = 0$—then
\[ (5.7) \int dx H(x,V) = 0 \]
and so $H_0(x,V) = 0$ for almost every and therefore every $x$ since $H_0(x,V)$ is
harmonic. Thus the measures $H_0(x,dz)$, $x$ in $D$ are all absolutely continuous with
respect to harmonic measure $\nu$ and so for $f$ in $F^r$ the random variable $f(X_\sigma)$
is well defined a.e. $dP_x$, $x$ in $D$. Also if $f_k \to f$ in $F^r$ then for a subsequence
$f_k(X_\sigma) \to f(X_\sigma)$ a.e. $dP_x$, $x$ in $D$.

**LEMMA 5.4.** For $f$ in $F^r$, $x$ in $D$ and a.e. $dP_x$
\[ (5.8) \lim_{t \to \sigma} f(X_t) = f(X_\sigma). \]

**PROOF.** For $f$ in $F^0$ the lemma is the same as Lemma 4.2 and so by Theorem
4.5 we need only consider the case when $f$ is in $H_0$. The proof of Lemma 5.2 implies
\[ (5.9) f(x) = H_0f(x) = E_x f(X_\sigma) \]
for $f$ in $Q$. The extension to general $f$ in $H_0$ follows by the same approximation
used for $(5.6)$. Then $(5.8)$ follows from the martingale convergence theorem (p. 305
in Durrett (1984)). Q.E.D.

6. Normal derivative and boundary conditions.

**LEMMA 6.1.** Let $D'$ be an open set with closure contained in $D$, and let $f$ be
in $F^0$. Then
\[ (6.1) E_\infty I(\sigma(D') < \infty) f^* < \infty \]
where $f^* = \sup_t |f(X_t)|$. License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
PROOF. Let $K = \{x \in D: |f(x)| > t\}$ for $t > 0$. By (3.0) in SMP, $\text{Cap}(K) \leq t^{-2}E(f, f)$. By (5.9) in SMP, $\text{Cap}(K) = P_\infty(\sigma(K) < \infty)$. Thus

$$P_\infty(f^* > t) \leq t^{-2}E(f, f)$$

and so

$$E_\infty I(\sigma(D') < \infty)f^* \leq P_\infty(\sigma(D') < \infty) + E_\infty(f^* - 1)^+$$

$$= P_\infty(\sigma(D') < \infty) + \int_0^\infty P_\infty(\{f^* - 1\} > t) dt$$

$$= P_\infty(\sigma(D') < \infty) + E(f, f) \int (t + 1)^{-2} dt. \quad \text{Q.E.D.}$$

**Lemma 6.2.** Let $f$ be in the reflected space $F^r$ and assume that $\Delta f$ is in $L^2$. Then there exists a polar set $N$ such that for $x$ in the complement $D \setminus N$ and a.e. $dP_x$

$$f(X_{t\wedge \sigma}) = f(X_0) + \int_0^{t\wedge \sigma} \nabla f(X_s) \cdot dX_s + \frac{1}{2} \int_0^{t\wedge \sigma} \Delta f(X_s) ds$$

for all $t \geq 0$.

**Proof.** Of course (6.3) is a standard formula in the “Ito calculus” when $f$ is $C^2$. By Theorem 4.5 there exists (harmonic and therefore $C^\infty$) $h$ in $H_0$ such that $f - h$ is in $F^0$. Therefore we need only consider $f$ in $F^0$. Extend $f$ to $R^n$ by setting $f(x) = 0$ for $x$ in $R^n \setminus D$.

Let $\varphi(y) \geq 0$ be radial (depends only on $|y|$), $C^\infty$, supported in the unit ball centered at the origin and such that $\int_{R^n} \varphi(y) dy = 1$. Assume also that $\varphi(|y|)$ is nonincreasing in $|y|$. For $k \geq 1$ let

$$\varphi_k(y) = k^n \varphi(ky),$$

$$f_k(y) = f \ast \varphi_k(y) = \int_{R^n} dy \varphi_k(y)f(x - y).$$

Each $f_k$ is $C^\infty$ and so (6.3) is true with $f$ replaced by $f_k$.

Certainly $f_k \rightarrow f$ in $L^2$ on $R^n$ and by the description of $R^0$ given in the Introduction $\nabla f$ is in $L^2$ on $R^n$ and therefore also $\nabla f_k \rightarrow \nabla f$ in $L^2$ on $R^n$. (Unfortunately it is not true in general that $\Delta f_k \rightarrow \Delta f$ in $L^2$ on $R^n$ and this limits applicability of the argument below.) For dimension $n \geq 3$ Brownian motion on $R^n$ is transient and so Theorem 3.13 in SMP guarantees the existence of a subsequence of the $f_k$ which converges to $f$ pointwise modulo a polar set. (It is easy to see that a subset of $D$ is polar for Brownian motion on $R^n$ if and only it is for the absorbed process on $D$. For $n = 2$ the analogous argument works with $R^n$ replaced by a sufficiently large disk, obtaining a transient process. Thus in every case there exists a polar set $N'$ of $D$ and a subsequence $f_k^\wedge$, $k \geq 1$, of $f_k$, $k \geq 1$, such that

$$f_k^\wedge(x) \rightarrow f(x), \quad x \in D \setminus N'.$$
In completing the proof we use the elementary fact that for \( g \geq 0 \) in \( L^1 \) convolutions \( \varphi_k \ast g \) are uniformly pointwise dominated by the Hardy-Littlewood maximal function \( Mg \), treated, for example, in Stein (1970a). This domination follows from

\[
\varphi_k \ast g(x) = \int_{\mathbb{R}^n} dy \varphi_k(y)g(x - y)
= \int_0^\infty dr r^{n-1}\varphi_k(r) \int_{S} d\sigma(z)g(x - rz)
\]

(\( S \) being the unit sphere in \( \mathbb{R}^n \) and \( d\sigma \) uniform measure (surface measure) on \( S \))

\[
= \int_0^\infty dr (-\varphi_k'(r)) \int_0^r ds s^{n-1} \int_{S} d\sigma(z)g(x - sz)
\leq c \int_0^\infty dr (-\varphi_k'(r)) r^n Mg(x)
= cMg(x)
\]

with usual convention that unspecified constant \( c \) may vary from line to line.

Also we need \( D_l, l \geq 1 \), an increasing sequence of open sets with closures contained in \( D \) and with union \( D \).

For a given \( l \) and for \( k \) sufficiently large and for \( x \) in \( D_l \) the convolution \( f_k^\wedge \) depends only on the values of \( f \) in some compact subset of \( D \) and so \( \Delta f_k^\wedge(x) \) is not effected by contributions to \( \Delta f \) on the boundary of \( D \). So by the inequalities for \( Mg \) proved in §1.1 of Stein (1970a), the functions

\[
|\nabla f|^* = \sup_{k \geq k_0} |\nabla f_k^\wedge|, \quad |\Delta f|^* = \sup_{k \geq k_0} |\Delta f_k^\wedge|
\]

(\( k_0 \) depending on \( D_l \)) are in \( L^2 \) on \( D_l \). By §1 in SMP

(6.6) \( S(x) = E_x \int_0^{\sigma(D_l)} dt \{|\nabla f|^2(X_t) + |\Delta f|^*(X_t)\} < \infty \)

for almost every \( x \) in \( D_l \). But \( S(x) \) is superharmonic in \( D_l \) and therefore \( S(X_{t\wedge\sigma(D_l)}) \), \( t \geq 0 \), is a supermartingale. This implies that if

\[
N_l = \{x \in D_l : S(x) = +\infty\}
\]

then \( P_x(\sigma(N_l) < \infty) = 0 \) for \( x \) in \( D_l \) such that \( S(x) < \infty \). Therefore \( N_l \) is polar.

By the corollary at the bottom of p. 5 in Stein (1970a)

(6.7) \( \nabla f_k^\wedge(x) \to \nabla f(x), \quad \Delta f_k^\wedge(x) \to \Delta f(x) \)

as \( k \to \infty \) for almost every \( x \) in \( D_l \). (This also depends on domination by the Hardy-Littlewood maximal function established above.) If Borel \( K \) is \( dx \) null then for every \( x \) in \( D \)

\[
E_x \int_0^\sigma I(X_s \text{ is in } K) \, ds = 0.
\]

Thus (6.7) implies that for \( x \) in \( D_l \) and a.e. \( dP_x \)

(6.8) \( \nabla f_k^\wedge(X_s) \to \nabla f(X_s), \quad \Delta f_k^\wedge(X_s) \to \Delta f(X_s) \)
for almost every $s$ such that $0 \leq s < \sigma(D_t)$. This combines with (6.6) to imply that for $x$ in $D_t \setminus N_t$

\[(6.9) \int_0^{t \wedge \sigma(D_t)} \Delta f_k^t(X_s) \, ds \to \int_0^{t \wedge \sigma(D_t)} \Delta f(X_s) \, ds\]

a.e. $dP_x$ and that

\[(6.10) \mathbb{E}_x \int_0^{t \wedge \sigma(D_t)} |\nabla f_k^t(X_s) - \nabla f(X_s)|^2 \, ds \to 0.\]

By §2.1 and Appendix A.5 in Durrett (1984), (6.10) implies

\[(6.11) \int_0^{t \wedge \sigma(D_t)} \nabla f_k^t(X_s) \cdot dX_s \to \int_0^{t \wedge \sigma(D_t)} \nabla f(X_s) \cdot dX_s\]

almost everywhere $dP_x$. Now (6.5), (6.9) and (6.11) imply (6.3) with $\sigma$ replaced by $\sigma(D_t)$ for $x$ not in the polar set $N' \cup N_t$.

Finally

\[(6.12) \mathbb{E}_x \int_0^{\sigma} ds \{\Delta f(X_s) + |\nabla f|^2(X_s)\} < \infty\]

a.e. $dP_x$ with $x$ in $D \setminus N''$ with $N''$ polar by the same argument which gave (6.6) modulo a polar set. This allows us to pass to the limit $l \uparrow \infty$ in (6.3) with $\sigma$ replaced by $\sigma(D_t)$ to obtain (6.3) as written a.e. $dP_x$ for $x$ in $D \setminus N$ with $N$ the polar set $N' \cup N'' \cup \bigcup_{i=1}^{\infty} N_i$. Q.E.D.

**LEMMA 6.3.** Let $h$ be in $H_0$. Then

1. $\langle h \rangle^* = \sup_{\sigma \leq t \leq \sigma \leq s} |h(X_t) - h(X_s)|$ is in $L^2(\Omega_\infty, dP_\infty)$.
2. $E_\infty \{h(X_\sigma) - h(X_{\sigma^*})\}^2 = \int dx |\nabla h|^2(x)$.
3. For $f$ in $F^\gamma$ with $\Delta f$ in $L^2$

\[\int dx \nabla h(x) \cdot \nabla f(x) = E_\infty \{h(X_\sigma) - h(X_{\sigma^*})\} \{f(X_\sigma) - f(X_{\sigma^*})\}\]

**PROOF.** Let $D'$ be open with closure contained in $D$. Then

\[E_\infty \sup_{t \geq \sigma(D')} |h(X_t) - h(X_{\sigma(D')})|^2 I(\sigma(D') < \infty) \leq 4 E_\infty |h(X_\sigma) - h(X_{\sigma(D')})|^2 I(\sigma(D') < \infty)\]

(by Doob's maximal inequality for $p = 2$, Appendix A.6 in Durrett (1984))

\[= 4 E_\infty I(\sigma(D') < \infty) \int_{\sigma(D')}^{\sigma} dt |\nabla h|^2(X_t)\]

(§2.1, ibid.)

\[\leq 4 \int dx |\nabla h|^2(x)\]

by (2.3). By time reversal invariance, conclusion 3 in Theorem 2.1, the same is true with $\sigma(D')$ replaced by the last exit time from $D'$, $\sigma^*(D')$ and with $t \geq \sigma(D')$ replaced by $t \leq \sigma^*(D')$ in the interval of integration. Conclusions 1 and 2 follow from these estimates.
For conclusion 3 note first that Lemma 6.2 implies

\[
\frac{1}{2} I(\sigma(D') < \infty) \int_{\sigma(D')}^{\sigma} \Delta f(X_t) \, dt
\]

\[
= I(\sigma(D) < \infty) \left\{ f(X_\sigma) - f(X_{\sigma(D')}) - \int_{\sigma(D')}^{\sigma} \nabla f(X_t) \cdot dX_t \right\}
\]
a.e. \(dP_\infty\). By Lemma 5.4 and Theorem 2.1

\[
I(\sigma(D') < \infty) f(X_{\sigma(D')}) \to f(X_\sigma)
\]
a.e. \(dP_\infty\) as \(D'\) increases to \(D\). Also

\[
I(\sigma(D') < \infty) \int_{\sigma(D')}^{\sigma} \Delta f(X_t) \, dt \to \int_{\sigma(D')}^{\sigma} \Delta f(X_t) \, dt
\]
a.e. \(dP_\infty\) since

\[
E_x \int_{\sigma(D')}^{\sigma} |\Delta f|(X_t) \, dt < \infty.
\]

Also

\[
I(\sigma(D') < \infty) \int_{\sigma(D')}^{\sigma} \nabla f(X_t) \cdot dX_t
\]
converges in \(L^2(\Omega_\infty, dP_\infty)\) since

\[
E_\infty \int_{\sigma(D')}^{\sigma} |\nabla f|^2(X_t) \, dt = \int dx |\nabla f|^2(x) < \infty.
\]

This together with (6.14) and (6.15) allows us to pass to the limit \(D' \uparrow D\) in (6.13) and obtain

\[
\frac{1}{2} \int_{\sigma(D')}^{\sigma} \Delta f(X_t) \, dt = f(X_\sigma) - f(X_{\sigma(D')}) - \int_{\sigma(D')}^{\sigma} \nabla f(X_t) \cdot dX_t.
\]

This implies in particular that the left-hand side is in \(L^2(\Omega_\infty, dP_\infty)\) since the terms on the right are. Conclusion 3 follows since by time reversal invariance

\[
E_\infty \left( \int_{\sigma(D')}^{\sigma} \Delta f(X_t) \, dt \right) \left( \int_{\sigma(D')}^{\sigma} \nabla h(X_t) \cdot dX_t \right)
\]

\[
= 2E_\infty \int_{\sigma(D')}^{\sigma} \Delta f(X_t) \int_t^\sigma \nabla h(X_s) \cdot dX_s = 0. \quad \text{Q.E.D.}
\]

As noted in the introduction \(f\) belongs to domain\((A^r)\) with \(A^r f = g\) if and only if the following two conditions are satisfied.

\text{GEN1.} \; f \text{ is in } F^r.

\text{GEN2.} \; \text{For all } h \text{ in } F^r

(6.18) \quad E(f, h) + \int dx g(x)h(x) = 0.

For \(h\) in \(C^\infty_{\text{com}}(D)\) integration by parts gives \(g = \frac{1}{2} \Delta f\). Therefore Gen2 can be replaced by the following two conditions.

\text{GEN2.1.} \; \Delta f \text{ is in } L^2.

\text{GEN2.2.} \; \text{For all } h \text{ in } F^r

(6.19) \quad \int dx \nabla f(x) \cdot \nabla h(x) + \int dx \Delta f(x)h(x) = 0.
If \( f \) satisfies conditions Gen1 and Gen2.1 then Gen2.2 is equivalent to (6.19) for bounded \( h \) in \( F^r \). By Theorem 4.5 there exists bounded \( h_1 \) in \( F^0 \) such that \( h - h_1 \) is in \( H_0 \). We have seen that (6.19) is true for \( h_1 \) and so we need only verify (6.19) for bounded \( h \) in \( H_0 \).

**Definition.** \( \varphi \) defined on \( \partial^#D \) is a test function if it is the boundary value (restriction to \( \partial^#D \)) of some \( f \) in \( F^r \).

**Definition.** For \( f \) in \( F^r \) its normal derivative is the function \( \partial f/\partial n \) defined on test functions \( \varphi \) by

\[
\frac{\partial f}{\partial n}(\varphi) = \lim_{k \to \infty} E_\infty \varphi(X_{\sigma^*}) I(\sigma(D_k) < \infty) \{ f(X_{\sigma^*}) - f(X_{\sigma(D_k)}) \}.
\]

The \( D_k, k \geq 1 \) are as in the proof of Lemma 6.2.

Now we prove the theorem stated in the Introduction. From (6.3) follows

\[
E_\infty \varphi(X_{\sigma^*}) I(\sigma(D_k) < \infty) \{ f(X_{\sigma^*}) - f(X_{\sigma(D_k)}) \} = E_\infty \varphi(X_{\sigma^*}) I(\sigma(D_k) < \infty) \{ f(X_{\sigma^*}) - f(X_{\sigma}) \} \\
+ E_\infty \varphi(X_{\sigma^*}) I(\sigma(D_k) < \infty) \int_{\sigma(D_k)} \nabla f(X_t) \cdot dx_t \\
+ E_\infty \varphi(X_{\sigma^*}) I(\sigma(D_k) < \infty) \int_0^{\sigma(D_k)} \frac{1}{2} \Delta f(X_t) dt.
\]

By time reversal invariance and Lemma 6.3 the first term converges to

\[
\frac{1}{2} E_\infty \{ \varphi(X_{\sigma}) - \varphi(X_{\sigma^*}) \} \{ f(X_{\sigma}) - f(X_{\sigma^*}) \} = \frac{1}{2} \int dx \nabla H_0 \varphi(x) \cdot \nabla f(x).
\]

The second term vanishes by conclusion 1 in Theorem 2.1. Passage to the limit in the third term gives

\[
E_\infty \varphi(X_{\sigma^*}) \int_{\sigma^*} \frac{1}{2} \Delta f(X_t) dt
\]

which by time reversal invariance

\[
= E_\infty \int_{\sigma^*} \frac{1}{2} \Delta f(X_t) dt \varphi(X_{\sigma}).
\]

Replacing \( \varphi(X_{\sigma}) \) by its conditional expectation \( H_0 \varphi(X_t) \) and using (2.3), we get

\[
\frac{1}{2} \int dx H_0 \varphi(x) \Delta f(x).
\]

Thus

\[
(6.22) \quad \frac{\partial f}{\partial n}(\varphi) = \frac{1}{2} \int dx \left\{ \nabla f(x) \cdot \nabla H_0 \varphi(x) + \Delta f(x) H_0 \varphi(x) \right\}
\]

and we are done since every \( h \) in \( H_0 \) can be represented \( h = H_0 \varphi \). Q.E.D.

**7. Direct reduction to the classical formula for \( D \) smooth.** In this section \( D \) is bounded and has \( C^\infty \) boundary. We begin by showing that the abstract boundary \( \partial^#D \) of \( \S 5 \) can be constructed so that it is identical to the Euclidean boundary \( \partial D \).

In Lemma 5.1 choose \( Q_0 \) so that each \( f \) in \( Q_0 \) is \( C^\infty \) on the closure of \( D \) and so that the corresponding algebra \( Q \), restricted to the boundary \( \partial D \), is dense in
Every point \( w \) in \( \partial^\# D \) is a bounded linear functional on \( Q \) which is nonnegative on nonnegative functions. So by the Riesz theorem there is a unique Radon measure \( d\nu \) on \( \partial D \) such that

\[
w(f) = \int_{\partial D} d\nu(z)f(z), \quad f \in Q.
\]

The multiplicative property \( w(fg) = w(f)w(g) \) in (5.3) implies that \( d\nu \) has mass one and is concentrated at a point \( \psi w \) in \( \partial D \). This map \( \psi : \partial^\# D \to \partial D \) is easily seen to be bijective. Finally, it is easy to see that the Euclidean topology on the closure \( D^- = D \cup \partial D \) agrees with the topology specified in §5. This completes the identification of \( \partial^\# D \) with \( \partial D \).

Since \( D \) is \( C^\infty \) the exterior unit normal \( N_z \) is well defined and a smooth function of \( z \) in \( \partial D \). For \( f \) defined on the closure \( D^- \) its “classical” normal derivative is

\[
D_n f(z) = \nabla f(z) \cdot N_z.
\]

The potential density \( G^0(x, y) \) is \( C^\infty \) for \( x \neq y \) in \( D^- \). (See the subsection on the “method of layer potentials” in Jerison and Kenig (1982).) In particular, for \( x \) in \( D \)

\[
D_n G^0(x, y) = \nabla_z G^0(x, z) \cdot N_z
\]

is well defined and continuous for \( z \) in \( \partial D \).

Let \( f \) be continuous on \( D^- \) and harmonic in \( D \). For \( x \) in \( D \)

\[
-\frac{1}{2} \Delta_y G^0(x, y) = \delta_x(dy)
\]

the unit point mass at \( x \), and so

\[
f(x) = -\frac{1}{2} \int dy f(y) \Delta_y G^0(x, y)
\]

\[
= \frac{1}{2} \int dy \{G^0(x, y)\Delta f(y) - f(y)\Delta_y G^0(x, y)\}
\]

\[
= \frac{1}{2} \int dy \text{div}\{G^0(x, y)\nabla f(y) - f(y)\nabla_y G^0(x, y)\}
\]

\[
= -\frac{1}{2} \int_{\partial D} d\sigma(z)f(z)D_n G^0(x, z)
\]

by the classical divergence theorem since \( G^0(x, z) = 0 \) for \( z \) in \( \partial D \). Here \( d\sigma \) denotes surface measure which can be defined in the usual way since \( D \) is smooth.

Here and below the operator \( D_n \) in the expression \( D_n G^0(x, z) \) acts on the second variable.

Let \( H_0(x, dy) \) denote the exit measure on \( \partial D \) with starting point \( x \). That is, for \( \phi \geq 0 \) on \( \partial D \)

\[
\int_{\partial D} H_0(x, dz)\phi(z) = E_x \phi(X_\sigma).
\]

In particular for \( f \) continuous on \( D^- \) and harmonic on \( D \)

\[
f(x) = \int H_0(x, dz)f(z).
\]

Together with (7.2), this implies

\[
H_0(x, dz) = -\frac{1}{2} D_n G^0(x, z) d\sigma(z).
\]
Assume that the origin 0 is in $D$. In this section it is convenient to take

(7.4) \[ H(dz) = H_0(0, dz) \]

as “harmonic measure”. Then

\[ H(dz) = -\frac{1}{2} D_n G^0(0, z) \, d\sigma(z). \]

By Hunt and Wheeden (1970) the Martin kernel $K(x, y)$, which is continuous and strictly positive for $x$ in $D$ and $y$ in $D^-$, satisfies $H_0(x, dz) = K(x, z) \, dH(z)$ and so for $x$ in $D$ and $z$ in $\partial D$

(7.5) \[ K(x, z) = D_n G^0(x, z) / D_n G^0(0, z) \]

except possibly for $z$ sets of harmonic measure 0. By continuity, (7.5) is true whenever $D_n G^0(0, z) \neq 0$. We verify now that this is true for all $z$, and indeed that

(7.6) \[ D_n G^0(x, z) < 0, \quad x \text{ in } D, \quad z \text{ in } \partial D. \]

From (7.5) it follows that if $D_n G^0(0, z) = 0$ for a given $z$ in $\partial D$, then $D_n G^0(x, z) = 0$ for all $x$ in $D$. But this is impossible. Using local coordinates it is easy to construct smooth $f$ vanishing on $\partial D$ such that $D_n f(z) \neq 0$. Such $f$ belongs to the domain of the absorbed generator $A_0$ and therefore

\[ f(x) = \frac{-1}{2} \int G^0(x, y) \Delta f(y). \]

This implies

\[ D_n f(z) = \frac{-1}{2} \int dy \Delta f(y) D_n G^0(y, z) = 0, \]

a contradiction which proves (7.6).

Let $D_1$ be open with closure contained in $D$ and denote by $h$ its equilibrium potential, that is, $h(x) = P_x(\sigma(D_1) < \infty)$. Then $h(z) = \int d\mu(y) G^0(y, z)$ with $d\mu$ equilibrium measure for $D_1$. For $z$ in $\partial D$

\[ D_n h(z) = \int d\mu(y) D_n G^0(y, z) \]

and since $d\mu$ is bounded, the above arguments imply that $D_n h$ is continuous and bounded away from zero on $\partial D$. Also $\nabla h$ is continuous on $D^-$ since $D$ is smooth.

Consider now $f$ and $\phi$ as in §6—only assume in addition that $\Delta f$ is bounded and $\nabla f$ continuous on $D^-$. We use now the idea of Gundy referred to in the introduction.

It is convenient to specify

\[ D_k = \{ x \in D : h(x) > 1/k \}. \]

Then

(7.7) \[ E_\infty \phi(X_{\sigma^-}) I(\sigma(D_k) < \infty) \{ f(X_{\sigma^-}) - f(X(\sigma(D_k))) \} \]

\[ = E_\infty \phi(X_{\sigma^-}) I(\sigma(D_k) < \infty) E_\infty \{ f(X_{\sigma^-}) - f(X(\sigma(D_k))) \} | X_{\sigma^-}, \sigma(D_k) < \infty, \]

the last occurrence of $E_\infty$ denoting conditional expectation given $X_{\sigma^-}$ and the event $\sigma(D_k) < \infty$. 

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The thus conditioned process $X_t$, $\sigma^* \leq t \leq \sigma(D_k)$, is absorbed Brownian motion on the complement $D \setminus D_k$, conditioned on exiting via $\partial D_k$ rather than $\partial D$. Since

$$kh(x) = P_x(\sigma(D_k) < \infty), \quad x \in D \setminus D_k,$$

the conditioned generator is

$$\frac{1}{2h} \Delta h = \frac{1}{2} \Delta + \frac{1}{h} \nabla h \cdot \nabla$$

and we can replace the conditional expectation on the right-hand side in (7.7) by

$$- \int_{\sigma^*}^{\sigma(D_k)} \left\{ \frac{1}{2} \Delta f(X_t) + \frac{1}{h} \nabla h \cdot \nabla f(X_t) \right\} dt$$

to get

$$-E_\infty \phi(X_{\sigma^*}) I(\sigma(D_k) < \infty) \int_{\sigma^*}^{\sigma(D_k)} \left\{ \frac{1}{2} \Delta f(X_t) + \frac{1}{h} \nabla h \cdot \nabla f(X_t) \right\} dt$$

and then

(7.8) $$-E_\infty \phi(X_{\sigma^*}) \int_{\sigma^*}^{\sigma(D_k)} \left\{ \frac{kh}{2} \Delta f(X_t) + \nabla k h \cdot \nabla f(X_t) \right\} dt$$

since $I(\sigma(D_k) < \infty)$ can be replaced by $kh(t)$ inside the integral by (2.1). The same argument can be made with $f$ replaced by $h$ to give

(7.9) $$E_\infty \phi(X_{\sigma^*}) I(\sigma(D_k) < \infty) \{ h(X_{\sigma^*}) - h(X(\sigma(D_k))) \}$$

$$= E_\infty \phi(X_{\sigma^*}) \int_{\sigma^*}^{\sigma(D_k)} k |\nabla h|^2(X_t) dt.$$ 

With the help of the additive functional determined by $d\mu$ in SMP, the proof in §6 can be adapted to show that $\partial h/\partial n(\phi)$ exists and is equal to

(7.10) $$\frac{1}{2} \int \nabla h \cdot \nabla H_0 \phi(x) \, dx - \int d\mu(x) H_0 \phi(x) = - \int d\mu(x) H_0 \phi(x)$$

since the $dx$ integral vanishes by (4.12) because $h$ is in $D_0$. From the definition of $\partial h/\partial n(\phi)$ it follows that (7.9) converges to (7.10) and in particular

(7.11) $$\lim_{k \to \infty} E_\infty \phi(X_{\sigma^*}) \int_{\sigma^*}^{\sigma(D_k)} |\nabla h|^2(X_t) dt$$

$$= - \int d\mu(x) H_0 \phi(x) = \int_{\partial D} d\sigma(z) \phi(z) D_n h(z),$$

the last step following from (7.3) integrated against $d\mu(x)$. Since $\nabla h$ is bounded away from 0 in any neighborhood of $\partial D$, the expression

$$\frac{h}{2} \Delta f(X_t) + \nabla h \cdot \nabla f(X_t)$$

in (7.8) is asymptotic as $t \downarrow \sigma^*$ to

$$(1/|\nabla h|^2)(X_{\sigma^*}) \nabla f \cdot \nabla h(X_{\sigma^*}) |\nabla h|^2(X_t)$$

and it follows that

(7.12) $$\partial f/\partial n(\phi) = \int_{\partial D} d\sigma(z)(1/|\nabla h|^2) \nabla h \cdot \nabla f(z) \phi(z) D_n h(z).$$
Since \( h \) vanishes on \( \partial D \), \( \nabla h = -D_n h \), \( \nabla h \cdot \nabla f = D_n h D_n f \) on \( \partial D \). Substituting this into (7.12) and applying (7.11), we get

\[
(7.13) \quad \frac{\partial f}{\partial n}(\phi) = \int_{\partial D} \phi(z) D_n f(z) \, d\sigma(z),
\]

the desired result.

We do not believe that (7.13) is valid for general \( f \) satisfying the hypotheses of our main theorem but certainly weaker additional assumptions would suffice. However questions about the domain \( D \) seem more promising. We know that (7.13) can be extended to Lipschitz domains using the techniques in Jerison and Kenig (1982)—especially Lemmas 3.3 and 3.4. Is (7.13) true whenever \( d\sigma \) is well defined? Is there a version of (7.13) which is true more generally?

8. Two examples done another way.

EXAMPLE 1. The dimension \( n = 1 \) and \( D = (0,1) \), the unit interval. The potential density is

\[
G^0(x,y) = 2xny(1-x+y).
\]

The approximating sets are

\[
D_\varepsilon = (\varepsilon, 1-\varepsilon) \quad \text{with} \quad 0 < \varepsilon < 1/2.
\]

The equilibrium measure \( \mu_\varepsilon \) is

\[
\mu_\varepsilon = (1/2\varepsilon)\{\delta_\varepsilon + \delta_{1+\varepsilon}\}
\]

with \( \delta_x \) denoting the usual point mass at \( x \).

Suppose that \( f \) is in \( F^r \) with \( \Delta f = f'' \) in \( L^2 \). Then \( \nabla f = f' \) is continuous on the closure of \( D \). The points \( \varepsilon, 1-\varepsilon \) can be reached only from 0, 1 respectively at the first entrance time \( \sigma(D_\varepsilon) \). Thus for \( \varphi \) a test function

\[
E_0 \varphi(X_{\sigma^*})\{f(X_{\sigma^*}) - f(X_{\sigma(D_\varepsilon)})\}
\]

\[
= (1/2\varepsilon)(\varphi(0)\{f(0) - f(\varepsilon)\} + \varphi(1)\{f(1) - f(1-\varepsilon)\})
\]

\[
- \frac{1}{2} \varphi f'(0) + \frac{1}{2} \varphi f'(1)
\]

in agreement with (0.10).

EXAMPLE 2. The dimension \( n = 2 \) and \( D \) is the upper half plane \( R^2_+ = \{(x,y) : x \text{ is real and } y > 0\} \). The approximating sets are shifted versions \( D_\varepsilon = D + (0,\varepsilon) \). It is convenient to write here \( B_\varepsilon = D \setminus D_\varepsilon \).

The main step is to find the Poisson kernel \( K_\varepsilon(z,t) \) for \( B_\varepsilon \) conditioned on hitting the upper boundary \( y = \varepsilon \) before the lower one \( y = 0 \). Here \( z = (x,y) \) in \( D_\varepsilon \) and \( t \) labels \((t,\varepsilon)\) in the upper boundary. Using the complex structure \((z = x + iy)\) on \( R^2 \), we define the map \( w = e^{\pi x/\varepsilon} \) which takes \( B_\varepsilon \) into \( R^2_+ \) in such a way that the upper boundary of \( B_\varepsilon \) corresponds to the negative half axis in the boundary of \( R^2_+ \). Let \( z = (x,y) \) in \( B_\varepsilon \) correspond to \( w = (u,v) \) in \( R^2_+ \) and let \( t \) labeling a point in the upper boundary of \( B_\varepsilon \) correspond to \( s < 0 \). Then

\[
u = e^{\pi x/\varepsilon} \cos(\pi y/\varepsilon), \quad v = e^{\pi x/\varepsilon} \sin(\pi y/\varepsilon), \quad s = -e^{\pi t/\varepsilon}.
\]

Let \( h \) be bounded and harmonic on \( B_\varepsilon \) and continuous on the closure. Then \( h \) is harmonic also as a function of \( w \) and so by the theory of harmonic functions on \( R^2_+ \).
(Stein (1970a)),

\[ h(z) = h((\varepsilon/\pi) \log w) \]
\[ = \frac{1}{\pi} \int_{-\infty}^{0} ds \frac{h((\varepsilon/\pi) \log(-s))v}{(s-u)^2 + v^2} \]
\[ = \frac{1}{\pi} e^{\pi z/\varepsilon} \sin \left( \frac{\pi y}{\varepsilon} \right) \int_{-\infty}^{0} ds \frac{h((\varepsilon/\pi) \log(-s))}{\{(s-e^{\pi z/\varepsilon} \cos(\pi y/\varepsilon))^2 + e^{2\pi z/\varepsilon} \sin^2(\pi y/\varepsilon)\}} \]
\[ = \frac{1}{\pi} \sin \left( \frac{\pi y}{\varepsilon} \right) \int_{-\infty}^{0} ds \frac{h((\varepsilon/\pi) \log(-s))e^{\pi z/\varepsilon}}{\{e^{2\pi z/\varepsilon} + s^2 - 2se^{\pi z/\varepsilon} \cos(\pi y/\varepsilon)\}}. \]

Substituting \( t = (s/\pi) \log(-s) \) for \( s \) and multiplying numerator and denominator by \( e^{-2\pi t/\varepsilon} \) gives

\[ h(z) = \frac{1}{\varepsilon} \sin \left( \frac{\pi y}{\varepsilon} \right) \int_{-\infty}^{+\infty} dt \frac{e^{(\pi z-t)/\varepsilon}}{e^{(\pi z-t)/\varepsilon} + e^{-\pi(x-t)/\varepsilon} + 2 \cos(\pi y/\varepsilon)}. \]

Finally, division by the exponential in the numerator gives

\[ K_{\varepsilon}(z,t) = \frac{(1/\varepsilon)}{\sin(\pi y/\varepsilon)} \{e^{(\pi z-t)/\varepsilon} + e^{-(\pi z-t)/\varepsilon} + 2 \cos(\pi y/\varepsilon)\}. \]

Equilibrium measure for \( D_{\varepsilon} \) is \( (1/2\varepsilon) \times (\text{Lebesgue measure}) \) on the line \( y = \varepsilon \). Extend \( \varphi \) to \( R_+^2 \) so that \( \varphi(x,y) = \varphi(0,y) \). Then for \( 0 < a < \varepsilon \)

\[ E_{\infty} I(\sigma(D_{\varepsilon}) < \infty) \varphi(X_{\sigma(D_{\varepsilon})}) \{f(X_{\sigma(D_{\varepsilon})}) - f(X_{\sigma(D_{\varepsilon})})\} \]
\[ = \frac{1}{2ae} \sin \left( \frac{\pi a}{\varepsilon} \right) \int_{-\infty}^{+\infty} dt \varphi(t) \int_{-\infty}^{+\infty} dx \frac{f(t,a) - f(x,\varepsilon)}{\{e^{(\pi z-t)/\varepsilon} + e^{-(\pi z-t)/\varepsilon} + 2 \cos(\pi z/\varepsilon)\}}. \]

Assume now that \( \varphi \) is bounded and continuous with compact support and that \( f \) is bounded and \( C^1 \) on the closure of \( R_+^2 \) with \( \nabla f \) bounded. Then passage to the limit \( a \downarrow 0 \) gives

\[ E_{\infty} \varphi(X_{\sigma}) I(\sigma(D_{\varepsilon}) < \infty) \{f(X_{\sigma}) - f(X_{\sigma(D_{\varepsilon})})\} \]
\[ = \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} ds \left( \frac{1}{2\varepsilon} \right) \varphi(t) \{f(t,0) - f(s,\varepsilon)\} Q_{\varepsilon}(t-s) \]

with \( Q_{\varepsilon}(t) = (1/\varepsilon)Q(t/\varepsilon) \) and \( Q(t) = (\pi/4)\text{sech}^2(\pi t/2) \). Replacing \( (t-s)/\varepsilon \) by \( s \) in the \( ds \) integral, we get

\[ \int_{-\infty}^{+\infty} dt \varphi(t) \int ds Q(s) \{f(t,0) - f(t + s\varepsilon,\varepsilon)\}. \]

Finally

\[ f(t,0) - f(t + \varepsilon s,\varepsilon) = \varepsilon \partial f/\partial x(t,0) - \varepsilon \partial f/\partial y(t,0) + O(\varepsilon^2) \]

and since \( \int Q(s)ds = 0 \) and \( \int Q(s)ds = 1 \), passage to the limit \( \varepsilon \downarrow 0 \) yields

\[ \frac{\partial f}{\partial n(\varphi)} = -\frac{1}{2} \int dt \varphi(t) \frac{\partial f}{\partial y}(t,0) \]

in agreement with (0.10).
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