CARLESON MEASURES AND MULTIPLIERS
OF DIRICHLET-TYPE SPACES

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ABSTRACT. A function \( \rho \) from \([0,1]\) onto itself is a Dirichlet weight if it is increasing, \( \rho'' \leq 0 \) and \( \lim_{x \to 0^+} x/\rho(x) = 0 \). The corresponding Dirichlet-type space, \( D_\rho \), consists of those bounded holomorphic functions on \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) such that \( |f'(z)|^2 \rho(1 - |z|) \) is integrable with respect to Lebesgue measure on \( U \). We characterize in terms of a Carleson-type maximal operator the functions in the set of pointwise multipliers of \( D_\rho \), \( M(D_\rho) = \{ g : U \to \mathbb{C} : gf \in D_\rho, \forall f \in D_\rho \} \).

I. Introduction. Let \( H(U) \) denote the set of functions holomorphic in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). A function \( \rho \) mapping \([0,1]\) onto itself is called a Dirichlet weight if it is increasing, concave in the strong sense that \( \rho'' \leq 0 \) and \( \lim_{x \to 0^+} x/\rho(x) = 0 \). Given such a weight, the Dirichlet-type space \( D_\rho \) consists of those \( f \in H(U) \) for which the norm

\[
\|f\|_\rho = |f(0)| + \left( \iint_U |f'(z)|^2 \rho(1 - |z|) \, dz \right)^{1/2}
\]

is finite; here \( dz \) denotes Lebesgue measure on \( U \). The main purpose of this paper is to characterize the set \( M(D_\rho) \) of pointwise multipliers of \( D_\rho \), where

\[
M(D_\rho) = \{ g : U \to \mathbb{C} : f \in D_\rho \text{ for all } f \in D_\rho \}.
\]

Previous results concerning pointwise multipliers of these spaces dealt with the special case

\[
D_\alpha = \left\{ f \in H(U), \ f = \sum_{n=0}^{\infty} a_n z^n : \|f\|_\alpha = \left[ \sum_{n=0}^{\infty} (1 + n^2)^\alpha |a_n|^2 \right]^{1/2} < \infty \right\},
\]

\(-\infty < \alpha < \infty\); in case \( \alpha = 1/2 \) this is the classical Dirichlet space of functions in \( H(U) \) whose derivatives are square-integrable on \( U \). Taylor [10] described \( M(D_\alpha) \) when \( \alpha \) lies outside \((0,1/2)\): for \( a \leq 0 \), \( M(D_\alpha) = H^\infty(U) = H(U) \cap L^\infty(U) \) while for \( \alpha > 1/2 \), \( M(D_\alpha) = D_\alpha \) (so that \( D_\alpha \) is an algebra). D. Stegenga [8] showed that for \( \rho_\alpha(r) = r^{1-2\alpha} \), \( \alpha < 1 \),

\[
\|f\|_\alpha \approx |f(0)| + \left( \iint_U |f'(z)|^2 \rho_\alpha(1 - |z|) \, dz \right)^{1/2};
\]
that is, the two terms in (1.2) are equivalent in the sense that each is no larger than a constant multiple of the other, the constants being independent of $f$. (We observe $[0,1/2]$ is precisely the range of $\alpha$ for which $\rho_\alpha$ is concave.) Stegenga proved $g \in M(D_\alpha)$, $\alpha \in (0,1/2]$, if and only if $g \in H^\infty(U)$ and

$$\int \bigcup_j S(I_j) |g'(z)|^2 \rho_\alpha(1-|z|) \, dz \leq C \text{Cap}_\alpha \left( \bigcup_j I_j \right)$$

for all finite collections of pairwise disjoint subarcs $\{I_j\}$ on the unit circle $T = \{z \in \mathbb{C}: |z| = 1\}$. Here $S(I)$ denotes the square $\{z = re^{i\theta}: e^{i\theta} \in I \text{ and } (1-|I|)_+ < r < 1\}$, $|I|$ is the arclength of the subarc $I$, and $\text{Cap}_\alpha(E)$ denotes the Bessel capacity of order $\alpha$ of the set $E$.

Our characterization of $M(D_\rho)$ differs from (1.3) in that we replace the capacity by a Carleson-type maximal operator and so are able to test a certain inequality, (1.4), over subarcs of $T$ rather than finite unions of subarcs. This maximal operator, $M_\rho$, associates to nonnegative $h$ on $U$ a function $M_\rho h$ on $T$ by

$$(M_\rho h)(e^{i\theta}) = \sup |[I]\rho(|I|)]^{-1/2} \int_{S(I)} h(z) \, dz,$$

the supremum being over all subarcs $I$ of $T$ containing $e^{i\theta}$. Adopting the usual notation, $\chi_E$, for the characteristic function of the set $E$, we can now state

**Theorem A.** Suppose $\rho$ is a Dirichlet weight. Then $g \in M(D_\rho)$ if and only if $g \in H^\infty(U)$ and there exists $C > 0$ such that

$$\int_I [M_\rho(\chi_{S(I)}|g'|^2 \rho(1-|\cdot|))(e^{i\theta})]^2 \, d\theta \leq C \int \int_{S(I)} |g'(z)|^2 \rho(1-|z|) \, dz < \infty$$

for all subarcs $I$ of $T$.

We now outline in some detail the proof of Theorem A. There are two main steps. First, in §2, Plancherel's theorem is used to show the norm defined at $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $D_\rho$ by

$$\|f\|_{D_\rho}^* = \left[ |a_0|^2 + \sum_{n=1}^{\infty} n \rho \left( \frac{1}{n} \right) |a_n|^2 \right]^{1/2}$$

is equivalent to the one in (1.1). We next observe that, $\rho$ being concave, $n \rho(1/n)$ is a nondecreasing function of $n$, so $D_\rho$ must be contained in the Hardy space $H^2(T) = \{f \in L^2(T): \hat{f}(n) = 0 \text{ for } n < 0\}$. Thus, defining the function $K_\rho$ on $T$ in terms of its Fourier coefficients

$$\hat{K}_\rho(n) = \begin{cases} 2, & n = 0, \\ \frac{n}{|n|\rho(1/|n|)}^{-1/2}, & n \neq 0, \end{cases}$$

we can identify $D_\rho$ with the space $S_\rho$ of Poisson integrals of $K_\rho$-potentials of functions in $H^2(T)$,

$$S_\rho = \{f(z) = (P_r * K_\rho * h)(e^{i\theta}) : z = re^{i\theta}, h \in H^2(T)\};$$
here, as usual,
\[
(g * h)(e^{i\theta}) = \frac{1}{2\pi} \int_T g(e^{i(\theta - t)})h(e^{it})\,dt
\]
and \(P_r(\theta) = (1 - r^2)/(1 - 2r \cos \theta + r^2)\). Indeed, if \(f \in D_\rho\), then \(f = P_r * K_\rho * h\) where
\[
\hat{h}(n) = \begin{cases} 
  a_0/2, & n = 0, \\
  0, & n < 0, \\
  n\rho(1/n)a_n, & n > 0.
\end{cases}
\]

Second, by [5, Proposition 1.6/4], \(M(D_\rho)\) is embedded in \(H^\infty(U)\). As pointed out on p. 178 of [5], this fact, together with the product rule of differentiation shows \(g \in M(D_\rho)\) if and only if \(g \in H^\infty(U)\) and
\[
\int_U |f(z)|^2 |g'(z)|^2 \rho(1 - |z|)\,dz < \infty
\]
for all \(f \in D_\rho\). The identification of \(D_\rho\) with \(S_\rho\) described above then says \(g \in M(D_\rho)\) if and only if \(g \in H^\infty(U)\) and
\[
\int_U |(P_r * K_\rho * h)(e^{i\theta})|^2 |g'(re^{i\theta})|^2 \rho(1 - r)\,d(re^{i\theta}) < \infty
\]
for all \(h \in H^2(T)\). Taking complex conjugates, it is seen (1.7) would, in fact, hold for all \(h \in L^2(T)\). By the closed graph theorem, this is equivalent to the existence of \(C > 0\) such that
\[
\int_U |(P_r * K_\rho * h)(e^{i\theta})|^2 \,d\mu(re^{i\theta}) \leq C \int_T |h(e^{i\theta})|^2\,d\theta
\]
for all \(h \in L^2(T)\), with \(d\mu = |g'(z)|^2 \rho(1 - |z|)\,dz\). Any \(\mu \in B(U)\), the class of positive Borel measures on \(U\), which satisfies (1.8) will be called a Carleson measure on \(S_\rho\).

The proof of Theorem A is completed in §III by a characterization of such Carleson measures. This characterization is also used in §IV to give another approach to the "Féjer-Riesz inequality" of Nagel, Rudin and Shapiro [6].

II. Dirichlet-type spaces. In Theorem 2.2 below we require the following result, whose proof can be found, for example, in [2, p. 183].

**Lemma 2.1.** Suppose \(\{c_n\}_{n=\infty}^{n=-\infty}\) is an even, nonnegative sequence on \(Z\) which is nonincreasing and convex on \(Z_+\) with \(\lim_{n\to\infty} c_n = 0\). Then the even function \(K(e^{i\theta}) = \sum_{n=-\infty}^{\infty} c_n e^{-in\theta}\) is nonnegative and integrable on \(T\).

**Theorem 2.2.** Suppose \(\rho\) is a Dirichlet weight. Then the function \(K_\rho\) given through its Fourier coefficients in (1.6) is even, nonnegative and integrable on \(T\). The mapping \(T_\rho\) defined in terms of \(K_\rho\) by
\[
(T_\rho h)(re^{i\theta}) = (P_r * K_\rho * h)(e^{i\theta}), \quad h \in H^2(T),
\]
is 1-1 from \(H^2(T)\) onto \(D_\rho\). Further, the norm \(\|\|_\rho^*\) of (1.5) induced on \(T_\rho h \in D_\rho\) by taking the \(H^2(T)\)-norm of \(h\) is equivalent to the norm \(\|\|_\rho\) in (1.1).

**Proof.** The assertions concerning \(K_\rho\) follow from Lemma 2.1 once it is observed \(c_n = [n\rho(1/n)]^{-1/2}, \ n \geq 1,\) is nonincreasing and convex as a consequence of the concavity of \(\rho\).
We next prove $\| \cdot \|_\rho$ and $\| \cdot \|_\rho^*$ are equivalent, which implies $D_\rho \subset T_\rho(H^2(T))$, as was pointed out in §1. Now, Plancherel’s theorem shows that for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $H(U)$,

$$\int_U |f'(z)|^2 \rho(1 - |z|) \, dz = \int_0^1 \rho(1 - r) r \, dr \int_0^{2\pi} \left| \sum_{n=1}^{\infty} n a_n r^{n-1} e^{i(n-1)\theta} \right|^2 \, d\theta = 2\pi \sum_{n=1}^{\infty} n^2 |a_n|^2 \left[ \int_0^1 r^{2n-1} \rho(1 - r) \, dr \right].$$

We must prove

$$\int_0^1 r^{2n-1} \rho(1 - r) \, dr = \int_0^1 \rho(r)(1 - r)^{2n-1} \, dr \approx n^{-1} \rho(1/n).$$

Since $\rho$ is increasing,

$$\left[ \frac{(1 - 1/n)^{2n}}{2} \right] n^{-1} \rho(1/n) = \rho(1/n) \int_{1/n}^1 (1 - r)^{2n-1} \, dr \leq \int_{1/n}^1 \rho(r)(1 - r)^{2n-1} \, dr \leq \int_0^1 \rho(r)(1 - r)^{2n-1} \, dr$$

and

$$\int_{1/n}^1 \rho(r)(1 - r)^{2n-1} \, dr \leq \rho(1/n) \int_{1/n}^1 (1 - r)^{2n-1} \, dr \leq n^{-1} \rho(1/n).$$

Moreover, $\rho(r)/r$ nonincreasing implies

$$\int_{1/n}^1 \rho(r)(1 - r)^{2n-1} \, dr \leq n \rho(1/n) \int_{1/n}^1 r(1 - r)^{2n-1} \, dr.$$

But, integration by parts, followed by elementary estimates, yields

$$\int_{1/n}^1 r(1 - r)^{2n-1} \, dr \leq n^{-2}.$$

This completes the proof of the equivalence of the norms.

To see $T_\rho$ is 1-1 onto $D_\rho$ we note that as

$$(T_\rho h)(re^{i\theta}) = (P_r * K_\rho * h)(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n r^n e^{in\theta},$$

then

$$[\rho(1/|n|)]^{1/2} a_n = [\rho(1/|n|)]^{1/2} \hat{K}_\rho(n) \hat{h}(n) = \begin{cases} \hat{h}(n), & n > 0, \\ 0, & n < 0. \end{cases}$$

III. Carleson measures. In this section we obtain a characterization of the Carleson measures of $S_\rho$ and then use it to complete the proof of Theorem A.

Let $K$ be any even, nonnegative function in $L^1(T)$. We wish to determine those $\mu \in B(U)$ for which the operator $T_K : h \to P_r * K * h$ is bounded from $L^2(T)$ to $L^2(\mu)$. Indeed, we consider the more general problem in which the index 2 is
replaced by any fixed $p$, $1 < p < \infty$. Such a measure $\mu$ will be called a Carleson measure on

$$S_K^p = \{P_r * K * h : h \in L^p(T)\}.$$ 

It turns out to be easier to deal with the equivalent problem for the dual operator $T_K'$ defined at $\nu \in B(U)$ by

$$(T_K'\nu)(e^{i\theta}) = \int_U (P_r * K)(e^{i(\theta - \phi)}) \, d\nu(e^{i\phi}).$$

Our result, Theorem 3.1, is given in terms of the maximal operator $M_K$ which sends a positive Borel measure $\nu$ on $U$ to a function $M_K\nu$ on $T$, with

$$(M_K\nu)(e^{i\theta}) = \sup \left[ |I|^{-1} \int_0^{|I|} K(e^{i\phi}) \, d\phi \right] \int_{S(I)} d\nu,$$

the supremum being over all subarcs $I$ of $T$ containing $e^{i\theta}$. For $K$ even and non-negative on $T$, let $\tilde{K}(e^{i\theta}) = \sup_{|\theta| \leq \phi \leq \pi} K(e^{i\phi})$ denote the least even nonincreasing majorant of $K$. For $1 < p < \infty$, $p' = p/(p - 1)$.

**THEOREM 3.1.** Fix $p \in (1, \infty)$ and suppose $K$ is an even, nonnegative function in $L^1(T)$ such that

$$(3.1) \int_0^\theta \tilde{K}(e^{i\phi}) \, d\phi \leq C \int_0^\theta K(e^{i\phi}) \, d\phi, \quad 0 < \theta < \pi.$$ 

Finally, let $\mu \in B(U)$. Then $T_K : L^p(T) \rightarrow L^p(\mu)$ if and only if $C > 0$ exists such that

$$(3.2) \int_I [M_K(\chi_{S(I)}\mu)(e^{i\theta})]^{p'} \, d\phi \leq C \int_{S(I)} d\mu < \infty$$

for subarcs $I$ of $T$.

The proof of Theorem 3.1 follows the line of argument used by the authors in [4] (see also [3]) to study the $L^p$ trace inequality for convolution operators with radially decreasing kernels. The first step is to relate $T_K'$ and $M_K$ (cf. Bonami-Johnson [1]).

**THEOREM 3.2.** Let $p$, $K$ and $\tilde{K}$ be as in Theorem 3.1. Then,

(a) There is $C > 0$ such that $(M_K\nu)(e^{i\theta}) \leq C M(T_K'\nu)(e^{i\theta})$, $e^{i\theta} \in T$, for all $\nu \in B(U)$. Here $M$ is the classical Hardy-Littlewood maximal operator on $T$, 

$$(Mg)(e^{i\theta}) = \sup_{e^{i\theta} \in I} |I|^{-1} \int_I |g(e^{i\phi})| \, d\phi.$$ 

(b) There exists $\gamma > 1$ and $C' > 0$ such that for all $\lambda > 0$, all $\beta \in (0, 1]$ and all $\nu \in B(U)$,

$$|\{e^{i\theta} \in T : (T_K'\nu)(e^{i\theta}) > \gamma \lambda, (M_K\nu)(e^{i\theta}) \leq \beta \lambda\}| \leq C' \beta |\{e^{i\theta} \in T : M(T_K'\nu)(e^{i\theta}) > \lambda\}|.$$
PROOF. (a) Fix a subarc $I$ of $T$. Then,
\begin{equation}
\int_I (T_K \nu)(e^{i\theta}) d\theta = \int_I d\theta \int_U (P_{r} \ast K)(e^{i(\theta - \phi)}) d\nu(r e^{i\phi})
\end{equation}
\begin{equation}
= \int_U d\nu(r e^{i\phi}) \int_I (P_{r} \ast K)(e^{i(\theta - \phi)}) d\theta
\end{equation}
\begin{equation}
\geq \left[ \inf_{r e^{i\phi} \in S(I)} \int_I (P_{r} \ast K)(e^{i(\theta - \phi)}) d\theta \right] \int_{S(I)} d\nu(r e^{i\phi}).
\end{equation}

However, for $r e^{i\phi} \in S(I)$,
\begin{equation}
\int_I (P_{r} \ast K)(e^{i(\theta - \phi)}) d\theta = \int_I d\theta \int_T P_{r}(e^{i(\theta - \phi - t)}) K(e^{it}) dt
\end{equation}
\begin{equation}
= \int_T K(e^{it}) dt \int_I P_{r}(e^{i(\theta - \phi - t)}) d\theta
\end{equation}
\begin{equation}
\geq \int_{-|I|}^{|I|} K(e^{it}) dt \int_I P_{r}(e^{i(\theta - \phi - t)}) d\theta
\end{equation}
\begin{equation}
\geq C \int_0^{|I|} K(e^{it}) dt,
\end{equation}
since
\begin{equation}
\int_I P_{r}(e^{i(\theta - \phi - t)}) d\theta \geq \int_{|I|}^{2|I|} P_{r}(e^{is}) ds \geq C,
\end{equation}
when $(1 - |I|) + \leq r < 1$ and $\phi + t \in 3I$. Combining (3.3) and (3.4) yields
\begin{equation}
|I|^{-1} \int_I (T_K \nu)(e^{i\theta}) d\theta \geq C \left[ |I|^{-1} \int_0^{|I|} K(e^{it}) dt \right] \int_{S(I)} d\nu(r e^{i\theta})
\end{equation}
for all $I$. Now fix $e^{i\theta} \in T$ and take the supremum in (3.5) over all $I$ containing $e^{i\theta}$ to complete the proof of (a).

(b) Let $\lambda > 0$ be given and set
\begin{equation}
\Omega_{\lambda} = \{ e^{i\theta} \in T : M(T_K \nu)(e^{i\theta}) > \lambda \}.
\end{equation}
Let \{I_{K}\} be those component subarcs of $\Omega_{\lambda}$ for which $e^{i\theta_k} \in I_k$ exists such that $(M_K)(e^{i\theta_k}) \leq \beta \lambda$. Fix such a subarc and denote it by $I$. Let $3I$ be the subarc with the same centre as $I$ but 3 times the length. We have
\begin{equation}
(3|I|)^{-1} \left[ \int_0^{3|I|} K(e^{i\theta}) d\theta \right] \left[ \int_{S(3I)} d\nu(r e^{i\phi}) \right] \leq \beta \lambda;
\end{equation}
\begin{equation}
|I|^{-1} \int_I (T_{K}^{'} \nu)(e^{i\theta}) d\theta \leq \lambda,
\end{equation}
since $(M_K)(e^{i\theta}) \leq \beta \lambda$ for some $e^{i\theta} \in I$ and since $M(T_{K}^{'} \nu) \leq \lambda$ at each end of $I$. Define $\nu_1 = \nu|_{S(3I)}$ and $\nu_2 = \nu - \nu_1 = \nu|_{S(3I)^c}$. It will be sufficient to obtain
\begin{equation}
(T_{K}^{'} \nu_2)(e^{i\theta}) \leq C_1 \lambda, \quad e^{i\theta} \in I,
\end{equation}
for some \(C_1 > 0\) independent of \(I\). To see this, suppose (3.8) holds and \(\gamma > 2C_1\).

Then,
\[
\left\{e^{i\theta} \in I: (T_{T_K}^{\theta})'(e^{i\theta}) > \gamma \lambda \right\}
\leq \left\{e^{i\theta} \in I: (T_{T_K}^{\theta})'(e^{i\theta}) > C_1 \lambda \right\}
\leq \frac{1}{(C_1 \lambda)^{-1} \int_I (T_{T_K}^{\theta})'(e^{i\theta}) \, d\theta}.
\]

Now,
\[
\int_I (T_{T_K}^{\theta})'(e^{i\theta}) \, d\theta = \int_I d\theta \int \int_{S(3I)} (P_r * \tilde{K})(e^{i(\theta - \phi)}) \, d\nu(r e^{i\phi})
\]
\[
= \int \int_{S(3I)} d\nu(r e^{i\phi}) \int_I (P_r * \tilde{K})(e^{i(\theta - \phi)}) \, d\theta
\]
\[
\text{and}
\]
\[
\int_I (P_r * \tilde{K})(e^{i(\theta - \phi)}) \, d\theta = \int_I d\theta \int_T P_r(e^{it}) \tilde{K}(e^{i(\theta - \phi - t)}) \, dt
\]
\[
\leq \int_T P_r(e^{it}) \, dt \int_{|I|/2} |I|/2 \tilde{K}(e^{it}) \, ds \leq 2 \int_{0}^{|I|} \tilde{K}(e^{it}) \, ds,
\]

since \(\tilde{K}(e^{it})\) is even when \(s \in [-\pi, \pi]\) and nonincreasing when \(s \in [0, \pi]\). Combining (3.9), (3.10) and (3.11) we obtain
\[
\left\{e^{i\theta} \in I: (T_{T_K}^{\theta})'(e^{i\theta}) > \gamma \lambda \right\}
\leq 2(C_1 \lambda)^{-1} \left[ \int_{0}^{3|I|} \tilde{K}(e^{it}) \, ds \right] \left[ \int \int_{S(3I)} d\nu(r e^{i\phi}) \right]
\leq CC_1^{-1} \beta |I|,
\]

where the last inequality follows from (3.1) and (3.6). Summing (3.12) over all the \(I_k\) gives \(b\).

It remains to prove (3.8). We claim this follows from the fact that
\[
(P_r * \tilde{K})(e^{i(\theta - \phi)}) \leq C|I|^{-1} \int_{|\theta - \phi - t| \leq |I|} (P_r * \tilde{K})(e^{it}) \, dt
\]
whenever \(e^{i\theta} \in I\) and \(r e^{i\phi} \notin S(3I)\). For given (3.13), we have
\[
(T_{T_K}^{\theta_2})(e^{i\theta}) = \int \int_{S(3I)^c} (P_r * \tilde{K})(e^{i(\theta - \phi)}) \, d\nu(r e^{i\phi})
\leq C|I|^{-1} \int \int_{S(3I)^c} d\nu(r e^{i\phi}) \int_{|\theta - \phi - t| \leq |I|} (P_r * \tilde{K})(e^{it}) \, dt
\leq C|I|^{-1} \int_U d\nu(r e^{i\phi}) \int_{|s| \leq |I|} (P_r * \tilde{K})(e^{i(\theta - \phi - s)}) \, ds
\leq C|I|^{-1} \int_{|s| \leq |I|} (T_{T_K}^{\theta})(e^{i(\theta - s)}) \, ds
\leq C \lambda \text{ by (3.7)},
\]
as required by (3.8).
To see (3.13) observe that if $re^{i\phi} \notin S(3I)$, then either $e^{i\phi} \notin 3I$, in which case $|I| \leq |\theta - \phi|$ since $e^{i\phi} \in I$; or $0 \leq r < 1 - 3|I|$. In the former case (3.13) holds, since $(P_r \ast \tilde{K})(e^{it})$ is even for $s \in [-\pi, \pi]$ and nonincreasing for $s \in (0, \pi]$. In the latter case we use the inequality

$$P_r(e^{it}) \leq C|I|^{-1} \int_{|s| \leq |I|} P_r(e^{i(t-s)}) \, ds, \quad -\pi \leq t \leq \pi, \quad 0 < r < 1 - 3|I|.$$ 

(For the case $|t| \leq |I|$, use the estimate $P_r(e^{iu}) \approx 1/(1 - r)$, $|u| \leq 2|I|$; for the case $|t| > |I|$, use the fact that $u \to P_r(e^{iu})$ is decreasing away from 0.) Thus the left side of (3.13),

$$\int_T P_r(e^{i(\theta - \phi - u)})\tilde{K}(e^{iu}) \, du,$$

is dominated by

$$C \int_T |I|^{-1} \int_{|s| \leq |I|} P_r(e^{i(\theta - \phi - u - s)}) \, ds\tilde{K}(e^{iu}) \, du$$

$$= C|I|^{-1} \int_{|\theta - \phi - t| \leq |I|} dt \int_T P_r(e^{i(t-u)})\tilde{K}(e^{iu}) \, du$$

which equals the right side of (3.13).

The second step in the proof of Theorem 3.1 involves the following analogue of a two-weight norm inequality for maximal operators in [7]; the proof is a straightforward adaptation of ones given in [7] and so is omitted.

**Theorem 3.3.** Let $K$ be as in Theorem 3.1 and let $q \in (1, \infty)$. Suppose $\mu \in B(U), \nu \in B(T)$. Then, the inequality

$$\int_T [MK(f\mu)(e^{i\theta})]^q \, d\nu(e^{i\theta}) \leq C \int_U f(re^{i\phi})^q \, d\nu(re^{i\phi})$$

holds for all $f \geq 0$ on $U$ if and only if

$$\int_I [MK(\chi_S(I)\mu)(e^{i\theta})]^q \, d\nu(e^{i\theta}) \leq C \int_{S(I)} d\mu(re^{i\phi})$$

for all subarcs $I$ of $T$.

We are now ready to give the

**Proof of Theorem 3.1.** By duality, $T_K : L^p(T) \to L^p(\mu)$ if and only if

$$\int_T [T_K(f\mu)(e^{i\theta})]^p \, d\theta \leq C \int_U f(re^{i\phi})^p \, d\mu(re^{i\phi})$$

(3.14)

for all $f \geq 0$ in $L^p(\mu)$. Part (a) of Theorem 3.2, together with the Hardy-Littlewood maximal theorem (see [9, p. 5]), shows that

$$\int_T [MK(f\mu)(e^{i\theta})]^p \, d\theta \leq C \int_T [M(T_K(f\mu))(e^{i\theta})]^p \, d\theta$$

(3.15)

$$\leq C \int_T [T'_K(f\mu)(e^{i\theta})]^p \, d\theta.$$
Again, by part (b) of Theorem 3.2 and the maximal theorem in [9],

\[
\int_T |T'_K(f\mu(e^{i\theta}))|^{p'} d\theta
\]

\[
= p' \gamma p' \int_0^\infty \lambda^{p'-1} |\{\theta : T'_K(f\mu(e^{i\theta})) > \gamma \lambda\}| d\lambda
\]

\[
\leq C \int_0^\infty \lambda^{p'-1} |\{M_K(f\mu(e^{i\theta})) > \beta \lambda\}| d\lambda
\]

\[
+ C \beta \int_0^\infty \lambda^{p'-1} |\{M(T'_K f\mu(e^{i\theta})) > \lambda\}| d\lambda
\]

(3.16)

\[
\leq C \beta^{-p'} \int_T |M_K(f\mu(e^{i\theta}))|^{p'} d\theta + C \beta \int_T |M(T'_K (f\mu))(e^{i\theta})|^{p'} d\theta
\]

\[
\leq C \beta^{-p'} \int_T |M_K(f\mu(e^{i\theta}))|^{p'} d\theta + C \beta \int_T |(T'_K (f\mu))(e^{i\theta})|^{p'} d\theta.
\]

Choosing \(\beta\) so small that \(C \beta < 1/2\) and subtracting \(C \beta \int_T |T'_K(f\mu(e^{i\theta}))|^{p'} d\theta\) from both sides of (3.16) yields

\[
\int_T |T'_K(f\mu(e^{i\theta}))|^{p'} d\theta \leq \int_T |T'_K(f\mu(e^{i\theta}))|^{p'} d\theta
\]

(3.17)

\[
\leq C \int_T |M_K(f\mu(e^{i\theta}))|^{p'} d\theta.
\]

From (3.15) and (3.17) we conclude that (3.14) holds if and only if

\[
\int_T |M_K(f\mu(e^{i\theta}))|^{p'} d\theta \leq C \int_U |f(re^{i\phi})|^{p'} d\mu(re^{i\phi})
\]

for all \(f \geq 0\). Finally, Theorem 3.3 with \(d\nu(e^{i\theta}) = d\theta\) yields the conclusion of Theorem 3.1.

As pointed out in §1, \(g \in M(D_{\rho})\) if and only if \(g \in H^\infty(U)\) and \(T_{K_{\rho}} : L^2(T) \rightarrow L^2(\mu)\), where \(d\mu(z) = |g'(z)|^2 \rho(1 - |z|) dz\). We now know the boundedness of \(T_{K_{\rho}}\) is equivalent to (3.2) holding for \(K = K_{\rho}\) and \(p = 2\), provided it can be shown that \(K_{\rho}\) satisfies (3.1). We complete the proof of Theorem A by showing (3.1) for \(K = K_{\rho}\), as well as the equivalence of \(M_{K_{\rho}}\) and \(M_{\rho}\) in Lemma 3.4.

**Lemma 3.4.** Suppose \(\rho\) is a Dirichlet weight and let \(K = K_{\rho}\) be defined by (1.6). If \(\tilde{K}\) denotes the least nonincreasing even majorant of \(K\), then one has the equivalences

\[
x^{-1} \int_0^x K(y) dy \approx x^{-1} \int_0^x \tilde{K}(y) dy \approx |x \rho(x)|^{-1/2}, \quad x \in [0, \pi].
\]

**Proof.** By the definition of \(\tilde{K}\), \(x^{-1} \int_0^x K(y) dy \leq x^{-1} \int_0^x \tilde{K}(y) dy\). Summing by parts we obtain \(K(x) = \sum_{n=0}^\infty (n+1) \Delta^2 \tilde{K}(n) \phi_n(x)\), where

\[
\phi_n(x) = (n+1)^{-1} \left[ \frac{\sin[(n+1)x/2]}{\sin(x/2)} \right]^2
\]

is the Féjer kernel of order \(n\). Thus, recalling that \(\{\tilde{K}(n)\}\) is convex, we obtain

\[
K(y) \leq \sum_{n=0}^\infty (n+1) \Delta^2 \tilde{K}(n) \sup_{|y| \leq x \leq \pi} \phi_n(z).
\]
But, it is easily seen that
\[
\int_0^x \left[ \sup_{|y| \leq z \leq \pi} \phi_n(z) \right] dy \leq C \int_0^x \phi_n(y) dy
\]
for $C > 0$ independent of $n$. Hence, the averages of $K$ and $\tilde{K}$ are equivalent.

With $N = [1/x], \quad (3.18) \quad x^{-1} \int_0^x K(y) dy \approx \int_{-\pi}^{\pi} \phi_N(y) K(y) dy \approx \sum_{|n| \leq [1/x]} \left( 1 - \frac{|n|}{N + 1} \right) \tilde{K}(n);
the last equivalence is a consequence of the general form of Parseval's formula. To complete the proof of the lemma it will be sufficient to prove $x^{-1} \int_0^x K(y) dy \approx [x \rho(x)]^{-1/2}$ for $x$ near 0, say $x \in (0, 1/4)$. For such $x$, we have from (1.6) and (3.18)
\[
C^{-1} \sum_{n=1}^{[1/x]/2} [n \rho(1/n)]^{-1/2} \leq x^{-1} \int_0^x K(y) dy \leq C \sum_{n=1}^{[1/x]} [n \rho(1/n)]^{-1/2},
\]
or, equivalently (since $\rho$ concave implies $\rho(z)/z$ nonincreasing)
\[
C^{-1} \int_1^{1/2x} [y \rho(1/y)]^{-1/2} dy \leq x^{-1} \int_0^x K(y) dy \leq C \int_1^{1/x} [y \rho(1/y)]^{-1/2} dy.
\]
But, since $\rho(z)/z$ nonincreasing,
\[
\int_1^{1/2x} [y \rho(1/y)]^{-1/2} dy \geq [\rho(2x)/2x]^{-1/2} \left( \frac{1}{2x} - 1 \right) \geq 4^{-1} [x \rho(x)]^{-1/2},
\]
while
\[
\int_1^{1/x} [y \rho(1/y)]^{-1/2} dy \leq \rho(x)^{-1/2} \int_1^{1/x} y^{-1/2} dy \leq 2 [x \rho(x)]^{-1/2}.
\]

**IV. The F{\'e}jer-Riesz Inequality.** Finally, we specialize Theorem 3.3 to the case in which $K = K_\rho$, so that $M_{K_\rho}$ is equivalent to $M_\rho$; $\nu$ is Lebesgue measure on $T$; $\mu$ is carried by the line segment $L = \{z \in U: \text{Im} \ z = 0, 0 \leq \text{Re} \ z < 1\}$.

**COROLLARY 4.5.** Suppose $\rho$ is a Dirichlet weight and let $\mu \in B(U)$ be carried by $L$. Then, $\mu$ is a Carleson measure on $S_\rho$ if and only if
\[
(4.1) \quad \int_0^t \left[ \sup_{x \leq s \leq t} [s \rho(s)]^{-1/2} \int_{1-s}^1 d\mu \right]^2 dx \leq C \int_1^1 d\mu < \infty,
\]
whenever $0 < t < 1$.

Condition (4.1) suggests a natural way to construct a Carleson measure for $S_\rho$ which is absolutely continuous with respect to Lebesgue measure on $L$. The idea
is to suppose equality holds in (4.1) and that the supremum in square brackets on
the left side is attained at \( s = x \). With \( F(t) = \int_{1-t}^1 d\mu, \) this means
\[
\int_0^t [x\rho(x)]^{-1}F(x)^2 \, dx = CF(t), \quad 0 < t < 1.
\]
If we further normalize \( \mu \) so that \( F(1) = 1 \) and set \( C = 1 \), then \( F \) satisfies the
boundary value problem
\[
\begin{cases}
F''(t) = [t\rho(t)]^{-1}F(t)^2, \\
F(0+) = 0, \quad F(1) = 1
\end{cases}
\]
whose solution is
\[
F(t) = \left[ 1 + \int_t^1 [s\rho(s)]^{-1} \, ds \right]^{-1}.
\]
Let
\[
(4.2) \quad d\mu(t) = dF(t) = [t\rho(t)]^{-1} \left[ 1 + \int_t^1 [s\rho(s)]^{-1} \, ds \right]^{-2} \, dt
\]
on \( L \). Then (4.1) will hold provided the supremum in square brackets on the left
side is attained when \( s = x \); that is, provided the function
\[
G(y) = \left[ y\rho(y) \right]^{-1/2} F(y) = \left[ y\rho(y) \right]^{-1/2} \left[ 1 + \int_y^1 [s\rho(s)]^{-1} \, ds \right]^{-1}
\]
is nonincreasing. While this is not the case, it is true that \( G \) is almost decreasing;
that is,
\[
(4.3) \quad G(y) \leq 2G(x), \quad 0 < x \leq y < 1,
\]
and this is enough to force (4.1). Since \( G(1) = 1 \) and, by L'Hôpital's rule,
\[
\lim_{y \to 0+} G(y) = \lim_{y \to 0+} \left[ \rho(y) + y\rho'(y) \right] / 2 \left[ y\rho(y) \right]^{1/2}
\]
\[
\geq \lim_{y \to 0+} 2^{-1} \left[ \rho(y) / y \right]^{1/2} = \infty,
\]
it suffices to prove (4.3) when both \( x \) and \( y \) are critical points of \( G \) or \( x \) is critical
and \( y = 1 \). However, if \( G'(z) = 0 \), then
\[
1 + \int_x^1 [t\rho(t)]^{-1} \, dt = 2 / [\rho(z) + z\rho'(z)].
\]
So, for \( x \) and \( y \) critical,
\[
G(y) = 2^{-1} [\rho(y) + y\rho'(y)] / [y\rho(y)]^{1/2}
\]
\[
\leq [\rho(y) / y]^{1/2} \quad (y\rho'(y) \leq \rho(y))
\]
\[
\leq [\rho(x) / x]^{1/2} \quad (x \leq y)
\]
\[
\leq [\rho(x) + x\rho'(x)] / [x\rho(x)]^{1/2} \quad (x\rho'(x) \geq 0)
\]
\[
= 2G(x),
\]
and for \( x \) critical and \( y = 1 \),
\[
G(y) = 1 \leq [\rho(x) / x]^{1/2} \leq 2G(x).
\]
To summarize, we have proved

**COROLLARY 4.6 (CF. NAGEL, RUDIN AND SHAPIRO [6]).** Suppose $\rho$ is a Dirichlet weight and let $\mu$ be the measure carried by $L$ given by (4.2). Then $\mu$ is a Carleson measure for $S_\rho$.

**REFERENCES**


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