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v_1-PERIODIC EXT OVER THE STEENROD ALGEBRA

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ABSTRACT. For a large family of modules M over the mod 2 Steenrod algebra A, $\mathrm{Ext}_A^{s,t}(M, \mathbf{Z}_2)$ is periodic for $t < 4s$ with respect to operators $v_1^{2^n}$ of period $(2^n, 3 \cdot 2^n)$ for varying n. $v_1^{-1}\,\mathrm{Ext}_A^{s,t}(M, \mathbf{Z}_2)$ can be defined by extending this periodic behavior outside this range. We calculate this completely when $M = H^*(Y)$, where Y is the suspension spectrum of $\mathbf{R}P^2 \wedge \mathbf{C}P^2$.

1. Introduction. Let A denote the mod 2 Steenrod algebra, and A_n the subalgebra generated by $\{\mathrm{Sq}^j : j \leq 2^n\}$. The Adams periodicity theorem [**A**] states that if M is an A-module which is free as an A_0-module, then $\mathrm{Ext}_A^{s,t}(M, \mathbf{Z}_2)$ is periodic in the range $t < 4s$. In this paper, we give a complete calculation of $\mathrm{Ext}_A^{s,t}(H^*Y, \mathbf{Z}_2)$ for $t < 4s$, where H^*Y is the A-module $\langle 1, \mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^3 = \mathrm{Sq}^2\,\mathrm{Sq}^1\rangle$. Here Y is a suitably indexed suspension spectrum of $\mathbf{R}P^2 \wedge \mathbf{C}P^2$, which played an important role in [**M2**]. We view this as an important first step toward the analogous calculation when $M = A_0$, and ultimately perhaps $M = \mathbf{Z}_2$. The result for A_0 was announced in [**M1**].

We recall more precisely the Adams periodicity theorem.

THEOREM 1.1 [**A**]. *For $n \geq 2$, there exists $\omega_n \in \mathrm{Ext}_{A_n}^{2^n, 3\cdot 2^n}(\mathbf{Z}_2, \mathbf{Z}_2)$ such that*

(i) *restriction $\mathrm{Ext}_{A_{n+1}}(\mathbf{Z}_2, \mathbf{Z}_2) \to \mathrm{Ext}_{A_n}(\mathbf{Z}_2, \mathbf{Z}_2)$ sends ω_{n+1} to ω_n^2;*

(ii) *if M is a connected A_0-free A_n-module, then*

$$\cdot\omega_n\colon\ \mathrm{Ext}_{A_n}^{s,t}(M, \mathbf{Z}_2) \to \mathrm{Ext}_{A_n}^{s+2^n, t+3\cdot 2^n}(M, \mathbf{Z}_2)$$

is an isomorphism if $t < 4s$;

(iii) *if M is a connected A_0-free A-module, then restriction*

$$\mathrm{Ext}_A^{s,t}(M, \mathbf{Z}_2) \to \mathrm{Ext}_{A_n}^{s,t}(M, \mathbf{Z}_2)$$

is an isomorphism if $0 < s$ and $t < 2^{n+1} + 3s - 5$.

This says that for such a module M, in the usual depiction of $\mathrm{Ext}_A^{s,t}(M, \mathbf{Z}_2)$ in position $(t - s, s)$, there is $(t - s, s)$-periodicity as indicated in Figure 1.2 below. (The 0-part follows from another result of [**A**].)

For an A_0-free module M we define $v_1^{-1}\,\mathrm{Ext}_A^{*,*}(M, \mathbf{Z}_2)$ to be the bigraded $\mathbf{Z}_2$-vector space obtained by extending the periodic behavior of $\mathrm{Ext}_A(M, \mathbf{Z}_2)$ from

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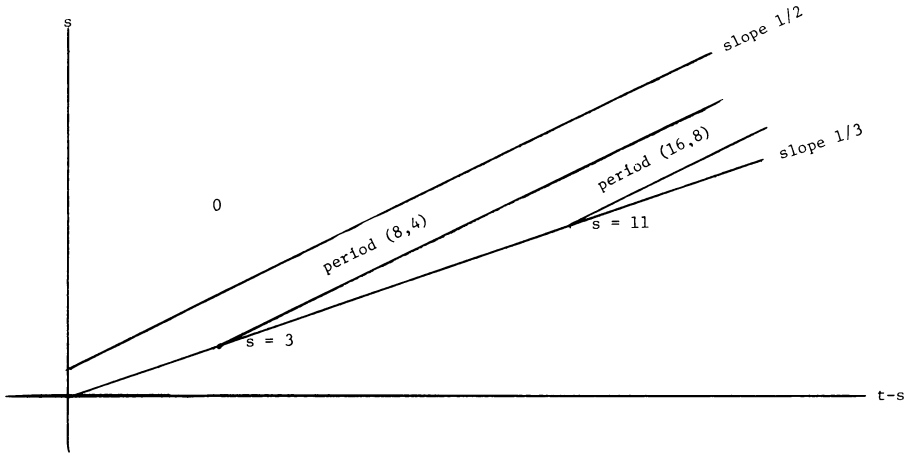


FIGURE 1.2

above the $\frac{1}{3}$-line to the region below the $\frac{1}{3}$-line (including negative gradings). More precisely, for any integers s and t,

$$v_1^{-1}\operatorname{Ext}_A^{s,t}(M,\mathbf{Z}_2) \equiv \operatorname{Ext}_A^{s+k\cdot 2^n,t+3k\cdot 2^n}(M,\mathbf{Z}_2) \qquad \text{if } 2^{n+1} > t+5-3s \text{ and } k\cdot 2^n > t-4s.$$

By 1.1, this is independent of k and n.

An alternative formulation, due to Mahowald and Shick, explains the reason for the name $v_1^{-1}\operatorname{Ext}(\ \ )$. If E_n denotes the exterior subalgebra generated by the Milnor primitives $Q_0,\dots,Q_n$, then $\operatorname{Ext}_{E_n}(\mathbf{Z}_2,\mathbf{Z}_2) \approx \mathbf{Z}_2[v_0,\dots,v_n]$, where $v_i \in \operatorname{Ext}^{1,2^{i+1}-1}$ corresponds roughly to a generator of $\pi_*(BP)$, where BP is the Brown-Peterson spectrum. In [**MS**] it was proved that for $n \geq i$ there are integers $L(n,i)$ and elements $v_i^{L(n,i)} \in \operatorname{Ext}_{A_n}(\mathbf{Z}_2,\mathbf{Z}_2)$ which restrict to the element of Ext_{E_n} with the same name. These can be chosen compatibly with respect to increasing n, so that for any A-module M one can define

$$v_i^{-1}\operatorname{Ext}_A(M,\mathbf{Z}_2) \equiv \varprojlim v_i^{-L(n,i)}\operatorname{Ext}_{A_n}(M,\mathbf{Z}_2).$$

It is easy to see that for $i=1$ this agrees with the definition using Adams periodicity if M is A_0-free. We will use the two definitions interchangeably.

Our main result is

THEOREM 1.3. *$v_1^{-1}\operatorname{Ext}_A^{*,*}(H^*Y,\mathbf{Z}_2) \approx \mathbf{Z}_2[v_1^{\pm 1}][h_{j,1}: j \geq 2]$, where*

$$h_{j,1} \in v_1^{-1}\operatorname{Ext}^{1,2^{j+1}-2}(\ ).$$

A generator $h_{j,1}$ of this polynomial algebra corresponds to the element ς_j^2 in the dual of the Steenrod algebra. It is not generally the case that $v_1^{-1}\operatorname{Ext}_A(M)$ is a module over $\mathbf{Z}_2[v_1^{\pm 1}]$; usually higher powers of v_1 are involved.

This result allows us to write $\operatorname{Ext}_A^{*,*}(H^*Y,\mathbf{Z}_2)$ above the $\frac{1}{3}$-line in a nice form, with elements expressed as products of elements that may not exist in Ext, but do exist in $v_1^{-1}\operatorname{Ext}$. We illustrate for $t-s \leq 37$ in Figure 1.4 below, with numbers at the end of a sequence indicating the subscripts of the $h_{j,1}$ elements represented.

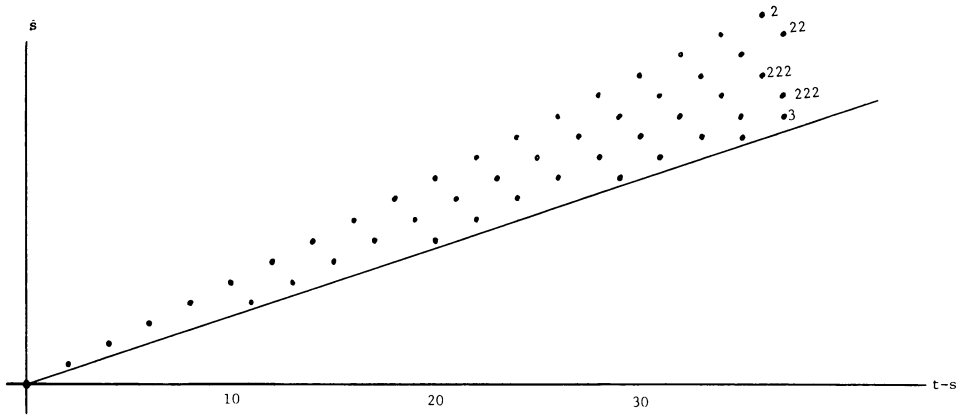


FIGURE 1.4

$v_1^{-1}\operatorname{Ext}_A^{*,*}(H^*Y, \mathbf{Z}_2)$ can be interpreted as the E_2-term of an Adams-type spectral sequence (SS) converging to $\pi_*(v_1^{-1}Y)$, where $v_1^{-1}Y$ is the mapping telescope of

$$Y \xrightarrow{v_1} \Sigma^{-2}Y \xrightarrow{v_1} \Sigma^{-4}Y \xrightarrow{v_1} \cdots,$$

where Y is the spectrum mentioned in the first paragraph, which was featured in [**M2**], and v_1 is the self-map constructed in [**DM1**]. However, this is not especially useful, since $\pi_*(v_1^{-1}Y)$ was calculated by other methods in [**M2**]; it is, in fact, the exterior algebra over $\mathbf{Z}_2[v_1^{-1}]$ on $h_{2,1} \in \pi_5(v_1^{-1}Y)$. To justify this interpretation one shows that the above definitions of $v_1^{-1}\operatorname{Ext}_A^{s,t}(H^*Y, \mathbf{Z}_2)$ are equivalent to

$$\varinjlim\left(\operatorname{Ext}_A^{s,t}(H^*Y, \mathbf{Z}_2) \xrightarrow{v_1} \operatorname{Ext}_A^{s+1,t+3}(H^*Y, \mathbf{Z}_2) \xrightarrow{v_1} \operatorname{Ext}_A^{s+2,t+6}(H^*Y, \mathbf{Z}_2) \xrightarrow{v_1} \cdots\right),$$

where the homomorphisms may be thought of as a Yoneda product, or as the Ext homomorphism induced by the filtration-1 map v_1. Then as in [**DM3**] one can show that the direct limit of the Adams spectral sequences converges to $\varinjlim \pi_*(\Sigma^{-2i}Y)$. This Yoneda product defines an action of $\mathbf{Z}_2[v_1]$ on $\operatorname{Ext}_A(H^*Y, \mathbf{Z}_2)$, providing a third definition of $v_1^{-1}\operatorname{Ext}_A(H^*Y, \mathbf{Z}_2)$, which can be shown compatible with the others. This is one way of seeing that one does not have to wait for large powers of v_1 to act on $\operatorname{Ext}_A(H^*Y, \mathbf{Z}_2)$, but this is special to H^*Y.

While this paper was being refereed, David Eisen, in his Princeton Ph.D. thesis, generalized our result to $v_n^{-1}\operatorname{Ext}$ and simplified the proof.

2. Sketch of proof. We prefer to work with the dual object H_*Y, which is an A^*-comodule, where A^* is the dual of the Steenrod algebra. We shall denote this simply by Y. Recall that $A^* = \mathbf{Z}_2[\varsigma_1, \varsigma_2, \ldots]$, where $\varsigma_i = \chi(\xi_i)$ with ξ_i the Milnor element of degree $2^i - 1$, and χ the canonical antiautomorphism. A^* is a Hopf algebra with

$$\psi(\varsigma_i) = \sum \varsigma_j \otimes \varsigma_{i-j}^{2^j}$$

and a right A-module with

$$(\varsigma_i)\chi\,\mathrm{Sq} = \varsigma_i + \varsigma_{i-1}^2.$$

Then Y is the subcoalgebra of A^* spanned by $1, \varsigma_1, \varsigma_1^2$, and ς_1^3. Now $\mathrm{Ext}_{A^*}(\mathbf{Z}_2, Y)$ is naturally isomorphic to $\mathrm{Ext}_A(H^*Y, \mathbf{Z}_2)$, and so we define $v_1^{-1}\,\mathrm{Ext}_{A^*}(\mathbf{Z}_2, Y)$ by periodicity as in §1.

In [**Mi**], H. Miller considered odd primary analogs of much of our work. He points out there that his methods do not seem readily adaptable to $p = 2$. We have adopted some ideas from his paper, and have benefited from conversations with him. He works with $\mathrm{Cotor}_{A^*}(\mathbf{Z}_2, \ )$ rather than $\mathrm{Ext}_{A^*}(\mathbf{Z}_2, \ )$. As these are naturally isomorphic [**R**, p. 320], we lose nothing by working with Ext.

We will prove 1.3 by calculating a certain SS. The hard parts are (i) showing that the SS is multiplicative and (ii) showing that its differentials d_r for $r \geq 2$ are all zero. The SS is the dual of the $A//A_1$-SS utilized, for example, in [**D**].

Let Γ denote the subalgebra $\mathbf{Z}_2[\varsigma_1^4, \varsigma_2^2, \varsigma_3, \dots]$ of A^*. This is the dual of $A//A_1$. Let $\overline{\Gamma}$ denote the quotient $\Gamma/\langle 1\rangle$. Then for any A^*-comodule M, there is an exact cobar resolution $B_\Gamma(M)$ of A^*-comodules

$$0 \to M \to \Gamma \otimes M \to \Gamma \otimes \overline{\Gamma} \otimes M \to \Gamma \otimes \overline{\Gamma}^2 \otimes M \to \cdots \tag{2.1}$$

with $\delta\big(a_0[a_1|\cdots|a_s]m\big) = [\bar{a}_0|a_1|\cdots|a_s]m$. Here the A^*-comodule structure is diagonal, and superscripts denote iterated $\otimes$. The more familiar form of the cobar resolution, with left coaction, requires a comultiplication in its δ, and Γ is not an A^*-coalgebra.

We will obtain from (2.1) a SS converging to $\mathrm{Ext}_{A^*}(M)$ with

$$E_1^{\sigma,s,t} = \mathrm{Ext}_{A^*}^{s,t}(\Gamma \otimes \overline{\Gamma}^\sigma \otimes M).$$

Here and henceforth we delete $\mathbf{Z}_2$ from the first component of Ext-groups. We apply a change-of-rings theorem to simplify this E_1-term. From [**ABP**],

$$A_1^* \approx A^*/(\varsigma_1^4, \varsigma_2^2, \varsigma_3, \varsigma_4, \dots)$$

and hence $\Gamma = A^* \square_{A_1^*} \mathbf{Z}_2$, so that by [**R**, A1.3.13]

$$\mathrm{Ext}_{A^*}(\Gamma \otimes \overline{\Gamma}^\sigma \otimes M) \approx \mathrm{Ext}_{A_1^*}(\overline{\Gamma}^\sigma \otimes M).$$

Before we specialize to $M = Y$, we construct this SS in a slightly more general setting, and consider pairings of such spectral sequences. This result is adapted from [**R**, pp. 323–330].

PROPOSITION 2.2. *Suppose Λ is a Hopf algebra over $\mathbf{Z}_2$ and*

$$0 \to M \to C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} \cdots$$

is an exact sequence of Λ-comodules, denoted C. There is a natural SS of $\mathrm{Ext}_\Lambda(\mathbf{Z}_2)$-modules converging to $\mathrm{Ext}_\Lambda^{*,*}(M)$ with

$$E_1^{\sigma,s,t} = \mathrm{Ext}_\Lambda^{s,t}(C_\sigma) \text{ and } d_r : E_r^{\sigma,s,t} \to E_r^{\sigma+r,s-r+1,t}.$$

$E_\infty^{\sigma,s,t}$ contributes to $\mathrm{Ext}_\Lambda^{s+\sigma,t}(M)$. There is a pairing of $\mathrm{Ext}_\Lambda(\mathbf{Z}_2)$-modules

$$E_r^{***}(C) \otimes E_r^{***}(C') \to E_r^{***}(C \otimes C').$$

PROOF. Let $B(C) = B_\Lambda(C)$ denote the bigraded cobar resolution

$$B(C)_{p,q} = \Lambda \otimes \overline{\Lambda}^p \otimes C_q.$$

Filtering the total complex with respect to second grading and applying $H^*(\mathrm{Hom}_\Lambda(\mathbf{Z}_2, \quad))$ yields a SS $E_r^{***}(C)$ with $E_1^{\sigma,*,*} = \mathrm{Ext}_\Lambda(\mathbf{Z}_2, C_\sigma)$. Filtering instead with respect to first grading shows that SS converges to $\mathrm{Ext}_\Lambda(\mathbf{Z}_2, M)$.

Standard methods of homological algebra imply the existence of pairings

$$B(C) \otimes B(C') \to B(C \otimes C'),$$

inducing the desired pairing of SS's. These pairings are associative and commutative up to chain homotopy. Thus if $\mathbf{Z}_2$ denotes the complex

$$0 \to \mathbf{Z}_2 \to \mathbf{Z}_2 \to 0 \to 0 \to \cdots,$$

then the diagram

$$\begin{array}{ccc}
B(\mathbf{Z}_2) \otimes B(C) \otimes B(C') & \longrightarrow & B(\mathbf{Z}_2) \otimes B(C \otimes C') \\
\Big\downarrow{\scriptstyle 1\otimes T} & & \Big\downarrow{\scriptstyle 1\otimes B(T)} \\
B(\mathbf{Z}_2) \otimes B(C') \otimes B(C) & \longrightarrow & B(\mathbf{Z}_2) \otimes B(C' \otimes C) \\
\Big\downarrow & & \Big\downarrow \\
B(\mathbf{Z}_2 \otimes C') \otimes B(C) & \longrightarrow & B(\mathbf{Z}_2 \otimes C' \otimes C) \\
\Big\downarrow{\scriptstyle T} & & \Big\downarrow{\scriptstyle B(T)} \\
B(C) \otimes B(\mathbf{Z}_2 \otimes C') & \longrightarrow & B(C \otimes \mathbf{Z}_2 \otimes C')
\end{array}$$

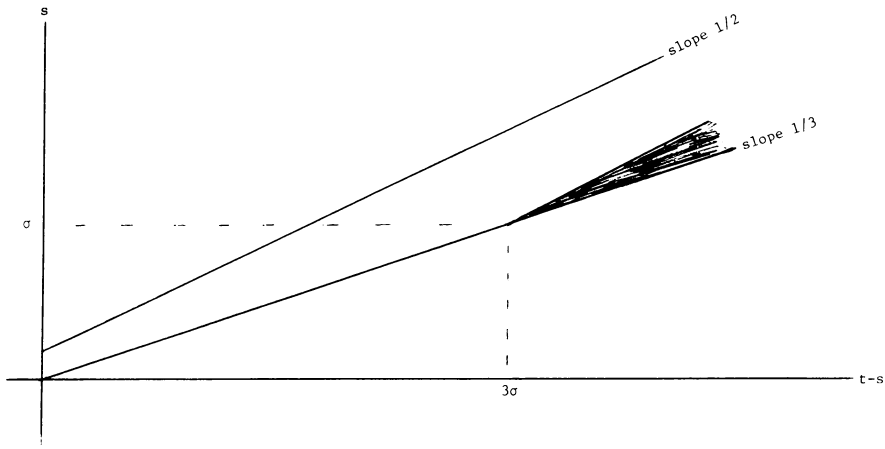
commutes up to chain homotopy, showing the pairing is linear with respect to the action of $\mathrm{Ext}_\Lambda(\mathbf{Z}_2)$ on the right factor. Linearity with respect to action on the left factor is similar, but easier, completing the proof of 2.2.

If $v \in \mathrm{Ext}_\Lambda(\mathbf{Z}_2)$, the module structure implies that the SS of 2.2 passes to a SS converging to $v^{-1}\,\mathrm{Ext}_\Lambda(M)$ with $E_1^{\sigma,s,t} = v^{-1}\,\mathrm{Ext}_\Lambda^{s,t}(C_\sigma)$, and pairings of complexes induce pairings of these SS's. We apply this to the complex of (2.1) with $M = Y = A_1^* \square_{E[\varsigma_2]} \mathbf{Z}_2$ and $\Lambda = A^*$, and using again the change-of-rings theorem [**R**, A1.3.13], we obtain the following result. Here $E[\varsigma_2]$ denotes the exterior algebra.

THEOREM 2.3. *There is a SS of algebras converging to* $v_1^{-1}\,\mathrm{Ext}_{A^*}^{**}(Y)$ *with* $E_1^{\sigma,s,t} = v_1^{-1}\,\mathrm{Ext}_{E[\varsigma_2]}^{s,t}(\overline{\Gamma}^\sigma)$.

PROOF. The SS is that of the preceding paragraph. If the Mahowald-Shick definition of $v_1^{-1}\,\mathrm{Ext}_A(\ )$ is used, then Proposition 2.2 must be applied with $\Lambda = A_n^*$ for all $n \geq 2$, inverting the appropriate power of v_1 each time, and noting compatibility.

If the Adams-periodicity definition is used, then for each of the regions of periodicity of Figure 1.2 the appropriate $\mathrm{Ext}_{A_n^*}(\ )$-SS of 2.2 can be used. Because $\overline{A_n//A_1}$ begins in degree 4, $E_1^{\sigma,s,t}$ will contribute to Figure 1.2 only in the region shaded below and by 1.1 if this overlaps a region of period $(2^{n+1}, 2^n)$ in 1.2, the same (or shorter) period will be present in the E_1-term. The linearity of the differentials with respect to the action of $v_1^{2^n} = \omega_n$ is necessary in order to deduce that the SS can be extended in the negative direction by periodicity in a well-defined manner.



Thus we obtain from 2.2 a SS $v_1^{-1}E_r^{***}(C)$ for a complex C as in 2.2. To see that this is a SS of algebras when $C = B_\Gamma(Y)$ of (2.1), we first note that there is a pairing of the complexes (2.1)

$$B_\Gamma(M) \otimes B_\Gamma(N) \xrightarrow{\mu} B_\Gamma(M \otimes N). \tag{2.4}$$

This is a standard result of relative homological algebra, and will be proved in §4, where we in fact give an explicit formula. Relative homological algebra is required since Γ is not an A^*-coalgebra, so that the pairing of [**R**, A1.2.15] does not apply.

Now 2.2 and the remarks after its proof give the first morphism below, the second follows from (2.4), and the third from Lemma 2.6 below.

$$\begin{aligned} v_1^{-1}E_r^{***}(B_\Gamma(Y)) \otimes v_1^{-1}E_r^{***}(B_\Gamma(Y)) &\to v_1^{-1}E_r^{***}(B_\Gamma(Y) \otimes B_\Gamma(Y)) \\ &\to v_1^{-1}E_r^{***}(B_\Gamma(Y \otimes Y)) \to v_1^{-1}E_r^{***}(B_\Gamma(Y')) \approx v_1^{-1}E_r^{***}(B_\Gamma(Y)) \end{aligned} \tag{2.5}$$

This completes the proof of 2.3.

Lemma 2.6, which is proved in §3, gets around the problem that the action of Sq^4 (or coaction of ς_1^4) prevents an A^*-homomorphism $Y \otimes Y \to Y$. It is actually done for spectra, and so, as the referee points out, the higher Adams differentials respect the multiplicative structure.

LEMMA 2.6. *There is a connected A^*-comodule Y' of finite type satisfying*

(i) *there is an A^*-morphism $Y \to Y'$ inducing an isomorphism in $v_1^{-1}\operatorname{Ext}_{A^*}(- \otimes M)$ for any A^*-comodule M;*

(ii) *there is an A^*-morphism $Y \otimes Y \to Y'$ which is an isomorphism in degree* 0.

Next we make more explicit the E_1-term of 2.3. If M is any $E[\varsigma_2]$-comodule, then we define a boundary homomorphism ∂ on M by $\partial(m) = m'$ if $\psi(m) = 1 \otimes m + \varsigma_2 \otimes m'$. Since ς_2 is primitive in $E[\varsigma_2]$ (our $E[\varsigma_2]$ is actually a quotient of A_1^*), $\partial(\partial(m)) = 0$, so that we can consider $H_*(M;\partial)$. Equivalent to this is the right action of the Milnor primitive $Q_1 \in A$ on M, and $H_*(M;Q_1)$, which is the way we shall write it. Since $\operatorname{Ext}_{E[\varsigma_2]}(\mathbf{Z}_2) \approx \mathbf{Z}_2[v_1]$, where $v_1 \in \operatorname{Ext}^{1,3}$, and $\operatorname{Ext}_{A_n^*}(\mathbf{Z}_2) \to \operatorname{Ext}_{E[\varsigma_2]}(\mathbf{Z}_2)$ sends the Mahowald-Shick or Adams elements to

powers of v_1, it follows that the change-of-rings isomorphism

$$\operatorname{Ext}_{A^*}(A^*\square_{A_1^*}(A_1^*\square_{E[\varsigma_2]}M)) \approx \operatorname{Ext}_{E[\varsigma_2]}(M)$$

induces an isomorphism of bigraded vector spaces

$$v_1^{-1}\operatorname{Ext}^{**}_{A^*}(A^*\square_{A_1^*}(A_1^*\square_{E[\varsigma_2]}M)) \approx v_1^{-1}\operatorname{Ext}^{**}_{E[\varsigma_2]}(M).$$

This is an isomorphism of v_1-modules to the extent that this makes sense, i.e., in regions in which $v_1^{2^n}$ acts on the LHS, this action is compatible with the 2^nth power of v_1 on the RHS. This RHS is a legitimate $v_1^{-1}\operatorname{Ext}$, without requiring any $\varprojlim$ or varying of powers of v_1.

Since an $E[\varsigma_2]$-comodule M splits as the sum of a free and a trivial comodule, one deduces that

$$v_1^{-1}\operatorname{Ext}^{**}_{E[\varsigma_2]}(M) \approx \mathbf{Z}_2[v_1^{\pm 1}] \otimes H_*(M;Q_1). \tag{2.7}$$

The right action of Q_1 on A^* is given by $(\varsigma_i)Q_1 = \varsigma_{i-2}^4$, from which we obtain for $\Gamma = \mathbf{Z}_2[\varsigma_1^4, \varsigma_2^2, \varsigma_3, \dots]$

$$H_*(\Gamma;Q_1) \approx E[\varsigma_j^2 : j \geq 2], \tag{2.8}$$

where E denotes an exterior algebra [**ABP**].

Theorem 1.3 follows from the following result, which calculates the SS of 2.3.

THEOREM 2.9. *There is a SS of algebras over* $\mathbf{Z}_2[v_1^{\pm 1}]$ *converging to* $v_1^{-1}\operatorname{Ext}_{A^*}(Y)$ *such that*

(i) E_1^{***} *is (at least additively) a tensor algebra over* $\mathbf{Z}_2[v_1^{\pm 1}]$ *on* $\overline{E}[\varsigma_j^2 : j \geq 2]$, *where* $\prod \varsigma_{j_i}^2 \in E_1^{1,0,t}$ *with* $t = 2\sum(2^{j_i}-1)$, *and* $\overline{E}$ *is the quotient of the exterior algebra* $\operatorname{mod}\langle 1\rangle$.

(ii) E_2^{***} *is a polynomial algebra* $\mathbf{Z}_2[v_1^{\pm 1}][h_{j,1} : j \geq 2]$, *where* $h_{j,1} = [\varsigma_j^2] \in E_2^{1,0,2^{j+1}-2}$.

(iii) *The SS collapses from* E_2, *i.e.* $E_\infty^{***} = E_2^{***}$.

We have already proved (i). Part (ii) will be proved in §4 by explicit calculation of d_1 and the pairing. The proof of (iii) involves a comparison with a sequence of Koszul SS's, which will be done in §5.

3. Existence of Y'. The main result of this section is a topological result which implies 2.6. In *this* section, Y denotes the spectrum defined in §1, not its homology.

THEOREM 3.1. *There is a spectrum* Y' *and a map* $Y \wedge Y \to Y'$ *such that the composite* $S^0 \wedge Y \to Y \wedge Y \to Y'$ *induces an isomorphism in* $v_1^{-1}\operatorname{Ext}_A(H^*(\ ))$.

The remainder of this section is devoted to the proof. Let X_2 denote the ring spectrum of [**DM1**, 3.5], and $X_2^{(n)}$ its n-skeleton. We recall that X_2 is a Thom spectrum over ΩS^2, and $H_*(X_2) \approx \mathbf{Z}_2[\varsigma_1]$. Through dimension 9, $\operatorname{Ext}_A(H^*X_2)$ is free over $\mathbf{Z}_2[v_1]$ on elements in $(t-s,s)$ position (0,0), (5,1), and (6,1). There are no possible higher Adams differentials in this range, and so we obtain a corresponding description of $\pi_*(X_2)$, which is a $\mathbf{Z}_2$-vector space since it is a ring whose unit has order 2.

Obstruction theory using this information shows that there is a map $\Sigma^6 Y \xrightarrow{v_2} X_2$ which extends a filtration-1 element of $\pi_6(X_2)$. One easily checks that $\pi_i(X_2/X_2^{(7)})$ has no elements of positive filtration for $6 \le i \le 9$, and so the composite

$$\Sigma^6 Y \xrightarrow{v_2} X_2 \to X_2/X_2^{(7)}$$

is trivial, and hence there is a compression $\Sigma^6 Y \xrightarrow{\widetilde{v_2}} X_2^{(7)}$. Our spectrum Y' is the cofiber of this map $\widetilde{v_2}$. Our multiplication is obtained as

$$Y \wedge Y = X_2^{(3)} \wedge X_2^{(3)} \xrightarrow{m} X_2^{(7)} \to Y'.$$

An easy diagram chase shows that the cofiber C of $Y \to Y'$ is equivalent to the cofiber of the composite

$$\Sigma^6 Y \xrightarrow{\widetilde{v_2}} X_2^{(7)} \to X_2^{(7)}/X_2^{(3)} = \Sigma^4 Y.$$

This composite is v_1, and the cofiber of v_1 realizes A_1 by [**DM1**, 2.1]. Thus by [**AD**], $\operatorname{Ext}_A(H^*C)$ vanishes above a line of slope 1/5, and hence $v_1^{-1}\operatorname{Ext}_A(H^*C) = 0$, establishing the claim that $Y \to Y'$ induces an isomorphism in $v_1^{-1}\operatorname{Ext}$.

4. E_1 and E_2. In this section we calculate the d_1-differential in the SS of 2.9, yielding the additive structure in 2.9(ii), and show that the product structure in 2.9(ii) is as claimed by deriving an explicit formula for the pairing.

The calculation of d_1 seems somewhat simpler if we work in the category of A-modules and then dualize. Then the d_1-differential is the homomorphism in $v_1^{-1}\operatorname{Ext}_A(\ ,\mathbf{Z}_2)$ induced by

$$\begin{array}{ccc}
A//A_1 \otimes A_1//E \otimes \overline{A//A_1} \otimes M & \xrightarrow{d} & A//A_1 \otimes A_1//E \otimes M \\
\updownarrow & & \updownarrow \\
A \otimes_{A_1} (A_1//E \otimes \overline{A//A_1} \otimes M) & & A \otimes_{A_1} (A_1//E \otimes M) \\
\updownarrow & & \updownarrow \\
A \otimes_{A_1} (A_1 \otimes_E (\overline{A//A_1} \otimes M)) & & A \otimes_{A_1} (A_1 \otimes_E M) \\
\updownarrow & & \updownarrow \\
A \otimes_E (\overline{A//A_1} \otimes M) & \cdots \xrightarrow{d'} & A \otimes_E M
\end{array} \tag{4.1}$$

where M could be any A-module, but is $\overline{A//A_1}^{\,\sigma}$, A-action is diagonal over unsubscripted $\otimes$, but is on the left of subscripted $\otimes$, and

$$d(a_0 \otimes y \otimes a_1 \otimes m) = \varepsilon(a_0) a_1 \otimes y \otimes m.$$

The isomorphisms are instances of

$$\begin{aligned}
A \otimes_B M &\underset{h_2}{\overset{h_1}{\rightleftarrows}} A//B \otimes M, \\
h_1(\overline{a \otimes m}) &= \sum \overline{a_i'} \otimes a_i'' m, \\
h_2(\bar{a} \otimes m) &= \sum \overline{a_i' \otimes \chi(a_i'') m}
\end{aligned} \tag{4.2}$$

if $\psi(a) = \sum a_i' \otimes a_i''$, and $^-$ denotes the class in the quotient. Tracing yields

$$d'(1 \otimes a \otimes m) = \sum a_i' \chi(\tilde{a}_i'') \otimes \tilde{a}_i''' \chi(a_i'''') m \tag{4.3}$$

.f $\psi^3(a) = \sum a_i' \otimes a_i'' \otimes a_i''' \otimes a_i''''$, and

$$\tilde{a} = \begin{cases} a & \text{if } |a| \le 3, \\ 0 & \text{if } |a| > 3. \end{cases}$$

This latter comes into play as the A-action on $A_1//E$.

From (2.7) we have $v_1^{-1}\operatorname{Ext}_A(A \otimes_E M, \mathbf{Z}_2) \approx \mathbf{Z}_2[v_1^{\pm 1}] \otimes H_*(M^*; Q_1)$, and so d_1 is determined as dual to the $1 \otimes -$ -terms in (4.3). Using (2.8), this becomes (cf. [**D**])

$$\begin{gathered} \overline{E[\varsigma_j^2 : j \ge 2]}^{\sigma} \xrightarrow{d_1} (\Gamma/(\varsigma_j^4))^{\sigma+1}, \\ d_1\big([p_1|\cdots|p_\sigma]\big) = \sum \big[\chi(p_1' \cdots p_\sigma')|p_1''|\cdots|p_\sigma''\big] \end{gathered} \tag{4.4}$$

if $\psi(p_i) = \sum p_i' \otimes p_i''$ (omitting summation index). (Dual to $a \otimes m \mapsto \sum \tilde{a}_i' \chi(a_i'')m$ of (4.3) is

$$\begin{gathered} M^* \to A^* \otimes M^*, \\ \phi \mapsto \sum \hat{\alpha}_i \otimes \phi_i \end{gathered}$$

if $\psi(\phi) = \sum \alpha_i \otimes \phi_i$ and $\hat{\alpha} \equiv \sum \overline{\alpha}_j' \chi(\alpha_j'')$,when $\psi(\alpha) = \sum \alpha_j' \otimes \alpha_j''$ and $\overline{\alpha}_j$ is the equivalence class in A_1^*. If we posit that $\hat{\alpha}_i \in \overline{\mathbf{Z}_2[\varsigma_1^4, \varsigma_2^2, \ldots]}$, then only $\overline{\alpha}_j' = 1$ can contribute.)

If S is a finite subset of $2, 3, \ldots$, let

$$\varsigma_S = \prod_{i \in S} \varsigma_i^2.$$

As (4.4) sets 4th powers to 0, it behaves as if

$$\psi(\varsigma_S) = \sum_{T \subset S} \varsigma_T \otimes \varsigma_{S-T}$$

and hence

$$d_1\big([\varsigma_{S_1}|\cdots|\varsigma_{S_\sigma}]\big) = \sum \big[\varsigma_{\cup S_j - T_j}|\varsigma_{T_1}|\cdots|\varsigma_{T_\sigma}\big],$$

with $T_1, \ldots, T_\sigma$ nonempty, and $S_1 - T_1, \ldots, S_\sigma - T_\sigma$ disjoint. This is isomorphic to the ordinary reduced cobar complex of a primitively generated exterior algebra, and so its cohomology is additively isomorphic to a polynomial algebra on $[\varsigma_j^2]$, $j \ge 2$, establishing 2.9(ii) additively.

Of course, the Ext groups relevant to E_1 and E_2 have products of the type claimed in 2.9, but it is important that this product be induced by a product on the entire SS, so that we can use it to draw deductions about higher differentials. As we have seen in 2.2 and 2.3, it will suffice to have a pairing of complexes (2.4) inducing these products. We will first review the facts of relative homological algebra [**Mac**, Chapter IX or **L**, 2.1] which guarantee the existence of such a pairing, and then show how these are used to construct the explicit pairing.

Let $B \to C$ be a homomorphism of Hopf algebras.

DEFINITION 4.5. (i) An exact sequence of B-comodules is (B, C)-*allowable* if it is contractible as a complex of C-comodules, or, equivalently, if the image-kernels are split summands as C-comodules;

(ii) A B-comodule N is *relative injective* if $\operatorname{Hom}_B(\ \ , N)$ is exact on (B, C)-allowable sequences.

PROPOSITION 4.6. (i) *If M is a C-comodule, then $B\Box_C M$ is relative injective.*

(ii) *If $0 \leftarrow M \leftarrow D_*$ is (B,C)-allowable and $0 \leftarrow N \leftarrow E_*$ is a complex with each E_i relative injective, then a B-comodule morphism $M \to N$ extends to a morphism of complexes, unique up to chain homotopy.*

If $\Gamma \approx A^*\Box_{A_1^*}\mathbf{Z}_2$ is as in §2, then the complex $B_\Gamma(M)$ of (2.1) is an (A^*, A_1^*)-allowable sequence of relative injectives, as is $B_\Gamma(M)\otimes B_\Gamma(N)$. (For (2.1) the second criterion of 4.5(i) is easily verified, and for the tensor product the first criterion of 4.5(i) follows from [**Mac**, V9.1].) Thus the pairing (2.4) follows from 4.6(ii).

As in the derivation of the formula for d_1 above, it seems somewhat easier to derive the product formula using left A-modules, and then dualize. We will fill in the following diagram with A-homomorphisms g_i. Here, and throughout the remainder of this section, $Q = A//A_1$, and some $\otimes$'s are omitted.

(4.7)

$$\begin{array}{ccccccc}
N & \leftarrow & A\otimes_{A_1} N & \xleftarrow{d_1} & A\otimes_{A_1}(\bar{Q}\otimes N) & \xleftarrow{d_2} & A\otimes_{A_1}(\bar{Q}\bar{Q}N) & \xleftarrow{d_3} \\
\downarrow g & & \downarrow g_0 & & \downarrow g_1 & & \downarrow g_2 & \\
M_1\otimes M_2 & \leftarrow & QM_1\otimes QM_2 & \xleftarrow{\partial_1} & \begin{array}{c} Q\bar{Q}M_1\otimes QM_2 \\ +QM_1\otimes Q\bar{Q}M_2 \end{array} & \xleftarrow{\partial_2} & \begin{array}{c} Q\bar{Q}\bar{Q}M_1\otimes QM_2 \\ +Q\bar{Q}M_1\otimes Q\bar{Q}M_2 \\ +QM_1\otimes Q\bar{Q}\bar{Q}M_2 \end{array} & \xleftarrow{\partial_3}
\end{array}$$

Here g is a given A-homomorphism, and $g_0(1\otimes n) = 1\otimes g(n)$. The lower sequence is $B(M_1)\otimes B(M_2)$, where ∂ in $B(M)$ satisfies

$$\partial(a_0[a_1|\cdots|a_s]m) = \varepsilon(a_0)a_1[a_2|\cdots|a_s]m.$$

The upper sequence agrees with $B(N)$ under the isomorphisms h_1 and h_2 of (4.2). It satisfies

$$d_s(1\otimes a_1\otimes\cdots\otimes a_s\otimes n) = \sum a_1'\otimes\chi a_1''(a_2\otimes\cdots\otimes a_s\otimes n)$$

if $\psi(a_1) = \sum a_1'\otimes a_1''$. Here we use abusive notation similar to that of (4.4) of omitting subscripts of terms in ψ-formulas; this will be continued throughout this section. The subscripts are instead used to indicate the subscript of the variable whose ψ was taken.

The reason for selecting the somewhat different forms for the sequences in (4.7) is that the (A,A_1)-projectivity is clear in the top sequence, while the (A,A_1)-allowability is clear in the bottom. Here we are using notions dual to 4.5 and 4.6. If s is an A_1-linear chain contraction for the lower sequence, then $g_\sigma(1\otimes[a_1|\cdots|a_\sigma]n)$ can be chosen to be $s_\sigma g_{\sigma-1}d_\sigma(1\otimes[a_1|\cdots|a_\sigma]n)$, and of course g_σ is extended over all of $A\otimes_{A_1}(\bar{Q}^\sigma N)$ by A-linearity (with left A-action on the top sequence and diagonal A-action on the bottom). If s is a chain contraction for $B(M_1)$, and $B(M_2)$ is chain contractible, then $s\otimes 1$ is a chain contraction for $B(M_1)\otimes B(M_2)$. An A_1-linear chain contraction for $B(M_1)$ is given by

$$s_\sigma(a_0[a_1|\cdots|a_{\sigma-1}]m) = [a_0|\cdots|a_{\sigma-1}]m$$

(which is 0 if $a_0 = 1$). Also, $s_0 : M \to Q\otimes M$ satisfies $s_0(m) = 1\otimes m$. Combining these ingredients, and making frequent use of coassociativity and cocommutativity

of ψ on A, and the defining property

$$\sum a_i'\chi(a_i'') = \begin{cases} 0 & \text{if } a \neq 1, \\ 1 & \text{if } a = 1 \end{cases}$$

one obtains inductively (if $g(n) = m_1 \otimes m_2$ and $\psi(a_i) = \sum a_i' \otimes a_i''$)

$$\begin{aligned} g_\sigma(1 \otimes [a_1|\cdots|a_\sigma]n) &= m_1 \otimes [a_1|\cdots|a_\sigma]m_2 \\ &+ \sum_{i=1}^{\sigma}[a_1|\cdots|a_i']m_1 \otimes a_i''[a_{i+1}|\cdots|a_\sigma]m_2 \\ &+ \sum_{i=1}^{\sigma}\sum_{j=1}^{i-1}[a_1|\cdots|a_{j-1}|a_j'|a_j''|a_{j+1}|\cdots|a_{i-1}]m_1 \otimes a_i[a_{i+1}|\cdots|a_\sigma]m_2 \end{aligned} \tag{4.8}$$

with summation signs over primed terms suppressed from notation.

We then follow these g_σ's of (4.8) by the isomorphisms h_2 of (4.1). When $\mathrm{Hom}_A(\ \ ,\mathbf{Z}_2)$ is applied, we obtain (Γ as before is dual to $A//A_1$, and $\mathrm{Hom}(M)$ is short for $\mathrm{Hom}(M,\mathbf{Z}_2)$.)

$$\begin{array}{ccc} \mathrm{Hom}_A(A \otimes_{A_1} (\bar{Q}^\sigma M_1)) \otimes \mathrm{Hom}_A(A \otimes_{A_1} (\bar{Q}^\tau M_2)) & \longrightarrow & \mathrm{Hom}_A(A \otimes_{A_1} (\bar{Q}^{\sigma+\tau} N)) \\ \updownarrow & & \updownarrow \\ \mathrm{Hom}_{A_1}(\bar{Q}^\sigma M_1) \otimes \mathrm{Hom}_{A_1}(\bar{Q}^\tau M_2) & \cdots \xrightarrow{\mu} & \mathrm{Hom}_{A_1}(\bar{Q}^{\sigma+\tau} N) \\ \cap & & \cap \\ \overline{\Gamma}^\sigma M_1^* \otimes \overline{\Gamma}^\tau M_2^* & & \overline{\Gamma}^{\sigma+\tau} N^* \end{array}$$

(4.9)

$$\begin{aligned} &\mu\big([\phi_1|\cdots|\phi_\sigma]\hat{m}_1 \otimes [\phi_{\sigma+1}|\cdots|\phi_{\sigma+\tau}]\hat{m}_2\big) \\ &= \Big([\phi_1|\cdots|\phi_{\sigma-1}|\phi_\sigma\chi(\phi_{\sigma+1}'\cdots\phi_{\sigma+\tau}'\hat{\phi}_2')|\phi_{\sigma+1}''|\cdots|\phi_{\sigma+\tau}''] \\ &\quad + \sum_{i=1}^{\sigma-1}[\phi_1|\cdots|\phi_i\phi_{i+1}|\cdots|\phi_\sigma|\chi(\phi_{\sigma+1}'\cdots\phi_{\sigma+\tau}'\hat{\phi}_2')|\phi_{\sigma+1}''|\cdots|\phi_{\sigma+\tau}'']\Big)\hat{m}_1 m_2' \end{aligned}$$

where $\psi(\phi_i) = \sum \phi' \otimes \phi_i''$ and $\psi(\hat{m}_2) = \sum \hat{\phi}_2' \otimes m_2'$. Although Γ does not admit a comultiplication, if the formula for $\psi(\phi_i)$ in $A^* \otimes A^*$ is used, this product will land in $\overline{\Gamma}^{\sigma+\tau}N^*$ when applied to bonafide elements of $\mathrm{Hom}_{A_1}(\ \ ,\mathbf{Z}_2)$. For example, if M_2 is H^*Y, then to be in $\mathrm{Hom}_{A_1}(\bar{Q}M_2,\mathbf{Z}_2) \subset \overline{\Gamma} \otimes Y$, $[\varsigma_2^2] \otimes 1$ must be completed to $[\varsigma_2^2] \otimes 1 + [\varsigma_1^4] \otimes \varsigma_1^2$, on μ of which $[\varsigma_1^2|\varsigma_1^4]$ appears twice.

To take this into account properly, the pairing (4.9) should be composed with isomorphisms such as those of (4.1) in order to interpret it when M_1, M_2, and N are $A_1//E$ as

$$\mathrm{Hom}_E(\bar{Q}^\sigma,\mathbf{Z}_2) \otimes \mathrm{Hom}_E(\bar{Q}^\tau,\mathbf{Z}_2) \to \mathrm{Hom}_E(\bar{Q}^{\sigma+\tau},\mathbf{Z}_2).$$

A formula similar to (4.9) is obtained, complicated by additional $\tilde{}$-terms such as those of (4.3). When we restrict attention to the v_1-periodic elements of Hom_E, i.e. the elements of $H_*(\bar{Q}^{*\sigma};Q_1)$, the $\tilde{}$-terms are not seen, nor are 4th powers of ς_i's. This simplifies greatly the pairing formula (4.9). When we look at the

induced pairing at E_2, where terms with $\varsigma_j^2\varsigma_k^2$-components are no longer present, (4.9) reduces to

$$\mu\Big([\varsigma_{j_1}^2|\cdots|\varsigma_{j_\sigma}^2]\otimes[\varsigma_{j_{\sigma+1}}^2|\cdots|\varsigma_{j_{\sigma+\tau}}^2]\Big)=[\varsigma_{j_1}^2|\cdots|\varsigma_{j_{\sigma+\tau}}^2],$$

i.e. the multiplicative structure on E_2 is as claimed in 2.9(ii).

5. Koszul spectral sequences. To prove Theorem 1.3 it now suffices to prove 2.9(iii)—that the SS of 2.3 collapses from E_2. This is not immediate. For example, it is conceivable that $d_7([\varsigma_4^2])$ could equal $v_1^{-6}[\varsigma_2^2|\cdots|\varsigma_2^2]$ (8 factors).

In order to prove this collapsing, we calculate $v_1^{-1}\operatorname{Ext}_{A^*}(Y)$ by a sequence of Koszul SS's (KSS's), and compare this calculation with 2.9 to deduce differentials in each SS. The point of using the KSS's is that a generator $[\varsigma_j^2]$ of E_2^{***} of 2.9 will first occur in the KSS converging to $\operatorname{Ext}_{A_j^*}(Y)$, where it has filtration 1. All elements which it could possibly hit by a differential in the SS of 2.9 will have already occurred in the sequence of KSS's; they will have filtration 0 in the KSS converging to $\operatorname{Ext}_{A_j^*}(Y)$, and so there can be no differential in the KSS, and hence not in the SS of 2.9.

The KSS has been presented in [**DM2** and **MS**], so we merely sketch. Let $n\geq 1$, and R denote $\mathbf{Z}_2[\varsigma_1^{2^n},\varsigma_2^{2^{n-1}},\dots,\varsigma_{n+1}]$ with right A_n-module structure as in A^*,

$$\varsigma_{n+1-j}^{2^j}\operatorname{Sq}^{2^j}=\varsigma_{n-j}^{2^{j+1}}.$$

Let R_σ denote the space of homogeneous polynomials of degree σ in the generators, also an A_n-module, as is E, the exterior algebra on these generators. There is an exact sequence of A_n-modules

$$\begin{aligned}(5.1)\qquad 0\to\mathbf{Z}_2\to E\otimes R_0\xrightarrow{d}E\otimes R_1\xrightarrow{d}E\otimes R_2\\ \to d\Big(\prod_{j\in S}\varsigma_j^{2^{n+1-j}}\otimes p\Big)=\sum_{j\in S}\Big(\prod_{k\in S-\{j\}}\varsigma_k^{2^{n+1-k}}\Big)\otimes\varsigma_j^{2^{n+1-j}}p.\end{aligned}$$

Applying $\otimes M$ to (5.1), from 2.2 one obtains a SS coverging to $\operatorname{Ext}_{A_n^*}(\mathbf{Z}_2,M)$ with

$$E_1^{\sigma,s,t}=\operatorname{Ext}^{s,t}_{A_n^*}(\mathbf{Z}_2,E\otimes R_\sigma\otimes M)\approx\operatorname{Ext}^{s,t}_{A_{n-1}^*}(\mathbf{Z}_2,R_\sigma\otimes M),$$

where the isomorphism follows from the change of rings theorem and

$$E\approx A_n^*\square_{A_{n-1}^*}\mathbf{Z}_2,$$

which follows from the well-known ([**ABP**])

$$A_n^*=A^*/(\varsigma_1^{2^{n+1}},\dots,\varsigma_{n+2},\dots).$$

Multiplying E-elements and multiplying R_σ-elements is easily seen to induce a multiplication on the complex (5.1), and arguing as in 2.3 we obtain

THEOREM 5.2. *There is a SS of algebras converging to* $v_1^{-1}\operatorname{Ext}_{A_n^*}(\mathbf{Z}_2,Y)$ *with*

$$E_1^{\sigma,s,t}=v_1^{-1}\operatorname{Ext}^{s,t}_{A_{n-1}^*}(\mathbf{Z}_2,R_\sigma\otimes Y).$$

The argument suggested in the second paragraph of this section could probably be formalized into a complete argument for 2.9(iii) without bothering with

the complete calculation of the KSS's. In the KSS converging to $v_1^{-1}\operatorname{Ext}_{A_j^*}(Y)$, we have

$$E_1^{0,*,*} = v_1^{-1}\operatorname{Ext}^{**}_{A_{j-1}^*}(Y),$$

which contains all elements built from $\varsigma_2^2, \dots, \varsigma_{j-1}^2$, which are the only elements ς_j^2 could possibly hit in the SS of 2.9, for reasons of dimension and filtration. Since in this KSS ς_j^2 has positive filtration, and differentials increase filtration, such a differential cannot occur. The multiplicative structure of 2.9 implies that if all ς_j^2 are permanent cycles, then everything is.

We include the complete calculation of the KSS's in part to jusify the above argument, and in part because we hope to use it in a later calculation of $v_1^{-1}\operatorname{Ext}_A(A_0)$.

THEOREM 5.3. *Let* $h_{i,j} = [\varsigma_i^{2^j}] \in \operatorname{Ext}^{1,2^{i+j}-2^j}$ *(where appropriate) and* $v_i = h_{i+1,0}$.

(i) *The KSS converging to* $v_1^{-1}\operatorname{Ext}_{A_n^*}(Y)$ *has*

$$E_1 = v_1^{-1}\operatorname{Ext}_{A_{n-1}^*}(Y)[v_n^2, h_{j,n+1-j} : 1 \le j \le n, j \ne n-1]$$

with $h_{j,n+1-j} \in E_1^{1,0,*}$ *and* $v_n^2 \in E_1^{2,0,*}$. *For* $2 \le j \le n-1$, *we have*

$$d_1(v_j^{2^{n-j}}) = v_1^{2^{n-j}} h_{j-1,n-j+2}.$$

(ii) $v_1^{-1}\operatorname{Ext}_{A_n^*}(Y) = \mathbf{Z}_2[v_1^{\pm 1}][h_{2,1}, \dots, h_{n,1}, v_2^{2^{n-1}}, v_3^{2^{n-2}}, \dots, v_n^2]$.

This result formalizes the argument for 2.9(iii) of the preceding paragraph. The proof of 5.3 is by induction on n, the case $n = 2$ following easily from [**DM2**]. An outline of the induction step is

(i) Verification that E_1 is as claimed.

(ii) Verification that the asserted d_1-differentials are present.

Since d_1 is a v_1-linear derivation, it follows that E_2 is as in 5.3(ii).

(iii) Proof that there are no other differentials.

Step (i). By definition $E_1^{\sigma,s,t} = v_1^{-1}\operatorname{Ext}^{s,t}_{A_{n-1}^*}(R_\sigma \otimes Y)$. Filtering by the cells of R_σ, one obtains a preliminary SS converging to $v_1^{-1}\operatorname{Ext}^{s,t}_{A_{n-1}^*}(R_\sigma \otimes Y)$ with initial term $v_1^{-1}\operatorname{Ext}_{A_{n-1}^*}(Y) \otimes R_\sigma$. If $(a)Q_1 = b$ in R_σ, then in this SS $d(x \otimes a) = v_1 x \otimes b$. (This is proved by noting that the KSS implies that $v_1^{-1}\operatorname{Ext}_{A_{n-1}^*}(M) = 0$ if $H_*(M; Q_1) = 0$.) This leaves $v_1^{-1}\operatorname{Ext}_{A_{n-1}^*}(Y) \otimes H_*(R_\sigma; Q_1)$, which is as claimed for E_1 in 5.3(i) since $\varsigma_{n+1}Q_1 = \varsigma_{n-1}^4$ is the only nontrivial Q_1-action on a generator.

Step (ii). These differentials should be directly derivable, similarly to [**DM1**, 3.5], but the most elementary way to see them here seems to be comparison with the SS of 2.9, whose E_2-term we know provides an upper bound for $v_1^{-1}\operatorname{Ext}_{A^*}(Y)$. The description of the E_1-term of the KSS in 5.3(i) implies that $v_1^{-1}\operatorname{Ext}^{s,t}_{A_n^*}(Y) \approx v_1^{-1}\operatorname{Ext}^{s,t}_{A^*}(Y)$ for $t - 3s < 2^{n+1} - 3$. (Of all new elements in all subsequent KSS's, the one with smallest $t - 3(s+\sigma)$ is $h_{1,n+1}$.) The elements in the asserted differentials in the KSS converging to $v_1^{-1}\operatorname{Ext}_{A_n^*}(Y)$ all satisfy $t - 3(s+\sigma) < 2^{n+1} - 11$, and by 2.9(ii) these elements cannot be present in $v_1^{-1}\operatorname{Ext}_{A^*}(Y)$. Thus the asserted

differentials must be as claimed. The following diagram of polynomial generators may help.

$$\begin{array}{llllll}
\sigma = 0 & h_{2,1}, & \ldots, h_{n-1,1}, & v_2^{2^{n-2}}, & \ldots, & v_{n-1}^2 \\
 & & & \downarrow & & \downarrow \\
\sigma = 1 & & & h_{1,n}, & \ldots, & h_{n-2,3}, \quad h_{n,1} \\
\sigma = 2 & v_n^2 & & & &
\end{array}$$

Step (iii). To see that there are no other nonzero differentials, we use that d_r increases σ by r but decreases $t - 3(\sigma + s)$ by 3. Using the multiplicative structure, it suffices to show

$$\begin{aligned}
&d_r(h_{n,1}) = 0 && \text{for } r \geq 1, \\
&d_r(v_n^2) = 0 && \text{for } r \geq 1, \text{ and} \\
&d_r(v_j^{2^{n+1-j}}) = 0 && \text{for } r \geq 2.
\end{aligned} \tag{5.4}$$

After elimination of $h_{1,n}, \ldots, h_{n-2,3}$, $E_1^{2,*,*}$ begins with $h_{n,1}^2$ in $t - 3(\sigma + s) = 2^{n+2} - 10$. This $t - 3(\sigma + s)$-degree is at most $2^{n+2} - 16$ for the $v_j^{2^{n+1-j}}$, and for $h_{n,1}$ it is much less. This establishes the first two parts of (5.4), and the third part follows similarly.

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