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LOCAL H-MAPS OF BU AND APPLICATIONS TO SMOOTHING THEORY

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ABSTRACT. When localized at an odd prime p, the classifying space PL/O for smoothing theory splits as an infinite loop space into the product $C \times N$ where $C = \text{Cokernel}(J)$ and N is the fiber of a p-local H-map $BU \to BU$. This paper studies spaces which arise in this latter fashion, computing the cohomology of their Postnikov towers and relating their k-invariants to properties of the defining self-maps of BU. If Y is a smooth manifold, the set of homotopy classes $[Y, N]$ is a certain subgroup of resmoothings of Y, and the k-invariants of N generate obstructions to computing that subgroup. These obstructions can be directly related to the geometry of Y and frequently vanish.

1. Introduction. When localized at an odd prime p, the classifying spaces for smoothings and for surgery theory can be written as products involving the p-local factors BO, C (the cokernel of J), and the fibers of certain H-maps $BU \to BU$. This paper studies the latter spaces, computing the cohomology of their Postnikov towers and relating the k-invariants to the underlying geometry they classify. In smoothing theory there is an infinite loop space decomposition $PL/O \approx C \times N$ where N is the fiber of a self-map of BU whose homotopy is the p-torsion in the groups of homotopy spheres bounding parallelizable manifolds. We show that for many smooth manifolds Y the resmoothings classified by N can be effectively computed and given a "characteristic variety" sort of description.

Any H-map $f\colon BU \to BU$ is determined up to homotopy by the characteristic sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$, where f_* is multiplication by λ_j on $\Pi_{2j}(BU) = Z_{(p)}$. Because of the infinite loop space decomposition $BU \approx W \times \Omega^2 W \times \cdots \times \Omega^{2p-4} W$ [4, 46] where $\Pi_i(W) = Z_{(p)}$ in dimensions $i = 2j(p-1)$, we may write f as a product of H-maps with corresponding factorization of the fiber

$$F \approx F(0) \times F(2) \times \cdots \times F(2p-4).$$

We can describe the cohomology in terms of a p-local basis for $H^*(BU)$. Let $\{c_{e_1}, c_{e_2}, \ldots\}$ be the usual basis for the primitives of $H^*(BU)$ (that is, c_{e_n} is the nth Newton polynomial in $c_1, c_2, \ldots, c_n$), and define $a^*_{n,j}$ inductively for n prime to p and $j \geq 0$ by

$$c_{e_{np^j}} = (a^*_{n,0})^{p^j} + p(a^*_{n,1})^{p^{j-1}} + \cdots + p^j a^*_{n,j}.$$

For any n prime to p let A^*_n denote the bipolynomial Hopf algebra

$$Z/p[a^*_{n,0}, a^*_{n,1}, \ldots],$$

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and let $E\{\ \}$ denote a primitively generated exterior algebra. In [**31**] it is shown that $H^*(F(2k), Z/p)$ is isomorphic as a Hopf algebra to

$$\bigotimes_{\substack{n \text{ prime to } p \\ n+k=0 \bmod (p-1)}} E\{\sigma a^*_{n,j} \mid 0 \le j < \delta_n\} \otimes A^*_n // \xi^{\delta_n} A^*_n$$

where δ_n is the smallest integer i such that λ_{np^i} is p-divisible by exactly p^i and ξ is the Frobenius map $x \to x^p$.

Using results of W. Singer [**48**], the methods of [**31**] also give the cohomology of the Postnikov tower. If $X = F(2n)$ for some n between 0 and $p-2$ such that $F(2n)$ is nontrivial, its homotopy groups vanish except in dimensions d, $d+r$, $d+2r, \ldots$, where $d = 2p-2n-3$ and $r = 2p-2$. For $m \ge 0$, construct $X[0,m]$ by adding cells of dimension $\ge m+2$ to kill homotopy in dimensions $\ge m+1$, and let $g_j \colon X \to X[0, d+jr]$ and $h_j \colon X[0, d+(j+1)r] \to X[0, d+jr]$ be the usual maps. For any nonnegative integer m let $\operatorname{tr}(m)$ equal the sum of the coefficients in a p-adic decomposition of m.

1.1. THEOREM. *$H^*(X[0, d+jr], Z/p)$ is isomorphic as a Hopf algebra to*

$$\left(\bigotimes_{\substack{k \text{ prime to } p \\ n+k=0 \bmod (p-1)}} E_{k,j} \otimes P_{k,j} // \xi^{\delta_k} P_{k,j} \right) \otimes L_{n,j},$$

where

$$E_{k,j} = E\{\sigma a^*_{k,i} \mid 0 \le i < \delta_k,\ \operatorname{tr}(2kp^i - 1) < (d+jr+1)/2\},$$
$$P_{k,j} = Z/p[a^*_{k,i} \mid \operatorname{tr}(2kp^i - 1) < (d+jr+1)/2],$$

and $L_{n,j}$ is a Hopf algebra which is mapped to 0 by $(g_j)^*$ and $(h_j)^*$.

In $\bmod p$ cohomology we can also get some idea of how X is constructed from Eilenberg-Mac Lane spaces using the k-invariants with the characteristic sequence as the blueprint. Suppose that $\Pi_{d+jr}(X)$ is cyclic of order p^{e_j} and define $\eta_j = (\lambda_{(d+jr+1)/2})/p$. Let $q_j \colon K(\Pi_{d+jr}(X), d+jr) \to X[0, d+jr]$ be the inclusion of the fiber of h_{j-1}. To simplify the description, suppose also that η_0 is a unit in $Z_{(p)}$ (i.e., $\lambda_{(d+1)/2}$ is p-divisible by exactly p) and that $d > 2$.

1.2. THEOREM. *For any $j > 0$ the $\bmod p$ reduction of k^{d+jr+1} restricts nontrivially to the fiber of h_{j-2}. This restriction is given by*

$$q^*_{j-1}(k^{d+jr+1}) = (\eta_j P^1 \beta_{e_{j-1}} - \eta_{j-1} \beta_1 P^1) \iota_{d+(j-1)r},$$

where $\iota_{d+(j-1)r} \in H^*(K(\Pi_{d+(j-1)r}(X), d+(j-1)r), Z/p)$ is the $\bmod p$ reduction of the fundamental class, β_i is the ith-order Bockstein, and P^1 is the Steenrod operation. Furthermore,

$$(\eta_{j+1} P^1 \beta_{e_j} - \eta_j \beta_1 P^1) k^{d+jr+1} = 0.$$

Computation of the $Z_{(p)}$ cohomology of F or its Bockstein spectral sequence is in general very complicated, and more must be known about the characteristic sequence than the values of the δ_n. For example, the characteristic sequence of $\psi^k - 1$ is $(k-1, k^2-1, k^3-1, \ldots)$. If k generates the units in Z/p^2, then the p-divisibility of this sequence is simple enough that we can write down the familiar Bockstein spectral sequence of the fiber J (a space frequently called Image(J)).

1.3. THEOREM. *In the Bockstein spectral sequence for J we have*

$$E_{r+1} \cong \bigotimes_{\substack{n \text{ prime to } p \\ n=0 \bmod (p-1)}} \sigma A^*_{n,r} \otimes A^*_{n,r},$$

where $A^*_{n,r} = Z/p[a^*_{n,r}, a^*_{n,r+1}, \dots]$ and $\sigma A^*_{n,r} = E\{\sigma a^*_{n,r}, \sigma a^*_{n,r+1}, \dots\}$.

Perhaps less familiar is the Bockstein spectral sequence of N, the fiber of $\rho^k \circ (2\psi^2 - \psi^4)\colon BO \to BO$ where ρ^k is the Adams-Bott cannibalistic class. We may regard N as the fiber of a self-map of BU with characteristic sequence $(1, \lambda_2, 1, \lambda_4, \dots)$ where

$$\lambda_{2j} = (-1)^{j-1} 2^{2j}(k^{2j} - 1)(1 - 2^{2j-1})B_j/(2j)$$

and B_j is the jth Bernoulli number. Suppose $n_1, \dots, n_k$ are the positive integers $\leq (p-3)/2$ for which B_n is p-divisible, and let S denote the set of all n prime to p which are congruent to some $n_j \bmod (p-1)/2$. These indices will reflect the contribution of ρ^k to the cohomology of N. We must similarly keep track of the contribution of $(2\psi^2 - \psi^4)$. Let n_p denote the smallest positive integer n such that $2^{2n-1} \equiv 1 \bmod p$ (if one exists), and suppose that $2^{p-1} - 1$ is p-divisible by exactly p^{d_p}. Let T denote the subset of all integers $2n_p$, $2(3n_p - 1)$, $2(5n_p - 2), \dots$ which are prime to p, and for each $n \in T$ suppose that $n - 1$ is p-divisible by p^{j_n}. We can write down the Bockstein spectral sequence of N for any normal p. This is a prime such that the numbers B_n/n, B_{pn}/pn, and $B_n/n - B_{n+(p-1)/2}/(n + (p-1)/2)$ are all nonzero $\bmod p^2$ for $n = n_1, \dots, n_k$, a condition probably satisfied by all odd primes.

1.4. THEOREM. *If p is normal, then for n prime to p the index δ_n of N equals 1 if $n \in S - T$, d_p if $n \in T - S$, $d_p + 1$ if $n \in S \cap T$, and 0 otherwise. The E_r term of the Bockstein spectral sequence of N is the tensor product over all $n \in S \cup T$ of Hopf algebras of the form*

$$E\{\sigma a^*_{n,j} \mid 0 \leq j < \delta_n - r\} \otimes A^*_n // \langle \xi^{\delta_n} A^*_n, (a^*_{n,0})^{\delta_n - r} \rangle$$

if $r < \delta_n$, where $\langle\ \rangle$ means "the Hopf algebra generated by". If $r \geq \delta_n$, the Hopf algebras have the form

$$\begin{aligned} &E\{\sigma^*_{n,0}\} \otimes A^*_n // \langle \xi^{\delta_n} A^*_n, (a^*_{n,0})^p \rangle && \text{if } r < \delta_n + e_n + j_n - 1, \quad \text{and} \\ &A^*_{n,r-\delta_n - j_n - e_n + 2} // \xi^{\delta_n} A^*_{n,r-\delta_n - j_n - e_n + 2} && \text{if } r \geq \delta_n + e_n + j_n - 1. \end{aligned}$$

The assumption of normality ensures that in any subsequence $\lambda_n, \lambda_{pn}, \lambda_{p^2 n}, \dots$ of the characteristic sequence the highest p-divisibility occurs in the first term, and it is this property that makes the Bockstein spectral sequence easily computable.

For any smooth manifold Y the set of homotopy classes $[Y, N]$ is a subgroup of the group of all resmoothings of Y with the k-invariants generating the obstructions to computing that subgroup. These obstructions can be related directly to the geometry of Y and frequently vanish.

1.5. THEOREM. *Suppose p is such that, on any finite skeleton, $\rho^k \circ (2\psi^2 - \psi^4)$ equals the 0 component of an infinite loop map of the spectrum $\dots, BU \times Z_{(p)}, U, BU \times Z_{(p)}, U, \dots$. Then for any $m > 0$ there is a splitting map S in the universal* *coefficient exact sequence*

$$
\begin{aligned}
0 \to \operatorname{Ext}(H_{4m-1}(N[0,4m-2]), \Pi_{4m-1}) \xhookrightarrow{S} H^{4m}(N[0,4m-2], \Pi_{4m-1}) \xrightarrow{\mu} \\
\operatorname{Hom}(H_{4m}(N[0,4m-2]), \Pi_{4m-1}) \to 0
\end{aligned}
$$

(where $\Pi_{4m-1} = \Pi_{4m-1}(N)$) such that $S(k^{4m}) = 0$. Furthermore, $\mu(k^{4m})$ vanishes on any homology class x which is Steenrod representable (i.e. $x = f_*([X])$ where $[X] \in H_{4m}(X)$ is an orientation class of some closed, smooth $4m$-manifold X and $f \colon X \to N[0,4m-2]$).

The theorem actually holds if p satisfies a slightly weaker condition which once again is probably satisfied by all odd primes. To apply this result, recall that given a handle decomposition of Y we can define an exact couple and a spectral sequence converging to $[\Sigma^* Y, N]$. But the Postnikov tower of N also gives us an exact couple whose associated spectral sequence is equivalent to that coming from the handle decomposition. If all of the homology of Y and ΣY is Steenrod representable, for example, then the portion of the E_2 spectral sequence converging to $[Y, N]$ must collapse. There may be extension problems, though, which can be determined explicitly by looking at certain preferred handle decompositions of Y. Given a geometric description of the differentials $d_r^{0,t}$ and $d_r^{-1,t}$, the idea of looking at special handle decompositions can be applied to any smooth Y, and the representable homology accounts for all of $[Y, N]$ in the following way. There is a family of manifolds $\{X_j\}$ and maps $X_j \to Y$ with the top cell of X_j mapped diffeomorphically onto the left-hand disk of some m_j-handle h_j. Then $[Y, N]$ is generated by extensions to Y of smoothings of Y^{m_j} of the form $Y^{m_j} \# \Sigma_j^{m_j}$ (defined in §5) where $\Sigma_j^{m_j}$ is a homotopy m_j-sphere bounding a parallelizable manifold. It is in this sense that we have a "characteristic variety" description of $[Y, N]$.

The paper is organized as follows. §2 reviews results from [**31** and **32**] which are applied in §3 to compute the mod p cohomology and the Bockstein spectral sequences of N and several other spaces of geometric interest. The Postnikov tower of a space X is introduced in §4, and the mod p cohomology of $X[0, m]$ is computed when X is the fiber of an H-map $BU \to BU$. Finally, the exact couples converging to $[\Sigma^* Y, N]$ are discussed in §5, and the proof of 1.5 and a description of the differentials is given. The extension problem as well as the computation of $[Y, N]$ for an arbitrary smooth manifold Y will be considered in a subsequent paper.

This paper presents ideas I have thought about for some time, and I have benefitted from the help of many people. For their support I am grateful to Peter May, Jim Stasheff, Bill Browder, Mike Chisolm, John McCleary, and especially to my wife Anne.

2. Properties of self-maps of BU**.** We begin with a review of some of the definitions and computations of [**31** and **32**] that will be needed later. Assume all spaces are localized at some fixed odd prime p. Then there exist infinite loop space equivalences [**4**, **46**]

$$
\begin{aligned}
&BU \to W \times \Omega^2 W \times \cdots \times \Omega^{2p-4} W \quad \text{and} \\
&BO \to W \times \Omega^4 W \times \cdots \times \Omega^{2p-6} W,
\end{aligned}
$$

where $\Pi_{2k(p-1)}(W) = Z_{(p)}$, $k = 1, 2, 3, \dots$, and $\Pi_i(W) = 0$ otherwise. In particular, $BU \approx BO \times \Omega^2 BO$ and any map $f\colon BO \to BO$ may be regarded as a factor of the map $f \times 1\colon BU \to BU$ with the same fiber, so we concentrate on self-maps of BU.

For any map $f\colon BU \to BU$ we define a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ in $Z_{(p)}$, the characteristic sequence of f, by the condition that f_* equals multiplication by λ_j on $\Pi_{2j}(BU) = Z_{(p)}$. If BU has the p-local H-multiplication μ coming from the Whitney sum, then an H-map is determined up to homotopy by its characteristic sequence [**38**, p. 100]. Given H-maps $f, g\colon BU \to BU$ with characteristic sequences λ and η we may form new H-maps $f+g = \mu\circ(f\times g)\circ\Delta$ and $f\circ g$ whose characteristic sequences are the pointwise sum $\lambda + \eta$ and product $\lambda \cdot \eta$, respectively. The loop space inverse -1 has characteristic sequence $(-1, -1, \dots)$, so that $-f = (-1)\circ f$ has characteristic sequence $-\lambda$.

There is a very useful factorization of the fiber F of any H-map $f\colon BU \to BU$. From the decomposition of BU above there exist maps $p_{2j}\colon BU \to \Omega^{2j}W$ and $q_{2j}\colon \Omega^{2j}W \to BU$ such that the composite $p_{2i}\circ q_{2j}$ is the identity or constant depending on whether or not $i = j$. Let $\tilde{f}_{2k}$ denote the composite $p_{2k}\circ f \circ q_{2k}\colon \Omega^{2k}W \to \Omega^{2k}W$, and write $f_{2k}\colon BU \to BU$ for the unique H-map with characteristic sequence η defined by $\eta_n = \lambda_n$ or 1 depending on whether or not $2k + 2n \equiv 0 \bmod 2(p-1)$. Then $f = f_0 \circ f_2 \circ \cdots \circ f_{2p-4}$ with the fiber factoring as $F = F(0)\times F(2)\times\cdots\times F(2p-4)$ where $F(2k)$ is the fiber of both f_{2k} and $\tilde{f}_{2k}$. We will see in 2.10 below that each of these factors is indecomposable (that is, cannot be written nontrivially as a product).

For any $\beta \in Z_{(p)}$ let $\nu(\beta)$ denote the exponent of p in a prime power decomposition of the numerator of β (let $\nu(0) = \infty$). We say that $\beta_1 \equiv \beta_2 \bmod p^k$ if $\nu(\beta_1 - \beta_2) \geq k$.

2.1. THEOREM [**31**, **19**]. *If $f\colon BU \to BU$ is any map with characteristic sequence λ, then $\lambda_m \equiv \lambda_n \bmod p^{j+1}$ whenever $m \equiv n \bmod p^j(p-1)$.*

One way to prove this congruence is to note that an H-map $BU \to BU$ agrees on finite skeleta with a linear combination of Adams operations, and for these the result is trivially verified. In fact, this approach suggests the following more general congruence, which was proved by F. Clarke in [**19**].

2.2. THEOREM. *If $f\colon BU \to BU$ has characteristic sequence λ, then*

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}\lambda_{n+ik} \equiv 0 \bmod p^{(j+1)m}$$

whenever $p^j(p-1)$ divides k and $n > (j+1)m$.

For any $f\colon BU \to BU$ with characteristic sequence λ define the surplus sequence $s(f)$ by $s(f)_n = \nu(\lambda_n) - \nu(n)$. For each n prime to p, the index $\delta_n(f) = \delta_n$ is the smallest nonnegative integer j such that $s(f)_{np^j} = 0$, and $\delta_n = \infty$ if no such j exists. By 2.1 the index δ_n is constant as n varies over an equivalence class $\bmod(p-1)$, and if for some n the surplus vanishes, the sequence $s(f)_n$, $s(f)_{np}$, $s(f)_{np^2}, \dots$ necessarily equals $0, -1, -2, \dots$ [**31**].

Let R be a commutative, p-local, principal ideal domain (so that $R \cong R \otimes_Z Z_{(p)}$ as a group), and write $c_n \in H^{2n}(BU, R)$ for the class associated to the nth Chern class. Then the cup product and the coproduct induced by the Whitney sum map μ give $h^*(BU, R)$ the following familiar Hopf algebra structure [**34**, **42**].

2.3. THEOREM. *$H^*(BU, R)$ is a polynomial Hopf algebra $R[c_1, c_2, \dots]$ with coproduct $\mu^* c_n = \sum c_i \otimes c_{n-i}$. If $d_n \in H_{2n}(BU, R)$ is dual (in the basis of monomials) to c_1^n, then the correspondence $c_n \to d_n$ defines an isomorphism of Hopf algebras $H^*(BU, R) \to H_*(BU, R)$.*

For any n-tuple $\alpha = (\alpha_1, \alpha_2, \dots)$ of weight $w(\alpha) = \alpha_1 + 2\alpha_2 + \cdots + n\alpha_n$ write c^α for the cup product $c_1^{\alpha_1} c_2^{\alpha_2} \cdots c_n^{\alpha_n} \cdots c_n^{\alpha_n} \in H^{2w(\alpha)}(BU, R)$ with dual class $d_\alpha \in H_{2w(\alpha)}(BU, R)$, and let $d^\alpha = d_1^{\alpha_1} \cdots d_n^{\alpha_n}$ be the Pontrjagin product induced by μ_*. Going full circle, write c_α for the dual of d^α in the monomial basis in $H_*(BU, R)$ (the Chern class c_n is in fact dual to d_1^n and the notation is consistent). By 2.3 it follows that the primitives in $H^*(BU, R)$ and $H_*(BU, R)$ are generated by $\{c_{e_1}, c_{e_2}, \dots\}$ and $\{d_{e_1}, d_{e_2}, \dots\}$ respectively (where $e_n = (0, 0, \dots, 1)$ is the nth unit vector), and $c_{e_n} \to d_{e_n}$ under the isomorphism of 2.3. An explicit formula for the primitives and their relation with the Hurewicz map is also well known [**10**, **12**, **34**]. For any α let

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \quad \text{and} \quad \{\alpha\} = (\alpha_1 + \cdots + \alpha_n)!/\alpha_1! \cdots \alpha_n!.$$

2.4. THEOREM. *$d_{e_n} = \sum_{w(\alpha)=n} (-1)^{|\alpha|+n} n\{\alpha\} d^\alpha / |\alpha|$, with an analogous formula for c_{e_n}. There exists a generator $\omega_n \in \Pi_{2n}(BU) = Z_{(p)}$ which is carried by the Hurewicz homomorphism to $(n-1)!\, d_{e_n}$.*

In this p-local setting it is very fruitful to work inductively using the Witt polynomials. Given indeterminates $t_0, t_1, \dots$, define the kth Witt polynomial T_k by $T_k(t) = t_0^{p^k} + p t_1^{p^{k-1}} + \cdots + p^k t_k$, where we abbreviate $t = (t_0, t_1, \dots)$. Let $t^p = (t_0^p, t_1^p, \dots)$. The basic miracle that makes the Witt polynomials so useful is the following integrality theorem.

2.5. LEMMA [**31**]. *Let $P_0, P_1, \dots$ be polynomials in $t_0, t_1, \dots$ with coefficients in R (respectively Z), such that $P_k(t) - P_{k-1}(t^p)$ vanishes $\operatorname{mod} p^k$ for $k \geq 1$. Then the equations*

$$P_k(t) = T_k(\phi_0(t), \phi_1(t), \dots, \phi_k(t))$$

inductively define polynomials $\phi_0, \phi_1, \dots$ with coefficients in R (respectively, Z).

2.6. THEOREM [**31**]. *For each n prime to p and $k \geq 0$ there exist $a_{n,k} \in H_{2np^k}(BU, R)$ defined inductively by*

$$d_{e_{np^k}} = T_k(a_{n,0}, \dots, a_{n,k}).$$

The set of all such $a_{n,k}$ forms a polynomial basis for $H_*(BU, R)$, and $a_{n,k} = (-1)^{n+1} n d_{np^k} +$ decomposables.

By identical formulas one can define elements of a polynomial basis in cohomology, denoted $a_{n,k}^* \in H^{2np^k}(BU, R)$. If we let A_n^* denote the polynomial Hopf subalgebra $R[a_{n,0}^*, a_{n,1}^*, \dots]$, then we have an isomorphism $H^*(BU) \cong \bigotimes_{n \text{ prime to } p} A_n^*$. One can similarly define a Hopf subalgebra A_n in homology, and the maps $a_{n,k} \to a_{n,k}^*$ give an alternate description of the isomorphism of 2.3 as a tensor product of isomorphisms of bipolynomial Hopf algebras.

If $f\colon BU \to BU$ is an H-map, the characteristic sequence λ gives the action of f on homotopy and hence on primitives by 2.4. The action of f^* is then given inductively by

$$\lambda_{np^k} c_{e_{np^k}} = T_k(f^*(a^*_{n,0}), \dots f^*(a^*_{n,k})).$$

By applying 2.5 [**31**] it follows that $f^*(a^*_{n,k})$ vanishes mod p for $k < \delta_n$, while $f^*(a^*_{n,k}) = u_{n,k}(\alpha^*_{n,k-\delta_n})^{p^{\delta_n}} +$ decomposables, where $u_{n,k}$ is a unit in $Z_{(p)}$ and $k \geq \delta_n$. Using this and one of the beautiful collapse theorems for the Eilenberg-Moore spectral sequence (e.g., [**23**, **44**, **58**, or **27**] we get strong information about the fiber.

2.7. THEOREM [**31**]. *Let $f\colon BU \to BU$ be an H-map with fiber F. Then*

$$H^*(F, Z/p) \cong \bigotimes_{n \text{ prime to } p} E\{\sigma a^*_{n,j} \mid 0 \leq j < \delta_n\} \otimes (A^*_n // \xi^{\delta_n} A^*_n)$$

as Hopf algebras where $E\{\ \}$ denotes an exterior algebra and the generators $\sigma a^*_{n,k}$ are primitive. The induced homomorphism of the inclusion $F \to BU$ is given by the natural projections $A^*_n \to A^*_n // \xi^{\delta_n} A^*$, while $\sigma a^*_{n,j}$ pulls back via the canonical map to the cohomology suspension of $a^*_{n,j}$ in $H^{2np^j-1}(U, Z/p)$. (ξ denotes the Frobenius map $x \to x^p$.)

To compute the action of the Steenrod algebra on F we first use 2.5 to define an integral lift of the Steenrod operations on BU.

2.8. THEOREM [**32**]. *There is a map of polynomial Hopf algebras $P = P^0 + P^1 + \cdots\colon H^*(BU, Z_{(p)}) \to H^*(BU, Z_{(p)})$ defined inductively for n prime to p and $k \geq 0$ by*

$$\sum_{s=0}^{\infty} \binom{np^k}{s} c_{e_{np^k+s(p-1)}} = T_k(Pa^*_{n,0}, \dots, Pa^*_{n,k}).$$

The component homomorphisms $P^s\colon H^m(BU, Z_{(p)}) \to H^{m+2s(p-1)}(BU, Z_{(p)})$ satisfy multiplicative and comultiplicative Cartan formulas, vanish if $2s > m$, and reduce mod p *to the Steenrod operations.*

The recursion above yields a series of explicit, but increasingly unwieldy, formulas for $P^s(a^*_{n,k})$. However, we do easily get some useful information about the action mod decomposables.

2.9. COROLLARY. *Let $f\colon BU \to BU$ be an H-map with fiber F. Suppose that $np^k + s(p-1) = mp^j$ where m, n are prime to p and $j, k, s \geq 0$. Then in $H^*(F, Z/p)$ we have*

$$P^s a^*_{n,k} = p^{j-k}\binom{np^k}{s} a^*_{m,j} + \textit{decomposables}, \quad \textit{and}$$

$$P^s \sigma a^*_{n,k} = p^{j-k}\binom{np^k}{s} \sigma a^*_{m,j} \quad (\textit{where we assume } \sigma a^*_{m,j} = 0 \textit{ if } j \geq \delta_n).$$

The coefficients are integral even if $j - k < 0$.

PROOF. The formula for $P^s a^*_{n,k}$ follows immediately from 2.8 and the definition of $a^*_{n,j}$ in terms of the primitive given in 2.6. The second formula follows from the first (since it also holds in $H^*(BU, Z/p)$), the primitivity of $\sigma a^*_{n,k}$, and the fact that P^s commutes with suspension. $\square$

For any H-map $f\colon BU \to BU$, let $F(2n)$ denote as before the fiber of the induced map $\tilde{f}_{2n}\colon \Omega^{2n}W \to \Omega^{2n}W$. If $0 \le 2n \le 2p-6$, then using the calculation above we can show that $H^*(F(2n), Z/p)$ is indecomposable as a module over the Steenrod algebra $\mathscr{A}(p)$. We say that a set of indecomposable elements $x_1, x_2, \ldots, x_k$ is "connected by P^I" iff $P^I x_j = u_{j+1}x_{j+1}$ mod decomposables for every $j = 1, \ldots, k-1$ where u_{j+1} is a unit in Z/p. For example, for any k prime to p such that $k + n \equiv 0 \bmod (p-1)$ it follows that P^1 connects the sequence $a^*_{k,1}, a^*_{pk+p-1,0}, a^*_{pk+2(p-1),0}, \ldots, a^*_{pk+(p-1)(p-1),0}$, but it extends no further on the right. From the congruence $\binom{pm}{pi} \equiv \binom{m}{i} \bmod p^{j+1}$ whenever p^j divides m [32, 5.2], it follows inductively that P^{p^j} connects the sequence $a^*_{k,j+1}, a^*_{pk+(p-1),j}, a^*_{pk+2(p-1),j}, \ldots, a^*_{pk+(p-1)^2,j}$. Finally, suppose $k = p-1-n$. By assumption $k \ne 1$ so that $\binom{k+p^i-1}{p^i} \not\equiv 0 \bmod p$ for every i, and hence $P^{p^j}P^{p^{j-1}}\cdots P^pP^1$ connects $a^*_{k,0}$ and $a^*_{k+p^{j+1}-1,0}$. It follows that between the indecomposables $a^*_{k_1,i}$ and $a^*_{k_2,j}$, where $k_1 \equiv k_2 \equiv -n \bmod (p-1)$, there is a chain of P^I-connected indecomposables. If $F(2n)$ is homotopy equivalent to a nontrivial product $X \times Y$, then we may choose $x \in H^*(X, Z/p)$ and $y \in H^*(Y, Z/p)$ which pull back to elements $a^*_{k_1,i}$ and $a^*_{k_2,j}$ modulo decomposables. But this gives a contradiction since the operators P^I connecting the latter elements commute with the homomorphism induced by the projection $X \times Y \to X$.

The argument breaks down for $F(2p-4)$. Let M_0 be the Hopf subalgebra of $H^*(F(2p-4), Z/p)$ generated by $\sigma a^*_{1,0}$ and $a^*_{1,0}$, and let M_1 be generated by all $\sigma a^*_{n,j}, a^*_{n,j} \in H^*(F(2p-4), Z/p)$ other than $\sigma a^*_{1,0}$ and $a^*_{1,0}$. If $\delta_1 = 1$, then M_0 and M_1 are modules over the Steenrod algebra and $H^*(F(2p-4), Z/p) \cong M_0 \otimes M_1$ as $\mathscr{A}(p)$ modules. The argument above does show that M_1 is not the cohomology of a product.

Suppose, then, that $F(2p-4) = X \times Y$. Then $\Omega^2_0(F(2p-4) = \Omega^2_0 X \times \Omega^2_0 Y$ is the fiber of $\Omega^2_0 f_{2p-4}$ where the subscript 0 denotes restriction to the component of the basepoint. It follows from the indecomposability of M_1 that one of the factors of $\Omega^2_0 F(2p-4)$, say $\Omega^2_0 X$, must be trivial. Thus $X = K(Z,1)$ or $K(Z/p^j, 1)$. But it is clear that the first case cannot occur by looking at the cohomology of $F(2p-4)$ in dimensions 1 and 2. If $X = K(Z/p^j, 1)$, it follows that $\delta_1 = \infty$. We can then modify the argument above to connect the indecomposables $a^*_{n,j}$ and the decomposable primitives $(a^*_{n,0})^{p^j}$ to show that $F(2p-4)$ is not a product.

2.10. COROLLARY. *If $f\colon BU \to BU$ is an H-map with fiber F, then the factors $F(2n)$ are indecomposable, $k = 0, \ldots, 2p-4$.*

The Bocksteins require a more detailed knowledge of the characteristic sequence than $\delta(f)$. For any chain complex (C, ∂) of abelian groups the long exact sequence in homology associated to the coefficient sequence $0 \to Z \xrightarrow{p} Z \to Z/p \to 0$ may be regarded as an exact couple whose underlying spectral sequence is the Bockstein spectral sequence with E_0 term $H_*(C, Z/p)$. The rth-order Bockstein β_r may be described as follows. Given $\{c\} \in H_*(C, Z/p)$, where c is a chain satisfying $\partial c = p^r c'$, define $\beta_r\{c\} = \{c'\} \in H_*(C, Z/p)$. (This spectral sequence is discussed in detail in [**13** and **14**].)

We will apply this to the chain complex $C = \bigotimes_{n \text{ prime to } p} \sigma A_n^* \otimes A_n^*$ with underlying ring $Z_{(p)}$ and degree 1 differential by $\partial(\sigma a_{n,j}^*) = f^*(a_{n,j}^*)$ where $f\colon BU \to BU$ is an H-map with characteristic sequence λ and fiber F. (σA_n^* is the image $E\{\sigma a_{n,0}^*, \sigma a_{n,1}^*, \ldots\}$ in $H^*(U, Z_{(p)})$ of A_n^* under cohomology suspension.) By [**31**], $H^*(F, Z_{(p)}) \cong H_*(C)$ and $H^*(F, Z/p) \cong H_*(C, Z/p)$. In particular, each of the classes $a_{n,j}^*$ arises in $H^*(F, Z_{(p)})$, and so $\beta_r(a_{n,j}^*) = 0$ for all $r \geq 1$. Understanding the Bocksteins on the elements $\sigma a_{n,j}^*$ reduces to understanding p-divisibility in the image of f^*. This question seems to be very hard in general, but we can give some partial results. Using 2.5 we define polynomials $\rho_{k-\delta_n}(t_0, \ldots, t_{k-\delta_n})$ and $\xi_{k-\delta_n}(t_1, \ldots, t_k)$ for $k \geq \delta_n$ inductively by

$$(p^{\delta_n}/\lambda_{np^k}) T_{k-\delta_n}(t) = T_{k-\delta_n}(\rho_0(t), \ldots, \rho_{k-\delta_n}(t)) \quad \text{and}$$
$$- p^{k-\delta_n+1} T_{\delta_n - 1}(t_{k-\delta_n+1}, \ldots, t_k) = T_{k-\delta_n}(\xi_0(t), \ldots, \xi_{k-\delta_n}(t)).$$

The coefficient $p^{\delta_n}/\lambda_{np^k}$ is a well-defined unit in $Z_{(p)}$ for $k \geq \delta_n$.

2.11. THEOREM [**31**]. *For each n prime to p we have*

$$f^*(a_{n,k}^*) = (\lambda_{np^k}/p^k) T_k(a_{n,0}^*, \ldots, a_{n,k}^*) + px_1 \quad \textit{for } k \leq \delta_n, \textit{ and}$$
$$\begin{aligned}(a_{n,k-\delta_n})^{p^{\delta_n}} &= \rho_{k-\delta_n}(f^*(a_{n,\delta_n}^*), \ldots, f^*(a_{n,k}^*)) \\ &\quad + \xi_{k-\delta_n}(a_{n,1}^*, \ldots, a_{n,k}^*) + px_2 \quad \textit{if } k \geq \delta_n,\end{aligned}$$

where x_i lies in the ideal in A_n^* generated by all $f^*(a_{n,j}^*)$, $j < k$. If f satisfies the additional growth condition $s(f)_{np^k} < ps(f)_{np^{k-1}} - 1$ for $k \leq \delta_n$, then $\nu(f^*(a_{n,k}^*)) = s(f)_{np^k}$ for $k \leq \delta_n$.

The growth condition on the surplus seems to be the best general one available and will suffice for many of our examples. The problem is that below δ_n the p-divisibility of λ_{np^j} can vary quite strangely, the major restriction being the congruences 2.1 and 2.2. In the next section we will look at the Bockstein spectral sequence for a number of H-maps $BU \to BU$ and see that Browder's implication theorems gives some further information about the behavior of λ_{np^k} when $k < \delta_n$.

When $f\colon BU \to BU$ is an infinite loop map, as is the case at odd primes with many important geometric examples, the fiber F becomes an infinite loop space whose homology supports Dyer-Lashof operations. To describe them, we first apply 2.7, universal coefficients, and some simple duality to obtain the following.

2.12. THEOREM. *There is an isomorphism of Hopf algebras*

$$H_*(F, Z/p) \cong \bigotimes_{n \text{ prime to } p} E\{\sigma a_{n,j} \mid 0 \leq j < \delta_n\} \otimes Z/p[a_{n,0}, \ldots, a_{n,\delta_n - 1}],$$

where $\sigma a_{n,j}$ and d_{e_m} are primitive. The inclusion map $F \to BU$ induces the natural inclusion of polynomial Hopf algebras, while $\sigma a_{n,j}$ is the image of an element of $H_{2np^j-1}(U, Z/p)$ which maps to the primitive $d_{e_{np^j}}$ under homology suspension.

The notation $\sigma a_{n,j}$ is a little misleading since these elements lie in the domain of the homology suspension, but they are dual in the appropriate basis to the $\sigma a_{n,j}^*$ with which we will be dealing in the sequel. To compute the Dyer-Lashof operation $Q^r\colon H_m(F,Z/p) \to H_{m+2r(p-1)}(F,Z/p)$ on the fiber of an infinite loop map, we again construct a $Z_{(p)}$ lift as we did in 2.8. Since $Q^r x$ will in general be nonzero for infinitely many r, we work in the ring $H_{**}(BU, Z_{(p)})$ of formal series $x_0 + x_1 + x_2 + \cdots$ where $x_j \in H_{2j}(BU, Z_{(p)})$.

2.13. THEOREM [**32**]. *Let $Q = Q^0 + Q^1 + \cdots\colon H_*(BU, Z_{(p)}) \to H_{**}(BU, Z_{(p)})$ be the ring homomorphism defined inductively for n prime to p and $k \geq 0$ by*

$$\sum_{r=0}^{\infty}(-1)^{n+r}\binom{r-1}{np^k-1} d_{e_{np^k+r(p-1)}} = T_k(Qa_{n,0},\ldots,Qa_{n,k}).$$

The component maps $Q^r\colon H_m(BU, Z_{(p)}) \to H_{m+2r(p-1)}(BU, Z_{(p)})$ satisfy Cartan and co-Cartan formulae, vanish if $m > r$, and reduce modulo p to the Dyer-Lashof maps.

By an argument similar to that in 2.9 we now have the following.

2.14. COROLLARY. *Let $F\colon BU \to BU$ be an infinite loop map with fiber F. Suppose $np^k + r(p-1) = mp^j$ where m, n are prime to p and $r, j, k \geq 0$. Then in $H_*(F, Z/p)$ we have*

$$Q^r\sigma a_{n,k} = (-1)^{n+r}\binom{r-1}{np^k-1}\sigma a_{m,j} \quad \textit{and}$$

$$Q^r a_{n,k} = (-1)^{n+r}p^{j-k}\binom{r-1}{np^k-1}a_{m,j} + \textit{decomposables}.$$

Finally, the following technical result will help determine dimensions in which the Hurewicz map vanishes for various F.

2.15. LEMMA. *Let $f\colon BU \to BU$ be a map with fiber F. The Hurewicz map $\Phi\colon \Pi_{2n-1}(F) \to H_{2n-1}(F)$ sends a generator of $\Pi_{2n-1}(F)$ to $(n-1)!x$ for some element $x \in H_{2n-1}(F)$ of order dividing the order of $\Pi_{2n-1}(F)$.*

PROOF. We may regard F as the total space of a pullback fibration

$$\begin{array}{ccccc} \Omega BU & \longrightarrow & PBU & \xrightarrow{\xi} & BU \\ {\scriptstyle 1}\uparrow & & \uparrow & & \uparrow \\ \Omega BU & \longrightarrow & F & \xrightarrow{\xi'} & BU \end{array}$$

where the top row is the usual (p-local) path fibration. Then we can write

$$H_*(\Omega BU, Z_{(p)}) = E\{x_1, x_2, \ldots\},$$

where the homology suspension $\sigma(x_n)$ of $x_n \in H_{2n-1}(\Omega BU, Z_{(p)})$ is the primitive d_{e_n}. In particular, $d_{e_n} \in \xi_*(H_*(PBU, \Omega BU; Z_{(p)}))$. It follows that $d_{e_n} \in \xi'_*(H_*(F, \Omega BU; Z_{(p)}))$. For we may write $d_{e_n} = \sigma(x_n) = g_*(s(x_n))$ where $g\colon \Sigma\Omega BU \to BU$ is adjoint to 1 and s is the inverse of the Mayer-Vietoris map. But g lifts to a map $\tilde{g}$:

$$(\mathrm{cone}(\Omega BU), \Omega BU) \to (F, \Omega BU),$$

and the class in $H_{2n}(\mathrm{cone}(\Omega BU), \Omega BU; Z_{(p)})$ corresponding to $s(x_n)$ is mapped by $\xi'_* \circ \tilde{g}_*$ to d_{e_n}. Thus d_{e_n} is transgressive in the Serre spectral sequence for ξ'.

The Hurewicz map Φ sends some generator of $\Pi_{2n}(BU) = Z_{(p)}$ to $(n-1)!\, d_{e_n}$ [12]. But in the homotopy exact sequences of ξ and ξ' the boundary maps $\Pi_{2n}(BU) \to \Pi_{2n-1}(\Omega BU)$ are an isomorphism and order $(\Pi_{2n-1}(F))$ times an isomorphism, respectively. Since these boundaries correspond to transgression in homology, it follows first that Φ sends some generator of $\Pi_{2n-1}(\Omega BU)$ to $(n-1)!\, x_n$ and hence that $(n-1)!\, x_n$ transgresses in the Serre spectral sequence for ξ' to

$$u(\mathrm{order}(\Pi_{2n-1}(F))(n-1)!\, x_n$$

for some unit $u \in Z_{(p)}$. But d_{e_n} also transgresses, and hence it does so to $u(\mathrm{order}(\Pi_{2n-1}(F))x_n$. If $i\colon \Omega BU \to F$ is the inclusion, it follows that $i_*(x_n)$ has order dividing $\mathrm{order}(\Pi_{2n-1}(F))$ and the result follows. □

As was the case with the Bockstein spectral sequence, the real difficulty comes in determining $\mathrm{order}(\Pi_{2n-1}(F))$, that is, in knowing the p-divisibility in the characteristic sequence.

3. Examples. The most interesting maps $f\colon BU \to BU$ arise as a reflection of geometric operations on bundles, such as the Adams operations, or as a consequence of several beautiful connections between different types of bundle theories, particularly the Adams conjecture. In this section we discuss some important examples of such maps and the geometric structures which their fibers classify.

3.1. *The image of J.* For any $k \geq 0$ Adams [1] has defined an H-map $\psi^k\colon BU \to BU$ which, in the p-local setting, is an infinite loop map if k is prime to p. These maps arise from the K-theory operations

$$\psi^k(x) = \sum_{w(\alpha)=k} (-1)^{|\alpha|+k}(k/|\alpha|)\{\alpha\} \textstyle\bigwedge^{\alpha}(x)$$

where $x \in K(X)$ for some complex X, $\bigwedge^1, \bigwedge^2, \ldots$ are the exterior power operations, and $\bigwedge^{\alpha}(x) = (\bigwedge^1(x))^{\alpha_1}(\bigwedge^2(x))^{\alpha_2}\ldots(\bigwedge^j(x))^{\alpha_j}$ for $\alpha = (\alpha_1, \ldots, \alpha_j)$. (Compare this with 2.4.) Applying this operation in real K-theory yields Adams maps on BO which are factors of the above maps by the methods of the previous section. The characteristic sequence for $\psi^k\colon BU \to BU$ is $(k, k^2, k^3, \ldots)$ [1]. Thus ψ^k is an equivalence if k is prime to p, and $\delta_n = \infty$ if p divides k.

Define the Image of J to be the fiber of $\psi^k - 1$ where k is some fixed generator of the multiplicative group of units in Z/p^2. For any two such k the fibers are equivalent as infinite loop spaces (see [49] or 3.7 below), so we denote this space simply as J. The integer k has the property that $k^n - 1$ is prime to p unless $n \equiv 0 \bmod (p-1)$, and for any m prime to p, $k^{mp^r(p-1)} - 1$ is p-divisible by precisely p^{r+1} (see, e.g., [2]). Thus $\delta_n = \infty$ or 0 depending on whether or not $n \equiv 0 \bmod (p-1)$. In particular, the factorization of $\psi^k - 1$ in the previous section yields

$$J = J(0) = \mathrm{fiber}((\psi^k - 1)\colon W \to W).$$

Using 2.7 we have the following familiar computation.

3.2. THEOREM.

$$\begin{aligned} H^*(J, Z/p) &\cong \bigotimes_{\substack{n \text{ prime to } p \\ n=0 \bmod (p-1)}} (\sigma A_n^*) \otimes A_n^* \\ &\cong \sigma(H^*(W, Z/p)) \otimes H^*(W, Z/p). \end{aligned}$$

Because of the simplicity (at p) of the characteristic sequence for $\psi^k - 1$ we can write down the p-local cohomology of J as well. Perhaps the easiest way to do this is the following result, a special case of [**31**, 5.5].

3.3. THEOREM. *In the Bockstein spectral sequence for J we have*

$$E_r \cong \bigotimes_{\substack{n \text{ prime to } p \\ n=0 \bmod (p-1),}} \sigma A^*_{n,r} \otimes A^*_{n,r}$$

where $A^*_{n,r} = Z/p[a^*_{n,r}, a^*_{n,r+1}, \dots]$ and $\sigma A^*_{n,r} = E\{\sigma a^*_{n,r}, \sigma a^*_{n,r+1}, \dots\}$.

We need another product on BU in addition to the Whitney sum map μ. Let $\mathscr{B}\mathscr{U}_\otimes = \mathscr{B}\mathscr{U} \times 1 \subseteq \mathscr{B}\mathscr{U} \times Z = \Omega^2\mathscr{B}\mathscr{U}$ be the H-space which classifies the multiplicative units in complex K-theory. (We use script to denote the unlocalized classifying space.) This is well known to be an infinite loop space. The multiplication $\mu^\otimes$ is a natural extension to BU of the product m on CP^∞ which represents the tensor product of the canonical line bundles on $CP^\infty \times CP^\infty$. Let $t\colon BU \times BU \to BU$ be defined using the representing maps for the tensor product of virtual bundles $(\gamma^m - m) \otimes (\gamma^n - n)$, where γ^n is the classifying bundle. Then $\mu^\otimes = \mu \circ (\mu \times t) \circ T \circ (\Delta \times \Delta)$ where Δ is the diagonal and T interchanges the middle two terms of $BU \times BU \times BU \times BU$. Write $*$ and $\#$ for the homology products μ_* and $(\mu^\otimes)_*$, and let $x \circ y = t_*(x \otimes y)$ for $x, y \in H_*(BU)$. If x and y have coproducts $\sum x' \otimes x''$ and $\sum y' \otimes y''$, then $x \# y = \sum (x' \circ y') * x'' * y''$. This is a special case of the distributive law in [**20**, II, 1.5], where by abuse of notation we identify the spaces $BU \times 0$ and $BU \times 1$. With this identification the $*$ polynomial generators d_i span the cohomology of the $\mu^\otimes$ sub-H-space CP^∞. Since m^* induces the additive formal product on $H^*(CP^\infty)$, it follows that

$$m_*(d_i \otimes d_j) = d_i \# d_j = \binom{i+j}{i} d_{i+j}.$$

Applying this and the identity $x \circ 1 = \varepsilon(x)1$ (where ε is the augmentation) to the above distributive law, we can work backwards to compute the operation $t_* = \circ$ explicitly. For example, $d_1 \circ d_1 = 2d_2 - d_1 * d_1$, $d_1 \circ d_2 = 3d_3 - 3d_1 * d_2 + d_1 * d_1 * d_1 * d_1$, and $d_1 \circ d_1 \circ d_1 = 6d_3 - 10d_1 * d_2 + 4d_1 * d_1 * d_1$. Adams has described $\circ$ for general homology theories [**5**, II, 12.4]. In this setting, the formal product m^* and subsequent computations become much more complicated.

We can similarly define the H-space (and again an infinite loop space) $\mathscr{B}\mathscr{O}_\otimes = \mathscr{B}\mathscr{O} \times 1 \subseteq \mathscr{B}\mathscr{O} \times Z = \Omega^8\mathscr{B}\mathscr{O}$. However, explicit computations are much harder since neither the equivalence $BU \approx BO \times \Omega^4 BO$ nor the forgetful map $BU \to BO$ behaves well with respect to the map t and its BO analogue. Fortunately, by the Adams-Priddy Theorem [**6**] $BO_\oplus$ and $BO_\otimes$ are equivalent as infinite loop spaces ($BO \approx BSO$ since we are localized at an odd prime). May [**20**] describes some $\#$ polynomial generators explicitly as $*$ polynomials when $p = 2$. Similarly, the infinite loop spaces $BSU_\oplus$ and $BSU_\otimes$ are equivalent.

3.4. *The cannibalistic class $\rho^k\colon BO_\oplus \to BO_\otimes$.* For $\bigwedge = R$ or C, define ρ^k on oriented bundles ξ by $\rho^k(\xi) = \Phi^{-1}\psi^k\Phi(1) \in K_\Lambda(B)$ where the complex B is the base of ξ and Φ is the Thom isomorphism. This is an exponential operation which extends to virtual bundles if we localize at p prime to k, and we obtain infinite loop maps [**37**] $\rho^k\colon BO_\oplus \to BO_\otimes$ and $\rho^k\colon BU_\oplus \to BU_\otimes$. (See [**2**, **11**, or **38**] for a discussion of ρ^k.) Adams [**2**] computed the characteristic sequence for the cannibalistic class on BO.

3.5. THEOREM. *On* $\Pi_{4j}(BO)$, ρ^k *is multiplication by* $(-1)^{j+1}(k^{2j}-1)B_j/4j$.

Here B_j is the jth Bernoulli number defined by the series

$$z/(e^z-1) = 1 - z/2 + (B_1/2!)z^2 - (B_2/4!)z^4 + (B_3/6!)z^6 - \cdots.$$

For each $j = 4, 8, \ldots, 2p-6$ we also have maps $\rho^k \circ (\psi^k - 1)^{-1}$ on each $\Omega^{4j}W$, and the congruences of 2.1 reduce to the familiar congruences of Kummer:

$$(-1)^m B_m/m \equiv (-1)^n B_n/n \bmod p^{j+1}$$

whenever $m \equiv n \bmod p^j(p-1)/2$ and $m \not\equiv 0 \not\equiv n \bmod (p-1)/2$. Further congruences follow from 2.2.

Suppose as before that k generates the units in Z/p^2, and define $M = \text{fiber}(\rho^k\colon BO_\oplus \to BO_\otimes)$. Up to a unit in $Z_{(p)}$, $(k^{2j}-1)$ equals the denominator of $B_j/4j$ when written as a fraction in lowest terms (see [**2**, §2] or [**42**, Appendix B], and this is p-divisible precisely when $j \equiv 0 \bmod (p-1)/2$. Thus the fiber of the induced map $\rho^k\colon W \to W$ is always trivial, and if p is a regular prime (i.e., p does not divide the numerator of any B_n/n), then $\rho^k\colon BO \to BO$ is an equivalence and M is trivial.

In general, determining the p-divisibility of B_n/n is tedious. For a given p, some global information can be obtained by direct computation of B_n/n for low values of n and then applying the congruences 2.1 or 2.2. We say that an odd prime p is normal if the numbers B_n/n, B_{pn}/pn, and $B_n/n - B_{n+(p-1)/2}/(n+(p-1)/2)$ are nonzero mod p^2 for every index $n = 1, \ldots, (p-3)/2$ for which B_n is p-divisible. It seems likely that every odd prime is normal. All three conditions have been verified by computer for a large number of primes, and the third appears to follow by translating Ferrero and Washington's proof of the vanishing of the Iwasawa invariant [**22**] into the equivalent formulation using Bernoulli numbers [**28**]. For normal primes we can give $H^*(M, Z/p)$ explicitly.

3.6. THEOREM. *Suppose* p *is normal, and let* $n_1, \ldots, n_k$ *be the indices* $\leq (p-3)/2$ *for which* $B_n/n \equiv 0 \bmod p$. *Then*

$$H^*(M, Z/p) \cong \bigotimes_{n\in S} E\{\sigma a^*_{n,0}\} \otimes A^*_n//\xi A^*_n,$$

where $S = \{2n \mid n$ *is prime to* p *and* $n \equiv n_j \bmod (p-1)/2$ *for some* $n_j\}$.

This follows directly from 2.7, since M is the fiber of $(h \circ \rho^k)\colon BO_\oplus \to BO_\oplus$ where $h\colon BO_\otimes \to BO_\oplus$ is an H-equivalence. Composing maps as in §2, M can be written as the fiber of an H-map $BU_\oplus \to BU_\oplus$ with characteristic sequence $(1, \lambda_1, 1, \lambda_2, \ldots)$ where λ_j equals the numerator of B_n/n times some unit in $Z_{(p)}$.

The Steenrod operations P^s on M can be read off directly from 2.8, but the Bocksteins once again require direct numerical computation. For example, $p = 491$ is an irregular prime which is normal, and the indices $n_1, \ldots, n_k$ of 3.6 are 146, 168, and 169. Thus $M = F(396) \times F(308) \times F(304)$ is an infinite loop space where the first nontrivial homotopy group of $F(m)$ appears in dimension $2p - m - 3 = 979 - m$. By 3.6,

$$H^*(M, Z/491) \cong \bigotimes_{n\in S} E\{\sigma a^*_{n,0}\} \otimes A^*_n//\xi A^*_n,$$

where S consists of all n primes to 491 which are congruent to 292, 336, or 338 mod 490. The first Bockstein vanishes on $\sigma a^*_{n,0}$ precisely when n is congruent to 53556, 63868, or 14624 mod $491(491-1)$, the indices for vanishing β_1 in $F(396)$, $F(308)$, and $F(304)$, respectively.

The definitions of J and M required a choice of a generator of the units of Z/p^2. We use the cannibalistic class to show that, up to an infinite loop equivalence, these spaces are independent of this choice.

3.7. LEMMA. *If k, m generate the units in Z/p^2, then there is an infinite loop equivalence $g\colon BO_\oplus \to BO_\oplus$ such that g_* is multiplication by $(k^{2j}-1)/(m^{2j}-1)$ on $\Pi_{4j}(BO)$.*

PROOF. Since $(k^{2j}-1)$ and $(m^{2j}-1)$ are units in $Z_{(p)}$ unless $j \equiv 0 \bmod (p-1)$, $\psi^k - 1$ and $\psi^m - 1$ are infinite loop equivalences on $\Omega^{4j}W$, $j = 1, \ldots, (p-3)/2$. If $h\colon BO_\otimes \to BO_\oplus$ is any infinite loop equivalence, then $h \circ \rho^k$ induces an equivalence $(h \circ \rho^k)_0\colon W \to W$. The desired map g is obtained by composing (as in §2) the maps

$$(\psi^m - 1)_{4j}^{-1} \circ (\psi^k - 1)_{4j}\colon \Omega^{4j}W \to \Omega^{4j}W, \qquad j = 1, \ldots, (p-3)/2,$$

and

$$(h \circ \rho^m)_0^{-1}(h \circ \rho^k)_0\colon W \to W. \quad \square$$

Suppose f_k, f_m represents the pair of maps $(\psi^k - 1)$, $(\psi^m - 1)$ or the pair ρ^k, ρ^m. Looking at characteristic sequences it follows that there is a homotopy commutative diagram

$$\begin{array}{ccccc}
\mathrm{fiber}(f_k) & \longrightarrow & BO & \xrightarrow{f_k} & BO \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle 1} \\
\mathrm{fiber}(f_m) & \longrightarrow & BO & \xrightarrow{f_m} & BO
\end{array}$$

which induces an infinite loop equivalence $\mathrm{fiber}(f_k) \to \mathrm{fiber}(f_m)$.

Atiyah and Segal have given an elementary construction of an exponential equivalence $h_p\colon BO_\oplus \to BO_\otimes$ for each odd prime p. Their work, formulated p-adically in [7], also gives results about exponential maps on BSO at 2 and on BU. Let $\gamma^k\colon KO(X) \to KO(X)$, $k = 0, 1, \ldots,$ be the natural operations which on bundles satisfy $\gamma^k(x) = \Lambda^k(x + k - 1)$. Define the formal power series valued operations Λ_t, $\gamma_t\colon KO(X) \to KO(X)[[t]]$ by $\Lambda_t(x) = \sum_{k=0}^\infty \Lambda^k(x)t^k$ and $\gamma_t(x) = \sum_{k=0}^\infty \gamma^k(x)t^k$. These are related by $\gamma_t = \Lambda_{t/(1-t)}$. For each $t \in Z$ we obtain exponential operations $\gamma_t\colon KO(X) \to 1 + KO(X)$ for finite-dimensional complexes X, and consequently a unique H-map $\gamma_t\colon BO_\oplus \to BO_\otimes$ (localized at an odd prime, as usual). Then for each $j = 0, \ldots, (p-3)/2$ there exists a positive integer $t(j)$ such that the induced map $(\gamma_{t(j)})_{4j}\colon \Omega^{4j}W \to \Omega^{4j}W$ is an equivalence (see [**36**, 9.14]). The desired equivalence is the composite $h_p = (\gamma_{t(0)})_0 \circ \cdots \circ (\gamma_{t((p-3)/2)})_{(2p-6)}$.

These maps are also of interest because of the congruences among Stirling numbers that can be obtained by applying 2.1 or 2.2 to their characteristic sequences. On $\Pi_{4j}(BO)$, $(\gamma_t)_*$ is multiplication by $\sum_{i=1}^{2j}(-1)^{i-1}(i-1)!\,S(2j,i)t^i$ where $S(a,b)$

denotes a Stirling number of the second kind. These are defined explicitly by

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^k\binom{k}{j}(k-j)^n,$$

and recursively by $S(1,1) = 1$, $S(1,k) = 0$ if $k > 1$, and $S(n+1,k) = S(n,k-1) + kS(n,k)$. See **[9]**, for a further discussion of these numbers.

3.8. *Smoothing theory.* As a final example we study a factor of the classifying space for smoothings. Let BPL and BG denote the (localized) classifying spaces for stable PL bundles and stable spherical fiber spaces. Denote the fibers of the forgetful classifying maps $BO \to BG$, $BPL \to BG$, and $BO \to BPL$ by G/O, G/PL, and PL/O, respectively. The Adams conjecture **[47, 54]** implies that for k prime to p the composite $BO \xrightarrow{\psi^k - 1} BO \to BG$ is null homotopic. Thus $\psi^k - 1$ lifts to a map $\gamma\colon BO \to B/O$, and if k generates the units in Z/p^2 we define N to be the fiber of the composite $\tau^k\colon BO \to G/O \to G/PL$. We thus have a homotopy commutative diagram

$$\begin{tikzcd}
N \arrow[r] \arrow[d, "\beta"] & BO \arrow[d, "\gamma"] \arrow[dr, "\tau^k"] & \\
PL/O \arrow[r] & G/O \arrow[r] & G/PL
\end{tikzcd}$$

in which β is part of a factorization of PL/O: If C is the p-local cokernel of J (see e.g. **[38]**), then there is a map $\alpha\colon C \to PL/O$ such that $N \times C \xrightarrow{\beta\times\alpha} PL/O \times PL/O \xrightarrow{\text{mult}} PL/O$ is an infinite loop space equivalence.

In his thesis Sullivan defined a homotopy equivalence (in fact, an infinite loop space equivalence) $\sigma\colon G/PL \to BO_\otimes$. Here it is essential that we are localized at on *odd* prime. By studying the cannibalistic class he computed the characteristic sequence of the composite $\theta^k = (\sigma)^{-1} \circ \tau^k\colon BO_\oplus \to BO_\otimes$.

3.9. THEOREM **[53]**. *On* $\Pi_{4j}(BO)$, θ^k *is multiplication by*

$$(-1)^{j-1}2^{2j}(k^{2j}-1)(1-2^{2j-1})B_j/(2j).$$

In particular, by comparing characteristic sequences we have that θ^k is homotopic to $\rho^k \circ (2\psi^2 - \psi^4)$. By 3.7, N is independent of the choice of k. Letting L denote the fiber of $2\psi^2 - \psi^4$ we obtain a commutative diagram of fibrations:

(3.10)
$$\begin{tikzcd}
L \arrow[rr, bend left] \arrow[dr] & & BO_\oplus \arrow[rr, bend left, "\rho^k\circ(2\psi^2-\psi^4)"] \arrow[dr, "(2\psi^2-\psi^4)"] & & BO_\otimes \\
& N \arrow[ur] \arrow[dr] & & BO_\oplus \arrow[ur, "\rho^k"'] & \\
& & M \arrow[ur] & &
\end{tikzcd}$$

We used the Whitney sum multiplication to define the map $2\psi^2 - \psi^4$. Using instead the product $\mu^{\otimes}$ we obtain an H-map $(\psi^2)^2/\psi^4 \colon BO_{\otimes} \to BO_{\otimes}$ with fiber $L_{\otimes}$ which is infinite loop space equivalent to L. The composites $\rho^k \circ (2\psi^2 - \psi^4)$ and $((\psi^2)^2/\psi^4) \circ \rho^k$ have the same characteristic sequence and are homotopic, and we obtain a second commutative diagram of fibrations:

$$
\begin{tikzcd}
M \arrow[rr, bend left] \arrow[dr] & & BO_{\oplus} \arrow[dr, "\rho^k"] \arrow[rr, bend left, "((\psi^2)^2/\psi^4)\circ\rho^k"] & & BO_{\otimes} \\
& N \arrow[ur] \arrow[dr] & & BO_{\otimes} \arrow[ur, "(\psi^2)^2/\psi^4"'] & \\
& & L_{\otimes} \arrow[ur] & &
\end{tikzcd}
\tag{3.11}
$$

Suppose first that p is a regular prime. Thus ρ^k is an equivalence and N may be regarded as the fiber of the composite $-(1/2)(\psi^2)^{-1} \circ (2\psi^2 - \psi^4) \colon BO_{\oplus} \to BO_{\oplus}$ with BO characteristic sequence $(2-1,\ 2^3-1,\ 2^5-1, \ldots)$. N is trivial unless $2^{2n-1} \equiv 1 \bmod p$ for some n, and we let $2n_p - 1$ denote the smallest such odd exponent, if one exists. (We always have that $2^{k(p-1)} \equiv 1 \bmod p$, but frequently these are the only exponents for which this congruence holds.) The usual number-theoretic terminology is that "2 belongs to $2n_p - 1 \bmod p$." Let $d_p = v(2^{2n_p-1} - 1)$, the exponent of p in a prime power decomposition. By some simple number theory it follows that $2^m \equiv 1 \bmod p$ iff $2n_p - 1$ divides m, and if $v(m) = j$ then $v(2^m - 1) = j + d_p$. Let T denote the subset of all of the integers $2n_p$, $2(3n_p - 1)$, $2(5n_p - 2), \ldots$ which are prime to p. These are the indices m for which $m - 1$ is an odd multiple of $2n_p - 1$.

3.12. THEOREM. *If p is regular and N is nontrivial, then*

$$H^*(N, Z/p) \cong \otimes_{n \in T} E\{\sigma a^*_{n,j} | 0 \le j < d_p\} \otimes A^*_n // \xi^{d_p} A^*_n.$$

This follows immediately from 2.6 and the simple p-divisibility properties of the characteristic sequence for $-(1/2)(\psi^2)^{-1}(2\psi^2 - \psi^4)$. The even indices, as in $\sigma a^*_{2n,j}$, reflect the fact that N is being regarded as the fiber of a map $BU \to BU$ with characteristic sequence $(1,\ 2-1,\ 1,\ 2^3-1,\ 1, \ldots)$. Most of the time $v(2^{p-1} - 1) = 1$, the only exceptions among primes ≤ 200183 being 1093 and 3511 [**45**]. When $p = 1093$, $v(2^{1092} - 1) = 2$ but N is trivial. At 3511, N is nontrivial because $v(2^{1755} - 1) = 2$ and also because 3511 is irregular: It divides B_{708} and B_{862}.

We can also give a very simple description of the Bockstein spectral sequence of N when p is regular, the principal task being to keep track of dimensions where higher-dimensional torsion appears. For each $n \in T$, let j_n denote the largest nonnegative integer such that $n \equiv 1 \bmod p^{j_n}$. Then $\Pi_{2n-1}(N)$ is cyclic of order $p^{d_p + j_n}$, and all of the higher (than p^{d_p}) torsion in $\Pi_*(N)$ occurs in dimensions $2n-1$, $n \in T$, with $j_n > 1$. The higher torsion in cohomology is more plentiful.

3.13. THEOREM. *Suppose p is regular and N is nontrivial. Then in the Bockstein spectral sequence for N we have*

$$E_r \cong \bigotimes_{n\in T} E\{\sigma a^*_{n,0},\dots,\sigma a^*_{n,k_p-r-1}\} \otimes A^*_n//\langle \xi^{d_p}A^*_n,(a^*_{n,0})^{p^{d_p-r}}\rangle$$

if $r < d_p$ where $\langle\ \rangle$ denotes "the Hopf subalgebra generated by". If $r \geq d_p$ we have $E_r = \bigotimes_{n\in T} H(n,r,d_p,j_n)$ where $H(n,r,s,t)$ is the Hopf algebra defined as follows:

$$H(n,r,s,t) = E\{\sigma a^*_{n,0}\} \otimes A^*_n//\langle \xi^s A^*_n,(a^*_{n,0})^p\rangle \quad \textit{if } r < s+t, \textit{ and}$$
$$H(n,r,s,t) = A^*_{n,r-s-t+1}//\xi^s A^*_{n,r-s-t+1} \quad \textit{if } r \geq s+t.$$

The proof is a fairly straightforward but tedious application of 2.11 and is omitted. It is interesting to compare this result with 3.3. In the characteristic sequence for J the subsequences λ_n, λ_{pn}, $\lambda_{p^2n},\dots$ are either prime to p or (up to units in $Z_{(p)}$) equal to p, p^2, $p^3,\dots$. The corresponding subsequences for N are of the form $p^{d_p+j_n}$, $p^{d_p}, p^{d_p},\dots$. It is precisely this lack of p growth that causes the exterior part to disappear first in the Bockstein spectral sequence of $H^*(N,Z/p)$.

It is also interesting to consider 3.13 from the viewpoint of Browder's implication theorems (e.g., [**13, 14**]). The even-dimensional primitives in $H^*(N,Z/p)$ have as a Z/p basis the elements $(a^*_{n,0})^{p_j}$ where $n \in I$ and $j \leq d_n$. All of these are hit by Bocksteins ($\beta_j(\sigma a^*_{n,j}) = (a^*_{n,0})^{p_j}$) and so have infinite implications. This cannot be seen by looking only at the cohomology, as all these elements lie in a truncated polynomial algebra. The infinite implications are detected by the homology elements $a_{n,j}$, $n \in I$ and $j \leq d_p$, which lie in polynomial algebras (Theorem 2.11). Notice also that when $j_n > 0$ the subalgebra $E\{\sigma a^*_{n,0},\dots,\sigma a_{n,d_p-1}\} \otimes A^*_n//\xi^{d_p}A^*_n$ and its associated Bockstein spectral sequence cannot be realized as the cohomology of an H-space. The Bocksteins $\beta_{d_p},\dots,\beta_{d_p+j_n-1}$ vanish identically, while higher Bocksteins are nonzero. This would contradict 4.6 of [**14**]. We saw in 2.11 that $N(0)$, $N(4),\dots,N(2p-6)$ were the indecomposable pieces of N by looking at the Steenrod operations.

For an arbitrary prime p the cohomology and Bockstein spectral sequence of N depend first on numerical calculations. Most of the time, however, the fibrations $N \to M$ and $N \to L_\otimes$ are equivalent to the projection maps $L \times M \to M$ and $M \times L_\otimes \to L_\otimes$, respectively, because the maps $(2\psi^2-\psi^4)^\sim_{2j}$ and $\rho^{\sim k}_{2j}$ induced by $2\psi^2-\psi^4$ and ρ^k are rarely both nontrivial. For example, $(2\psi^2-\psi^4)^\sim_{452}$ and $\rho^{\sim k}_{452}$ are both nontrivial when $p = 631$, and this is the only example among primes < 8000.

Whether or not the maps $N \to M$ and $N \to L_\otimes$ are projections, if p is normal we end up with results very similar to those for regular primes. Suppose $f\colon BU \to BU$ is any H-map with characteristic sequence λ. Then by 2.2 we have congruences $\lambda_n - 2\lambda_{n+p^j(p-1)} + \lambda_{n+2p^j(p-1)} \equiv 0 \bmod p^{j+1}$, $n \geq 2(j+1)$. It follows inductively that $\lambda_{n+kp^j(p-1)} \equiv \lambda_n + k(\lambda_{n+p^j(p-1)} - \lambda_n) \bmod p^{j+1}$. Suppose now that f is the cannibalistic class ρ^k. Because p is normal, it follows that $\lambda_{n+p^j(p-1)} \not\equiv \lambda_n \bmod p^{j+1}$. In particular, if p is irregular, then for any $k > 0$ there exist infinitely many m such that $\lambda_m \equiv 0 \bmod p^k$. When p is normal this higher p-divisibility must occur at the start of a subsequence $\lambda_n, \lambda_{pn}, \lambda_{p^2n},\dots$ where n is prime to p. For such primes, we can give the mod p cohomology of N and the Bockstein spectral sequence of M and N explicitly. Let S and T be the index sets of 3.6 and 3.12, respectively, and write $S - T$ for the subset of elements of S which are not in T.

3.14. THEOREM. *Let p be normal and $n \in S \cup T$. Then $\delta_n = 1$ if $n \in S - T$, $\delta_n = d_p$ if $n \in T - S$, $\delta_n = d_p + 1$ for $n \in S \cap T$, and we have*

$$H^*(N, Z/p) \cong \bigotimes_{n \in S \cup T} E\{\sigma a^*_{n,j} \mid 0 \leq j < \delta_n\} \otimes A^*_n // \xi^{\delta_n} A^*_n.$$

Suppose p is a normal prime, and let λ be the characteristic sequence for an H-map having either M or N as fiber. Then the highest p-divisibility in any subsequence $\lambda, \lambda_{pn}, \lambda_{p^2 n}, \ldots$ must occur in λ_n, and we obtain Bockstein spectral sequences similar to 3.13. Recall from that theorem that $j_n = \nu(n-1)$ when $n \in T$. We set $j_n = 0$ if $n \notin T$, and let $e_n = \nu(B_n/n)$.

3.15. THEOREM. *Suppose p is normal. Then in the Bockstein spectral sequence for M, if $r \geq 1$ we have*

$$E_r \cong \bigotimes_{n \in S} H(n, r, 1, e_n - 1),$$

where H is the Hopf algebra defined in 3.13.

3.16. THEOREM. *Suppose p is normal. Then the E_r term of the Bockstein spectral sequence for N is the tensor product over all $n \in S \cup T$ of the Hopf algebras*

$$E\{\sigma a^*_{n,j} \mid 0 \leq j < \delta_n - r\} \otimes A^*_n // \langle \xi^{\delta_n} A^*_n, (a^*_{n,0})^{p^{\delta_n - r}} \rangle$$

if $r < \delta_n$, where δ_n is given in 3.14, *and the Hopf algebras*

$$H(n, r, \delta_n, j_n + e_n - 1)$$

if $r \geq \delta_n$ and H is as in 3.13.

4. The Postnikov tower. Suppose X is a connected CW complex which is simple (that is, $\Pi_1(X)$ is abelian and acts trivially on higher homotopy groups). For any integer $m \geq 0$ we denote by $X[0, m]$ a space obtained from X by attaching cells of dimension $\geq m + 2$ to kill homotopy groups in dimension $\geq m + 1$. There exist maps $X[0, m+1] \to X[0, m]$ which, like the natural inclusion $X \to X[0, m]$, induce isomorphisms on homotopy groups through dimension m. These maps form a homotopy commutative diagram

(4.1)
$$\begin{tikzcd}
& \vdots \arrow[d] & \\
& X[0,m+1] \arrow[r, "k^{m+3}"] \arrow[d, "\alpha_m"] & K(\Pi_{m+2}(X), m+3) \\
X \arrow[ur] \arrow[r] \arrow[ddr] & X[0,m] \arrow[r, "k^{m+2}"] \arrow[d, "\alpha_{m-1}"] & K(\Pi_{m+1}(X), m+2) \\
& \vdots \arrow[d] & \\
& X[0,0] = * \arrow[r, "k^2"] & K(\Pi_1(X), 2)
\end{tikzcd}$$

where $X[0,m+1] \to X[0,m]$ is a CW approximation of the pullback via k^{m+2} of the path fibration over $K(\Pi_{m+1}(X), m+2)$. The maps $X[0,m+1] \to X[0,m]$ are essentially unique, and the above diagram and the k-invariants k^m are natural. (See, e.g., [**43**, **50**, or **57**] for a general discussion.)

Suppose that $f\colon BU \to BU$ is an H-map. We saw in §2 that f is equivalent to a product of uniquely determined H-maps $\tilde{f}_{2n}\colon \Omega^{2n}W \to \Omega^{2n}W$, $n = 0, \dots, p-2$. Choose some index n such that the fiber $F(2n)$ of $\tilde{f}_{2n}$ is nontrivial. If $d = 2(p-1)-2n-1$ and $r = 2(p-1)$, then the nontrivial homotopy groups of $F(2n)$ occur in dimensions d, $d+r$, $d+2r, \dots$. If we abbreviate $X = F(2n)$ and $\Pi_j = \Pi_j(X)$, then diagram (4.1) above becomes

(4.2)
$$\begin{tikzcd}
& \vdots \arrow[d] & \\
& X[0, d+jr+r] \arrow[r, "k^{d+(j+2)r+1}"] \arrow[d, "h_j"] & K(\Pi_{d+jr+2r}, d+jr+2r+1) \\
X \arrow[ur, "g_{j+1}"] \arrow[r, "g_j"] \arrow[ddr, "g_0"'] & X[0, d+jr] \arrow[r, "k^{d+(j+1)r+1}"] \arrow[d, "h_{j-1}"] & K(\Pi_{d+jr+r}, d+jr+r+1) \\
& \vdots \arrow[d] & \\
& X[0,d] \arrow[r, "k^{d+r+1}"] \arrow[d] & K(\Pi_{d+r}, d+r+1) \\
& * &
\end{tikzcd}$$

where Π_{d+jr} is a cyclic group of order $p^{v(\lambda_{(d+jr+1)/2)})}$ and $\Pi_k = 0$ otherwise. We will study the cohomology of diagram (4.2).

As a first step we compute the mod p cohomology of the space $X[0, d + jr]$. This is a Hopf algebra since the Whitney sum multiplication μ on X induces one on $X[0, d + jr]$. For any nonnegative integer m define its p-trace by $\operatorname{tr}(m) = a_0 + a_1 + \cdots + a_k$, where m can be written $m = a_0 + a_1 p + \cdots + a_k p^k$ with $0 \leq a_i < p$.

4.3. THEOREM. *$H^*(X[0, d+jr], Z/p)$ is isomorphic as a Hopf algebra to*

$$\left(\bigotimes_{\substack{k \text{ prime to } p \\ n+k=0 \bmod (p-1)}} E_{k,j} \otimes P_{k,j} // \xi^{\delta_k} P_{k,j}) \otimes L_{n,j} \right),$$

where

$$E_{k,j} = E\{\sigma a^*_{k,i} \mid 0 \leq i < \delta_k,\ \operatorname{tr}(2kp^i - 1) < (d + jr + 1)/2\},$$
$$P_{k,j} = Z/p[a^*_{k,j} \mid \operatorname{tr}(2kp^i - 1) < (d + jr + 1)/2],$$

and $L_{k,j}$ is the Hopf algebra described below. The maps $g^*_j \colon H^*(X[0, d+jr], Z/p) \to H^*(X, Z/p)$ and $h^*_j \colon H^*(X[0, d + jr], Z/p) \to H^*(X[0, d + jr + r], Z/p)$ are the natural inclusions on $E_{k,j} \otimes P_{k,j} // \xi^{\delta_k} P_{k,j}$. Both g^*_j and h^*_j vanish on $L_{n,j}$.

To describe $L_{n,j}$ we note first that the Postnikov tower of the factor $\Omega^{2n}W$ of BU yields a diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & \Omega^{2n}W[0, d + jr + r + 1] \arrow[r] \arrow[d] & K(Z_{(p)}, d + jr + 2r + 2) \\
\Omega^{2n}W \arrow[ur] \arrow[r] & \Omega^{2n}W[0, d + jr + 1] \arrow[r] \arrow[d] & K(Z_{(p)}, d + jr + r + 2) \\
 & \vdots &
\end{tikzcd} \tag{4.4}$$

Any H-map $f_{2n} \colon \Omega^{2n}W \to \Omega^{2n}W$ induces a self-map of this diagram in which the individual maps are H-maps. In particular, the loop of the resulting map on $K(Z_{(p)}, d + jr + 2)$ induces a homomorphism ϕ on $H^*(K(Z_{(p)}, d + jr + 1), Z/p)$ which we can then regard as a module over itself via the operation $x * y = \phi(x)y$. Let F_m denote the primitively generated Hopf subalgebra of $H^*(K(Z_{(p)}, m), Z/p)$ which is generated over the Steenrod algebra $\mathscr{A}(p)$ by the single element $\beta_1 P^1 \iota_m$. Define $L_{n,j} = \bigoplus \operatorname{Tor}_{F_{d+jr+1}}(Z/p, F_{d+jr+1})$, where $\bigoplus$ Tor denotes the graded object associated to the bigraded Tor and F_{d+jr+1} has the module structure derived from ϕ as above. $L_{n,j}$ inherits the structure of a Hopf algebra from the multiplication on $X[0, d + jr]$.

One can calculate $L_{n,j}$ just as in [**31**] using the general Koszul complex for a free associative algebra. Although the generating set of F_{d+jr+1} may be complicated, all the generators vanish under the map induced by $\tilde{f}_{2n}$ so that Tor is in principle computable. For example, $L_{n,j}$ is 0 or Z/p and consists entirely of primitives (whatever the coproduct may be) at least through dimension $d+(j+2)r+2$, a fact that will be helpful in 4.7 below.

PROOF OF 4.3. In [**48**] W. Singer studied the cohomology of the connective covers and Postnikov towers of U and BU, and in particular obtained the following:

$$H^*(BU[0,2m],Z/p) \cong Z/p[x_{2j} \mid \operatorname{tr}(2j-1)<m] \otimes \left(\bigotimes_{i=0}^{p-2} F_{2m-2i}\right)$$

as algebras where the elements x_{2j} are certain inductively defined indecomposables in $H^*(BU)$. His proof respects the decomposition $BU \approx W \times \Omega^2 W \times \cdots \times \Omega^{2p-4}W$, and the generators x_{2j} may in fact be chosen to be the elements $a^*_{k,j}$ with

$$\operatorname{tr}(2kp^j-1)<m$$

which pull back in the obvious way under the maps $BU \to BU[0,m]$, $BU[0,m+1] \to BU[0,m]$. This choice also allows us to keep track of coproducts, and his theorem sharpens as follows: $H^*(\Omega^{2n}W[0,d+jr+1],Z/p)$ is isomorphic as a Hopf algebra to

$$\bigotimes_{\substack{k \text{ prime to } p\\ n+k=0 \bmod (p-1)}} Z/p[a^*_{k,j} \mid \operatorname{tr}(2kp^j-1)<(d+jr+1)/2] \otimes F_{d+jr+1}.$$

The H-map $\tilde{f}_{2n}$ induces a self-map $\tilde{f}_{2n}(j)$ of $\Omega^{2n}W[0,d+jr+1]$ with fiber $X[0,d+jr]$. We can then apply the methods of [**31**] to the Eilenberg-Moore spectral sequence of the fibration $\tilde{f}_{2n}(j)$. The E_2 term is

$$\operatorname{Tor}_H(Z/p,H) \otimes \operatorname{Tor}_{F_{d+jr+1}}(Z/p,F_{d+jr+1}),$$

where H denotes $H^*(\Omega^{2n}W[0,d+jr+1],Z/p)$ regarded as a module over itself with multiplication defined by $(\tilde{f}_{2n}(j))^*$ as before. ($\operatorname{Tor}_H(Z/p,H)$ may be computed just as in [**31**] using 2.11, the calculation of H described above, and the Koszul resolution.) Then applying the collapse theorem of Gugenheim and May [**23**, Theorem A] for the Eilenberg-Moore spectral sequence and applying some standard Hopf algebra arguments, we have the desired computation of $H^*(X[0,d+jr],Z/p)$. The behavior of the maps g^*_j and h^*_j on $E_{k,j}$ and $P_{k,j}//\xi^{\delta^k}P_{k,j}$ follows directly from the behavior of the corresponding maps in the Postnikov tower of BU. It also follows from Singer's computations that F_{d+jr+1} pulls back to 0 in

$$H^*(\Omega^{2n}W[0,d+jr+r+1],Z/p),$$

and hence $h^*_j(L_{k,j})=0$. □

We can see fairly directly some of the first contributions of k^{d+jr+1} to the cohomology of X by looking at the cohomology sequence of the $d+jr-1$ connected pair $(X[0,d+jr-r],\ X[0,d+jr])$. Let Y denote the quotient space $X[0,d+jr-r]/X[0,d+jr]$. It follows by homotopy excision (e.g., [**50**, 9.3.6]) that Y is a $K(\Pi_{d+jr},d+jr+1)$ through at least dimension $2d+jr-2$. If

$g\colon Y \to K(\Pi_{d+jr}, d+jr+1)$ represents the cohomology class corresponding to the identity map of Π_{d+jr} under universal coefficients, then the composite

$$h\colon X[0, d+jr-r] \xrightarrow{\text{collapse}} Y \xrightarrow{g} K(\Pi_{d+jr}, d+jr+1)$$

is the k-invariant k^{d+jr+1}. Substituting this into the cohomology sequence of the pair $(X[0,d+jr-r], X[0,d+jr])$, we obtain the exact sequence

$$\begin{aligned}(4.5)\qquad &\cdots \xrightarrow{h_{j-1}^*} H^{m-1}(X[0,d+jr],G) \to H^m(K(\Pi_{d+jr}, d+jr+1),G) \\ &\xrightarrow{(k^{d+jr+1})^*} H^m(X[0,d+jr-r],G) \xrightarrow{h_j^*} H^m(X[0,d+jr],G) \to \cdots\end{aligned}$$

for any abelian group G and $m \leq 2d+jr-2$, and we have an analogous sequence with X replacing $X[0,d+jr]$. Note that the composite

$$\begin{aligned}\Pi_{d+jr} = \Pi_{d+jr+1}(K(\Pi_{d+jr}, d+jr+1)) &\xrightarrow{\text{Hur}} H_{d+jr+1}(K(\Pi_{d+jr}, d+jr+1)) \\ \xleftarrow{g_*} H_{d+jr+1}(Y) &\xleftarrow{\text{excis.}} H_{d+jr+1}(X[0,d+jr-r],X) \to H_{d+jr}(X)\end{aligned}$$

is the Hurewicz map for X. If this vanishes, it follows easily from (4.5) that for any finitely generated abelian group there is a split short exact sequence

$$\begin{aligned}(4.6)\qquad &0 \to H^m(K(\Pi_{d+jr}, d+jr+1),G) \xrightarrow{(k^{d+jr+1})^*} H^m(X[0,d+jr-r],G) \\ &\xrightarrow{h_{j-1}^*} H^m(X[0,d+jr],G) \to 0\end{aligned}$$

with a corresponding sequence in homology. In the dimension range $m \leq 2d+jr-2$ the only nonzero cohomology of $K(\Pi_{d+jr}, d+jr+1)$ occurs in dimensions $d+jr+1$ and $d+jr+2$. Setting $G = Z/p$, it follows that $L_{n,j}$ vanishes below dimension $2d+jr-1$, while $(k^{d+jr+1})^*$ is killing $L_{n,j-1}$ in dimensions $d+jr+1$ and $d+jr+2$. Even with a nonzero Hurewicz map, the sequence (4.5) gives some low-dimensional information about $L_{n,j}$ and $L_{n,j-1}$.

We can use these sequences to give a direct description of the connection between the k-invariants and the characteristic sequence. Since $X = F(2n)$ is the fiber of the H-map $\tilde{f}\colon \Omega^{2n}W \to \Omega^{2n}W$, it is completely determined by the characteristic sequence elements $\lambda_{(d+1)/2}, \lambda_{(d+r+1)/2}, \lambda_{(d+2r+1)/2}, \ldots$, where we set $d = 2(p-1)-2n-1$ and $r = 2(p-1)$ as before. If $x = \lambda_{(d+jr+1)/2}$, then define $e_j = v(x), \eta_j = x/p$, and $u_j = xp^{-e_j}$ (the unit part of x). Let $q_j\colon K(\Pi_{d+jr}, d+jr) \to X[0,d+jr]$ denote the inclusion of the fiber of h_{j-1}. To stay with the dimension range of the above sequences we must also assume that $d > 2$.

4.7. THEOREM. *For any $j > 0$ the image*

$$(h_{j-1})^*(H^*(X[0,d+((j-1)r, Z/p] \subseteq H^*(X[0,d+jr],Z/p)$$

maps monomorphically via g_j^* to $H^*(X,Z/p)$. The mod p *reduction of k^{d+jr+1} restricts nontrivially to the cohomology of the fiber of h_{j-2}. If η_0 is a unit this restriction can be written*

$$q_{j-1}^*(k^{d+jr+1}) = (\eta_j P^1 \beta_{e_{j-1}} - \eta_{j-1}\beta_1 P^1)\iota_{d+(j-1)r}$$

where $\iota_{d+(j-1)r} \in H^{d+(j-1)r}(K(\Pi_{d+(j-1)r}, d+(j-1)r), Z/p)$ *is the* $\bmod p$ *reduction of the fundamental class, and*

$$(\eta_{j+1}P^1\beta_{e_j} - \eta_j\beta_1P^1)k^{d+jr+1} = 0.$$

As will be clear from the proof, even when η_0 is not a unit there is an explicit, but messier, expression for the pullback of k^{d+jr+1} and a relation that it satisfies, but this time involving the η_j and the units u_j as well as higher-order Bocksteins. As we saw earlier, for the spaces L, M, and N of geometric interest the coefficient η_0 is nearly always a unit. Similarly, the assumption that $d > 2$ is not a restriction for these spaces since $\Pi_i(N) = 0$ for $i < 7$.

PROOF. The first part of the theorem is an immediate consequence of 4.3. For the second part we proceed by induction on j. To begin, note that $X[0,d] = K(\Pi_d, d) = K(Z/p^{e_0}, d)$. The fundamental class ι_d clearly pulls back nontrivially to $H^*(X, Z/p)$, so we may assume that $g_0^*\iota_d = \sigma a^*_{(d+1)/2,0}$. The first few generators of the Hopf algebra $H^*(K(\Pi_d, d), Z/p)$ are ι_d, $\beta_{e_0}\iota_d$, $P^1\iota_d$, $\beta_1P^1\iota_d$, and $P^1\beta_{e_0}\iota_d$, and from the computations in §2 it follows that $(\eta_1P^1\beta_{e_0} - u_0\beta_1P^1)\iota_d \neq 0$ maps to 0 under g_0^*. But since k^{d+r+1} generates $\ker(g_0^*)$ by (4.5) we may, after multiplying by a unit, assume that $k^{d+r+1} = (\eta_1P^1\beta_{e_0} - u_0\beta_1P^1)\iota_d$. (By abuse of notation, we let k^{d+r+1} denote both the k-invariant and its image in $\bmod p$ cohomology.) It follows from the Adem relation $P^1P^1 = 2P^2$ and from $P^1\beta_{e_1}P^1\beta_{e_0}\iota_d = \beta_1P^2\beta_{e_0}\iota_d$ that $P^1\beta_{e_1}k^{d+r+1} = \eta_1\beta_1P^2\beta_{e_0}\iota_d$ and

$$\beta_1P^1k^{d+r+1} = (2\eta_1\beta_1P^2\beta_{e_0} - u_0\beta_1P^2\beta_1)\iota_d.$$

Then by checking separately the cases $e_0 = 1$ and $e_0 > 1$ and applying the congruence $\eta_0 - 2\eta_1 + \eta_2 \equiv 0 \bmod p$, a special case of 2.2, it follows that

$$(\eta_2P^1\beta_{e_1} - \eta_1\beta_1P^1)k^{d+r+1} = 0.$$

Consider now the special case when η_0 is a unit and assume inductively that $(\eta_{j+1}P^1\beta_{e_j} - \eta_j\beta_1P^1)k^{d+jr+1} = 0$ and that

$$q^*_{j-1}(k^{d+jr+1}) = \eta_jP^1\beta_{e_{j-1}}1 - \eta_{j-1}\beta_1P^1)\iota_{d+(j-1)r}.$$

In the Serre spectral sequence of $h_{j-1}\colon X[0, d+jr] \to X[0, d+(j-1)r]$, the fundamental class $\iota_{d+jr} \in H^{d+jr}(K(\Pi_{d+jr}, d+jr), Z/p)$ transgresses to the $\bmod p$ image of k^{d+jr+1}. Then since P^1 and β_s commute up to sign with transgression, it follows from the inductive assumptions that $(\eta_{j+1}P^1\beta_{e_j} - \eta_j\beta_1P^1)\iota_{d+jr}$ transgresses to 0. We will check that this element is nonzero and is the pullback of $k^{d+(j+1)r+1}$.

From the congruences $\eta_s - 2\eta_{s+1} + \eta_{s+2} \equiv 0 \bmod p$ it follows inductively that $\eta_s \equiv \eta_0 + s(\eta_1 - \eta_0)$. If $\eta_1 \equiv \eta_0 \bmod p$ then every η_s is a unit in $Z_{(p)}$ since η_0 is, and if $\eta_s \not\equiv \eta_0$ then η_{j+1} and η_j cannot both vanish $\bmod p$. Thus

$$x = (\eta_{j+1}P^1\beta_{e_j} - \eta_j\beta_1P^1)\iota_{d+jr+1} \neq 0.$$

Since $q^*_{j-1} \circ \tau$ sends $P^1\beta_{e_j}\iota_{d+jr}$ to $\eta_j\beta_1P^2\beta_{e_{j-1}}\iota_{d+(j-1)r}$ and sends $\beta_1P^1\iota_{d+jr}$ to $\eta_{j+2}\beta_1P^2\beta_{e_{j-1}}i_{d+(j-1)r}$ it also follows that x generates the kernel of the transgression in dimension $d + (j+1)r$. By looking at the Koszul complex it is clear that $L_{n,j} \cong Z/p$ in dimension $d + (j+1)r + 1$, and the cohomology class of x in $H^*(X[0, d+jr], Z/p)$ generates the primitives in $\ker g_j^*$. We may thus suppose that this class equals $k^{d+(j+1)r+1}$, and it has the desired pullback. From the congruence $\eta_{j+2} - 2\eta_{j+1} + \eta_j \equiv 0 \bmod p$ it follows as before that $q_j^* y = 0$ where

$$y = (\eta_{j+2} P^1 \beta_{e_{j+1}} - \eta_{j+1} \beta_1 P^1) k^{d+(j+1)r+1}.$$

But both $\eta_{j+2} P^1 \beta_{e_{j+1}} k^{d+(j+1)r+1}$ and $\eta_{j+1} \beta_1 P^1 k^{d+r+1}$ are primitive and restrict to 0 in $H^*(X[0, d + (j+1)r + 1], Z/p)$. Hence one is a multiple of the other, and $y = 0$. □

The proof for arbitrary η_0 is in principle the same except that 2.2 no longer gives such a nice description of the coefficients. If $e_0 > 1$, then η_1 and η_2 might both vanish $\bmod p$ (e.g., for the map $p^2 I$). If p^s is the highest power of p dividing both η_1 and η_2, then $p^{-s}(\eta_2 P^1 \beta_{e_1} - \eta_1 \beta_1 P^1) k^{d+r+1} = 0$. Applying the arguments above to the Serre spectral sequence of h_0, it follows that k^{d+2r+1} is the cohomology class of $p^{-s}(\eta_0 P^1 \beta_{e_1} - \eta_1 \beta_1 P^1)\iota_{d+r}$. If η_1 vanishes $\bmod p$ then $(2P^1\beta_{e_2} - \beta_1 P^1) k^{d+r+1} = 0$, and if $\eta_1 \not\equiv 0$ then $p^{-s}((2\eta_2 - \eta_1) P^1 \beta_{e_2} - \eta_2 \beta_1 P^1) k^{d+2r+1} = 0$. This can then be applied to determine k^{d+3r+1}. But the calculations are already very messy, and for want of a clearer approach to the general result the details are omitted. The identity $(\eta_{j+1} P^1 \beta_{e_j} - \beta_1 P^1) k^{k+jr+1} = 0$ is still valid, but it might be of no use in identifying $k^{d+(j+1)r+1}$. We did see in the previous section that with the spaces J, L, M, and N the coefficient η_0 is usually a unit, and in those cases where this is not so the p-divisibility in the characteristic sequence still permits a description of the k-invariants similar to that in 4.7.

Finally, it is interesting to note that the congruences 2.2 are closely related to the Adem relations. For example, $\eta_{j+2} - 2\eta_{j+1} + \eta_j \equiv 0 \bmod p$ follows from applying the Adem relation $P^1 \beta_1 P^1 = \beta_1 P^2 + P^2 \beta_1$ to $\sigma a^*_{m,k}$ (where $j = mp^k$) and comparing coefficients.

5. The geometry of the Postnikov tower. A number of H-maps $f\colon BU \to BU$ are of interest for the geometric structures they classify. For the fiber F of such a map the Postnikov decomposition can frequently be used both to compute the group of homotopy classes $[Y, F]$ for a suitable space Y, and to describe explicitly the geometry of an element of $[Y, F]$. In this section we apply the Postnikov tower in this way to the spaces L, M, and N which arise in smoothing theory.

Let X be a connected, simple, CW complex with Postnikov tower as in (4.1). Applying the usual fiber mapping sequence to the fibration $j_{t-1}\colon X[0,t] \to X[0, t-1]$, we obtain a sequence of maps

$$\cdots \xrightarrow{\Omega k^{t+1}} K(\Pi_t, t) \to X[0,t] \xrightarrow{j_{t-1}} X[0,t-1] \xrightarrow{k^{t+1}} K(\Pi_t, t+1),$$

where $\Pi_t = \Pi_t(X)$. If X is an infinite loop space this sequence may be extended indefinitely to the right. Applying the homotopy class functor $[Y, \ ]$ we obtain a long exact sequence of abelian groups for each t and can collect these into an exact couple

$$\begin{tikzcd} D_2 \arrow[rr, "j"] & & D_2 \arrow[dl, "k"] \\ & E_2 \arrow[ul] & \end{tikzcd}$$

with $D_2^{-s,t} = [Y, \Omega^s X[0, t-s]]$ and $E_2^{-s,t} = [Y, K(\Pi_{t-s}, t-2s)]$ and converging to $[Y, \Omega^* X]$.

Suppose now that Y is a smooth or PL manifold with handle decomposition $\phi \subseteq Y^0 \subseteq Y^1 \subseteq \cdots \subseteq Y^m = Y$. Then the Puppe exact sequences

$$\cdots \to [\Sigma^{s+1}Y^{t-2s-1}, X] \to [\Sigma^s Y^{t-2s}/\Sigma^s Y^{t-2s-1}, X] \to [\Sigma^s Y^{t-2s}, X] \to \cdots$$

can also be collected into an exact couple (the Puppe couple) with

$$E_1^{-s,t} = [\Sigma^s Y^{t-2s}/\Sigma^s Y^{t-2s-1}, X],$$

differentials d_r of bidegree $(1, r-2)$ as before, and

$$E_\infty^{-s,t} = \frac{\ker([\Sigma^s Y, X] \to [\Sigma^s Y^{t-2s-1}, X])}{\ker([\Sigma^s Y, X] \to [\Sigma^s Y^{t-2s}, X])}.$$

With an infinite loop space structure on X we may suppose that $[\Sigma^s Y^{t-2s}, X]$ is defined for any integer s. For $E_1^{-s,t}$ this may be interpreted geometrically since Y^{t-2s}/Y^{t-2s-1} is homotopy equivalent to a wedge of spheres and can be desuspended. The wonderful fact about these two constructions is that the spectral sequences have isomorphic E_2 terms. For $E_1^{-s,t} \cong C^{t-2s}(Y, \Pi_{t-s}(X))$, the handle-based $t-2s$ cochains of Y, and $d_1^{-s,t}$ equals the usual differential. Thus $E_2^{-s,t} = H^{t-2s}(Y, \Pi_{t-s}(X)) = [Y, K(\Pi_{t-s}(X), t-2s)]$, and the higher differentials agree. (See **[43]** for details.)

When X is one of the spaces L, M, or N the computation of $[Y, X]$ using the Postnikov tower of X can often be reduced to questions about the geometry of the manifold Y. A key ingredient in this reduction is the following.

5.1. THEOREM. *Let X be the fiber of an H-map $h\colon BU \to BU$ which, on any finite skeleton, equals the* 0 *component of an infinite loop map of the spectrum for $BU \times Z_{(p)}$. Then for $m > 0$ there is a splitting map S for the universal coefficient exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}(H_{2m-1}(X[0,2m-2]), \Pi_{2m-1}) \arrow[r] & H^{2m}(X[0,2m-2], \Pi_{2m-1}) \arrow[l, bend right, "S"'] \arrow[r, "\mu"] & \mathrm{Hom}(H_{2m}(X[0,2m-2], \Pi_{2m-1}) \arrow[r] & 0 \end{tikzcd}$$

(*where* $\Pi_{2m-1} = \Pi_{2m-1}(X)$) *such that* $S(k^{2m}) = 0$.

PROOF. The result is trivially true if $2m < 2p+2$; the k-invariants must vanish in that range since the fiber of any map $\Omega^{2n}W \to \Omega^{2n}W$ has at most one nonzero homotopy group below dimension $2p+1$. Thus let $m \geq 2p+2$ and suppose for the moment that the Hurewicz map $\Pi_{2m-1} \to H_{2m-1}(X)$ vanishes. Since k^{2m} is the image under

$$H^{2m}(X[0,2m-2], X; \Pi_{2m-1}) \to H^{2m}(X[0,2m-2], \Pi_{2m-1})$$

of the class corresponding to the identity map of Π_{2m-1} via the isomorphisms

$$\begin{aligned} H^{2m}(X[0,2m-2], X; \Pi_{2m-1} &\to \mathrm{Hom}(H_{2m}(X[0,2m-2], X), \Pi_{2m-1}) \\ &\to \mathrm{Hom}(\Pi_{2m-1}, \Pi_{2m-1}), \end{aligned}$$

it is represented by the cochain sending every $2m$-cell of X to 0 and each $2m$-cell of $X[0,2m-2] - X$ to the homotopy class of its attaching map. Since Π_{2m-1} is cyclic, we may suppose that $X[0, 2m-2] - X$ has a single $2m$-cell whose attaching map is homologically trivial because the Hurewicz map vanishes. Thus this cell plus some combination of $2m$-cells from X form a cycle, and $\mu(k^{2m})$ is a surjection. (In general, it maps onto the kernel of the Hurewicz map.)

Following [**35**, p. 77], a splitting map S may be constructed as follows. On the cellular chain level we first choose a right inverse $s\colon B_{2m-1}(X[0, 2m-2] \to C_{2m}(X[0, 2m-2])$ of the differential. Since the Hurewicz map vanishes, s can be chosen with all of its values in $C_{2n}(X)$. The corresponding splitting map S is the homomorphism assigning to a $2m$-cocycle $\phi\colon C_{2m}(X[0, 2m-2]) \to \Pi_{2m-1}$ the extension induced from

$$0 \to B_{2m-1}(X[0, 2m-2] \to Z_{2m-1}(X[0, 2m-2]) \to H_{2m-1}(X[0, 2m-2]) \to 0$$

by the composite $\phi \circ s\colon B_{2m-1}(X[0, 2m-2]) \to \Pi_{2m-1}$. But k^{2m} is represented by a cochain vanishing on the cells of X, so $s(k^{2m}) = 0$.

Suppose that $h\colon BU \to BU$ is any H-map as in the statement of the theorem. Then for fixed m there is a map $g\colon BU \to BU$ which is the 0 component of an infinite loop map of the spectrum

$$\ldots,\ Z_{(p)} \times BU,\ U,\ Z_{(p)} \times BU,\ U, \ldots$$

such that the characteristic sequences of h and g agree through the mth term. Choose an integer i such that $\nu((m+i-1)!) > \nu(\lambda_m)$, where λ is the characteristic sequence of h. By assumption g is the 0 component of $\Omega^i g'$ for some map $g'\colon BU \to BU$ whose characteristic sequence has $(m+i)$th entry equal to λ_m. Thus by 2.15 the Hurewicz map of the fiber Y of g' vanishes in dimension $2(m+i)-1$. By the work above, the universal coefficient sequence of Y has a splitting map

$$\begin{aligned} S\colon & H^{2(m+i)}(Y[0, 2(m+i-1)], \Pi_{2m-1}) \\ & \to \operatorname{Ext}(H_{2(m+i)-1}(Y[0, 2(m+i-1)], \Pi_{2m-1})) \end{aligned}$$

such that $s(k_Y^{2(m+i)}) = 0$, where $\Pi_{2m-1} = \Pi_{2m-1}(X) = \Pi_{2(m+i)-1}(Y)$. Since the cohomology suspension sends k-invariants to k-invariants, the result is a consequence of the following.

5.2. LEMMA. *Let T be a complex such that $\Pi_{n+2}(T) = 0$. Suppose Π is an abelian group, and let $S\colon H^{n+1}(T, \Pi) \to \operatorname{Ext}(H_n(T), \Pi)$ be a splitting for the universal coefficient sequence of T. Then there exists a splitting map*

$$S^{\Omega}\colon H^n(\Omega T, \Pi) \to \operatorname{Ext}(H_{n-1}(\Omega T), \Pi)$$

such that $S^{\Omega} \circ \sigma^* = -(\sigma_*)^* \circ S$, where σ_* and σ^* are the homology and cohomology suspensions, respectively.

We use the letter T as a reminder that in our application T is a term in a Postnikov tower, and the vanishing homotopy group follows from that.

PROOF. Let $s_{n-1}^{\Omega}\colon B_{n-1}(\Omega T) \to \Delta_n(\Omega T)$ and $s_n\colon B_n(T) \to \Delta_{n+1}(T)$ be right inverses for the singular chains of ΩT and T, respectively, and suppose that s_n arises by evaluation from a right inverse of the boundary map for $\Delta_{n+1}(PT)/\Delta_{n+1}(\Omega T)$

where PT is the space of paths starting at the basepoint of T. Consider diagram (5.3)

$$\begin{tikzcd}
 & \Delta_{n+1}(PT) \arrow[r, "q"] \arrow[d, "\partial"'] & \Delta_{n+1}(PT)/\Delta_{n+1}(\Omega T) \arrow[r, "\phi"] \arrow[d, "\partial"] \arrow[ddr, "\partial"'] & \Pi \\
\Delta_n(\Omega T) \arrow[r, "i"] \arrow[dd, "\partial"'] & \Delta_n(PT) \arrow[r, "q"] \arrow[u, bend right, "D"'] \arrow[dd, "\partial"'] & \Delta_n(PT)/\Delta_n(\Omega T) & \\
 & & & B_n(\Delta_n(PT)/\Delta_n(\Omega T)) \arrow[ul, "\subseteq"] \arrow[uul, bend right, "s_n"'] \\
B_{n-1}(\Omega T) \arrow[r, "i"] \arrow[uu, bend right, "s^{\Omega}_{n-1}"'] & B_{n-1}(PT) \arrow[uu, bend right, "D"'] & & \\
 & & \mathrm{Im}(i) \arrow[ul, "\subseteq"] \arrow[uur, "qD"'] &
\end{tikzcd}$$

$$\phi\colon \Delta_{n+1}(PT)/\Delta_{n+1}(\Omega T) \to \Pi$$

in which D is a chain null homotopy for the contractible space PT and is a cocycle. Both q and i commute with the boundary, but no general claim is being made about the commutativity of the diagram. We must show that $S^{\Omega} \circ \sigma^* = -(\sigma_*)^* \circ S$; that is, verify that $S^{\Omega} \circ \delta^{-1} \circ e^* = -(e_* \circ \partial^{-1})^* \circ S$ where $e\colon (PT, \Omega T) \to (Z, *)$ is the evaluation map and δ, ∂ are the cohomology and homology boundary maps of the pair $(PT, \Omega T)$. If S also denotes a splitting for the universal coefficient sequences of $(PT, \Omega T)$ with coefficients in Π arising, as described below, from right-hand inverses s_n of the boundary map on singular chains, then by our assumptions about s_n it follows that $(e_* \circ \partial^{-1})^* \circ S = (\partial^{-1})^* \circ S \circ e^*$.

To simplify notation, let Δ_n, Z_n, and B_n denote the chains, cycles, and boundaries of the pair $(PT, \Omega T)$, and write $\Delta_n^P, \Delta_n^{\Omega}$ (and so on) for the chains of PT and ΩT, respectively. Then $S(\{\phi\})$ is the extension $0 \to \Pi \to (\Pi \oplus Z_n)/A \to H_n \to 0$ where A is the subgroup generated by all elements of the form $(\phi \circ s_n(b), -i(b))$ for $b \in B_n$. If $[g, z_n]$ denotes a typical coset in the middle group of the sequence, then $(\partial^{-1})^* \circ S\{\phi\}$ is the extension

$$0 \to \Pi \to \{([g, z_n], h^{\Omega}_{n-1}) \mid \partial\{z_n\} = h^{\Omega}_{n-1}\} \to H^{\Omega}_{n-1} \to 0.$$

To describe $S^{\Omega} \circ \delta^{-1}\{\phi\}$, note first that $\delta^{-1}\{\phi\}$ is the cohomology class of the cocycle

$$\phi'\colon \Delta^{\Omega}_n \to \Delta^P_n \xrightarrow{D} \Delta^P_{n+1} \to \Delta^P_{n+1}/\Delta^{\Omega}_{n-1} \xrightarrow{\phi} \Pi,$$

where D is the chain homotopy in (5.3). Then $S^{\Omega} \circ \delta^{-1}\{\phi\}$ is the extension

$$0 \to \Pi \to (\Pi \oplus Z^{\Omega}_{n-1})/B \to H^{\Omega}_{n-1} \to 0,$$

where $B = \{(\phi' \circ s^{\Omega}_{n-1}(b), b) \mid b \in B^{\Omega}_{n-1}\}$.

Claim. If $s_j \circ q \circ D \circ i = q \circ D \circ i \circ s^{\Omega}_{n-1}$, then the correspondence

$$[g, z_{n-1}] \to ([g, -q \circ D \circ i(z_{n-1})], \{-z_{n-1}\})$$

is well defined and gives an equivalence between the two splittings, so that

$$S^{\Omega} \circ \delta^{-1} = -(\partial^{-1})^{*} \circ S.$$

The proof is a straightforward diagram chase in (5.3) and is omitted. To complete the proof of 5.2, then, we must show the following:

Given a right inverse $s_n\colon B_n(T) \to \Delta_{n+1}(T)$ for the singular boundary map for T, there exist right inverses $s_n\colon B_n \to \Delta_{n+1}$ and $s^{\Omega}_{n-1}\colon B^{\Omega}_{n-1} \to \Delta^{\Omega}_n$ such that $s_n \circ q \circ D \circ i = -q \circ D \circ i \circ s^{\Omega}_{n-1}$ and $e \circ s_n = s_n \circ e$.

The first equation we get for free once we have defined s_n on B_n. For if s_n, s^{Ω}_{n-1} are as in diagram (5.3) and $x \in B^{\Omega}_{n-1}$, then

$$\begin{aligned} &\partial \circ (q \circ D \circ i \circ s^{\Omega}_{n-1} + s_n \circ q \circ D \circ i)(x) \\ &\quad = q \circ (I - D \circ \partial) \circ i \circ s^{\Omega}_{n-1}(x) + q \circ D \circ i(x) \\ &\quad = -q \circ D \circ i \circ \partial \circ s^{\Omega}_{n-1}(x) + q \circ D \circ i(x) = 0. \end{aligned}$$

Choose a basis for B^{Ω}_{n-1} and let x be an element of that basis. By the computation above and a simple diagram chase, we may alter $s^{\Omega}_{n-1}x$ by a cycle so that $(q \circ D \circ i \circ s^{\Omega}_{n-1} + s_n \circ q \circ D \circ i)(x) = 0$. Extending by linearity we have the desired homomorphism s^{Ω}_{n-1}.

To construct $s_n\colon B_n \to \Delta_{n+1}$, we will define $s_n x$ for each element x of some basis for B_n. By assumption, $x = \partial y$ for some $y \in \Delta_{n+1}(PT)/\Delta_{n+1}(\Omega T)$. We may represent y as a finite linear combination $y = \sum n_j \alpha_j$ for suitable maps $\alpha_j\colon \Delta^{n+1} \to PT$. We will define $s_n(\partial\alpha_j)$, extend by linearity to define $s_n(x)$, and extend again by linearity to define s_n. (It could turn out that the final value of $s_n(\partial\alpha_j)$ is not that given in the construction.)

Fix some α_j and let $w = s_n(e(\partial\alpha_j))$; thus $w = \sum m_i\beta_i$, where $\beta_j\colon \Delta^{n+1} \to T$. Define X_w to be the disjoint union of $\sum |m_i|$ copies of Δ^{n+1} modulo the following face identifications: If Δ^{n+1}_1 and Δ^{n+1}_2 correspond to maps β_{i_1} and β_{i_2}, then we identify the kth n-dimensional faces of these two simplices if the restrictions of β_{i_1} and β_{i_2} to these faces are equal. Although the cell structure given here may not be that of a simplicial complex (too many identifications), all the identifications are linear and X_w has a barycentric refinement which is simplicial. The elements w and $\alpha_j|_{\dot{\Delta}^{n+1}}$ combine to give a map $(X_w \times \dot{I}) \cup (\dot{\Delta}^{n+1} \times I) \xrightarrow{\xi} T$ such that $\dot{\Delta}^{n+1} \times 0$ maps to the basepoint. We want to extend ξ to a map $X_w \times I \to T$. This then gives us a geometric template to define $s_n(\partial\alpha_j)$ with the correct evaluation.

We may clearly suppose that ξ sends all of $\dot{\Delta}^{n+1} \times I$ to the basepoint. By induction on dimension and on the number of cells of a given dimension in the barycentric refinement of X_w, it also follows that ξ deforms to a map in which all cells of dimension $< n+1$ in $X_w \times 1$ map to the basepoint. We can thus trivially extend ξ across (n-skeleton of X_w) $\times$ I, and since $\Pi_{n+2}(T) = 0$ we can further extend to the desired map $\xi\colon X_w \times I \to T$. $\square$

An H-map $h\colon BU \to BU$ equals a linear combination of Adams operations on any finite skeleton, and ψ^k is the 0 component of an infinite loop map of $BU \times Z_{(p)}$ provided k is prime to p. Thus one approach to showing that h satisfies the criteria in 5.1 is to write the operations ψ^{p^j} p-locally in terms of ψ^k with k prime to p. In fact, for any j there is a $Z_{(p)}$ linear combination of such ψ^k where the characteristic sequences agree for a while, but the process eventually breaks down. The proof of

5.1 used less than approximation by infinite loop maps, though, and we record this slightly more general result for use below.

5.4. COROLLARY. *The conclusion of* 5.1 *holds for the fiber X of any map $h\colon BU \to BU$ satisfying the following:*

For any $m > 0$ there is an integer $j \geq 0$ and a map $g\colon BU \to BU$ with fiber Y such that the characteristic sequences of h and $\Omega^j g$ agree through the mth term, and such that, in the universal coefficient sequence for

$$H^{2(m+j)}(Y[0, 2(m+j-1)], \Pi_{2m-1}(X)),$$

there is a splitting map S with $S(k_y^{2(m+j)}) = 0$.

We apply this work to the classifying spaces of smoothing theory. Recall that the infinite loop spaces L, M, and N can be defined as fibers in the diagram of fibrations (3.1), and they are related to smoothing theory by the p-local infinite loop equivalence $\Psi\colon C \times N \xrightarrow{\alpha\times\beta} PL/O \times PL/O \xrightarrow{\text{mult}} PL/O$. Using Ψ we obtain projections $\Pi_C\colon PL/O \to C$ and $\Pi_N\colon PL/O \to N$. If Y is a smooth manifold, a resmoothing of Y is a smooth structure on the underlying topological manifold which has a smooth triangulation in common with the original smoothing. (We adopt the notation and terminology of [**24**], and use script as before to denote the unlocalized classifying spaces in the Kervaire-Milnor braid.) If $\mathscr{PL}/\mathscr{O}$ denotes the fiber of the forgetful map $\mathscr{BO} \to \mathscr{BPL}$, then there is a bijection (which depends on the original choice of smoothing of Y) between the concordance classes of resmoothings of Y and the set of homotopy classes $[Y, \mathscr{PL}/\mathscr{O}]$ under which the original smooth structure corresponds to the constant map ([**24**, Part II, 4.2] or [**33**]). This puts the structure of a finite abelian group on the set of concordance classes of resmoothings, and $[Y, PL/O]$ equals the p-torsion in that group. For any map $h\colon Y \to N$ there is a resmoothing of Y associated to the composite $\beta \circ h\colon Y \to PL/O$ whose concordance class is uniquely determined by the homotopy class of h.

If Y is an n-sphere, $n \geq 5$, then the terms "concordant", "h-cobordant", and "orientation-preserving diffeomorphic" are equivalent, and when we refer to a particular resmoothing we ordinarily mean its equivalence class. The set of all such classes for the n-sphere is the finite abelian group of homotopy n-spheres whose p-primary subgroup we denote Θ_n. Let $\Phi\colon \Pi_n(PL/O) \to \Theta_n$ be the p-localization of the Hirsch-Mazur bijection [**24**] based on the usual smoothing of S^n. The composite $\Phi \circ \beta_*\colon \Pi_n(N) \to \Theta_n$ has image bP^{n+1}, the p-torsion in the group of all homotopy spheres which bound parallelizable manifolds. The latter group is 0 or $Z/2$ unless $n = 4m-1$ when it is cyclic of order

$$b_m = (3-(-1)^m)2^{2m-3}(2^{2m-1}-1)\text{numerator}(B_m/4m).$$

In fact, for any homotopy $4m-1$-sphere Σ bounded by a parallelizable manifold Y, the signature of Y is unique modulo $8b_m$. The correspondence $\Sigma \to \text{signature}(Y)/8$ is the isomorphism of Kervaire-Milnor with p-localization $\sigma\colon bP^{4m} \to Z/p^{v(b_m)}$. All this agrees up to units in $Z_{(p)}$ with the characteristic sequence for $\rho^k \circ (2\psi^2 - \psi^4)$ in 3.9 since the p-divisibility of $k^{2m}-1$ is precisely that of the denominator of $B_m/2m$ whenever k generates the units in Z/p^2.

We can think of σ as a boundary of the localized surgery obstruction map $\sigma\colon \Pi_{4m}(G/PL) \to Z_{(p)}$. It was extended by Brumfiel [**15**] to all homotopy $4m-1$-spheres: Any Σ^{4m-1} bounds a spin manifold Y all of whose decomposable Pontrjagin numbers vanish, and the signature of Y is again well defined modulo $8b_m$. This extension gives a splitting $\Theta_{4m-1} \cong \mathrm{Cok}(J)_{4m-1} \oplus bP^{4m}$ (with a nonlocal analogue) which agrees with the geometric splitting given by Π_N and Π_C as we see in the following. Let d, d' denote the boundary maps for the fibrations $N \to BO \to G/PL$ and $PL/O \to G/O \to G/PL$.

5.5. LEMMA. *The following diagram is commutative.*

$$\begin{tikzcd}
 & \Pi_{4m}(G/PL) \arrow[rr, "\sigma"] \arrow[dl, "d'"'] \arrow[dd, "d"] & & Z_{(p)} \arrow[dr, "q=\text{quotient}"] & \\
\Pi_{4m-1}(PL/O) \arrow[rr, "\Phi"] \arrow[dr, "(\Pi_N)_*"'] & & \Theta_{4m-1} \arrow[rr, "\sigma"] & & Z/p^{\nu(b_m)} \\
 & \Pi_{4m-1}(N) \arrow[rr] & & bP^{4m} \arrow[ul, "i"] \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

PROOF. In [**15**, Section §4], Brumfiel shows that $\sigma \circ \Phi \circ d = q \circ \sigma$. The bottom half of the diagram will commute provided the kernels of the various composites agree. In the homotopy exact sequence of the p-local fibration $PL/O \to BO \to BPL$, the torsion in $\Pi_{4m}(BPL)$ is carried by the boundary to $\Phi \circ \alpha(\Pi_{4m-1}(C))$. (See [**38**] or [**36**] for a general discussion of the p-local decomposition of the Kervaire-Milnor braid.) But in [**15**, Section §4], it is shown that the image of the boundary is precisely $\ker(\sigma\colon \Theta_{4m-1} \to Z/p^{\nu(b_m)})$. Thus $\ker(\sigma \circ \Phi \circ \beta_* \circ (\Pi_N)_*) = \ker(\sigma \circ \Phi)$, as desired. $\square$

For any space X an element $x \in H_j(X)$ is Steenrod representable iff there is a smooth oriented manifold Y and a map $h\colon Y \to X$ such that $h_*([Y]) = x$. We say that $x \in H_*(X, Z_{(p)})$ is p-locally representable if $h_*([Y]) = ux$ for some unit $u \in Z_{(p)}$.

5.6. THEOREM. *Let X be one of the classifying spaces L, M, or N. Then in the universal exact coefficient sequence*

$$0 \to \mathrm{Ext}(H_{4m-1}(X[0,4m-2]), \Pi_{4m-1}) \to H^{4m}(X[0,4m-2], \Pi_{4m-1}) \xrightarrow{\mu} \mathrm{Hom}(H_{4m}(X[0,4m-2]), \Pi_{4m-1}) \to 0$$

(*where* $\Pi_{4m-1} = \Pi_{4m-1}(X)$) *the homomorphism $\mu(k^{4m})$ vanishes on any homology class which is p-locally Steenrod representable.*

PROOF. It is clearly sufficient to check the result for representable classes, and because of the fibrations in (3.10) and (3.11) we may suppose that $X = N$. We must show that for any smooth closed orientable $4m$-manifold Y and map $h\colon Y \to N[0,4m-2]$, the composite $k_N^{4m} \circ h$ is null homotopic.

Fix some handle decomposition of Y with only one handle of dimension $4m$, and let Y^j denote the union of all handles of dimension $\leq j$. By the equivalence of the two spectral sequences described above, h corresponds to the homotopy class of a map h' in image($[Y^{4m-1}, N] \to [Y^{4m-2}, N]$). Choose some extension $\tilde{h}$ of h'. Let g denote the composite $S^{4m-1} \xrightarrow{j} Y^{4m-1} \xrightarrow{\tilde{h}} N \xrightarrow{\beta} PL/O$ where $j\colon S^{4m-1} \to Y^{4m-1}$ is the inclusion of the boundary. By [**18**, 5.6], $\sigma \circ \Phi([j]) = 0 \in Z_{(p)}$. (Brumfiel shows that the unlocalized smoothing has order a power of 2 and hence localizes to 0.) Thus by 5.5 it follows that

$$0 = \sigma \circ \Phi([g]) = \sigma \circ \Phi \circ \beta_* \circ (\Pi_N)_*([j]) = \sigma \circ \Phi \circ \beta_*([\tilde{h} \circ j])$$

But since $\sigma \circ \Phi \circ \beta_*$ is an isomorphism, $\tilde{h} \circ j$ is null homotopic. Thus if we regard $[\tilde{h} \circ j]$ as a class in $[\Sigma^{-1}(Y^{4m}/Y^{4m-1}), N] = E_1^{1,4m-2}$, it corresponds to the cochain sending the single $4m$-cell of Y to the composite of $\tilde{h}$ with the attaching map. The resulting cohomology class corresponds to $[h \circ k_N^{4m}]$ in the Postnikov system exact couple and the proof is complete. $\square$

Combining this with 5.4, we obtain a tool for computing, in certain cases, the smoothings classified by N.

5.7. COROLLARY. *Suppose N satisfies the criteria in 5.4 and Y is any space such that the homology of $\Sigma^s Y$ is p-locally Steenrod representable. Then in the Postnikov exact couple the differentials $d_r^{s,t}$ vanish for $r \geq 2$ and arbitrary t.*

For example, if Y is a smooth manifold such that both Y and ΣY have locally representable homology, then that portion of the Postnikov exact couple which converges to $[Y, N]$ collapses, provided N satisfies the conditions in 5.4. When p is regular this is clearly the case since $N = M = \text{fiber}(2\psi^2 - \psi^4)$, and it seems likely that the conditions are always satisfied. For all primes $p < 20{,}000$, for example, it is possible to approximate $\rho^k \circ (2\psi^2 - \psi^4)$ on finite skeleta of high enough dimension by a $Z(p)$ linear combination of Adams maps ψ^j with j prime to p so that 5.4 applies. Unfortunately, this is a long computer verification; what is still needed is a better hold on the Bernoulli numbers or a more global approximation theorem for maps $h\colon BU \to BU$.

We conclude with a geometric description of the differentials $d_r^{0,t}$ and $d_r^{-1,t}$ for the above exact couple when Y is a smooth manifold. These determine $[Y, N]$ up to group extensions. For suppose Y has some fixed smooth handle decomposition $\varnothing = Y^{-1} \subseteq Y^0 \subseteq Y^1 \subseteq \cdots \subseteq Y^m = Y$ upon which the Puppe exact couple is based. If

$$D^{s,t} = \bigcup_r d_r^{s-1,t-r+2}(E_r^{s-1,t-r+2}) \subseteq E_1^{s,t},$$

then by a simple diagram chase we obtain an exact sequence

$$0 \to E_1^{0,t}/D^{0,t} \to [Y^t, N] \to [Y^{t-1}, N] \to D^{-1,t-2} \to 0. \tag{5.8}$$

For any $t > 0$, Y^t is obtained from Y^{t-1} by attaching t-handles $D^t \times D^{m-t}$ via diffeomorphisms $\eta_i^t\colon S^{t-1} \times D^{m-t} \to \partial Y^{t-1}$ for $i = i, \ldots, j_t$. Given homotopy spheres $\Sigma_1^t, \ldots, \Sigma_{j_t}^t \in bP^{t+1}$, let $Y^t \# \Sigma_1^t \# \cdots \# \Sigma_{t_t}^t$ denote the smooth manifold obtained from Y^{t-1} by attaching j_t t-handles as before but with the ith handle attached via the embedding $\eta_i^t \circ (\varsigma_i \times 1)$ where $\varsigma_i \colon S^{t-1} \to S^{t-1}$ is a gluing diffeomorphism for the two hemispheres of Σ_i^t. The resulting manifold is a resmoothing of Y^t. Inclusion of the left-hand disks of the handles induces in the quotient Y^t/Y^{t-1} a homotopy equivalence with a wedge of j_t spheres S^t. Applying the functor $\Phi \circ (\Pi_N)_*([\,,\ N])$, we may identify $E_1^{-s,t}$, with a direct sum of j_{t-s} copies of bP^{t-s+1}. When $s=0$, for example, $(\Sigma_1^t, \dots, \Sigma_{j_t}^t)$ corresponds under this identification to $Y^{t-1} \# \Sigma_1^t \# \cdots \# \Sigma_{j_t}^t$.

5.9. PROPOSITION. *Let $\Sigma_1^{t+r-1}, \dots, \Sigma_{j_{t+r}}^{t+r-1} \in bP^{t+r}$ and $\Sigma_1^t, \dots, \Sigma_{j_t}^t \in bP^{t+1}$. Then the equation*

$$d_r^{0,t}(\Sigma_1^t, \dots, \Sigma_{j_t}^t) = (\Sigma_1^{t+r-1}, \dots, \Sigma_{j_{t+r}}^{t+r-1}) \in E_r^{1,t+r-1}$$

means that the resmoothing $Y^{t-1} \# \Sigma_1^t \# \cdots \# \Sigma_{j_t}^t$ of Y^t extends to a resmoothing of Y^{t+r-1}. The extension may be so chosen that the induced smoothings on the submanifolds $\eta_i^{t+r}(S^{t+r-1} \times D^{m-t-r})$, in the boundary of Y^{t+r-1} pull back via η_i^{t+r} to the product smoothing $\Sigma_i^{t+r-1} \times D^{m-t-r}$ for $i = 1, \dots, j_{t+r}$.

This follows directly from the definitions of the differentials in the Puppe exact couple and the correspondence Φ. To describe the differentials $d_r^{-1,t}$ we use the following description of $[\Sigma Y, \mathscr{PL}/\mathscr{O}]$ for a smooth manifold Y.

5.10. PROPOSITION. *For any smooth manifold Y and mapping $f \colon \Sigma Y \to \mathscr{PL}/\mathscr{O}$, there is a canonically induced diffeomorphism $\phi_f \colon Y \to Y$ which is pl concordant to 1_Y and unique up to a smooth concordance. Given disjoint codimension 0 submanifolds $W_1, \dots, W_k$, we may further suppose that $\phi_f(W_i) = W_i$ for each i. The correspondence $f \to \phi_f$ defines a bijection between $[\Sigma Y, \mathscr{PL}/\mathscr{O}]$ and the set of smooth concordance classes of diffeomorphisms pl concordant to 1_Y.*

PROOF. We sketch the definition of ϕ_f. The result then follows by standard smoothing theory arguments. Let Φ denote the bijection between $[S^1 \times Y, \mathscr{PL}/\mathscr{O}]$ and the concordance classes of resmoothings of $S^1 \times Y$ in which the original smoothing corresponds to the constant map. The composite

$$S^1 \times Y \xrightarrow{\text{collapse}} S^1 \wedge Y \xrightarrow{f} \mathscr{PL}/\mathscr{O}$$

corresponds to a resmoothing of $S^1 \times Y$ which, by the Hirsch Product Theorem, has the form $I \times Y/\approx$ where $(0, y) \approx (1, \phi_f(y))$ for some diffeomorphism ϕ_f. Using the relative version Hirsch's theorem, ϕ_f can be chosen so that $\phi_f(W_i) = W_i$. $\square$

5.11. PROPOSITION. *Suppose $\Sigma_1^{t-1}, \dots, \Sigma_{j_{t-2}}^{t-1} \in bP^t$, and $\Sigma_1^{t+r-2}, \dots, \Sigma_{j_{t+r-2}}^{t+r-2} \in bP^{t+r-1}$. Then the equation*

$$d_r^{-1,t}(\Sigma_1^{t-1}, \dots, \Sigma_{j_{t-2}}^{t-1}) = (\Sigma_1^{t+r-2}, \dots, \Sigma_{j_{t+r-2}}^{t+r-2})$$

means the following:

The diffeomorphism $\phi_f \colon Y^{t-2} \to Y^{t-2}$ corresponding to

$$f \colon \Sigma Y^{t-2} \to \Sigma Y^{t-2}/\Sigma Y^{t-3} \xrightarrow{(\Sigma_1^{t-1}, \dots, \Sigma_{j_{t-2}}^{t-1})} N$$

extends to a diffeomorphism $\tilde{\phi}_f$ of Y^{t+r-2} such that each of the submanifolds $\eta_i^{t+r-2}(S^{t+r-3} \times D^{m-t-r+2})$, $i = 1, \ldots, j_{t+r-2}$, is mapped diffeomorphically onto itself. The resulting resmoothing

$$D^{t+r-2} \times D^{m-t-r+2} \bigcup_{\tilde{\phi}_f|\mathrm{im}(\eta_i^{t+r-2})} D^{t+r-2} \times D^{m-t-r+2}$$

of $S^{t+r-2} \times D^{m-t-r+2}$ is equal to $\Sigma_i^{t+r-2} \times D^{m-t-r+2}$ for every i.

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