ABSTRACT. We study certain periodic phenomena in the cohomology of the mod $p$ Steenrod algebra which are related to the polynomial generators $v_n \in \pi_*BP$. A chromatic resolution of the $E_2$ term of the classical Adams spectral sequence is constructed.

One of the major goals of homotopy theory is the understanding of $\pi_*(S^0)$, the stable homotopy groups of spheres. A technique for studying these groups is by the construction of certain "systematic families" of classes, first due to M. G. Barratt [3]. One way to express this idea is as follows. Let $X$ be a finite complex. (All "spaces" and "complexes" are objects in the stable category localized at a prime $p$.) A self-map of degree $i$, $v: \Sigma^i X \to X$, is nonnilpotent if the $k$-fold composition $v^k = (v \circ v \circ \cdots \circ v): \Sigma^k X \to X$ is essential for all $k > 0$.

DEFINITION (1). For a given nonnilpotent map $v$, a class $\alpha \in \pi_j(S^0)$ is $v$-periodic if $\alpha$ can be decomposed as $S^t \leftrightarrow X/X^{(t-1)} \xrightarrow{\alpha} S^{t-j}$, where $X^{(k)}$ denotes the $k$-skeleton of $X$, and the composite $\Sigma^k X \xrightarrow{\alpha^k} X \xrightarrow{p} X/X^{(t-1)} \xrightarrow{\alpha} S^{t-j}$ is essential for all $k > 0$ [4].

A $v$-periodic class $\alpha \in \pi_*(S^0)$ determines an infinite "systematic family" in the following manner. For each $k > 0$ there exists an integer $\varepsilon$ with $0 \leq \varepsilon \leq \dim X$, such that the composite $\Sigma^{k+t+\varepsilon} \xleftarrow{\alpha^k} \Sigma^{k+t} X/X^{(t+\varepsilon-1)} \xrightarrow{\alpha^k} X \xrightarrow{p} X/X^{(t-1)} \xrightarrow{\alpha} S^{t-j}$ is essential (since the composite above is essential for all $k$), so that each $k > 0$ determines a class (or classes) in $\pi_{(k+t+\varepsilon-t+1)}(S^0)$. Here are several well-known examples of this sort of phenomenon.

EXAMPLE (2). Let $M_p$ denote the mod $p$ Moore space ($p \geq 3$). Then Adams [1] has constructed a nonnilpotent map $A: \Sigma^q M_p \to M_p$, where $q = 2(p-1)$. This map determines a family of nontrivial classes $\{\alpha_t\}$, $t \geq 1$, with $\alpha_t \in \pi_{q(t-1)}(S^0)$ given by the following diagram:

$$
\begin{array}{c}
\Sigma^q M_p \\
\alpha_t \\
S^q \\
\alpha_t \to S^1
\end{array}
\xrightarrow{A^t} \\
\xrightarrow{p} \\
M_p
$$

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EXAMPLE (3). Let \( V(1) \) denote the cofiber of the map \( A \) above. Then for \( p \geq 5 \), there is a nonnilpotent map \( B : \Sigma^{2(p^2-1)}V(1) \rightarrow V(1) \) which determines a family of nontrivial classes \( \{ \beta_t \} \), \( t \geq 1 \), with \( \beta_t \in \pi_{2(p^2-1)t-q-1}(S^0) \), by including \( \Sigma^{2(p^2-1)t} \) into the bottom cell of \( \Sigma^{2(p^2-1)t}V(1) \) and pinching out onto the top cell of \( V(1) \) \([17]\).

EXAMPLE (4). Let \( V(2) \) denote the cofiber of \( B \). Then for \( p \geq 7 \) there is a map \( C : \Sigma^{2(p^3-1)}V(2) \rightarrow V(2) \) which determines a family of nontrivial classes \( \{ \gamma_t \} \) in \( \pi_{*}(S^0) \) in a similar manner \([13]\).

Nonnilpotent self-maps of finite complexes have been classified by Devinatz, Hopkins and J. Smith \([6]\) as part of the affirmative solution of the Nilpotence Conjecture. Part of this result can be stated as follows.

THEOREM (5). (Nilpotence theorem) Let \( X \) be a finite complex. A self-map \( v : \Sigma X \rightarrow X \) is nonnilpotent if and only if the induced homomorphism \( BP_*v \) is nonnilpotent in \( BP_*(X) \).

Here \( BP \) is the mod \( p \) Brown-Peterson spectrum, where

\[
\pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \ldots],
\]

with \( |v_i| = 2(p^i - 1) \). The three examples above all represent multiplication by a generator in \( BP \)-homology. Here \( BP_*A \) is the map \( v_1 \) in \( BP_*M_p \), \( BP_*B = v_2 \), and \( BP_*C = v_3 \). Two other interesting maps representing \( v_i \)'s have been studied at the prime 2. These are \( v_1^4 : \Sigma^8M_2 \rightarrow M_2 \) and \( v_2^3 : \Sigma^48Y \rightarrow Y \), where \( Y \) is a certain four cell complex. Adams and Barratt have used the first map and Davis and Mahowald have used the second map to produce families in \( \pi_*(S^0) \) at the prime 2 in \([1\text{ and 4}]\).

Since these systematic families in \( \pi_*S^0 \) are associated with \( n \)-self-maps, one obvious way to investigate this sort of thing is by way of the Adams-Novikov spectral sequence. Here the \( E_2 \) term is \( \text{Ext}_{BP_*,BP_*}(BP_*M, BP_*) \), with the spectral sequence converging to \( \pi_*X \), completed at \( p \). For the sake of convenience, we denote \( \text{Ext}_{BP_*,BP_*}(BP_*, M) \) by \( \text{Ext}(M) \), for a \( BP_*BP \)-comodule \( M \). Let \( I_n \) denote the prime ideal \( (p, v_1, v_2, \ldots, v_{n-1}) \) in \( BP_* \). Then the connecting homomorphisms in \( \text{Ext} \) associated to the short exact sequences

\[
0 \rightarrow BP_*/I_{n-1} \xrightarrow{v_n^{-1}} BP_*/I_{n-1} \xrightarrow{BP_*/I_n} 0
\]

yield

\[
\text{Ext}^0(BP_*/I_n) \xrightarrow{\delta} \text{Ext}^1(BP_*/I_{n-1}) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \text{Ext}^n(BP_*).
\]

Clearly there is a class \( v_n^t \in \text{Ext}^0(BP_*/I_n) \). Denote the class \( (\delta \delta \cdots \delta)(v_n^t) \in \text{Ext}^n(BP_*) \) by \( gr_t^{(n)}(v_n) \), where \( gr^{(n)} \) is meant to represent the "\( n \)th Greek letter". It is shown in \([13]\) that for \( n = 1, 2 \) and 3, these classes in \( \text{Ext} \) survive the Adams-Novikov spectral sequence to represent the classes \( \alpha_t, \beta_t \) and \( \gamma_t \), respectively. The following conjecture generalizes these results.

CONJECTURE (6). For \( p \) a sufficiently large prime, depending on \( n \), \( gr_t^{(n)} \) is a nontrivial class in \( \text{Ext}^n(BP_*) \) which survives the Adams-Novikov spectral sequence to represent a nontrivial homotopy class in \( \pi_*S^0 \).
The process of investigating $\text{Ext}(BP)$ by means of the $n$-fold connecting homomorphism shown above can be set up formally as the Chromatic spectral sequence of [13], which filters the Adams-Novikov spectral sequence $E_2$ term into $v_n$-periodic subquotients, known as the "chromatic filtration". This can be geometrically realized by spectra [15].

A natural question to ask is: how does this machinery of $v_n$-self-maps of finite complexes and their associated systematic families in $\pi_* S^0$ appear in the classical Adams spectral sequence? At the prime 2, this question was answered in [9] and [16]. There, a fair amount of technical machinery was necessary to start the analysis. For odd primes, the question may be answered in a much simpler fashion.

Recall that the classical Adams spectral sequence (abbreviated by "clASS") at a prime $p$ has $E_2 \cong \text{Ext}^{*,*}_A(Z/p,Z/p) \Rightarrow \pi_*(-(S^0))$, where $A$ denotes the mod $p$ Steenrod algebra and $\pi_*$ denotes completion at $p$. Let $A_n$ denote the Hopf subalgebra generated by $\{\beta, p^1, \ldots, p^{p^n-1}\}$ if $p$ is odd, $\{S^1, \ldots, S^{2^n}\}$ if $p = 2$. Then $A = \lim_n A_n$, so that

$$\text{Ext}_A(Z/p,Z/p) \cong \lim_n \text{Ext}_{A_n}(Z/p,Z/p).$$

We can use information about the cohomology of the finite Hopf algebra $A_n$, then, to infer results about the clASS $E_2$ term.

Consider $E(n) = E(Q_0, Q_1, \ldots, Q_n)$, the $F_p$ exterior algebra on the first $n + 1$ Milnor generators. Then $E(n)$ is a Hopf subalgebra of $A_n$, where we denote the inclusion by $i: E(n) \hookrightarrow A_n$. Recall also that for $n \geq 0$ there is a spectrum $BP(n)$, known as the Baas-Sullivan spectrum [2] (or as the Johnson-Wilson spectrum in [14]), such that $\pi_*(BP(n)) \cong Z(p)[v_1, v_2, \ldots, v_n]$, where $|v_n| = 2p^n - 2$. Its cohomology is given by $H^*(BP(n)) \cong A \otimes_{E(n)} Z/p$, (where, as in the sequel, all cohomology groups are assumed to have $Z/p$ coefficients, unless otherwise specified). Then the clASS converging to $\pi_*(BP(n))$ has

$$E_2(BP(n)) = \text{Ext}_A(H^*(BP(n)), Z/p)$$
$$= \text{Ext}_A(A \otimes_{E(n)} Z/p, Z/p)$$
$$\cong \text{Ext}_{E(n)}(Z/p, Z/p)$$
$$\cong Z/p[v_0, v_1, \ldots, v_n],$$

converging to $\pi_*(BP(n)) \cong Z(p)[v_1, v_2, \ldots, v_n]$, where the class "$v_i" = \{Q_i\}$ in $\text{Ext}_{E(n)}^{1,2p^n-1}(Z/p, Z/p)$ represents the homotopy class $v_i$ and multiplication by $q_0$ corresponds to multiplication by $p$ in $\pi_* BP(n)$. Here the $E_2$ term is concentrated in even dimensions, so that the clASS collapses from that stage. The inclusion map $i: E(n) \hookrightarrow A_n$ given above induces the restriction map in cohomology

$$i^*: \text{Ext}_{A_n}(Z/p, Z/p) \rightarrow \text{Ext}_{E(n)}(Z/p, Z/p) = Z/p[q_0, v_1, \ldots, v_n].$$

**Definition (7).** A class $x \in \text{Ext}_{A_n}(Z/p, Z/p)$ is said to represent $v_i^k$ if the restriction $i^*(x)$ is $v_i^k \in \text{Ext}_{E(n)}(Z/p, Z/p)$.

With these conventions, we can state our first main result.
THEOREM A. For all $n \geq 1$, $p$ an odd prime, there exist classes $u_1, u_2, \ldots, u_n$ in $\text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)$ such that

(i) $\mathbb{Z}/p[q_0, u_1, u_2, \ldots, u_n] \subset \text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)$,

(ii) $i^*(u_n) = v_i^p \in \text{Ext}_{E(n)}(\mathbb{Z}/p, \mathbb{Z}/p)$,

(iii) $i^*(u_i) = v_i^{n-i+1} \in \text{Ext}_{E(n)}(\mathbb{Z}/p, \mathbb{Z}/p)$ for $1 \leq i \leq n$,

(iv) $u_n$ is a non-zero-divisor in $\text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)$.

Thus $u_i \in \text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)$ represents $v_i^{n-i+1}$. We hereafter abuse notation and write $\mathbb{Z}/p[q_0, v_1, v_2, \ldots, v_n] \subset \text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)$. At the prime 2, the best that one can show is that $\mathbb{Z}/2[h_0, v_1^{N_1}, v_2^{N_2}, \ldots, v_n^{2^{n+1}}] \subset \text{Ext}_{A_n}(\mathbb{Z}/2, \mathbb{Z}/2)$, where $N_i$ is some (possibly very large) integer [9]. The proof in the mod 2 case requires the use of Koszul-type resolutions [5, 9], together with a theorem of Lin and Wilkerson, rather than the simpler machinery used below. It should be noted that there are possibly many classes in $\text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)$ representing $v_i$, one of which we will explicitly produce in the proof of the theorem. For notational ease, we will let $w_i \subset \text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)$ denote the coset of classes which represent $v_i^{n-i+1}$. An easy inspection of the May spectral sequence converging to $\text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)$ shows that there is only one class in the same bigrading as the class $u_n$ of the theorem, so that $v_i^p$ is uniquely represented.

PROOF. Let $A_n^*$ denote the dual of the Hopf algebra $A_n$. Then there is an extension of Hopf algebras:

$$F_p \to P_n \to A_n^* \to E_n \to F_p,$$

where $P_n$ is the truncated polynomial algebra $\mathbb{Z}/p[\xi_1, \xi_2, \ldots, \xi_n]/(e_i^{p+1})$ and $E_n$ denotes the $F_p$ exterior algebra $E(\tau_0, \tau_1, \ldots, \tau_n)$. Here $|\xi_i| = 2p^i - 2$ and $|\tau_i| = 2p^i - 1$. Associated to this short exact sequence, we have a Cartan-Eilenberg spectral sequence (CESS) converging to $\text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)$, with $E_2$ term given by

$$\text{Ext}_{P_n}(\mathbb{Z}/p, \text{Ext}_{E_n}(\mathbb{Z}/p, \mathbb{Z}/p))$$

[14]. To analyze this spectral sequence, we first note that $\text{Ext}_{E_n}(\mathbb{Z}/p, \mathbb{Z}/p)$ is a polynomial algebra on $n + 1$ generators, which we denote by $\mathbb{Z}/p[a_0, a_1, \ldots, a_n]$, where $a_i$ has bigrading $(1, 2p^i - 1)$. The spectral sequence collapses from $E_2$ for odd primes [12], as one can see by filtering the dual of the Steenrod algebra by the number of $\tau$'s in a term. This filtration leads to an $E_2$ term filtration in terms of the $a_i$'s, which is preserved by the differentials in the CESS, for $p > 2$, so that there can be no nontrivial differentials. Hence the $E_2$ term gives a filtered version of $\text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)$. The $P_n$-coaction on $H^*E_n = \mathbb{Z}/p[a_0, a_1, \ldots, a_n]$ is given by $\psi(a_k) = \sum e_i^{p^i}a_k^{i-1} \otimes a_i$. Thus, the $P_n$-coaction on the class $a_n^p$ is $\psi(a_n^p) = \sum e_i^{p^i+1} \otimes a_i^p = 1 \otimes a_n^p \in P_n \otimes H^*E_n$. Since $a_n^p$ is primitive in $H^*E_n$, it yields a nontrivial cohomology class in $E_2 = \text{Ext}_{P_n}(\mathbb{Z}/p, H^*E_n)$.

Further, the map

$$(a_n^p): \mathbb{Z}/p[a_0, a_1, \ldots, a_n] \to \mathbb{Z}/p[a_0, a_1, \ldots, a_n]$$
is the inclusion of a direct summand as a map of of $P_n$-comodules, since $\psi(x)$ can have a term containing $a^n_p$ if and only if $a^n_p$ divides $x$. If we let $u_n$ denote the class in $\text{Ext}_{P_n}(\mathbb{Z}/p, H^*E_n) = E_0 \text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)$ given by the map $(a^n_p)$, then $u_n$ is a nontrivial class in bidegree $(p, 2p^{n+1} - 1)$. Further, the map $(a^n_p)$ induces

$$(a^n_p): \text{Ext}_{P_n}(\mathbb{Z}/p, H^*E_n) \to \text{Ext}_{P_n}(\mathbb{Z}/p, H^*E_n)$$

which is also the inclusion of a direct summand. Thus the class $u_n \in \text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)$ representing $(a^n_p)$ is a non-zero-divisor.

To produce the classes $u_i$ for $i < n$ of the theorem, one notes that $\psi(a_i^{p^{n-i+1}}) = 1 \otimes a_i^{p^{n-i+1}}$, so that $(a_i^{p^{n-i+1}})$ is a primitive in $H^*E_n$. Let $u_i$ denote the class in $E_0 \text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)$ representing $(a_i^{p^{n-i+1}})$. The $u_i$'s are not necessarily non-zero-divisors, however, since the map

$$(a_i^{p^{n-i+1}}): \mathbb{Z}/p[a_0, a_1, \ldots, a_n] \to \mathbb{Z}/p[a_0, a_1, \ldots, a_n]$$

is not the inclusion of a direct summand of $P_n$-comodules (a class $x \in H^*E_n$ might have $a_i^{p^{n-i+1}}$ as a factor of $\psi(x)$ even if $x$ is not divisible by $a_i^{p^{n-i+1}}$).

That $i^*(u_i) = v_i^{p^{n-i+1}}$ follows from the fact that the edge homomorphism of the CESS of an extension is the restriction map. Equivalently, the result is clear from the following commutative diagram of Hopf algebras and the naturality of the CESS:

$${\begin{array}{cccccc}
F_p & \to & P_n & \to & A_n^* & \to & E_n & \to & F_p \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_p & \to & E(n)^* & \to & E(n)^* & \to & F_p
\end{array}}$$

This completes the proof of the theorem.

We now use these classes representing $v_i^k$ in $\text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)$ to define what it means for elements in $\text{Ext}_A(\mathbb{Z}/p, \mathbb{Z}/p)$ to be $v_i$-periodic or $v_i$-torsion.

**Definition (8).** Let $S \subset \text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)$ be the multiplicative set consisting of the elements which represent $v_i^{p^k}$ for some $k$. Define $\text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)(v_i^{-1})$ to be the ring $S^{-1}\text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)$.

Note that this definition is independent of the power of $v_i^{p^{n-i+1}}$ chosen. Let

$$p_n: \text{Ext}_{A_{n+1}}(\mathbb{Z}/p, \mathbb{Z}/p) \to \text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)$$
denote the restriction map in cohomology. Then these localizations fit together into the following tower:

\[
\begin{array}{c}
\text{Ext}_A(Z/p, Z/p) \\
\downarrow \\
\vdots \\
\text{Ext}_{A_n}(Z/p, Z/p) \longrightarrow \text{Ext}_{A_n}(Z/p, Z/p)(u_i^{-1}) \\
\downarrow \\
\text{Ext}_{A_i}(Z/p, Z/p) \longrightarrow \text{Ext}_{A_i}(Z/p, Z/p)(u_i^{-1}).
\end{array}
\]  

(9)

Taking the inverse limit, we obtain a map

\[
f_i: \text{Ext}_A(Z/p, Z/p) \longrightarrow \lim_n \{\text{Ext}_{A_n}(Z/p, Z/p)(u_i^{-1})\}.
\]

(10)

This allows us to make the following definition.

**Definition (11).** A class \(a \in \text{Ext}_A(Z/p, Z/p)\) is \(u_i\)-periodic if \(f_i(a) \neq 0\) and is \(u_i\)-torsion if \(f_i(a) = 0\).

This definition is equivalent to the following. Let

\[
q_n: \text{Ext}_A(Z/p, Z/p) \rightarrow \text{Ext}_{A_n}(Z/p, Z/p)
\]

denote the restriction map. A class \(a \in \text{Ext}_A(Z/p, Z/p)\) is \(u_i\)-periodic if and only if for each \(n\) such that \(a = q_n(a) \neq 0\), we have \(\alpha(v_i^{n+s}) \neq 0\) in \(\text{Ext}_{A_n}(Z/p, Z/p)\) for all \(s \geq 0\), where we use the informal notation for any representative for a power of \(u_i\). A class \(a \in \text{Ext}_A(Z/p, Z/p)\) is \(u_i\)-torsion if and only if for each \(n\) such that \(a = q_n(a) \neq 0\), there is some \(s > 0\) such that \(\alpha(v_i^{n+s}) = 0\) in \(\text{Ext}_{A_n}(Z/p, Z/p)\). Note that for some \(N\) sufficiently large, \(a = q_n(a) \neq 0\) in \(\text{Ext}_{A_n}(Z/p, Z/p)\), for all \(n \geq N\).

**Theorem B.** If a class \(a \in \text{Ext}_A(Z/p, Z/p)\) is \(u_n\)-periodic, then \(a\) is also \(u_{n+k}\)-periodic for all \(k > 0\). Equivalently, if \(a\) is \(u_n\)-torsion, then \(a\) is also \(u_i\)-torsion for all \(i < n\).

This result is known in the setting of \(BP_\ast BP\)-comodules by a result of Johnson and Yosimura [7]. At the prime 2, this appears as Theorem C in [9]. The proof for odd primes is similar to that for the prime 2.

**Proof.** The proof uses a map of algebras which is essentially the total reduced power operation. Let \(t\) be an indeterminate of degree \(2(p - 1)\), and let

\[
P_t = \sum_{n \geq 0} p^n t^n.
\]
be the total reduced power operation. Let
\[ r: A^* \to A^*[t] \]
denote the action of \( P_t \) on the left. Then \( r \) is a map of right \( A \)-algebras given by
\begin{align*}
  r(\tau_{n+1}) &= \tau_{n+1} + \tau_n t^{p^n} & \text{if } n \geq -1, \\
  r(\xi_{n+1}) &= \xi_{n+1} + \xi_n t^{p^n} & \text{if } n \geq 0.
\end{align*}
(12)

Recall that \((A//A_n)^*\) is isomorphic to
\[ \mathbb{Z}/p[\xi^n_1, \xi^{n-1}_2, \ldots, \xi^n_{n+1}, \ldots] \otimes E(\tau_{n+1}, \tau_{n+2}, \ldots), \]
both as algebras and as right \( A \)-modules. Similarly,
\[ (A//E(n))^* \cong \mathbb{Z}/p[\xi_1, \xi_2, \ldots] \otimes E(\tau_{n+1}, \tau_{n+2}, \ldots). \]
The following lemma follows easily from (12).

**LEMMA 13.** There are inclusions
\[ r(A//A_{n+1})^* \subset (A//A_n)^*[t], \]
\[ r(E//E(n+1))^* \subset (E//E(n))^*[t]. \]

By Lemma 13, we have maps (after suitable change of rings)
\[ r^*: \text{Ext}_{A_{n+1}}(\mathbb{Z}/p, \mathbb{Z}/p) \to \text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)[t], \]
\[ r^*: \text{Ext}_{E(n+1)}(\mathbb{Z}/p, \mathbb{Z}/p) \to \text{Ext}_{E(n)}(\mathbb{Z}/p, \mathbb{Z}/p)[t], \]
which are ring homomorphisms, since \( r \) is given by a map of algebras. The image of \( \text{Ext}_{A_{n+1}}(\mathbb{Z}/p, \mathbb{Z}/p) \) is contained in the ideal generated by \( t^{p^n} \). Note that if a class \( a \in \text{Ext}_A(\mathbb{Z}/p, \mathbb{Z}/p) \) has nontrivial restriction \( \hat{a} \in \text{Ext}_{A_{n+1}}(\mathbb{Z}/p, \mathbb{Z}/p) \) and \( \text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p) \), then \( r(\hat{a}) = \hat{a} \). (Here, as in the rest of the paper, we use \( \hat{a} \) to denote any nontrivial restrictions of \( a \in \text{Ext}_A(\mathbb{Z}/p, \mathbb{Z}/p) \) in \( \text{Ext}_{A_m}(\mathbb{Z}/p, \mathbb{Z}/p) \), for all \( m > 0 \).)

**LEMMA 14.** The induced map
\[ r^*: \text{Ext}_{E(n+1)}(\mathbb{Z}/p, \mathbb{Z}/p) \to \text{Ext}_{E(n)}(\mathbb{Z}/p, \mathbb{Z}/p)[t] \]
has the values
\[ r^*(v_{i+1}) = v_{i+1} + v_i t^{p^i} \in \text{Ext}_{E(n)}(\mathbb{Z}/p, \mathbb{Z}/p)[t] \]
whenever \( 1 \leq i \leq n \).

**PROOF.** We compute in the bar construction. Let \( \pi: A^* \to E(n)^* \) be the natural restriction and \( \varepsilon \) denote the augmentation. Then the change of rings isomorphism
\[ \text{Ext}_A(\mathbb{Z}/p, (A//E(n))^*) \cong \text{Ext}_{E(n)}(\mathbb{Z}/p, \mathbb{Z}/p) \]
is given in terms of the bar construction by
\[ \sum [a'_i|a''_i] \mapsto \sum \varepsilon(a'_i)[\pi a''_i] \in E(n)^*, \]
where
\[ \sum [a'_i|a''_i] \in (A//E(n))^* \otimes A^*. \]
Recall that $H^*E(n) \cong \mathbb{Z}/p[q_0, v_1, \ldots, v_n]$, where $v_i = \{Q_i\}$, corresponding to $\partial\tau_i$ in the bar resolution for $E(n)^*$. Consider the element $\sum [\xi_{i-j}^p|\tau_j] = \partial[\tau_i]$. It is a cycle since $\partial^2 = 0$ and further, $\sum \varepsilon(\xi_{i-j}^p)[\pi\tau_j] = \partial[\tau_i]$, since $\varepsilon(\xi_{i-j}^p) = 0$ unless $j = i$. So it follows that $\sum [\xi_{i-j}^p|\tau_j]$ is a representative for $v_i$. The element $r^*(v_{i+1})$ is therefore represented by

$$\sum [r(\xi_{i+1-j}^p)|\tau_j] = \sum [(\xi_{i-j}^p t^{p^{i-j}})|\tau_j] = \sum [\xi_{i-j}^p|\tau_j] t^{p^i}$$

which represents $v_i t^{p^i}$.

**COROLLARY 15.** If $x \in \text{Ext}_{A_{n+1}}(\mathbb{Z}/p, \mathbb{Z}/p)$ represents $v_i^{p^k}$ then

$$r^*(x) = x + y t^{p^{k+i}} \in \text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)[t],$$

where $y$ represents $v_i^{p^k}$.

**PROOF.** This follows from naturality (Lemma 13 and Lemma 14).

**PROOF OF THEOREM B.** Let $a \in \text{Ext}_A(\mathbb{Z}/p, \mathbb{Z}/p)$ be $v_{i+1}$-torsion. It suffices to show that $a$ is $v_i$-torsion. Let $n$ be sufficiently large so that the restriction $q_n(a) \neq 0$ in $\text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)$. Let $\hat{a}$ denote both $q_{n+1}(a)$ and $q_n(a)$, as above. Since $a$ is $v_{i+1}$-torsion, there is some integer $s$ such that $x\hat{a} = 0$ in $\text{Ext}_{A_{n+1}}(\mathbb{Z}/p, \mathbb{Z}/p)$, where $x$ represents $v_i^{p^{i+1}}$. Then

$$0 = r^*(x\hat{a}) = (x + y t^{p^{k+i}})r^*(\hat{a}) = x\hat{a} + y\hat{a} t^{p^{k+i}} = 0 + y\hat{a} t^{p^{k+i}},$$

where $y$ represents $v_i^{p^k}$ by the above corollary. Thus $\hat{a}$ is $v_i$-torsion in $\text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)$, implying our result.

It should be remarked that the total reduced power operation $r$ can be factored through the Davis-Mahowald splitting, which decomposes $A \otimes_{A_n} \mathbb{Z}/p[x, x^{-1}]$ as a sum of $A \otimes_{A_{n-1}} \mathbb{Z}/p$'s [8].

As an easy consequence of Theorem B, we have the following corollary.

**COROLLARY C.** There is a filtration, which we call the chromatic filtration,

$$\text{Ext}_A(\mathbb{Z}/p, \mathbb{Z}/p) = F_{-1} \supset F_0 \supset F_1 \supset \cdots \supset F_n \supset F_{n+1} \supset \cdots$$

such that $F_n/F_{n+1}$ is the subquotient of classes that are $v_k$-torsion for all $k \leq n$ and $v_j$-periodic for all $j \geq n + 1$.

**PROOF.** Let $F_n = \ker (f_n)$, where the map $f_n$ is given in Definition (11). The result follows immediately from Theorem B.
One should think of this chromatic filtration in the following manner:

\[
\begin{align*}
\text{Ext}_A(\mathbb{Z}/p, \mathbb{Z}/p) & \longrightarrow (q_0\text{-periodic quotient}) \\
\cup & \\
(q_0\text{-torsion subgroup}) & \longrightarrow (v_1\text{-periodic subquotient}) \\
\cup & \\
(v_1\text{-torsion subgroup}) & \longrightarrow (v_2\text{-periodic subquotient}) \\
\cup & \\
(v_2\text{-torsion subgroup}) & \longrightarrow (v_3\text{-periodic subquotient}) \\
\cup & \\
\vdots & \\
\cup & \\
(v_n\text{-torsion subgroup}) & \longrightarrow (v_{n+1}\text{-periodic subquotient}) \\
\cup & \\
\vdots & 
\end{align*}
\]

**Proposition (16).** The chromatic filtration of \(\text{Ext}_A(\mathbb{Z}/p, \mathbb{Z}/p)\) is complete.

**Proof (Mahowald).** Recall that \(v_p^n\) is a non-zero-divisor in \(\text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)\). For each class \(a \in \text{Ext}_A(\mathbb{Z}/p, \mathbb{Z}/p)\), there is some integer \(n\) such that \(\hat{a} = q_n(a) \neq 0\) in \(\text{Ext}_{A_n}(\mathbb{Z}/p, \mathbb{Z}/p)\). So for each such \(n\), the class \(\hat{a}\) is \(v_n\)-periodic. Hence

\[
\bigcap_{n \geq 0} (v_n\text{-torsion subgroup}) = 0,
\]

completing the proof.

Haynes Miller has constructed a chromatic spectral sequence converging to \(\text{Ext}_A(\mathbb{Z}/p, \mathbb{Z}/p)\) using the collapsing of the CESS for \(p\) odd [11]. This allows one to define \(v_n\)-periodicity in \(\text{Ext}_A(\mathbb{Z}/p, \mathbb{Z}/p)\) in another manner. It is not hard to show that if a class \(a \in \text{Ext}_A(\mathbb{Z}/p, \mathbb{Z}/p)\) is \(v_n\)-torsion in Miller’s definition, then it is also \(v_n\)-torsion in the sense given above. The converse seems to be quite difficult to prove, because of the intractability of the chromatic SS differentials. It is conjectured that the two definitions of \(v_n\)-periodicity in \(\text{Ext}_A(\mathbb{Z}/p, \mathbb{Z}/p)\) agree.

The chromatic filtration given above is intimately tied in with the idea of “root invariants” in stable homotopy. See [10 or 16] for a partial explanation of this relationship.

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