TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 309, Number 1, September 1988

ON THE DUAL OF AN EXPONENTIAL SOLVABLE LIE GROUP

BRADLEY N. CURREY

ABSTRACT. Let G be a connected, simply connected exponential solvable Lie group with Lie algebra $\mathfrak{g}$. The Kirillov mapping $\eta\colon \mathfrak{g}^*/\operatorname{Ad}^*(G) \to \hat{G}$ gives a natural parametrization of $\hat{G}$ by co-adjoint orbits and is known to be continuous. In this paper a finite partition of $\mathfrak{g}^*/\operatorname{Ad}^*(G)$ is defined by means of an explicit construction which gives the partition a natural total ordering, such that the minimal element is open and dense. Given $\pi \in \hat{G}$, elements in the enveloping algebra of $\mathfrak{g}_c$ are constructed whose images under π are scalar and give crucial information about the associated orbit. This information is then used to show that the restriction of η to each element of the above-mentioned partition is a homeomorphism.

1. Introduction. Let G be a real, connected, simply connected exponential solvable Lie group with Lie algebra $\mathfrak{g}$. By a representation of G we shall mean a strongly continuous, unitary representation of G in some Hilbert space, and we denote the dual of G by $\hat{G}$, that is, the set of unitary equivalence classes of topologically irreducible representation of G. Denote by η the natural mapping of the set $\mathfrak{g}^*/\operatorname{Ad}^*(G)$ of co-adjoint orbits in the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ onto $\hat{G}$. When $\mathfrak{g}^*/\operatorname{Ad}^*(G)$ is given the quotient topology and $\hat{G}$ the hull kernel topology, η is continuous. It was first conjectured by A. A. Kirillov in [**8**] and proved by I. Brown in [**3**] that if G is nilpotent, η is a homeomorphism. K. Joy in a later paper [**7**] gives a much shorter proof of Brown's Theorem using results of J. M. G. Fell pertaining to the space $S(G)$ of subgroup representation pairs (π, H), where H is a closed connected subgroup of G and π is an unitary equivalence class of representations of H. Two results on the bicontinuity of η when G is exponential are due to J. Boidol [**2**] and H. Fujiwara [**6**]. Boidol shows that η^{-1} is continuous provided that G is $*$-regular; $*$-regularity is seen to fail however even for a completely solvable group of dimension four. On the other hand, Fijuwara proves the existence of a dense open subset U of $\hat{G}$ such that $V = \eta^{-1}(U)$ is dense and such that the restriction of η to V is a homeomorphism. However, Fujiwara's result provides no explicit characterization of U. Finally, it is known that η is a homeomorphism for all G of dimension less than six. Those cases which are not $*$-regular are handled by constructing elements W in the center of the enveloping algebra $U(\mathfrak{g}_c)$, and using the fact that the mapping ϕ_W on $\hat{G}$ given by $\rho(W) = \phi_W(\rho)I$ is continuous. $\phi_W \circ \eta$ can be regarded as an $\operatorname{Ad}^*(G)$-invariant polynomial function of $\mathfrak{g}^*$, and as such provides enough information to conclude convergence of the corresponding orbits. In the general case the center of $U(\mathfrak{g}_c)$ is not large enough to yield sufficient information about η^{-1}.

Now let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$, and let $\rho \in \hat{G}$ such that ρ is extended from $N = \exp(\mathfrak{n})$. A generalization of the construction mentioned above is given whereby

Received by the editors October 30, 1986 and, in revised form, June 19, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 22E27, Secondary 17B30.

elements $w_i \in U(\mathfrak{g}_c)$ are constructed such that $\{\rho(W_i)\}$ are scalar operators whose values allow one to systematically obtain $\eta^{-1}(\rho)$ from the orbit of $\rho|_N$. The Kirillov mapping has a natural generalization in the context of the space of subgroup representation pairs (ρ, H) such that $H \supset N$ and $\rho \in \hat{H}$, and a theorem regarding this mapping is proved which has as a corollary the following. There is a finite partition $\{U_\alpha\}$ of $\mathfrak{g}^*/\operatorname{Ad}^*(G)$—obtained by an explicit construction depending only on a choice of Jordan-Hölder sequence for η—on each element of which η is open.

2. Preliminaries. Let $\mathfrak{g}$ be a real, solvable Lie algebra of exponential type. For any subspace $\mathfrak{h}$ of $\mathfrak{g}$, let $\mathfrak{h}^*$ denote the dual space of $\mathfrak{h}$, and if $\mathfrak{j}$ is a subspace of $\mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{j}$ and $\lambda \in \mathfrak{j}^*$, denoted by B_λ the bilinear form defined on $\mathfrak{h}$ by $B_\lambda(X,Y) = \lambda([X,Y])$, $X, Y \in \mathfrak{h}$. For any subset $\mathfrak{s}$ of $\mathfrak{h}$, denote by $\mathfrak{s}^{\lambda,\mathfrak{h}}$ the orthogonal complement of $\mathfrak{s}$ in h with respect to B_λ. The radical $\mathfrak{h}^{\lambda,\mathfrak{h}}$ of B_λ will also be denoted by $R(\lambda, \mathfrak{h})$.

Let $\{\mathfrak{h}_n\}_{n=1}^\infty$ be a sequence of subspaces of $\mathfrak{g}$. We shall say that $\mathfrak{h}_n$ converges to a subspace $\mathfrak{h}$ (or write $\mathfrak{h}_n \to \mathfrak{h}$) if there are positive integers K and d such that for each $n > K$, there is a basis $X_1^{(n)}, X_2^{(n)}, \dots, X_d^{(n)}$ of $\mathfrak{h}_n$ and a basis $X_1, X_2, \dots, X_d$ of $\mathfrak{h}$ with $X_j = \lim_n X_j^{(n)}$, $1 \le j \le d$. Suppose that $\mathfrak{h}_n \to \mathfrak{h}$, and let $W_n \in \mathfrak{h}_n$, $n \ge 1$, such that for some $W \in \mathfrak{g}$, $W = \lim_n W_n$. Then $W \in h$, and it follows that if for some $\mathfrak{h}', \mathfrak{h}_n \to \mathfrak{h}'$, then $\mathfrak{h}' = \mathfrak{h}$, and if $\mathfrak{h}_n$ is a subalgebra (ideal) for infinitely many n, then $\mathfrak{h}$ is a subalgebra (ideal). Clearly every sequence $\{\mathfrak{h}_n\}$ of nontrivial subspaces of $\mathfrak{g}$ has a subsequence which converges, and it is easily seen that $\mathfrak{h}_n \to \mathfrak{h}$ if and only if every convergent subsequence of $\{\mathfrak{h}_n\}$ converges to $\mathfrak{h}$.

LEMMA 2.1. *Let $\{\mathfrak{j}_n\}_{n=1}^\infty$ be a sequence of subspaces of $\mathfrak{g}$ such that for each n, $\mathfrak{j}_n \subset \mathfrak{h}_n$, and suppose that $\mathfrak{j}_n \to \mathfrak{j}$ and $\mathfrak{h}_n \to \mathfrak{h}$. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence in $\mathfrak{g}^*$ such that for some $\lambda \in \mathfrak{g}^*$, $\lambda|_{[\mathfrak{g},\mathfrak{g}]} = \lim_n \lambda_n|_{[\mathfrak{g},\mathfrak{g}]}$, and $\dim_{\mathbf{R}}(\mathfrak{j}^{\lambda,\mathfrak{h}}) = \liminf_n \dim_{\mathbf{R}}(\mathfrak{j}_n^{\lambda_n,\mathfrak{h}_n})$. Then $\mathfrak{j}_n^{\lambda_n,\mathfrak{h}_n} \to \mathfrak{j}^{\lambda,\mathfrak{h}}$.*

PROOF. Let K and d be positive integers such that for each $n > K$, there is a basis $Y_1^{(n)}, Y_2^{(n)}, \dots, Y_d^{(n)}$ of $\mathfrak{j}_n$ with $\lim_n Y_j^{(n)} = Y_j$, $1 \le j \le d$, and $\{Y_j\}$ a basis of $\mathfrak{j}$. Let $\{\mathfrak{j}_k^{\lambda_k,h_k}\}$ be any convergent subsequence, $\mathfrak{j}_k^{\lambda_k,h_k} \to \mathfrak{j}_0$, and let $W \in \mathfrak{j}_0$. Then for each k, there is $W^{(k)} \in \mathfrak{j}_k^{\lambda_k,h_k}$ such that $W = \lim_k W^{(k)}$, and we have

$$\lambda([W, Y_j]) = \lim_k \lambda_k([W^{(k)}, Y_j^{(k)}]) = 0,$$

$1 \le j \le d$. Thus $\mathfrak{j}_0 \subset \mathfrak{j}_{\lambda,\mathfrak{h}}$. On the other hand,

$$\dim_{\mathbf{R}}(\mathfrak{j}_0) \ge \liminf_n \dim_{\mathbf{R}}(\mathfrak{j}_n^{\lambda_n,\mathfrak{h}_n})$$

so that $\mathfrak{j}_0 = \mathfrak{j}^{\lambda,\mathfrak{h}}$, and hence $\mathfrak{j}_n^{\lambda_n,\mathfrak{h}_n} \to \mathfrak{j}^{\lambda,\mathfrak{h}}$. $\square$

Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. We denote by $U(\mathfrak{h})$ the enveloping algebra of $\mathfrak{h}$ and regard $U(\mathfrak{h})$ as a subalgebra of $U(\mathfrak{g})$. We denote the complexification $\mathfrak{h} \otimes_{\mathbf{R}} \mathbf{C}$ by $\mathfrak{h}_c$ and regard $U(\mathfrak{h}_c)$ as a subalgebra of $U(\mathfrak{g}_c)$.

Let G be a connected, simply connected Lie group with Lie algebra $\mathfrak{g}$, and let H be the closed, connected subgroup of G with Lie algebra $\mathfrak{h}$. Denote by η_H Kirillov mapping $\mathfrak{h}^*/\operatorname{Ad}^*(H) \to \hat{H}$, let π be a representation of H, and let $\lambda \in \mathfrak{h}^*$. We shall say that π corresponds to λ if $\pi \in \eta_H(\operatorname{Ad}^*(H)\lambda)$. If $\mathfrak{p} \in \mathfrak{h}^*$ is a polarization at λ, we occasionally use the notation $\operatorname{ind}(\lambda, \mathfrak{p})$ for the irreducible representation

$\operatorname{ind}(\chi_\lambda, P, H)$ of H induced by the character χ_λ of $P = \exp(\mathfrak{p})$ with differential $i(\lambda|_\mathfrak{p})$.

Now let $\lambda \in \mathfrak{g}^*$, and let $\mathfrak{m}$ be a nilpotent subalgebra of $\mathfrak{g}$.

DEFINITION 2.2. A pair $(\mathfrak{m}_1, \mathfrak{m}_0)$ of $\mathfrak{m}$-ideals such that $\mathfrak{m}_0 \subset \mathfrak{m}_1$, $\dim_{\mathbf{R}}(\mathfrak{m}_1/\mathfrak{m}_0) = 1$, $\mathfrak{m}_0 \subset R(\lambda, \mathfrak{m})$ and $\mathfrak{m}_1 \not\subset R(\lambda, \mathfrak{m})$ will be called a Kirillov pair in $\mathfrak{m}$ at λ.

Let $(\mathfrak{m}_1, \mathfrak{m}_0)$ be a Kirillov pair in $\mathfrak{m}$ at λ, and let $\mathfrak{l} = \mathfrak{m}_1^{\lambda, \mathfrak{m}}$. Then $\mathfrak{l}$ is a codimension 1 subalgebra of $\mathfrak{m}$. Let π_1 be an irreducible representation of $L = \exp(\mathfrak{l})$ corresponding to $\lambda|_\mathfrak{l}$, and let $X \in \mathfrak{m} \sim \mathfrak{l}$. Then the representation $\pi = \pi(\pi_1, X)$ defined in $L^2(\mathbf{R}, H(\pi_1))$ by the formula

$$(1) \quad (\pi(y \exp sX)f)(t) = \pi_1(\exp tXy\exp -tX)f(t+s) \qquad (y \in L,\ s,t \in \mathbf{R})$$

corresponds to $\lambda|_\mathfrak{m}$. The primary representation $\tilde{\pi}_1$ defined in $L^2(\mathbf{R}, H(\pi_1))$ by $(\tilde{\pi}_1(y)f)(t) = \pi_1(y)f(t)$, $y \in L$, can be differentiated in the space $C^\infty(\pi)$ of smooth vectors for π, that is, $C^\infty(\tilde{\pi}_1) \supset C^\infty(\pi)$. The following lemma is more or less well known, but crucial in this paper.

LEMMA 2.3. *There is an explicit construction by which, given any $W \in U(\mathfrak{l}_c)$, one obtains $\tilde{W} \in U(\mathfrak{l}_c)$ such that $\pi(\tilde{W}) = \tilde{\pi}_1(W)$.*

PROOF. Let m be a positive integer such that $\operatorname{ad} X^{m+1} \equiv 0$, and let $W \in U(\mathfrak{l}_c)$. We construct an element $W_m \in U(\mathfrak{l}_c)$ as follows. Let (by abuse of notation) t denote the operator on $C^\infty(\pi)$ defined by $\phi(t) \to t\phi(t)$. We have $\pi(W) = \sum_{j=0}^m (t^j/j!)\tilde{\pi}_1(\operatorname{ad} X^j W)$ so that $\pi(\operatorname{ad} X^m W) = \tilde{x}_1(\operatorname{ad} X^m W)$. Let Y be the element in $\mathfrak{m}_1 \sim \mathfrak{m}_0$ such that $\lambda(Y) = 0$, and $B_\lambda(X, Y) = 1$, so that $\pi(Y) = it$. Define $W_1 \in U(\mathfrak{l}_c)$ by

$$W_1 = W - \frac{\operatorname{ad} X^m W \cdot (-iY)^m}{m!}.$$

Then $\pi(W_1) = \sum_{j=0}^{m-1}(t^j/j!)\pi_1(\operatorname{ad} X^j W)$ and $\pi(\operatorname{ad} X^{m-1} W_1) = \tilde{\pi}_1(\operatorname{ad} X^{m-1} W)$. If $m > 1$, set

$$W_2 = W_1 - \frac{\operatorname{ad} X^{m-1} W_1 \cdot (-iY)^{m-1}}{(m-1)!}$$

and we find that $\pi(\operatorname{ad} X^{m-2} W_2) = \tilde{\pi}_1(\operatorname{ad} X^{m-2} W)$. Continue in this way until $W_m = \tilde{W}$ is obtained. Q.E.D.

3. A partition of the dual of a nilpotent Lie group. Now let us assume that $\mathfrak{g}$ is nilpotent; fix $\lambda \in \mathfrak{g}^*$. By induction on the dimension of $\mathfrak{g}$ it is easily seen that there is a sequence of subalgebras $\mathfrak{g} = \mathfrak{m}_0 \supset \mathfrak{m}_1 \supset \cdots \supset \mathfrak{m}_d$ satisfying the conditions

(i) $\mathfrak{m}_d$ is a polarization at λ.

(ii) If $R(\lambda, \mathfrak{g}) \neq \mathfrak{g}$, then $d > 0$ and for each $k = 1, 2, \ldots, d-1$, there is a Kirillov pair $(\mathfrak{m}_{k1}, \mathfrak{m}_{k0})$ in $\mathfrak{m}_k$ at λ such that $\mathfrak{m}_{k+1} = \mathfrak{m}_{k1}^{\lambda, \mathfrak{m}_k}$. Thus if $R(\lambda, \mathfrak{g}) \neq \mathfrak{g}$, then $d = \frac{1}{2}\dim(\operatorname{Ad}^*(G))\lambda)$.

DEFINITION 3.1. A sequence of subalgebras satisfying conditions (i) and (ii) above will be called a Kirillov sequence for λ in $\mathfrak{g}$.

Let d be a nonnegative integer. Let us say that an operator D on $C^\infty(\mathbf{R}^d)$ ($C^\infty(\mathbf{R}^0) \equiv \mathbf{C}$) is a polynomial differential operator if there is a polynomial P in $2d$ indeterminants with complex coefficients such that

$$D = P(t_1, \ldots, t_d, \partial/\partial t_1, \ldots, \partial/\partial t_d).$$

A theorem of Kirillov (cf. [**8**, Theorem 7.1]) states that $\eta(\mathrm{Ad}^*(G)\lambda)$ has a realization π in a space of functions on $\mathbf{R}^d$ such that the image of $U(\mathfrak{g}_c)$ under π is the set of polynomial differential operators. In this section we shall determine when it is possible, given a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in $\mathfrak{g}^*$ such that $\lambda_n \to \lambda = \lambda_0$, to obtain a corresponding sequence $\{\pi_n\}_{n=0}^{\infty}$ of irreducible representations such that given any D as above, there is a sequence $\{W_n\}_{n=0}^{\infty}$ in $U(\mathfrak{g}_c)^{(m)}$ for some m, with $W_n \to W_0$ and $\pi_n(W_n) = D$ for each n.

Let $\mathfrak{g} = \mathfrak{g}_p \supset \mathfrak{g}_{p-1} \supset \cdots \supset \mathfrak{g}_0 = (0)$ be a Jordan-Hölder sequence for $\mathfrak{g}$. Define subsets $e(\lambda)$, $j(\lambda)$ and $i(\lambda)$ of $\{1, 2, \ldots, p\}$ as follows. Set

$$e(\lambda) = \{t | \mathfrak{g}_t + R(\lambda, \mathfrak{g}) \supsetneqq \mathfrak{g}_{t-1} + R(\lambda, \mathfrak{g})\}$$

and let $\mathfrak{p}(\lambda) = \sum_t R(\lambda, \mathfrak{g}_t)$. Define $j(\lambda) \subset e(\lambda)$ by

$$j(\lambda) = \{t | \mathfrak{g}_t + \mathfrak{p}(\lambda) \supsetneqq \mathfrak{g}_{t-1} + \mathfrak{p}(\lambda)\}$$

and let $i(\lambda) = e(\lambda) \sim j(\lambda)$. Then $\mathrm{card}(e(\lambda)) = \dim(\mathrm{Ad}^*(G)\lambda)$ and it is shown in [**1**] that $\mathfrak{p}(\lambda)$ is a polarization at λ, hence $\mathrm{card}(j(\lambda)) = \frac{1}{2}\,\mathrm{card}(e(\lambda))$. If $e(\lambda) \neq \{\phi\}$, we shall write $e(\lambda) = \{e_1 < e_2 < \cdots < e_{2d}\}$. We define a sequence of subalgebras $\mathfrak{g} = \mathfrak{g}^0(\lambda) \supset \mathfrak{g}^1(\lambda) \supset \cdots \supset \mathfrak{g}^d(\lambda)$ as follows. Setting $\mathfrak{g}^0(\lambda) = \mathfrak{g}$, assume that for some $k \geq 0$, $\mathfrak{g}^k(\lambda)$ is defined and $\mathfrak{g}^k(\lambda) \neq R(\lambda, \mathfrak{g}^k(\lambda))$. Let i_{k+1} be the smallest index such that $\mathfrak{g}_{i_{k+1}} \cap \mathfrak{g}^k(\lambda) \not\subset R(\lambda, \mathfrak{g}^k(\lambda))$ and set

$$\mathfrak{g}^{k+1}(\lambda) = \left(\mathfrak{g}_{i_{k+1}} \cap \mathfrak{g}^k(\lambda)\right)^{\lambda, \mathfrak{g}^k(\lambda)}.$$

Note that $\mathfrak{g}^{k+1}(\lambda)$ is codimension 1 in $\mathfrak{g}^k(\lambda)$. If $\mathfrak{g}^k(\lambda) = R(\lambda, \mathfrak{g}^k(\lambda))$, then let the sequence terminate at $\mathfrak{g}^k(\lambda)$, and set $k = d$. Thus $\mathfrak{g}^d(\lambda)$ is isotropic with respect to B_λ. If $e(\lambda) \neq \{\phi\}$, then in this way we obtain a sequence of indices $i_1, i_2, \ldots, i_d$. Note that if $R(\lambda, \mathfrak{g}) \subset \mathfrak{g}^k(\lambda)$, then $R(\lambda, \mathfrak{g}) \subset \mathfrak{g}^{k+1}(\lambda)$, $0 \leq k \leq d$; thus we have

$$R(\lambda, \mathfrak{g}) \subset R(\lambda, \mathfrak{g}^k(\lambda)) \subset \mathfrak{g}^k(\lambda), \qquad 0 \leq k \leq d.$$

Now for each $k = 1, 2, \ldots, d$, let j_k be the smallest index such that $\mathfrak{g}_{j_k} \cap \mathfrak{g}^{k-1}(\lambda) \not\subset \mathfrak{g}^k(\lambda)$.

LEMMA 3.2. *For each $k = 1, 2, 3, \ldots, d$, $i_k \in e(\lambda)$ and $j_k \in e(\lambda)$. If $k < d$, $i_k < i_{k+1}$, and for $k \leq d$, $i_k < j_k$. Moreover $\mathfrak{g}^d(\lambda) = \mathfrak{p}(\lambda)$, $i(\lambda) = \{i_k\}_{k=1}^d$, and $j(\lambda) = \{j_k\}_{k=1}^d$.*

PROOF. If $i_k \notin e(\lambda)$, there is $Y \in R(\lambda, \mathfrak{g})$ such that $\mathfrak{g}_{i_k} = \mathbf{R}Y + \mathfrak{g}_{i_k - 1}$. But since $R(\lambda, \mathfrak{g}) \subset R(\lambda, \mathfrak{g}^{k-1}(\lambda))$,

$$\mathfrak{g}_{i_k} \cap \mathfrak{g}^{k-1}(\lambda) = \mathbf{R}Y + (\mathfrak{g}_{i_k - 1} \cap \mathfrak{g}^{k-1}(\lambda)) \subset R(\lambda, \mathfrak{g}^{k-1}(\lambda))$$

a contradiction. If $j_k \notin e(\lambda)$, let $X \in R(\lambda, \mathfrak{g})$ such that $\mathfrak{g}_{j_k} = \mathbf{R}X + \mathfrak{g}_{j_k - 1}$. Since $R(\lambda, \mathfrak{g}) \subset \mathfrak{g}^k(\lambda)$,

$$\mathfrak{g}_{j_k} \cap \mathfrak{g}^{k-1}(\lambda) = \mathbf{R}X + (\mathfrak{g}_{j_k - 1} \cap \mathfrak{g}^{k-1}(\lambda)) \subset \mathfrak{g}^k(\lambda)$$

a contradiction. This proves the first statement of the lemma.

Now by definition of i_k,

$$\mathfrak{g}_{i_k} \cap \mathfrak{g}^{k-1}(\lambda) = \mathbf{R}Y + \mathfrak{g}_{i_k - 1} \cap \mathfrak{g}^{k-1}(\lambda) \subset \mathbf{R}Y + R(\lambda, \mathfrak{g}^{k-1}(\lambda)).$$

Thus

$$[\mathfrak{g}_{i_k} \cap \mathfrak{g}^{k-1}(\lambda), \mathfrak{g}_{i_k} \cap \mathfrak{g}^{k-1}(\lambda)] \subset [Y, R(\lambda, \mathfrak{g}^{k-1}(\lambda))] \\ + [R(\lambda, \mathfrak{g}^{k-1}(\lambda)), R(\lambda, \mathfrak{g}^{k-1}(\lambda))] \subset \operatorname{Ker}(\lambda).$$

By definition of $\mathfrak{g}^k(\lambda)$ and j_k, it follows that $\mathfrak{g}_{i_k} \cap \mathfrak{g}^{k-1}(\lambda) \subset \mathfrak{g}^k(\lambda)$, and hence that $i_k < j_k$, and that

$$\mathfrak{g}_{i_k} \cap \mathfrak{g}^k(\lambda) = \mathfrak{g}_{i_k} \cap \mathfrak{g}^{k-1}(\lambda) \subset R(\lambda, \mathfrak{g}^k(\lambda));$$

therefore $i_k < i_{k+1}$.

Next we show that $\mathfrak{g}^d(\lambda) = \mathfrak{p}(\lambda)$. For this, let $X \in R(\lambda, \mathfrak{g}_t)$, and suppose that $X \in \mathfrak{g}^k(\lambda)$ for some $k < d$. We show that $X \in \mathfrak{g}^{k+1}(\lambda)$. Suppose $t < i_{k+1}$. By choice of i_{k+1}, $X \in \mathfrak{g}_t \cap \mathfrak{g}^k(\lambda) \subset R(\lambda, \mathfrak{g}^k(\lambda)) \subset \mathfrak{g}^{k+1}(\lambda)$. Suppose $t \geq i_{k+1}$; then $X \in R(\lambda, \mathfrak{g}_t) \cap \mathfrak{g}^k(\lambda) \subset (\mathfrak{g}_t \cap \mathfrak{g}^k(\lambda))^{\lambda, \mathfrak{g}^k(\lambda)} \subset \mathfrak{g}^{k+1}(\lambda)$. Since $X \in \mathfrak{g}^0(\lambda) = \mathfrak{g}$, it follows that $X \in \mathfrak{g}^d(\lambda)$.

Let us show now that $i(\lambda) = \{i_k\}_{k=1}^d$. Note that for each k, $R(\lambda, \mathfrak{g}^k(\lambda)) \subset \mathfrak{g}^{k+1}(\lambda)$ so that $R(\lambda, \mathfrak{g}^k(\lambda)) \subset R(\lambda, \mathfrak{g}^{k+1}(\lambda))$ and hence $R(\lambda, \mathfrak{g}^k(\lambda)) \subset \mathfrak{p}(\lambda)$. Thus $\mathfrak{g}_{i_k} \subset \mathfrak{g}_{i_k-1} + \mathfrak{p}(\lambda)$ and $i_k \notin j(\lambda)$. It follows that $i(\lambda) = \{i_k\}_{k=1}^d$.

Finally, to see that $\{j_k\}_{k=1}^d = j(\lambda)$, note that $j_k \in j(\lambda)$, since if not, then $\mathfrak{g}_{j_k} \subset \mathfrak{g}_{j_k-1} + \mathfrak{g}^k(\lambda)$, hence $\mathfrak{g}_{j_k} \cap \mathfrak{g}^{k-1}(\lambda) \subset (\mathfrak{g}_{j_k-1} + \mathfrak{g}^k(\lambda)) \cap \mathfrak{g}^{k-1}(\lambda) = \mathfrak{g}_{j_k-1} \cap \mathfrak{g}^{k-1}(\lambda) + \mathfrak{g}^k(\lambda) = \mathfrak{g}^k(\lambda)$ a contradiction.

Let $j \in j(\lambda)$, and let k_0 be the smallest k, $1 \leq k \leq d$, such that $\mathfrak{g}_j \subset \mathfrak{g}_{j-1} + \mathfrak{g}^k(\lambda)$. We claim that $j = j_{k_0}$. Now $\mathfrak{g}_j \subset \mathfrak{g}_{j-1} + \mathfrak{g}^{k_0-1}(\lambda)$, so we may write $\mathfrak{g}_j = \mathbf{R}X + \mathfrak{g}_{j-1}$ where $X \in \mathfrak{g}^{k_0-1}(\lambda)$, and by choice of k_0, $X \in \mathfrak{g}^{k_0}(\lambda)$. Hence $\mathfrak{g}_j \cap \mathfrak{g}^{k_0-1}(\lambda) \subset \mathfrak{g}^{k_0}(\lambda)$ and $j \geq j_{j_{k_0}}$, by choice of $j \geq j_{j_{k_0}}$. If $j > j_{k_0}$, then choose $\tilde{X} \in \mathfrak{g}_{j_{k_0}}$ such that $\mathfrak{g}^{k_0-1}(\lambda) = \mathbf{R}\tilde{X} + \mathfrak{g}^{k_0}(\lambda)$. Since $\dim(\mathfrak{g}^{k_0-1}(\lambda)/\mathfrak{g}^{k_0}(\lambda)) = 1$, there are elements $a \neq 0$, $b \neq 0$ in such that $W = a\tilde{X} + bX \in \mathfrak{g}^{k_0}(\lambda)$. But $\mathfrak{g}_j = \mathbf{R}W + \mathfrak{g}_{j-1} \subset \mathfrak{g}_{j-1} + \mathfrak{g}^{k_0}(\lambda)$, contradicting our choice of k_0; therefore $j = j_{k_0}$, and the proof of the lemma is finished. □

Now let E denote the set of pairs

$$E = \{(e(\lambda), j(\lambda)) | \lambda \in \mathfrak{g}^*\}$$

and for d a positive integer, let

$$E_d = \{(e, j) \in E | \operatorname{card}(j) = d\}.$$

Let us regard elements of E_d as ordered $3d$-tuples of integers

$$(e, j) = (e_1, e_2, \ldots, e_{2d}, j_1, j_2, \ldots, j_d)$$

where $e_1 < e_2 < \cdots$, and $\{j_1, \ldots, j_d\} = j$ is indexed by the inductive process above. We define a total order on E in the following way. Let (ϕ, ϕ) be the maximal element of E, and regarding E_d as above, let E_d have the natural lexographic ordering. If $d > d'$, let us say that for any $\alpha \in E_d$, $\alpha' \in E_{d'}$, $\alpha < \alpha'$.

Now for each $\alpha \in E$, set $\Omega_\alpha = \{\lambda \in \mathfrak{g}^* | (e(\lambda), j(\lambda)) = \alpha\}$ and for each $e_0 = e(\lambda_0)$, $\Omega_{e_0} = \{\lambda \in \mathfrak{g}^* | e(\lambda) = e_0\}$ so that $\Omega_{e_0} = \bigcup\{\Omega_{(e,j)} | e = e_0\}$. If $s \in G$, and $\lambda \in \mathfrak{g}^*$, then $\mathfrak{g}^k(\operatorname{Ad}^*(s)\lambda) = \operatorname{Ad}(s)(\mathfrak{g}^k(\lambda))$ and it follows that each $\alpha \in E$, Ω_α is *G-invariant.* The sets Ω_{e_0} were first considered by Pukanszky in [**10**], and the sets Ω_α are considered by N. V. Pedersen in a paper to appear.

Let $\{Z_1, Z_2, \ldots, Z_p\}$ be a basis compatible with the Jordan-Hölder sequence chosen at the beginning of this section. Let $e = e(\lambda_0)$ for some λ_0, let $P_e^{ij}(\lambda) = \lambda([Z_{e_i}, Z_{e_j}])$, $e_i, e_j \in e$, and set $P_e(\lambda) = \det((P_e^{ij}(\lambda)))$. Letting the set $\{e(\lambda)|\lambda \in \mathfrak{g}^*\}$ have the total ordering inherited from E, it is shown in **[9]** that

$$\Omega_e = \{\lambda \in \mathfrak{g}^* | P_{e'}(\lambda) = 0,\ e' < e \text{ and } P_e(\lambda) \neq 0\}.$$

Now for each $e = e(\lambda)$, let $J_e = \{j | (e, j) \in E\}$, and let J_e have the total ordering inherited from E.

PROPOSITION 3.3. *There are polynomials $P_{(e,j)}$, $j \in J_e$, such that for each $j \in J_e$,*

$$\Omega_{(e,j)} = \{\lambda \in \Omega_e | P_{(e,j')}(\lambda) = 0, j' < j \text{ and } P_{(e,j')}(\lambda) \neq 0\}.$$

PROOF. Let $j \in J_e$, and write $j = \{j_1, \ldots, j_d\}$ and $i = e - j = \{i_1, \ldots, i_d\}$ as in the inductive process above. Let $\lambda \in \Omega_e$ and for each $k = 1, 2, \ldots, d$ set $e^{(k)} = e - \{i_1, \ldots, i_k, j_1, \ldots, j_k\}$ and define elements $Z_t^k(\lambda) \in g$, $t \in e^{(k)}$ as follows. Note that $\{Z_t\}_{t \in e}$ is a basis for $\mathfrak{g}$ modulo $R(\lambda, \mathfrak{g})$. Let $Z_t^1(\lambda) = Z_t$ if $t \in e^{(1)}$, $t < j_1$ and for $t > j_1$, set

$$Z_t^1(\lambda) = B_\lambda(Z_{j_1}, Z_{i_1})Z_t - B_\lambda(Z_t, Z_{i_1})Z_{j_1}.$$

Suppose that $\lambda \in \Omega_{(e,j')}$ with $j' \geq j$ and write $j' = \{j_1', \ldots, j_d'\}$, $i = e - j' = \{i_1', \ldots, i_d'\}$. Since $e_1 = i_1 = i_1'$, clearly $j_1' = j_1$ if and only if $B_\lambda(Z_{j_1}, Z_{i_1}) \neq 0$, and in this case, $\{Z_t^1(\lambda)\}_{t \in e^{(1)}}$ is a basis of $\mathfrak{g}^1(\lambda)$ modulo $R(\lambda, \mathfrak{g}^1(\lambda))$. Therefore, by definition of j_1, j_2, we have $j_1' = j_1$ and $j_2' = j_2$ if and only if, $B_\lambda(Z_{j_1}, Z_{i_1}) \neq 0$ and $B_\lambda(Z_{j_2}^1(\lambda), Z_{i_2}^1(\lambda)) \neq 0$. Now define $Z_t^2(\lambda)$, $t \in e^{(2)}$ in the same way as $\{Z_t^1(\lambda)\}_{t \in e^{(1)}}$, so that if $j_1' = j_1$, $j_2' = j_2$, then $\{Z_t^2(\lambda)\}_{t \in e^{(2)}}$ is a basis of $\mathfrak{g}^2(\lambda)$ modulo $R(\lambda, \mathfrak{g}^2(\lambda))$. Continuing in this way, set

$$P_{e,j}(\lambda) = B_\lambda(Z_{j_1}, Z_{i_1})B_\lambda(Z_{j_2}^1(\lambda), Z_{i_2}^1(\lambda)) \cdots B_\lambda(Z_{j_d}^{d-1}(\lambda), Z_{i_d}^{d-1}(\lambda))$$

and the proposition follows. Q.E.D.

COROLLARY 3.4. *Let α_0 be the minimal element of E. Then Ω_{α_0} is Zariski open in $\mathfrak{g}^*$. Moreover, for each $\alpha \in E$, Ω_α is open in $\bigcup_{\beta \geq \alpha} \Omega_\beta$.*

PROOF. That Ω_{α_0} is Zariski open is clear. As for the second statement, let $\alpha \in E$, $\alpha = (e, j)$. Ω_α is Zariski open in $\bigcup_{j' \geq j} \Omega_{(e,j')}$ by Proposition 3.3. But Ω_e is open in $\bigcup_{e' \geq e} \Omega_{e'}$, and hence $\bigcup_{j' \geq j} \Omega_{(e,j')} = \Omega_e \cap \bigcup_{\beta \geq \alpha} \Omega_\beta$ is open in $\bigcup_{\beta \geq \alpha} \Omega_\beta$, and it follows that Ω_α is open in $\bigcup_{\beta \geq \alpha} \Omega_\beta$. Q.E.D.

Now for each $\lambda \in \mathfrak{g}^*$ such that $\{\phi\} \neq e(\lambda) = i(\lambda) \cup j(\lambda)$, $i(\lambda) = \{i_1, i_2, \ldots, i_d\}$, $j(\lambda) = \{j_1, j_2, \ldots, j_d\}$, define for each $0 \leq k < d$, $\mathfrak{m}_{k1}(\lambda) = \mathfrak{g}^k(\lambda) \cap \mathfrak{g}_{i_{k+1}}$ and $\mathfrak{m}_{k0}(\lambda) = \mathfrak{g}^k(\lambda) \cap \mathfrak{g}_{i_{k+1}-1}$. Then $(\mathfrak{m}_{k1}(\lambda), \mathfrak{m}_{k0}(\lambda))$ is a Kirillov pair in $\mathfrak{g}^k(\lambda)$ at λ, and $\mathfrak{m}_{k1}^\lambda \mathfrak{g}^{k(\lambda)} = \mathfrak{g}^{k+1}(\lambda)$, $0 \leq k < d$. Thus the sequence $\mathfrak{g}^0(\lambda) \supset \mathfrak{g}^1(\lambda) \supset \cdots \supset \mathfrak{g}^d(\lambda)$ is a Kirillov sequence for λ in $\mathfrak{g}$.

THEOREM 3.5. *Let d be a positive integer and let $\alpha \in E_d$. Let $\{\lambda_n\}_{n=0}^\infty$ be a sequence in Ω_α which converges to λ_0. Then for each $n \geq 0$, there is an irreducible representation π_n corresponding to λ_n such that if*

$$D = P(t_1, t_2, \ldots, t_d, \partial/\partial t_1, \partial/\partial t_2, \ldots, \partial/\partial t_d)$$

is any polynomial differential operator, then there is an integer $m > 0$ and a sequence $\{W_n\}_{n=0}^{\infty}$ in $U(\mathfrak{g}_c)^{(m)}$ which converges to W_0 and such that $\pi_n(W_n) = D$, $n = 0, 1, 2, \ldots$.

PROOF. Clearly we may assume that for some $1 \leq k \leq d$, either $D = t_k$ or $D = \partial/\partial t_k$. For each $n \geq 0$, we have the data $\{\mathfrak{g}^k(\lambda_n)\}_{k=0}^{d}$, $\{(\mathfrak{m}_{k1}(\lambda_n), \mathfrak{m}_{k0}(\lambda_n))\}_{k=0}^{d-1}$ as in the remarks preceding the theorem. Note that for each $n \geq 0$, $1 \leq t \leq p$, and $1 \leq k \leq d$, $\dim(\mathfrak{g}_t \cap \mathfrak{g}^k(\lambda_n)) = \mathrm{card}(\{j_s | s \leq k, j_s < t\})$. Now by Lemma 1.1, we have that $\mathfrak{g}^1(\lambda_n) \to \mathfrak{g}^1(\lambda_0)$. Since $\dim_{\mathbf{R}}(\mathfrak{g}_t \cap \mathfrak{g}^1(\lambda_n)) = \dim_{\mathbf{R}}(\mathfrak{g}_t \cap \mathfrak{g}^1(\lambda_0))$, $n = 1, 2, 3, \ldots$, it follows that $\mathfrak{g}_t \cap \mathfrak{g}^1(\lambda_n) \to \mathfrak{g}_t \cap \mathfrak{g}^1(\lambda_0)$, and in particular, $\mathfrak{m}_{11}(\lambda_n) \to \mathfrak{m}_{11}(\lambda_0)$ and $\mathfrak{m}_{10}(\lambda_n) \to \mathfrak{m}_{10}(\lambda_0)$. But then Lemma 1.1 implies that $\mathfrak{g}^2(\lambda_n) \to \mathfrak{g}^2(\lambda_0)$. Continuing in this way, we obtain for each $k = 0, 1, \ldots, d-1$, $\mathfrak{g}^{k+1}(\lambda_n) \to \mathfrak{g}^{k+1}(\lambda_0)$, $\mathfrak{m}_{k1}(\lambda_n) \to \mathfrak{m}_{k1}(\lambda_0)$ and $\mathfrak{m}_{k0}(\lambda_n) \to \mathfrak{m}_{k0}(\lambda_0)$. Now, for each $0 \leq k \leq d$, $n \geq 0$, we shall define an irreducible representation $\pi_k^{(n)}$ of $G^k(\lambda_n) = \exp_G(\mathfrak{g}^k(\lambda_n))$. Let $\pi_d^{(n)}$ be the character of $G^d(\lambda_n)$ with differential $i(\lambda_n | \mathfrak{g}_d(\lambda_n))$. Choose $X_d^{(0)} \in \mathfrak{g}^{d-1}(\lambda_0) \sim \mathfrak{g}^d(\lambda_0)$, and since $\mathfrak{g}^{d-1}(\lambda_n) \to \mathfrak{g}^{d-1}(\lambda_0)$ and $\mathfrak{g}^d(\lambda_n) \to \mathfrak{g}^d(\lambda)$, we can choose $X_d^{(n)} \in \mathfrak{g}^{d-1}(\lambda_n) \sim \mathfrak{g}^d(\lambda_n)$, $n = 1, 2, 3, \ldots$, such that $X_d^{(n)} \to X_d^{(0)}$. Now for each n, define $\pi_{d-1}^{(n)} = \pi(\pi_d^{(n)}, X_d^{(n)})$ as in formula (1) above, that is, for each $f \in L^2(\mathbf{R}, H(\pi_d^{(n)})) = L^2(\mathbf{R})$, $y \in G^d(\lambda_n)$, and $s, t \in \mathbf{R}$,

$$(\pi_{d-1}^{(n)}(y \cdot \exp_G(sX_d^{(n)}))f)(t) = \pi_d^{(n)}(\exp_G(tX_d^{(n)}) \cdot y \cdot \exp(-tX_d^{(n)}))f(t+s).$$

We continue in this way, choosing $X_k^{(0)} \in \mathfrak{g}^{k-1}(\lambda_0) \sim \mathfrak{g}^k(\lambda_0)$ and $X_k^{(n)} \in \mathfrak{g}^{k-1}(\lambda_n) \sim \mathfrak{g}^k(\lambda_n)$, $n = 1, 2, 3, \ldots$, such that $X_k^n \to X_k^{(0)}$, for each k, so that

$$H(\pi_k^{(n)}) = L^2(\mathbf{R}, H(\pi_{k+1}^{(n)})), \qquad n \geq 0.$$

For each $k < d$, denote elements of $\mathbf{R}^{d-k}$ by $(t_{k+1}, t_{k+2}, \ldots, t_d)$, set $U_{d-1} =$ identity mapping on $L^2(\mathbf{R})$, and define for $k < d-1$, $U_k \colon H(\pi_k) \to L^2(\mathbf{R}^{d-k})$ by $U_k f(t_{k+1}, t_{k+2}, \ldots, t_d) = U_{k+1}(f(t_{k+1}))(t_{k+2}, \ldots, t_d)$. Set for each $n \geq 0$, $\pi_n = U_0 \pi_0^{(n)} U_0^{-1}$. Now suppose that $D = t_k$, and set $j = k - 1$. For each $n \geq 0$, let $Y_k^{(n)} \in \mathfrak{m}_{j1}(\lambda_n) \sim \mathfrak{m}_{j0}(\lambda_n)$ such that $\lambda(Y_k^{(n)}) = 0$ and $B_{\lambda_n}(X_k^{(n)}, Y_k^{(n)}) = 1$. It is easily seen that $Y_k^{(n)} \to Y_k^{(0)}$, and that for each n, $U_j \pi_j^{(n)} U_j^{-1}(-iY_k^{(n)}) = t_k$. If $j = 0$, we are done. Otherwise, we apply Lemma 2.3 to obtain, for each n, $W^{(n)} \in U(g^k(\lambda_n)_c)$ such that $\pi_{j-1}^{(n)}(W^{(n)}) = \pi_j^{(n)}(-iY_k^{(n)})$. The construction whereby $W^{(n)}$ is obtained involves only $\operatorname{ad} X_j^{(n)}$, $Y_j^{(n)} = Y_{k-1}^{(n)} \in \mathfrak{m}_{k1}(\lambda_n) \cap \ker(\lambda_n) \sim \mathfrak{m}_{k0}(\lambda_n)$ where $B_{\lambda_n}(X_j^{(n)}, Y_j^{(n)}) = 1$, $n = 0, 1, 2, 3, \ldots$, and we have $Y_j^{(n)} \to Y_j^{(0)}$ as well as $\operatorname{ad} X_j^{(n)} \to \operatorname{ad} X_j^{(0)}$. Hence it is clear that for some m, $W^{(n)} \in U(\mathfrak{g}_c)^{(m)}$, $n \geq 0$, and $W^{(n)} \to W^{(0)}$, and from the definition of U_{j-1} it is clear that for each n, $U_{j-1} \pi_{j-1}^{(n)} U_{j-1}^{-1}(W^{(n)}) = t_k$. If $j = 1$, then we are done. If $j > 1$, then we continue this process applying Lemma 2.3 at each step. This finishes the case $D = t_k$. If $D = \partial/\partial t_k$, the proof is similar. Q.E.D.

4. A theorem. We now drop the assumption that $\mathfrak{g}$ is nilpotent, that is, let $\mathfrak{g}$ be a real solvable Lie algebra of exponential type, and G a connected, simply connected Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$, and for the remainder of this paper, let $\mathfrak{n} = \mathfrak{n}_p \supset \mathfrak{n}_{p-1} \supset \cdots \supset \mathfrak{n}_0 = (0)$ be a Jordan-Hölder sequence for $\mathfrak{n}$ having the property that for each $t = 1, 2, 3, \ldots, p-1$, if $[\mathfrak{g}, \mathfrak{n}_t] \not\subset \mathfrak{n}_t$, then $[\mathfrak{g}, \mathfrak{n}_{t+1}] \subset \mathfrak{n}_{t+1}$. Let E be the index set and $\{\Omega_\alpha\}_{\alpha \in E}$ the $\mathrm{Ad}^*(N)$-invariant partition of $\mathfrak{n}^*$ corresponding to this Jordan-Hölder sequence as constructed in the previous section.

Let $\lambda \in \mathfrak{n}^*$, $\mathfrak{p}(\lambda) = \sum_t R(\lambda, \mathfrak{n}_t)$. Suppose that there is $A \in R(\lambda, \mathfrak{g}) \sim \mathfrak{n}$, and set $\mathfrak{h} = \mathbf{R}A + \mathfrak{n}$, $H = \exp(\mathfrak{h})$. It is shown in [**1**] that $\mathfrak{p}(\lambda)$ is invariant under $\mathrm{ad}\, A$, and we may extend the equivalence class of $\sigma = \mathrm{ind}(\lambda, \mathfrak{p}(\lambda))$ in $\hat{N}$ to H by setting

$$(2)\quad (\sigma(\exp sA)f)(y) = f(\exp(-sA) \cdot y \cdot \exp sA) \exp^{\{\frac{1}{2}\mathrm{tr}(\mathrm{ad}_{\mathfrak{n}/\mathfrak{p}(\lambda_n)} A_n)\}}, \qquad y \in N.$$

The corresponding extension of λ to $\mathfrak{h}$ is obtained by setting $\lambda(A) = 0$. Now let $\mathfrak{n} = \mathfrak{n}^0(\lambda) \supset \mathfrak{n}^1(\lambda) \supset \cdots \supset \mathfrak{n}^d(\lambda) = \mathfrak{p}(\lambda)$ be the Kirillov sequence in λ as constructed in the previous section, let $X_k \in \mathfrak{n}^{k-1}(\lambda) \sim \mathfrak{n}^k(\lambda)$, $k = 1, 2, \ldots, d$, and let $\pi = \pi_0$ be the irreducible representation of N corresponding to λ as constructed in Theorem 3.5. Then $\pi_0 = U\sigma U^{-1}$ where $U \colon H(\pi_0) \to L^2(\mathbf{R}^d)$ is defined by

$$Uf(t_1, t_2, \ldots, t_d) = f(\exp t_1 X_1 \cdot \exp t_2 X_2 \cdots \exp t_d X_d).$$

We extend π_0 as indicated above (that is, so as to be isomorphic with the above extension of σ).

Now let $\alpha \in E$ such that $\lambda \in \Omega_\alpha$, and suppose that $\{\lambda_n\}_{n=1}^\infty$ is a sequence in Ω_α such that $\lambda_n \to \lambda$. By Theorem 3.5, we have a corresponding sequence $\{\pi_n\}_{n=1}^\infty$ of irreducible representations of N such that if D is a polynomial differential operator on $\mathbf{R}^d$, then there is $W_n \in U(\mathfrak{n}_c)^m$, $n = 0, 1, 2, \ldots$, for some m, such that $W_n \to W_0$ and $\pi_n(W_n) = D$ for each n. Recall that for each n, we have $X_k^{(n)} \in \mathfrak{n}^{k-1}(\lambda_n) \sim \mathfrak{n}^k(\lambda_n)$, $k = 1, 2, \ldots, d$ such that $\mathrm{ind}(\lambda, \mathfrak{p}(\lambda_n))$ is equivalent to π_n via the isomorphism

$$Uf(t_1, t_2, \ldots, t_d) = f(\exp t_1 X_1^{(n)} \cdot \exp t_2 X_2^{(n)} \cdots \exp t_d X_d^{(n)})$$

and for each k, $X_k^{(n)} \to X_k$. Suppose that we have $A_n \in R(\lambda_n, \mathfrak{g}) \sim \mathfrak{n}$, $n = 1, 2, 3, \ldots$, such that $A_n \to A$, and set $h_n = \mathbf{R}A_n + \mathfrak{n}$, $H_n = \exp h_n$ for each n. Extend π_n to H_n, as above, so that the corresponding extension of λ_n is obtained by setting $\lambda_n(A_n) = 0$. Let the algebra $\mathfrak{D}$ of polynomial differential operators on $\mathbf{R}^d$ have the (obvious) filtration $\mathfrak{D}^{(0)} \subset \mathfrak{D}^{(1)} \subset \mathfrak{D}^{(2)} \subset \cdots$ so that $D \in \mathfrak{D}^{(m)}$ if and only if there is a polynomial P of degree $\leq m$ such that $D = P(t_1, t_2, \ldots, t_d, \partial/\partial t_1, \partial/\partial t_2, \ldots, \partial/\partial t_d)$, and for each m, let $\mathfrak{D}^{(m)}$ have the usual topology as a finite dimensional vector space over $\mathbf{C}$.

LEMMA 4.1. *There is an integer $m > 0$ such that $\pi_0(A) \in \mathfrak{D}^{(m)}$, $\{\pi_n(A_n)\}_{n=0}^\infty \subset \mathfrak{D}^{(m)}$, and $\pi_n(A_n) \to \pi_0(A)$ in $\mathfrak{D}^{(m)}$.*

PROOF. Clearly we may assume that $d > 0$. Let us use the notation $T = (t_1, t_2, \ldots, t_d)$, $U = (u_1, u_2, \ldots, u_p)$ and $Z = (z_{11}, z_{12}, \ldots, z_{ij}, \ldots, z_{pp})$ for elements of $\mathbf{R}^d$, $\mathbf{R}^p$, and $\mathbf{R}^{p^2}$, respectively, and denote the objects associated with π_0 by λ_0, A_0, etc. For each $n \geq 0$, let $\{X_k^{(n)}\}_{k=d+1}^p$ be a basis of $\mathfrak{p}(\lambda_n)$ such that $X_k^{(n)} \to X_k^{(0)}$, $d < k \leq p$, and for each $i, j = 1, 2, \ldots, p$ and $s \in \mathbf{R}$, let $a_{ij}^{(n)}(s)$ denote the coefficient of $X_j^{(n)}$ in the expansion of $e^{s\,\mathrm{ad}A_n}(X_i^{(n)})$ in terms of the ordered basis $X_1^{(n)}, X_2^{(n)}, \ldots, X_p^{(n)}$ of $\mathfrak{g}$. Denote the element $(a_{11}^{(n)}(s), a_{12}^{(n)}(s), \ldots, a_{ij}^{(n)}(s), \ldots, a_{pp}^{(n)}(s))$ of $\mathbf{R}^{p^2}$ by $a^{(n)}(s)$.

By the Campbell-Hausdorff formula, we have for each n, polynomials $P_1^{(n)}, P_2^{(n)}, \ldots, P_p^{(n)}$ in T such that

$$\prod_{j=1}^{p} \exp t_j X_j^{(n)} = \exp\left(\sum_{j=1}^{p} P_j^{(n)}(T) X_j^{(n)}\right).$$

Let $q > 0$ and such that N is step q. Then for each n, j, $\deg(P_j^{(n)}) < q$, and the coefficients of $P_j^{(n)}$ depend only on the structure constants $(b_k^{ij})^{(n)}$, $[X_i^{(n)}, X_j^{(n)}] = \sum (b_k^{ij})^{(n)} X_k^{(n)}$. Clearly for each i, j, k, $(b_k^{ij})^{(n)} \to (b_k^{ij})^{(0)}$ and hence $P_j^{(n)} \to P_j^{(0)}$ in the vector space $\mathbf{C}[T]^{(q)}$, $1 \le j \le p$. Now let the polynomials $\tilde{P}_j^{(n)}$ in T and Z be defined by $\tilde{P}_j^{(n)}(T, Z) = \sum_i P_i^{(n)}(T) z_{ij}$; then we have

$$\exp -sA_n \left[\prod_{j=1}^{d} \exp t_j X_j^{(n)}\right] \exp sA_n = \exp \sum_{j=1}^{p} \tilde{P}_j^{(n)}(T, a^{(n)}(s)) X_j^{(n)},$$
$$s \in \mathbf{R},\ n = 0, 1, 2, 3, \ldots.$$

On the other hand, there are polynomials $R_j^{(n)}$, $1 \le j \le p$, in U such that

$$\exp \sum_{j=1}^{p} u_j X_j^{(n)} = \exp \sum_{j>d} R_j^{(n)}(U) X_j^{(n)} \cdot \prod_{j=1}^{d} \exp R_j^{(n)}(U) X_j^{(n)}.$$

As with $P_j^{(n)}$, we see that for each n, j, $\deg(R_j^n) \le q$, and for each j, $R_j^{(n)} \to R_j^{(0)}$ in $\mathbf{C}[U]^{(q)}$. Now let $Q_j^{(n)} = R_j^{(n)}(\tilde{P}_1^{(n)}, \ldots, \tilde{P}_p^{(n)})$, $1 \le j \le p$. Then $Q_j^{(n)} \to Q_j^{(0)}$ in $\mathbf{C}[T, Z]^{(q^2)}$ and from the definition of π_n we have, for each $\phi \in C^\infty(\pi_n)$,

$$\begin{aligned}(\pi_n(A_n)\phi)(T) = \frac{d}{ds}\bigg|_{s=0} & \exp^{i \sum_{i>d} Q_j^{(n)}(T, a^{(n)}(s)) \lambda_n(X_j^{(n)})} \\ & \cdot \phi\left(Q_1^{(n)}(T, a^{(n)}(s)), Q_2^{(n)}(T, a^{(n)}(s)), \ldots, Q_d^{(n)}(T, a^{(n)}(s))\right) \\ & \cdot \exp^{\{-\frac{1}{2}\operatorname{tr}(\mathrm{ad}_{\mathfrak{n}/\mathfrak{p}(\lambda_n)} A_n)\}}\end{aligned}$$

for each n. Let $\tilde{Q}_j^{(n)}$ be the polynomial in $\mathbf{C}[T]^{(q)}$ such that

$$\frac{d}{ds}\bigg|_{s=0} Q_j^{(n)}(T, a^{(n)}(s)) = \tilde{Q}_j^{(n)}(T), \qquad 1 \le j \le p,\ n \ge 0.$$

Note that $a_{ij}^{(n)}(0) = \delta_{ij}$ for each i, j and n, and for each i, j, $d/ds|_{s=0}\, a_{ij}^{(n)}(s) \to d/ds|_{s=0}\, a_{ij}^{(0)}(s)$, whence $\tilde{Q}_j^{(n)} \to \tilde{Q}_j^{(0)}$. Since for each n,

$$\pi_n(A_n) = i \sum_{j>d} \tilde{Q}_j^{(n)}(T) \lambda_n(X_j^{(n)}) + \sum_{j=1}^{d} \tilde{Q}_j^{(n)}(T) \frac{\partial}{\partial t_j} - \frac{1}{2}\operatorname{tr}(\mathrm{ad} n/P(\lambda_n) A_n)$$

the result follows. Q.E.D.

For each $n > 0$, define A_n^* in $\mathfrak{h}_n^*$ by setting $A_n^*(A_n) = 1$, $A_n^*|_{\mathfrak{n}} \equiv 0$.

COROLLARY 4.2. *There is an integer $m > 0$ and for each $n = 0, 1, 2, \ldots$, there is $W_n \in U((\mathfrak{h}_n)_c)$ such that $\{W_n\}_{n=0}^{\infty} \subset U(\mathfrak{g}_c)^{(m)}$, $W_n \to W_0$ in $U(\mathfrak{g}_c)^{(m)}$, and such that for any real sequence $\{c_n\}_{n=0}^{\infty}$, $(\chi_{c_n} \otimes \pi_n)(W_n) = c_n$ where χ_{c_n} is the character of H_n with differential $ic_n A_n^*$.*

PROOF. By Lemma 4.1, for each n we may write $\pi_n(A_n) = \sum_\mu a_\mu^{(n)} D_\mu$ where $\{D_\mu\}$ is a finite collection of polynomial differential operators and for each μ, $\{a_\mu^{(n)}\}_{n=0}^{\infty}$ is a sequence of complex numbers such that $a_\mu^{(0)} = \lim_n a_\mu^{(n)}$. By Theorem 3.5, for each μ, there is a sequence $\{V_\mu^{(n)}\}_{n=0}^{\infty} \subset U(\mathfrak{n}_c)^{(m_\mu)}$ such that $V_\mu^{(n)} \to V_\mu^{(0)}$ in $U(\mathfrak{n}_c)^{(m_\mu)}$ and such that for each n, $\pi_n(V_\mu^{(n)}) = D_\mu$. Thus $\pi_n(A_n - \sum_\mu a_\mu^{(n)} V_\mu^{(n)}) = 0$, $n = 0, 1, 2, \ldots$, and we may take $m = \max_\mu\{m_\mu\}$ and

$$W_n = -i\left(A_n - \sum_\mu a_\mu^{(n)} V_\mu^{(n)}\right), \qquad n = 0, 1, 2, \ldots. \quad \text{Q.E.D.}$$

Let $K(G)$ be the space of all closed, connected subgroups of G (with the compact-open topology), and let $S(G)$ be the space of all pairs (ρ, H) where $H \in K(G)$ and ρ is an unitary equivalence class of representations of H with the topology of Fell (cf. [**5**]). Let $K_N(G)$ be the set of all $H \in K(G)$ such that $N \subset H$, and $S_N(G)$ the set of all $(\rho, H) \in S(G)$ such that $H \in K_N(G)$ and $\rho \in \hat{H}$. For each $H \in K_N(G)$ we have a topological embedding of $\hat{H}$ in $S_N(G)$. The proof that η_G is continuous (cf. [**10**, Proposition 2]) is easily generalized to show that the mapping $\Theta\colon \mathfrak{g}^* \times K_N(G) \to S_N(G)$ given by

$$\Theta((l, H)) = (\eta_H(\mathrm{Ad}^*(H))(l|_{\mathfrak{h}}), H)$$

is continuous, where $\mathfrak{g}^* \times K_N(G)$ has the product topology. If $(\rho, H) \in S_N(G)$ we denote the $\mathrm{Ad}^*(H)$-orbit $\eta_H^{-1}(\rho)$ by O_ρ, and if $J \subset H$, $\mathfrak{j} = \log(J)$, let $O_\rho|_{\mathfrak{j}} = \{l|_{\mathfrak{j}} \mid l \in O_\rho\}$.

The following two facts are well-known consequences of the general theory (cf. [**4** and **5**]).

LEMMA 4.3. *Let $(\rho, H) \in S_N(G)$ and let $J \in K_N(G)$ be a subgroup of H. Then the set of all $\sigma \in \hat{J}$ such that $O_\sigma \subset O_\rho|_{\log(J)}$ is a dense subset of* $\mathrm{Sp}(\rho|_J)$.

LEMMA 4.4. *Let $(\rho, H) \in S_N(G)$, and let $\{(\rho_n, H_n)\}_{n=1}^{\infty}$ be a sequence in $S_N(G)$ such that $(\rho_n, H_n) \to (\rho, H)$. Let $J \in K_N(G)$ and for each n, $J_n \in K_N(G)$ such that $J \subset H$, $J_n \subset H_n$, and $J_n \to J$. Let $(\sigma, J) \in S_N(G)$ such that $O_\sigma \subset O_\rho|_{\log(J)}$. Then for each n, there is $\sigma_n \in \hat{J}_n$ such that $O_{\sigma_n} \subset O_{\rho_n}|_{\log(J_n)}$ and such that the sequence $\{(\sigma_n, J_n)\}_{n=1}^{\infty}$ converges to (σ, J).*

We define a partition of $S_N(G)$ as follows. For each $(\rho, H) \in S_N(G)$, let $\alpha(\rho)$ be the smallest index in E such that $O_\rho|_{\mathfrak{n}} \cap \Omega_\alpha \neq \{\phi\}$. For each $\alpha \in E$, let

$$\tilde{V}_\alpha = \{(\rho, H) \in S_N(G) | \alpha(\rho) = \alpha\}.$$

From Brown's Theorem [**3**] and Lemma 4.3 above it follows that $(\rho, H) \in \tilde{V}_\alpha$ if and only if $\mathrm{Sp}(\rho|_N) \cap \eta_N(\Omega_\alpha) \neq \{\phi\}$ and $\mathrm{Sp}(\rho|_N) \cap \eta_N(\Omega_\beta) = \{\phi\}$ for all $\beta < \alpha$. For each α, set $V_\alpha = \tilde{V}_\alpha \cap \hat{G}$, and $U_\alpha = \eta_G^{-1}(V_\alpha)$. Then $U_\alpha = \{O \in \mathfrak{g}^*/\mathrm{Ad}^*(G) | \; O|_{\mathfrak{n}} \cap \Omega_\alpha \neq \{\phi\}$ and $O|_{\mathfrak{n}} \cap \Omega_\beta = \{\phi\}$, for all $\beta < \alpha\}$.

LEMMA 4.5. *For each* $\alpha, \tilde{V}_\alpha$ *is open in* $\bigcup_{\beta \geq \alpha} \tilde{V}_\beta$. *If* α_0 *is the smallest element of* E, *then* $\tilde{V}_{\alpha_0}$ *is dense in* $S_N(G)$.

PROOF. Let $(\rho, H) \in \tilde{V}_\alpha$ and suppose that $\{(\rho_n, H_n)\}_{n=1}^{\infty}$ is a sequence in $\bigcap_{\beta \geq \alpha} \tilde{V}_\rho$ such that $(\rho_n, H_n) \to (\rho, H)$. Let $\sigma_0 \in \mathrm{Sp}(\rho|_N)$ such that $O_{\sigma_n} \subset O_\rho|_{\mathfrak{n}} \cap \Omega_\alpha$. By Lemma 4.4, there is $\{\sigma_n\} \subset \hat{N}$ such that $\sigma_n \to \sigma$ and for each n, $O_{\sigma_n} \subset O_{\rho_n}|_N$. By Brown's Theorem, $O_{\sigma_n} \to O_\sigma$. Since $\{O_{\sigma_n}\} \subset \bigcup_{\beta \geq \alpha} \Omega_\alpha$ and Ω_α is open in $\bigcup_{\beta \geq \alpha} \Omega_\beta$, $\{O_{\sigma_n}\}$ is eventually in Ω_α, thus $\{(\rho_n, H_n)\}_{n=1}^{\infty}$ is eventually in $\tilde{V}_\alpha$.

Let α_0 be the minimal element of E and let $(\rho, H) \in S_N(G)$, $\mathfrak{h} = \log(H)$. The set $\{O \in \mathfrak{h}^*/\mathrm{Ad}^*(H) |\ O|_{\mathfrak{n}} \cap \Omega_{\alpha_0} \neq \{\phi\}\}$ is dense in $\mathfrak{h}^*/\mathrm{Ad}^*(H)$, hence the set $\{\rho \in \hat{H} |\ O_\rho|_{\mathfrak{n}} \cap \Omega_\alpha \neq \{\phi\}\}$ is dense in $\hat{H}$ (by continuity of η_H). The embedding of this set in $S_N(G)$ is contained in $\tilde{V}_{\alpha_0}$ and (ρ, H) is contained in its closure. □

Let $(\rho, H) \in S_N(G)$, $\mathfrak{h} = \log(H)$, and let $\lambda \in O_\rho|_{\mathfrak{n}}$. Let $\mathfrak{j} = R(\lambda, h) + \mathfrak{n}$, $J = \exp(\mathfrak{j})$, and let $\sigma \in \hat{J}$ such that $O_\sigma \subset O_\rho|_{\mathfrak{j}}$. Suppose that $\mathfrak{j} \neq \mathfrak{n}$, and let $\{A_1, A_2, \ldots, A_r\} \subset R(\lambda, h)$ be a basis for $\mathfrak{j}$ mod $\mathfrak{n}$. Define $A_j^* \in \mathfrak{h}^*$ by $A_j^*(A_i) = \delta_{ij}$ and $A_j^*|_{\mathfrak{n}} \equiv 0$. Recall then that $\sigma|_N \in \hat{N}$, and that if λ is extended to $\mathfrak{j}^*$ by setting $\lambda(A_j) = 0$, $1 \leq j \leq r$, then there is a unique $t = (t_1, t_2, \ldots, t_r) \in \mathbf{R}^r$ such that $\lambda + \sum_{j=1}^r t_j A_j^* \in O_\sigma$. For any j, $1 \leq j \leq r$, if $\nu = \sigma|_N$ is extended to $H_j = \exp(\mathbf{R}A_i + N)$ by formula (2), and χ_{t_i} is the character of H_j having differential $it_j(A_j^*|_{h_j})$, then $\sigma|_{H_j} = \chi_{t_i} \otimes \nu$.

LEMMA 4.6. *Let* $\alpha \in E$ *and let* $\lambda_n \in \Omega_\alpha$, $n = 0, 1, 2, \ldots$, *such that the sequence* $\{\lambda_n\}_{n=1}^{\infty}$ *converges to* λ_0, *for each* $n \geq 0$, *let* $A_n \in R(\lambda_n, \mathfrak{g}) \sim \mathfrak{n}$, $\mathfrak{h}_n = \mathbf{R}A_n + \mathfrak{n}$, $H_n = \exp(\mathfrak{h}_n)$, *extend* λ_n *to* $\mathfrak{h}_n$ *by setting* $\lambda_n(A_n) = 0$, *define* $A_n^* \in \mathfrak{h}_n^*$ *by* $A_n^*(A_n) = 1$, $A_n^*|_{\mathfrak{n}} \equiv 0$, *let* $t_n \in \mathbf{R}$, *and let* $\rho_n = \eta_{H_n}^{-1} \mathrm{Ad}^*(H_n)(\lambda_n + t_n A_n^*)$. *Assume that* $A_n \to A_0$ *as* $n \to \infty$. *Then* $(\rho_n, H_n) \to (\rho_0, H_0)$ *if and only if* $t_n \to t_0$.

PROOF. We need only prove the "only if" part. Suppose that $(\rho_n, H_n) \to (\rho_0, H_0)$. Let π_0 an irreducible representation corresponding to λ_0 and let $\{\pi_n\}_{n=1}^{\infty}$ a sequence of representations corresponding to $\{\lambda_n\}_{n=1}^{\infty}$ as obtained in Theorem 3.5, so that $\pi_n \in \rho_n|_N$, $n \geq 0$. Extend π_n to H_n as in formula (2) so as to correspond to λ_n, and let χ_n be the character of H_n such that $\gamma_n = \chi_n \otimes \pi_n \in \rho_n$. Then by Corollary 4.2, there is $m > 0$ and $\{W_n\}_{n=0}^{\infty} \subset U(\mathfrak{g}_c)^{(m)}$ such that $W_n \to W_0$ and such that for each n, $W_n \in U((\mathfrak{h}_n)_c)$ and $\gamma_n(W_n) = t_n$. Now the general theory implies that $t_n \to t_0$. To see this, let $\Psi_0 \in C_c^\infty(G)$ and $v_0 \in H(\gamma_0)$ such that $\langle \gamma_0(\Psi_0)v_0, v_0 \rangle \neq 0$. For each n, let Γ_n be the representation of $C_s^*(G)$ lifted from γ_n. Note that any $\Psi \in C_c^\infty(G)$ defines in a natural way an element $\tilde{\Psi}$ in $C_s^*(G)$ by setting $\tilde{\Psi}((K, x)) = \Psi(x)$, $K \in K(G)$, $x \in K$ such that for any $v \in H(\gamma_n)$, $\langle \Gamma_n(\tilde{\Psi})v, v \rangle = \langle \gamma_n(\Psi)v, v \rangle$. Now let $V_1, V_2, \ldots, V_q \in U(\mathfrak{g}_c)$ such that for each n, $W_n = \sum_{j=1}^q a_j^{(n)} V_j$ with $a_j^{(n)} \in C$, $1 \leq j \leq q$, for each j, $a_j^{(0)} = \lim_n a_j^{(n)}$. Set $\Psi_j = V_j \Psi_0$, $1 \leq j \leq q$. Then by Lemma 2.2 of [4], there is, for each $n > 0$, $v_n \in H(\gamma_n)$ such that $\langle \Gamma_n(\tilde{\Psi}_j)v_n, v_n \rangle \to \langle \Gamma_0(\tilde{\Psi}_j)v_0, v_0 \rangle$ as $n \to \infty$, $0 \leq j \leq q$. Thus $\langle \Gamma_n(W_n \Psi_0)v_n, v_n \rangle \to \langle \Gamma_0(W_0 \Psi_0)v_0, v_0 \rangle$ as $n \to \infty$, and we have

$$t_n = -\frac{\langle \gamma_n(W_n \Psi_0)v_n, v_n \rangle}{\langle \gamma_n(\Psi_0)v_n, v_n \rangle} \to -\frac{\langle \gamma_0(W_0 \Psi_0)v_0, v_0 \rangle}{\langle \gamma_0(\Psi_0)v_0, v_0 \rangle} = t_0. \quad \text{Q.E.D.}$$

For each $\alpha \in E$, set $\tilde{U}_\alpha = \Theta^{-1}(\tilde{V}_\alpha)$.

THEOREM 4.7. *$\Theta|_{\tilde{U}_a}: \tilde{U}_\alpha \to \tilde{V}_\alpha$ is open, for each $\alpha \in E$.*

PROOF. Let $(\rho_0, H_0) \in \tilde{V}_\alpha$, and suppose that $\{(\rho_n, H_n)\}_{n=1}^\infty$ is a sequence in $\tilde{V}_\alpha$ which converges to (ρ_0, H_0). Let $\mathfrak{h}_n = \log(H_n)$, $n = 0, 1, 2, \ldots$, and let $l_0 \in \mathfrak{g}^*$ such that $l|_\mathfrak{h} \in O_{\rho_0}$. It is enough to show that there is a subsequence $\{(\rho_k, H_k)\}_{k=1}^\infty$ of $\{(\rho_n, H_n)\}_{n=1}^\infty$ and a corresponding sequence $\{l_k\}_{k=1}^\infty$ in $\mathfrak{g}^*$ such that $l_k|_{\mathfrak{h}_k} \in O_{\rho_k}$ for each k and $l_k \to l_0$. Note that we may assume that $\lambda_0 = l_0|_\mathfrak{n} \in \Omega_\alpha$. Let $\nu \in \hat{N}$ such that $\lambda_0 \in O_\nu$. By Lemma 4.3, there is $\nu_n \in \hat{N}$ such that $O_{\nu_n} \subset O_{\rho_n}|_\mathfrak{n}$, $n = 1, 2, 3, \ldots$, and such that the sequence $\{\nu_n\}_{n=1}^\infty$ converges to ν. Thus we have $\lambda_n \in O_{\nu_n} \subset O_{\rho_n}|_\mathfrak{n}$, $n = 1, 2, 3, \ldots$, such that $\{\lambda_n\}$ converges to λ_0. Now by restriction to a subsequence, we may assume that $\dim(\mathfrak{h}_n) = m$, $n = 0, 1, 2, \ldots$, and since Ω_α is open in $\bigcup_{\beta \geq \alpha} \Omega_\beta$ and $\{\lambda_n\} \in \bigcup_{\beta \geq \alpha} \Omega_\alpha$, we may assume that $\lambda_n \in \Omega_\alpha$ for all n. We proceed by induction on $\dim(\mathfrak{h}_n/\mathfrak{n}) = m - p$.

The case $m-p = 0$ is now trivial due to the above, so assume that $m > p$ and that the theorem is valid for sequences in $S_N(G)$ whose subgroup have dimension less than m. Let $\{\lambda_k\}_{k=1}^\infty$ be a subsequence of $\{\lambda_n\}_{n=1}^\infty$ such that for some subalgebra $\mathfrak{j}_0$, the sequence $\mathfrak{j}_k = R(\lambda_k, \mathfrak{h}_k) + \mathfrak{n}$, $k = 1, 2, 3, \ldots$, converges to $\mathfrak{j}_0$. Let $J_k = \exp(\mathfrak{j}_k)$, $k \geq 0$. By Lemma 4.3, we have $\sigma_k \in \hat{J}_k$, $k = 0, 1, 2, \ldots$, such that $l_0|\mathfrak{j}_0 \in O_{\sigma_0}$, $O_{\sigma_k} \subset O_{\rho_k}|_{\mathfrak{j}_k}$, $k \geq 1$, and the sequence $\{(\sigma_k, J_k)\}_{k=1}^\infty$ converges to (σ_0, J_0). Suppose that $\dim J_0 < m$. By induction there is $l_k \in \mathfrak{g}^*$ such that $l_k|_{\mathfrak{j}_k} \in O_{\sigma_k}$, $k = 1, 2, 3, \ldots$, and such that the sequence $\{l_k\}_{k=1}^\infty$ converges to l_0. Now if $\rho_k: \mathfrak{h}_k^* \to \mathfrak{j}_k^*$ is the restriction mapping, then $\rho_k^{-1}(O_{\sigma_k}) \subset O_{\rho_k}$, $k \geq 1$ (cf. [**1**, Chapter II, §4.2]). Therefore $l_k|_{\mathfrak{h}_k} \in O_{\rho_k}$, and we are done. Hence by induction we have reduced to the case $\mathfrak{j}_k = \mathfrak{h}_k$, $k \geq 0$.

For each k, let $\{A_1^{(k)}, A_2^{(k)}, \ldots, A_r^{(k)}\} \subset R(\lambda_k, \mathfrak{h}_k)$ be a basis for $\mathfrak{h}_k$ mod $\mathfrak{n}$ such that for each $1 \leq j \leq r$, $A_j^{(0)} = \lim_k A_j^{(k)}$. Extending λ_k to $\mathfrak{h}_k$ by setting $\lambda_k(A_j^{(k)}) = 0$, $1 \leq j \leq r$, let $t_1^{(k)}, t_2^{(k)}, \ldots, t_r^{(k)}$ be real numbers such that $\lambda_k + \sum_{j=1}^r t_j^{(k)} A_k^{(k)^*} \in O_{\rho_k}$ (where $A_j^{(k)^*}$ is defined by $A_j^{(k)^*}(A_i^{(k)}) = \delta_{ij}$, $A_j^{(k)^*}|_\mathfrak{n} \equiv 0$). For each j, $i \leq j \leq r$, apply Lemma 4.6 to the sequence $\{(\rho_k|_{\exp(\mathbf{R}A_j^{(k)}+N)}, \exp(\mathbf{R}A_j^{(k)} + N))\}_{k=1}^\infty$ which converges to $(\rho_0|_{\exp(\mathbf{R}A_j^0+N)}, \exp(\mathbf{R}A_j^0 + N))$, and we obtain $t_j^{(0)} = \lim_k t_j^{(k)}$. Since $A_j^{(k)} \to A_j^{(0)}$, $1 \leq j \leq q$, it is clear that we may extend $\lambda_k + \sum_{j=1}^q t_j^{(k)} A_j^{(k)^*}$ to an element $l_k \in \mathfrak{g}^*$ such that the sequence $\{l_k\}_{k=1}^\infty$ converges to l_0. This finishes the proof. Q.E.D.

The following corollary is immediate.

COROLLARY 4.8. *$\eta_G|_{U_a}: U_\alpha \to V_\alpha$ is a homeomorphism, for each α.*

Note that for each $\alpha \in E$, the dimensions of the orbits in U_α may vary, and the relative topology in U_α may not be T_1. Indeed, if N is abelian, then $E = \{\alpha_0\}$ and $U_{\alpha_0} = \mathfrak{g}^*$. Examples indicate that for each $\alpha \in E$, there is a finite partition of U_α, each element of which is T_2. Finally, the subsets U_α may be describable as Zariski-open subsets of algebraic varieties in $\mathfrak{g}^*$.

The author would like to thank his thesis advisor, Professor Richard Penney, for his help and advice. The author's indebtedness to him goes far beyond the present work. Thanks go also to Professor Jeffrey Fox for some instructive and stimulating conversations and to the referee for his attention to this work and his helpful suggestions.

REFERENCES

1. P. Bernat et al., *Représentations des groups de Lie résoluble*, Dunod, Paris, 1972.
2. J. Boidol, **-regularity of exponential Lie groups*, Invent. Math. **56** (1980), 231–238.
3. I. Brown, *Dual topology of a nilpotent Lie group*, Ann. Sci. Ecole Norm. Sup. **6** (1973), 407–411.
4. J. M. G. Fell, *Weak containment and induced representations of groups*, Canad. J. Math. **14** (1964), 237–268.
5. ——, *Weak containment and induced representations of groups*. II, Trans. Amer. Math. Soc. **110** (1964), 424–447.
6. H. Fujiwara, *Sur le dual d'un groupe de Lie résoluble exponential*, J. Math. Soc. Japan **36** (1984), 629–636.
7. K. T. Joy, *A description of the topology on the dual space of a nilpotent Lie group*, Pacific J. Math. **112** (1984), 135–139.
8. A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Russian Math. Surveys **17** (1962), 53–104.
9. N. V. Pedersen, *On the characters of exponential solvable Lie groups*, Ann. Sci. Ecole Norm. Sup. (4) **17** (1984), 1–29.
10. L. Pukanszky, *On the unitary representations of exponential groups*, J. Funct. Anal. **2** (1968), 73–113.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, SAINT LOUIS UNIVERSITY, ST. LOUIS, MISSOURI 63103