v₁-PERIODIC EXT OVER THE STEENROD ALGEBRA

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ABSTRACT. For a large family of modules M over the mod 2 Steenrod algebra $A$, $\text{Ext}^{s,t}_{A}(M, \mathbb{Z}_2)$ is periodic for $t < 4s$ with respect to operators $v_1^{2n}$ of period $(2^n, 3 \cdot 2^n)$ for varying $n$. $v_1^{-1}\text{Ext}^{s,t}_{A}(M, \mathbb{Z}_2)$ can be defined by extending this periodic behavior outside this range. We calculate this completely when $M = H^{*}(Y)$, where $Y$ is the suspension spectrum of $RP^2 \wedge CP^2$.

1. Introduction. Let $A$ denote the mod 2 Steenrod algebra, and $A_n$ the subalgebra generated by $\{Sq^j : j \leq 2^n\}$. The Adams periodicity theorem [A] states that if $M$ is an $A$-module which is free as an $A_0$-module, then $\text{Ext}^{s,t}_{A}(M, \mathbb{Z}_2)$ is periodic in the range $t < 4s$. In this paper, we give a complete calculation of $\text{Ext}^{s,t}_{A}(H^{*}Y, \mathbb{Z}_2)$ for $t < 4s$, where $H^{*}Y$ is the $A$-module $(1, Sq^1, Sq^2, Sq^3 = Sq^2 Sq^1)$. Here $Y$ is a suitably indexed suspension spectrum of $RP^2 \wedge CP^2$, which played an important role in [M2]. We view this as an important first step toward the analogous calculation when $M = A_0$, and ultimately perhaps $M = \mathbb{Z}_2$. The result for $A_0$ was announced in [M1].

We recall more precisely the Adams periodicity theorem.

THEOREM 1.1 [A]. For $n \geq 2$, there exists $\omega_n \in \text{Ext}^{2^n, 3 \cdot 2^n}_{A_n}(\mathbb{Z}_2, \mathbb{Z}_2)$ such that

(i) restriction $\text{Ext}_{A_{n+1}}(\mathbb{Z}_2, \mathbb{Z}_2) \to \text{Ext}_{A_n}(\mathbb{Z}_2, \mathbb{Z}_2)$ sends $\omega_{n+1}$ to $\omega_n$;

(ii) if $M$ is a connected $A_0$-free $A_n$-module, then

$$\omega_n: \text{Ext}^{s,t}_{A_n}(M, \mathbb{Z}_2) \to \text{Ext}^{s+2^n, t+3 \cdot 2^n}_{A_n}(M, \mathbb{Z}_2)$$

is an isomorphism if $t < 4s$;

(iii) if $M$ is a connected $A_0$-free $A$-module, then restriction

$$\text{Ext}^{s,t}_{A}(M, \mathbb{Z}_2) \to \text{Ext}^{s,t}_{A_n}(M, \mathbb{Z}_2)$$

is an isomorphism if $0 < s$ and $t < 2^{n+1} + 3s - 5$.

This says that for such a module $M$, in the usual depiction of $\text{Ext}^{s,t}_{A}(M, \mathbb{Z}_2)$ in position $(t - s, s)$, there is $(t - s, s)$-periodicity as indicated in Figure 1.2 below. (The 0-part follows from another result of [A].)

For an $A_0$-free module $M$ we define $v_1^{-1}\text{Ext}^{s,t}_{A}(M, \mathbb{Z}_2)$ to be the bigraded $\mathbb{Z}_2$-vector space obtained by extending the periodic behavior of $\text{Ext}_{A}(M, \mathbb{Z}_2)$ from...
above the $\frac{1}{3}$-line to the region below the $\frac{1}{3}$-line (including negative gradings). More precisely, for any integers $s$ and $t$,

$$v_1^{-1} \text{Ext}^s_A(M, \mathbb{Z}_2) \equiv \text{Ext}^{s+k \cdot 2^n, t+3k \cdot 2^n}_A(M, \mathbb{Z}_2)$$

if $2^{n+1} > t + 5 - 3s$ and $k \cdot 2^n > t - 4s$.

By 1.1, this is independent of $k$ and $n$.

An alternative formulation, due to Mahowald and Shick, explains the reason for the name $v_1^{-1} \text{Ext}(\ )$. If $E_n$ denotes the exterior subalgebra generated by the Milnor primitives $Q_0, \ldots, Q_n$, then $\text{Ext}_{E_n}(\mathbb{Z}_2, \mathbb{Z}_2) \approx \mathbb{Z}_2[v_0, \ldots, v_n]$, where $v_i \in \text{Ext}(1,2^{i+1}-1)$ corresponds roughly to a generator of $\pi_*(BP)$, where $BP$ is the Brown-Peterson spectrum. In [MS] it was proved that for $n \geq i$ there are integers $L(n,i)$ and elements $v_i^{L(n,i)} \in \text{Ext}_{A_n}(\mathbb{Z}_2, \mathbb{Z}_2)$ which restrict to the element of $\text{Ext}_{E_n}$ with the same name. These can be chosen compatibly with respect to increasing $n$, so that for any $A$-module $M$ one can define

$$v_i^{-1} \text{Ext}_A(M, \mathbb{Z}_2) \equiv \lim_{\rightarrow} v_i^{-L(n,i)} \text{Ext}_{A_n}(M, \mathbb{Z}_2).$$

It is easy to see that for $i = 1$ this agrees with the definition using Adams periodicity if $M$ is $A_0$-free. We will use the two definitions interchangeably.

Our main result is

**THEOREM 1.3.** $v_1^{-1} \text{Ext}_A^*(H^Y, \mathbb{Z}_2) \approx \mathbb{Z}_2[v_1^{\pm 1}][h_{j,1}: j \geq 2]$, where

$$h_{j,1} \in v_1^{-1} \text{Ext}(1,2^{j+1}-2)(\ ).$$

A generator $h_{j,1}$ of this polynomial algebra corresponds to the element $s_j^2$ in the dual of the Steenrod algebra. It is not generally the case that $v_1^{-1} \text{Ext}_A(M)$ is a module over $\mathbb{Z}_2[v_1^{\pm 1}]$; usually higher powers of $v_1$ are involved.

This result allows us to write $\text{Ext}_A^*(H^Y, \mathbb{Z}_2)$ above the $\frac{1}{3}$-line in a nice form, with elements expressed as products of elements that may not exist in $\text{Ext}$, but do exist in $v_1^{-1} \text{Ext}$. We illustrate for $t - s \leq 37$ in Figure 1.4 below, with numbers at the end of a sequence indicating the subscripts of the $h_{j,1}$ elements represented.
\( v_1^{-1} \text{Ext}^*_{A^*}(H^* Y, \mathbb{Z}_2) \) can be interpreted as the \( E_2 \)-term of an Adams-type spectral sequence (SS) converging to \( \pi_*(v_1^{-1}Y) \), where \( v_1^{-1}Y \) is the mapping telescope of

\[
Y \xrightarrow{v_1} \Sigma^{-2} Y \xrightarrow{v_1} \Sigma^{-4} Y \xrightarrow{v_1} \ldots,
\]

where \( Y \) is the spectrum mentioned in the first paragraph, which was featured in [M2], and \( v_1 \) is the self-map constructed in [DM1]. However, this is not especially useful, since \( \pi_*(v_1^{-1}Y) \) was calculated by other methods in [M2]; it is, in fact, the exterior algebra over \( \mathbb{Z}_2[v_1^{-1}] \) on \( h_{2,1} \in \pi_5(v_1^{-1}Y) \). To justify this interpretation one shows that the above definitions of \( v_1^{-1} \text{Ext}^*_{A^*}(H^* Y, \mathbb{Z}_2) \) are equivalent to

\[
\lim \text{Ext}^*_{A^*}(H^* Y, \mathbb{Z}_2) \xrightarrow{v_1} \text{Ext}^*_{A^*}(H^* Y, \mathbb{Z}_2) \xrightarrow{v_1} \text{Ext}^*_{A^*}(H^* Y, \mathbb{Z}_2) \xrightarrow{v_1} \ldots,
\]

where the homomorphisms may be thought of as a Yoneda product, or as the Ext homomorphism induced by the filtration-1 map \( v_1 \). Then as in [DM3] one can show that the direct limit of the Adams spectral sequences converges to \( \lim \pi_*(\Sigma^{-2} Y) \).

This Yoneda product defines an action of \( \mathbb{Z}_2[v_1] \) on \( \text{Ext}_{A^*}(H^* Y, \mathbb{Z}_2) \), providing a third definition of \( v_1^{-1} \text{Ext}_{A^*}(H^* Y, \mathbb{Z}_2) \), which can be shown compatible with the others. This is one way of seeing that one does not have to wait for large powers of \( v_1 \) to act on \( \text{Ext}_{A^*}(H^* Y, \mathbb{Z}_2) \), but this is special to \( H^* Y \).

While this paper was being refereed, David Eisen, in his Princeton Ph.D. thesis, generalized our result to \( v_1^{-1} \text{Ext} \) and simplified the proof.

2. Sketch of proof. We prefer to work with the dual object \( H_* Y \), which is an \( A^* \)-comodule, where \( A^* \) is the dual of the Steenrod algebra. We shall denote this simply by \( Y \). Recall that \( A^* = \mathbb{Z}_2[\xi_1, \xi_2, \ldots] \), where \( \xi_i = \chi(\xi_i) \) with \( \xi_i \) the Milnor element of degree \( 2^i - 1 \), and \( \chi \) the canonical antiautomorphism. \( A^* \) is a Hopf algebra with

\[
\psi(\xi_i) = \sum \xi_j \otimes \xi_{i-j}^{2^j}
\]

and a right \( A \)-module with

\[
(\xi_i) \chi \text{Sq} = \xi_i + \xi_{i-1}^2.
\]
Then $Y$ is the subcoalgebra of $A^*$ spanned by $1, \zeta_1, \zeta_2^2,$ and $\zeta_3^3$. Now $\text{Ext}_{A^*}(\mathbb{Z}_2, Y)$ is naturally isomorphic to $\text{Ext}_A(H^*Y, \mathbb{Z}_2)$, and so we define $v_1^{-1}\text{Ext}_{A^*}(\mathbb{Z}_2, Y)$ by periodicity as in §1.

In [Mi], H. Miller considered odd primary analogs of much of our work. He points out there that his methods do not seem readily adaptable to $p = 2$. We have adopted some ideas from his paper, and have benefited from conversations with him. He works with $\text{Cotor}_{A^*}(\mathbb{Z}_2, \ )$ rather than $\text{Ext}_{A^*}(\mathbb{Z}_2, \ )$. As these are naturally isomorphic [R, p. 320], we lose nothing by working with $\text{Ext}$.

We will prove 1.3 by calculating a certain SS. The hard parts are (i) showing that the SS is multiplicative and (ii) showing that its differentials $d_r$ for $r \geq 2$ are all zero. The SS is the dual of the $A/A_1$-SS utilized, for example, in [D].

Let $\Gamma$ denote the subalgebra $\mathbb{Z}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \ldots]$ of $A^*$. This is the dual of $A/A_1$. Let $\bar{\Gamma}$ denote the quotient $\Gamma/(1)$. Then for any $A^*$-comodule $M$, there is an exact cobar resolution $B_{\Gamma}(M)$ of $A^*$-comodules

$$0 \to M \to \Gamma \otimes M \to \Gamma \otimes \bar{\Gamma} \otimes M \to \Gamma \otimes \bar{\Gamma}^2 \otimes M \to \cdots$$

with $\delta(a_0|a_1| \cdots |a_s)m = [a_0|a_1| \cdots |a_s]m$. Here the $A^*$-comodule structure is diagonal, and superscripts denote iterated $\otimes$. The more familiar form of the cobar resolution, with left coaction, requires a comultiplication in its $\delta$, and $\Gamma$ is not an $A^*$-coalgebra.

We will obtain from (2.1) a SS converging to $\text{Ext}_{A^*}(M)$ with

$$E_1^{s,t} = \text{Ext}_{A^*}^{s,t}(\Gamma \otimes \bar{\Gamma}^s \otimes M).$$

Here and henceforth we delete $\mathbb{Z}_2$ from the first component of Ext-groups. We apply a change-of-rings theorem to simplify this $E_1$-term. From [ABP],

$$A_1^s \approx A^*/(\zeta_1^4, \zeta_2^2, \zeta_3^3, \ldots)$$

and hence $\Gamma = A^* A_1^s \mathbb{Z}_2$, so that by [R, A1.3.13]

$$\text{Ext}_{A^*}(\Gamma \otimes \bar{\Gamma}^s \otimes M) \approx \text{Ext}_{A_1^s}(\bar{\Gamma}^s \otimes M).$$

Before we specialize to $M = Y$, we construct this SS in a slightly more general setting, and consider pairings of such spectral sequences. This result is adapted from [R, pp. 323–330].

**Proposition 2.2.** Suppose $A$ is a Hopf algebra over $\mathbb{Z}_2$ and

$$0 \to M \to C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} \cdots$$

is an exact sequence of $A$-comodules, denoted $C$. There is a natural SS of $\text{Ext}_A(\mathbb{Z}_2)$-modules converging to $\text{Ext}^*_{A^*}(M)$ with

$$E_1^{s,t} = \text{Ext}_A^{s,t}(C_\sigma) \text{ and } d_r : E_r^{s,t} \to E_r^{s+r,s-r+1,t}.$$

$E_\infty^{s,t}$ contributes to $\text{Ext}_{A^*}^{s+r,s-r}(M)$. There is a pairing of $\text{Ext}_{A}(\mathbb{Z}_2)$-modules

$$E_r^{s+t}(C) \otimes E_r^{s+s}(C') \to E_r^{s+t}(C \otimes C').$$

**Proof.** Let $B(C) = B_A(C)$ denote the bigraded cobar resolution

$$B(C)_{p,q} = \Lambda \otimes \bar{\Lambda}^p \otimes C_q.$$
Filtering the total complex with respect to second grading and applying 
\( H^*(\text{Hom}_A(Z_2, \_)) \) yields a SS \( E^{\gamma, s,t}_r(C) \) with \( E^{\gamma, s,t}_1 = \text{Ext}_A(Z_2, C_\sigma) \). Filtering instead with respect to first grading shows that SS converges to \( \text{Ext}_A(Z_2, M) \).

Standard methods of homological algebra imply the existence of pairings

\[ B(C) \otimes B(C') \rightarrow B(C \otimes C'), \]

inducing the desired pairing of SS’s. These pairings are associative and commutative up to chain homotopy. Thus if \( Z_2 \) denotes the complex

\[ 0 \rightarrow Z_2 \rightarrow Z_2 \rightarrow 0 \rightarrow 0 \rightarrow \cdots, \]

then the diagram

\[
\begin{array}{ccc}
B(Z_2) \otimes B(C) \otimes B(C') & \rightarrow & B(Z_2) \otimes B(C \otimes C') \\
\downarrow 1 \otimes T & & \downarrow 1 \otimes B(T) \\
B(Z_2) \otimes B(C') \otimes B(C) & \rightarrow & B(Z_2) \otimes B(C' \otimes C) \\
\downarrow & & \downarrow \\
B(Z_2 \otimes C') \otimes B(C) & \rightarrow & B(Z_2 \otimes C' \otimes C) \\
\downarrow T & & \downarrow B(T) \\
B(C) \otimes B(Z_2 \otimes C') & \rightarrow & B(C \otimes Z_2 \otimes C')
\end{array}
\]

commutes up to chain homotopy, showing the pairing is linear with respect to the action of \( \text{Ext}_A(Z_2) \) on the right factor. Linearity with respect to action on the left factor is similar, but easier, completing the proof of 2.2.

If \( v \in \text{Ext}_A(Z_2) \), the module structure implies that the SS of 2.2 passes to a SS converging to \( v^{-1} \text{Ext}_A(M) \) with \( E^{\gamma, s,t}_r = v^{-1} \text{Ext}_A^s\cdot t(C_\sigma) \), and pairings of complexes induce pairings of these SS’s. We apply this to the complex of (2.1) with \( M = Y = A^*DB[[Z_2]] \) and \( A = A^* \), and using again the change-of-rings theorem [R, A1.3.13], we obtain the following result. Here \( E[52] \) denotes the exterior algebra.

**Theorem 2.3.** There is a SS of algebras converging to \( v^{-1}_1 \text{Ext}_A^s(Y) \) with \( E^{\gamma, s,t}_1 = v^{-1}_1 \text{Ext}_A^s\cdot t(E[52]) \).

**Proof.** The SS is that of the preceding paragraph. If the Mahowald-Shick definition of \( v^{-1}_1 \text{Ext}_A(\_ ) \) is used, then Proposition 2.2 must be applied with \( \Lambda = A^*_n \) for all \( n \geq 2 \), inverting the appropriate power of \( v_1 \) each time, and noting compatibility.

If the Adams-periodicity definition is used, then for each of the regions of periodicity of Figure 1.2 the appropriate \( \text{Ext}_{A^*_n}(\_ ) \)-SS of 2.2 can be used. Because \( A_n/A_1 \) begins in degree 4, \( E^{\gamma, s,t}_1 \) will contribute to Figure 1.2 only in the region shaded below and by 1.1 if this overlaps a region of period \( (2^{n+1}, 2^n) \) in 1.2, the same (or shorter) period will be present in the \( E_1 \)-term. The linearity of the differentials with respect to the action of \( v_1^{2^n} = \omega_n \) is necessary in order to deduce that the SS can be extended in the negative direction by periodicity in a well-defined manner.

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Thus we obtain from 2.2 a SS $v_1^{-1}E_r^{***}(C)$ for a complex $C$ as in 2.2. To see that this is a SS of algebras when $C = B_\Gamma(Y)$ of (2.1), we first note that there is a pairing of the complexes (2.1)

$$B_\Gamma(M) \otimes B_\Gamma(N) \xrightarrow{\mu} B_\Gamma(M \otimes N).$$

This is a standard result of relative homological algebra, and will be proved in §4, where we in fact give an explicit formula. Relative homological algebra is required since $\Gamma$ is not an $A^*$-coalgebra, so that the pairing of [R, A1.2.15] does not apply.

Now 2.2 and the remarks after its proof give the first morphism below, the second follows from (2.4), and the third from Lemma 2.6 below.

$$v_1^{-1}E_\Gamma^{***}(B_\Gamma(Y) \otimes B_\Gamma(Y)) \rightarrow v_1^{-1}E_\Gamma^{***}(B_\Gamma(Y))$$

This completes the proof of 2.3.

Lemma 2.6, which is proved in §3, gets around the problem that the action of $\text{Sq}^4$ (or coaction of $s_1^4$) prevents an $A^*$-homomorphism $Y \otimes Y \rightarrow Y$. It is actually done for spectra, and so, as the referee points out, the higher Adams differentials respect the multiplicative structure.

**Lemma 2.6.** There is a connected $A^*$-comodule $Y'$ of finite type satisfying

(i) there is an $A^*$-morphism $Y \rightarrow Y'$ inducing an isomorphism in $v_1^{-1}\text{Ext}_{A^*}(- \otimes M)$ for any $A^*$-comodule $M$;

(ii) there is an $A^*$-morphism $Y \otimes Y \rightarrow Y'$ which is an isomorphism in degree 0.

Next we make more explicit the $E_1$-term of 2.3. If $M$ is any $E[\zeta_2]$-comodule, then we define a boundary homomorphism $\partial$ on $M$ by $\partial(m) = m + \zeta_2 \otimes m'$. Since $\zeta_2$ is primitive in $E[\zeta_2]$ (our $E[\zeta_2]$ is actually a quotient of $A^*_1$), $\partial(\partial(m)) = 0$, so that we can consider $H_* (M; \partial)$. Equivalent to this is the right action of the Milnor primitive $Q_1 \in A$ on $M$, and $H_* (M; Q_1)$, which is the way we shall write it. Since $\text{Ext}_{E[\zeta_2]}(Z_2) \approx Z_2[v_1]$, where $v_1 \in \text{Ext}^{1,3}$, and $\text{Ext}_{A^*_1}(Z_2) \rightarrow \text{Ext}_{E[\zeta_2]}(Z_2)$ sends the Mahowald-Shick or Adams elements to
powers of \( v_1 \), it follows that the change-of-rings isomorphism
\[
\text{Ext}_A^*(A^n \boxtimes A^*_1(A^n \boxtimes E[\{\xi\}] M)) \approx \text{Ext}_{E[\{\xi\}]}(M)
\]
induces an isomorphism of bigraded vector spaces
\[
v_1^{-1} \text{Ext}_{E[\{\xi\}]}^\bullet(A^n \boxtimes A^*_1(A^n \boxtimes E[\{\xi\}] M)) \approx v_1^{-1} \text{Ext}_{E[\{\xi\}]}^\bullet(M).
\]
This is an isomorphism of \( v_1 \)-modules to the extent that this makes sense, i.e., in regions in which \( v_1^n \) acts on the LHS, this action is compatible with the \( 2^n \)th power of \( v_1 \) on the RHS. This RHS is a legitimate \( v_1^{-1} \text{Ext} \), without requiring any or varying of powers of \( v_1 \).

Since an \( E[\{\xi\}] \)-comodule \( M \) splits as the sum of a free and a trivial comodule, one deduces that
\[
(2.7) \quad v_1^{-1} \text{Ext}_{E[\{\xi\}]}^\bullet(M) \approx \mathbb{Z}_2[v_1^{\pm 1}] \otimes H_\bullet(M; \mathbb{Q}_v).
\]
The right action of \( \mathbb{Q}_v \) on \( A^* \) is given by \( (\Omega) Q_1 = \xi_{\Omega}^4 - 2 \), from which we obtain for \( \Gamma = \mathbb{Z}_2[\xi_1^4, \xi_2^2, \xi_3, \ldots] \)
\[
(2.8) \quad H_\bullet(\Gamma; Q_1) \approx E[\xi_j^2 : j \geq 2],
\]
where \( E \) denotes an exterior algebra \([\text{ABP}]\).

Theorem 1.3 follows from the following result, which calculates the SS of 2.3.

**Theorem 2.9.** There is a SS of algebras over \( \mathbb{Z}_2[v_1^{\pm 1}] \) converging to \( v_1^{-1} \text{Ext}_A^*\) (\( Y \)) such that

(i) \( E_1^{\bullet \bullet} \) is (at least additively) a tensor algebra over \( \mathbb{Z}_2[v_1^{\pm 1}] \) on \( E[\xi_j^2 : j \geq 2] \), where \( \prod \xi_{j_1}^t \in E_1^{1,0,t} \) with \( t = 2 \sum (2^{j_1} - 1) \), and \( E \) is the quotient of the exterior algebra mod \( \langle 1 \rangle \).

(ii) \( E_2^{\bullet \bullet} \) is a polynomial algebra \( \mathbb{Z}_2[v_1^{\pm 1}][h_{j,1} : j \geq 2] \), where \( h_{j,1} = [\xi_j^2] \in E_2^{1,0,2^{j+1} - 2} \).

(iii) The SS collapses from \( E_2 \), i.e. \( E_\infty^{\bullet \bullet} = E_2^{\bullet \bullet} \).

We have already proved (i). Part (ii) will be proved in §4 by explicit calculation of \( d_1 \) and the pairing. The proof of (iii) involves a comparison with a sequence of Koszul SS's, which will be done in §5.

### 3. Existence of \( Y' \)

The main result of this section is a topological result which implies 2.6. In this section, \( Y \) denotes the spectrum defined in §1, not its homology.

**Theorem 3.1.** There is a spectrum \( Y' \) and a map \( Y \wedge Y \to Y' \) such that the composite \( S^0 \wedge Y \to Y \wedge Y \to Y' \) induces an isomorphism in \( v_1^{-1} \text{Ext}_A^*(\wedge^*( ) \). 

The remainder of this section is devoted to the proof. Let \( X_2 \) denote the ring spectrum of \([\text{DM1}, 3.5] \), and \( X_2^{(n)} \) its \( n \)-skeleton. We recall that \( X_2 \) is a Thom spectrum over \( \Omega S^2 \), and \( H_\bullet(X_2) \approx \mathbb{Z}_2[\xi_1] \). Through dimension 9, \( \text{Ext}_A(H_\bullet X_2) \) is free over \( \mathbb{Z}_2[v_1] \) on elements in \((t-s, s) \) position \((0,0), (5,1)\), and \((6,1)\). There are no possible higher Adams differentials in this range, and so we obtain a corresponding description of \( \pi_\bullet(X_2) \), which is a \( \mathbb{Z}_2 \)-vector space since it is a ring whose unit has order 2.

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Obstruction theory using this information shows that there is a map \( \Sigma^6 Y \map X_2 \) which extends a filtration-1 element of \( \pi_6(X_2) \). One easily checks that \( \pi_i(X_2/X_2^{(7)}) \) has no elements of positive filtration for \( 6 \leq i \leq 9 \), and so the composite

\[
\Sigma^6 Y \map X_2 \to X_2/X_2^{(7)}
\]

is trivial, and hence there is a compression \( \Sigma^6 Y \map X_2^{(7)} \). Our spectrum \( Y' \) is the cofiber of this map \( v_2 \). Our multiplication is obtained as

\[
Y \wedge Y = X_2^{(3)} \wedge X_2^{(3)} \map X_2^{(7)} \to Y'.
\]

An easy diagram chase shows that the cofiber \( C \) of \( Y \to Y' \) is equivalent to the cofiber of the composite

\[
\Sigma^6 Y \map X_2^{(7)} \to X_2^{(7)}/X_2^{(3)} = \Sigma^4 Y.
\]

This composite is \( v_1 \), and the cofiber of \( v_1 \) realizes \( A_1 \) by [DM1, 2.1]. Thus by [AD], \( \text{Ext}_A(H^\bullet C) \) vanishes above a line of slope 1/5, and hence \( v_1^{-1} \text{Ext}_A(H^\bullet C) = 0 \), establishing the claim that \( Y \to Y' \) induces an isomorphism in \( v_1^{-1} \text{Ext} \).

4. \( E_1 \) and \( E_2 \). In this section we calculate the \( d_1 \)-differential in the SS of 2.9, yielding the additive structure in 2.9(ii), and show that the product structure in 2.9(ii) is as claimed by deriving an explicit formula for the pairing.

The calculation of \( d_1 \) seems somewhat simpler if we work in the category of \( A \)-modules and then dualize. Then the \( d_1 \)-differential is the homomorphism in \( v_1^{-1} \text{Ext}_A(\cdot, \mathbb{Z}_2) \) induced by

\[
\begin{array}{cccccc}
A//A_1 \otimes A_1//E \otimes A//A_1 \otimes M & \to & A//A_1 \otimes A_1//E \otimes M \\
\downarrow & & \downarrow \\
A \otimes A_1 (A_1//E \otimes A//A_1 \otimes M) & \to & A \otimes A_1 (A_1//E \otimes M) \\
\downarrow & & \downarrow \\
A \otimes A_1 (A_1 \otimes E (A//A_1 \otimes M)) & \to & A \otimes A_1 (A_1 \otimes E M) \\
\downarrow & & \downarrow \\
A \otimes E (A//A_1 \otimes M) & \to & A \otimes E M
\end{array}
\]

where \( M \) could be any \( A \)-module, but is \( A//A_1^\sigma \), \( A \)-action is diagonal over unsubscripted \( \otimes \), but is on the left of subscripted \( \otimes \), and

\[
d(a_0 \otimes y \otimes a_1 \otimes m) = \varepsilon(a_0)a_1 \otimes y \otimes m.
\]

The isomorphisms are instances of

\[
\begin{array}{c}
A \otimes_B M \xrightarrow{h_1} A//B \otimes M, \\
h_1(a \otimes m) = \sum a'_i \otimes a''_i m,
\end{array}
\]

\[
\begin{array}{c}
h_2(a \otimes m) = \sum a'_i \otimes \chi(a''_i)m
\end{array}
\]

if \( \psi(a) = \sum a'_i \otimes a''_i \), and \( \chi \) denotes the class in the quotient. Tracing yields

\[
d'(1 \otimes a \otimes m) = \sum a'_i \chi(\tilde{a}'_i) \otimes \tilde{a}''_i \chi(a''_i)m
\]
\[ \psi^3(a) = \sum a'_i \otimes a''_i \otimes a'''_i \otimes a''''_i, \]
and
\[ \tilde{a} = \begin{cases} a & \text{if } |a| \leq 3, \\ 0 & \text{if } |a| > 3. \end{cases} \]

This latter comes into play as the \( A \)-action on \( A_1/E \).

From (2.7) we have \( v_1^{-1} \operatorname{Ext}_A(A \otimes_E M, Z_2) \cong Z_2[v_1^{-1}] \otimes H_*(M^*; Q_1) \), and so \( d_1 \) is determined as dual to the \( 1 \otimes \)-terms in (4.3). Using (2.8), this becomes (cf. \[D\])

\[ \overline{E}[\xi_j^2 : j \geq 2] \xrightarrow{\delta} (\Gamma/\langle \xi_j^4 \rangle)^{\sigma+1}, \]

(4.4)
\[ d_1([p_1 | \cdots | p_\sigma]) = \sum [\chi(p'_1 \cdots p'_\sigma)p''_1 | \cdots | p''_\sigma] \]
if \( \psi(p_i) = \sum p'_i \otimes p''_i \) (omitting summation index). (Dual to \( a \otimes m \mapsto \sum \tilde{a}_i \chi(a''_i)m \) of (4.3) is
\[ M^* \rightarrow A^* \otimes M^*, \]
\[ \phi \mapsto \sum \tilde{\alpha}_i \otimes \phi_i \]
if \( \psi(\phi) = \sum \alpha_i \otimes \phi_i \) and \( \tilde{\alpha}_i = \sum \tilde{\alpha}_j \chi(a''_j), \) when \( \psi(\alpha) = \sum \alpha'_j \otimes \alpha''_j \) and \( \tilde{\alpha}_j \) is the equivalence class in \( A^*_1 \). If we posit that \( \tilde{\alpha}_i \in \mathbb{Z}_2[\xi^4_1, \xi^2_2, \ldots] \), then only \( \tilde{\alpha}_j^2 = 1 \) can contribute.)

If \( S \) is a finite subset of \( 2, 3, \ldots, \), let
\[ \xi_S = \prod_{i \in S} \xi_i^2. \]

As (4.4) sets 4th powers to 0, it behaves as if
\[ \psi(\xi_S) = \sum T \otimes \xi_{S-T} \]
and hence
\[ d_1([\xi_{S_1} | \cdots | \xi_{S_\sigma}]) = \sum [\xi_{S_1 \cup S_j - T_j} | T_{S_j} | \cdots | T_{S_\sigma}], \]
with \( T_1, \ldots, T_\sigma \) nonempty, and \( S_1 - T_1, \ldots, S_\sigma - T_\sigma \) disjoint. This is isomorphic to the ordinary reduced cobar complex of a primitively generated exterior algebra, and so its cohomology is additively isomorphic to a polynomial algebra on \( [\xi^j_j], j \geq 2, \) establishing 2.9(ii) additively.

Of course, the Ext groups relevant to \( E_1 \) and \( E_2 \) have products of the type claimed in 2.9, but it is important that this product be induced by a product on the entire \( SS \), so that we can use it to draw deductions about higher differentials. As we have seen in 2.2 and 2.3, it will suffice to have a pairing of complexes (2.4) inducing these products. We will first review the facts of relative homological algebra [Mac, Chapter IX or L, 2.1] which guarantee the existence of such a pairing, and then show how these are used to construct the explicit pairing.

Let \( B \rightarrow C \) be a homomorphism of Hopf algebras.

**Definition 4.5.** (i) An exact sequence of \( B \)-comodules is \( (B, C) \)-allowable if it is contractible as a complex of \( C \)-comodules, or, equivalently, if the image-kernels are split summands as \( C \)-comodules;

(ii) A \( B \)-comodule \( N \) is relative injective if \( \operatorname{Hom}_B(\ , N) \) is exact on \( (B, C) \)-allowable sequences.

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PROPOSITION 4.6. (i) If $M$ is a $C$-comodule, then $B\square_C M$ is relative injective.

(ii) If $0 \to M \to D_*$ is $(B, C)$-allowable and $0 \to N \to E_*$ is a complex with each $E_i$ relative injective, then a $B$-comodule morphism $M \to N$ extends to a morphism of complexes, unique up to chain homotopy.

If $\Gamma \cong A^*\Box_{A_i} Z_2$ is as in §2, then the complex $B_\Gamma (M)$ of (2.1) is an $(A^*, A_i^*)$-allowable sequence of relative injectives, as is $B_\Gamma (M) \otimes B_\Gamma (N)$. (For (2.1) the second criterion of 4.5(i) is easily verified, and for the tensor product the first criterion of 4.5(i) follows from [Mac, V9.1].) Thus the pairing (2.4) follows from 4.6(ii).

As in the derivation of the formula for $d_1$ above, it seems somewhat easier to derive the product formula using left $A$-modules, and then dualize. We will fill in the following diagram with $A$-homomorphisms $g_i$. Here, and throughout the remainder of this section, $Q = A/A_1$, and some $\otimes$'s are omitted.

(4.7)

\[
\begin{array}{ccccccccc}
N & \to & A \otimes A_1 N & \xrightarrow{d_1} & A \otimes A_1 (Q \otimes N) & \xrightarrow{d_2} & A \otimes A_1 (QQN) & \xrightarrow{d_3} \\
\downarrow g & & \downarrow g_0 & & \downarrow g_1 & & \downarrow g_2 & \\
M_1 \otimes M_2 & \xrightarrow{\partial_1} & QM_1 \otimes QM_2 & \xrightarrow{\partial_2} & QQQM_1 \otimes QM_2 & & \\
\end{array}
\]

Here $g$ is a given $A$-homomorphism, and $g_0(1 \otimes n) = 1 \otimes g(n)$. The lower sequence is $B(M_1) \otimes B(M_2)$, where $\partial$ in $B(M)$ satisfies

$$\partial(a_0[a_1| \cdots |a_s|m) = \varepsilon(a_0)a_1[a_2| \cdots |a_s|m).$$

The upper sequence agrees with $B(N)$ under the isomorphisms $h_1$ and $h_2$ of (4.2).

It satisfies

$$d_s(1 \otimes a_1 \otimes \cdots \otimes a_s \otimes n) = \sum a_1' \otimes a_1''(a_2 \otimes \cdots \otimes a_s \otimes n)$$

if $\psi(a_1) = \sum a_1' \otimes a_1''$. Here we use abusive notation similar to that of (4.4) of omitting subscripts of terms in $\psi$-formulas; this will be continued throughout this section. The subscripts are instead used to indicate the subscript of the variable whose $\psi$ was taken.

The reason for selecting the somewhat different forms for the sequences in (4.7) is that the $(A, A_1)$-projectivity is clear in the top sequence, while the $(A, A_1)$-allowability is clear in the bottom. Here we are using notions dual to 4.5 and 4.6. If $s$ is an $A_1$-linear chain contraction for the lower sequence, then $g_\sigma(1 \otimes [a_1| \cdots |a_\sigma]|n)$ can be chosen to be $s_\sigma g_{\sigma - 1}d_\sigma(1 \otimes [a_1| \cdots |a_\sigma]|n)$, and of course $g_\sigma$ is extended over all of $A \otimes A_1 (\tilde{Q}^n N)$ by $A$-linearity (with left $A$-action on the top sequence and diagonal $A$-action on the bottom). If $s$ is a chain contraction for $B(M_1)$, and $B(M_2)$ is chain contractible, then $s \otimes 1$ is a chain contraction for $B(M_1) \otimes B(M_2)$. An $A_1$-linear chain contraction for $B(M_1)$ is given by

$$s_\sigma(a_0[a_1| \cdots |a_\sigma-1|m) = [a_0| \cdots |a_\sigma-1|m)$$

(which is 0 if $a_0 = 1$). Also, $s_0 : M \to Q \otimes M$ satisfies $s_0(m) = 1 \otimes m$. Combining these ingredients, and making frequent use of coassociativity and cocommutativity.
of $\psi$ on $A$, and the defining property

$$\sum a'_i x(a''_i) = \begin{cases} 0 & \text{if } a \neq 1, \\ 1 & \text{if } a = 1 \end{cases}$$

one obtains inductively (if $g(n) = m_1 \otimes m_2$ and $\psi(a_i) = \sum a'_i \otimes a''_i$)

$$g_\sigma(1 \otimes [a_1| \cdots |a_\sigma] n) = m_1 \otimes [a_1| \cdots |a_\sigma] m_2$$

(4.8) + $\sum_{i=1}^{\sigma} [a_1| \cdots |a'_i|m_1 \otimes a''_i[a_{i+1}| \cdots |a_\sigma] m_2$

+ $\sum_{i=1}^{\sigma-1} \sum_{j=1}^{i} [a_1| \cdots |a_j|a'_j[a_{j+1}| \cdots |a_{i-1}] m_1 \otimes a_i[a_{i+1}| \cdots |a_\sigma] m_2$

with summation signs over primed terms suppressed from notation.

We then follow these $g_\sigma$'s of (4.8) by the isomorphisms $h_2$ of (4.1). When $\text{Hom}_A(,\mathbb{Z}_2)$ is applied, we obtain ($\Gamma$ as before is dual to $A//A_1$, and $\text{Hom}(M)$ is short for $\text{Hom}(M, \mathbb{Z}_2)$.)

$$\text{Hom}_A(A \otimes A_1 (Q^\sigma M_1)) \otimes \text{Hom}_A(A \otimes A_1 (Q^\sigma M_2)) \longrightarrow \text{Hom}_A(A \otimes A_1 (Q^{\sigma+r} N))$$

$$\uparrow$$

$$\text{Hom}_A(Q^\sigma M_1) \otimes \text{Hom}_A(Q^\sigma M_2) \quad \longrightarrow \quad \text{Hom}_A(Q^{\sigma+r} N)$$

$$\cap$$

$$\Gamma^\sigma M_1^* \otimes \Gamma^\sigma M_2^* \cap \Gamma^{\sigma+r} N^*$$

(4.9)

$$\mu([\phi_1| \cdots |\phi_\sigma]|\hat{m}_1 \otimes [\phi_{\sigma+1}| \cdots |\phi_{\sigma+r}]|\hat{m}_2)$$

$$= \left( [\phi_1| \cdots |\phi_{\sigma-1}| \phi_\sigma x(\phi'_{\sigma+1} \cdots \phi'_{\sigma+r} \hat{\phi}_2)|\phi''_{\sigma+1}| \cdots |\phi''_{\sigma+r}] \\
+ \sum_{i=1}^{\sigma-1} [\phi_1| \cdots |\phi_i \phi_{i+1}| \cdots |\phi_\sigma x(\phi'_{\sigma+1} \cdots \phi'_{\sigma+r} \hat{\phi}_2)|\phi''_{\sigma+1}| \cdots |\phi''_{\sigma+r}] \right) \hat{m}_1 \hat{m}_2$$

where $\psi(\phi_i) = \sum \phi' \otimes \phi''_i$ and $\psi(\hat{m}_2) = \sum \hat{\phi}_2 \otimes m_2'$. Although $\Gamma$ does not admit a comultiplication, if the formula for $\psi(\phi_i)$ in $A^* \otimes A^*$ is used, this product will land in $\Gamma^{\sigma+r} N^*$ when applied to bona fide elements of $\text{Hom}_A(,\mathbb{Z}_2)$. For example, if $M_2$ is $H^*Y$, then to be in $\text{Hom}_A(Q^\sigma M_2, \mathbb{Z}_2) \subset \Gamma \otimes Y$, $[\sigma_2^2] \otimes 1$ must be completed to $[\sigma_2^2] \otimes 1 + [\sigma_1^4] \otimes \sigma_1^2$, on $\mu$ of which $[\sigma_2^2] \otimes [\sigma_1^4]$ appears twice.

To take this into account properly, the pairing (4.9) should be composed with isomorphisms such as those of (4.1) in order to interpret it when $M_1, M_2$, and $N$ are $A_1//E$ as

$$\text{Hom}_E(Q^\sigma, \mathbb{Z}_2) \otimes \text{Hom}_E(Q^*, \mathbb{Z}_2) \longrightarrow \text{Hom}_E(Q^{\sigma+r}, \mathbb{Z}_2).$$

A formula similar to (4.9) is obtained, complicated by additional $\sim$-terms such as those of (4.3). When we restrict attention to the $v_1$-periodic elements of $\text{Hom}_E$, i.e. the elements of $H_*(Q^\sigma; Q_1)$, the $\sim$-terms are not seen, nor are 4th powers of $\sigma_i$'s. This simplifies greatly the pairing formula (4.9). When we look at the
induced pairing at $E_2$, where terms with $s_{j}^{2}s_{k}^{2}$-components are no longer present, (4.9) reduces to
$$
\mu \left( \left[ s_{j_1}^{2} \cdots | s_{j_r}^{2} \right] \otimes \left[ s_{j_{r+1}}^{2} \cdots | s_{j_{r+r}}^{2} \right] \right) = \left[ s_{j_1}^{2} \cdots | s_{j_{r+r}}^{2} \right],
$$
i.e. the multiplicative structure on $E_2$ is as claimed in 2.9(ii).

5. Koszul spectral sequences. To prove Theorem 1.3 it now suffices to prove 2.9(iii)—that the SS of 2.3 collapses from $E_2$. This is not immediate. For example, it is conceivable that $d_7(\{f|\})$ could equal $\nu_1^{-6}[s_2^{2}] \cdots [s_2^{2}]$ (8 factors).

In order to prove this collapsing, we calculate $\nu_1^{-1}\text{Ext}_{A^*}(Y)$ by a sequence of Koszul SS’s (KSS’s), and compare this calculation with 2.9 to deduce differentials in each SS. The point of using the KSS’s is that a generator $[s_j^2]$ of $E_2^{**}$ of 2.9 will first occur in the KSS converging to $\text{Ext}_{A^*}(Y)$, where it has filtration 1. All elements which it could possibly hit by a differential in the SS of 2.9 will have already occurred in the sequence of KSS’s; they will have filtration 0 in the KSS converging to $\text{Ext}_{A^*}(Y)$, and so there can be no differential in the KSS, and hence not in the SS of 2.9.

The KSS has been presented in [DM2 and MS], so we merely sketch. Let $n \geq 1$, and $R$ denote $\mathbb{Z}_2[s_1^{2n}, s_2^{2n-2}, \ldots, s_{n+1}]$ with right $A_n$-module structure as in $A^*$,
$$
\delta_{n+1-j} s_{k}^{2j+1} = s_{n-j}^{2j+1}.
$$
Let $R_\sigma$ denote the space of homogeneous polynomials of degree $\sigma$ in the generators, also an $A_n$-module, as is $E$, the exterior algebra on these generators. There is an exact sequence of $A_n$-modules
$$
0 \rightarrow \mathbb{Z}_2 \rightarrow E \otimes R_0 \xrightarrow{d} E \otimes R_1 \xrightarrow{d} E \otimes R_2 \rightarrow \cdots
$$

Applying $\otimes M$ to (5.1), from 2.2 one obtains a SS converging to $\text{Ext}_{A_2^*}(\mathbb{Z}_2, M)$ with
$$
E_1^{\sigma,s,t} = \text{Ext}_{A_n^*}^{s,t}(\mathbb{Z}_2, E \otimes R_\sigma \otimes M) \approx \text{Ext}_{A_{n-1}^*}^{s,t}(\mathbb{Z}_2, R_\sigma \otimes M),
$$
where the isomorphism follows from the change of rings theorem and
$$
E \approx A_n^* \square A_{n-1}^* \mathbb{Z}_2,
$$
which follows from the well-known ([ABP])
$$
A_n^* = A^*/(s_1^{2n+1}, \ldots, s_{n+2}, \ldots).
$$
Multiplying $E$-elements and multiplying $R_\sigma$-elements is easily seen to induce a multiplication on the complex (5.1), and arguing as in 2.3 we obtain

**Theorem 5.2.** There is a SS of algebras converging to $\nu_1^{-1}\text{Ext}_{A^*}(\mathbb{Z}_2, Y)$ with
$$
E_1^{\sigma,s,t} = \nu_1^{-1}\text{Ext}_{A_{n-1}^*}^{s,t}(\mathbb{Z}_2, R_\sigma \otimes Y).
$$
The argument suggested in the second paragraph of this section could probably be formalized into a complete argument for 2.9(iii) without bothering with
the complete calculation of the KSS's. In the KSS converging to $v_1^{-1} \text{Ext}_{A^*_n}(Y)$, we have

$$E_{1,1}^{0,0} = v_1^{-1} \text{Ext}_{A^*_n}^{*,*}(Y),$$

which contains all elements built from $s^2, \ldots, s^2_{n-1}$, which are the only elements $s^2$ could possibly hit in the SS of 2.9, for reasons of dimension and filtration. Since in this KSS $s^2$ has positive filtration, and differentials increase filtration, such a differential cannot occur. The multiplicative structure of 2.9 implies that if all $s^2$ are permanent cycles, then everything is.

We include the complete calculation of the KSS's in part to justify the above argument, and in part because we hope to use it in a later calculation of $v_1^{-1} \text{Ext}_A(A_0)$.

**Theorem 5.3.** Let $h_{i,j} = [s^j] \in \text{Ext}^{1,2i+j-2j}(\text{where appropriate})$ and $v_i = h_{i+1,0}$. 

(i) The KSS converging to $v_1^{-1} \text{Ext}_{A^*_n}(Y)$ has

$$E_1 = v_1^{-1} \text{Ext}_{A^*_n}^{*,*}(Y)[v^2_n, h_{j,n+1-j} : 1 \leq j \leq n, j \neq n - 1]$$

with $h_{j,n+1-j} \in E_1^{1,0,*}$ and $v^2_n \in E_2^{1,0,*}$. For $2 \leq j \leq n-1$, we have

$$d_1(v^2_{n-j}) = v_1^{2n-j} h_{j-1,n-j+2}.$$

(ii) $v_1^{-1} \text{Ext}_{A^*_n}(Y) = Z_2[v_1^{\pm 1}][h_2, \ldots, h_{n,1}, v^2_n, v^2_2, \ldots, v^2_n]$.

This result formalizes the argument for 2.9(iii) of the preceding paragraph. The proof of 5.3 is by induction on $n$, the case $n = 2$ following easily from [DM2]. An outline of the induction step is

(i) Verification that $E_1$ is as claimed.

(ii) Verification that the asserted $d_1$-differentials are present.

Since $d_1$ is a $v_1$-linear derivation, it follows that $E_2$ is as in 5.3(ii).

(iii) Proof that there are no other differentials.

**Step (i).** By definition $E_1^{s,s,t} = v_1^{-1} \text{Ext}_{A^*_n}^{s,t}(R_{\sigma} \otimes Y)$. Filtering by the cells of $R_{\sigma}$, one obtains a preliminary SS converging to $v_1^{-1} \text{Ext}_{A^*_n}^{s,t}(R_{\sigma} \otimes Y)$ with initial term $v_1^{-1} \text{Ext}_{A^*_n}^{s,t}(Y) \otimes R_{\sigma}$. If $(a)Q_1 = b$ in $R_{\sigma}$, then in this SS $d(x \otimes a) = v_1 x \otimes b$. (This is proved by noting that the KSS implies that $v_1^{-1} \text{Ext}_{A^*_n}^{s,t}(M) = 0$ if $H_*(M;Q_1) = 0$.) This leaves $v_1^{-1} \text{Ext}_{A^*_n}^{s,t}(Y) \otimes H_*(R_{\sigma};Q_1)$, which is as claimed for $E_1$ in 5.3(i) since $s_{n+1}Q_1 = s_4$ is the only nontrivial $Q_1$-action on a generator.

**Step (ii).** These differentials should be directly derivable, similarly to [DM1, 3.5], but the most elementary way to see them here seems to be comparison with the SS of 2.9, whose $E_2$-term we know provides an upper bound for $v_1^{-1} \text{Ext}_{A^*_n}(Y)$. The description of the $E_1$-term of the KSS in 5.3(i) implies that $v_1^{-1} \text{Ext}_{A^*_n}^{s,t}(Y) \approx v_1^{-1} \text{Ext}_{A^*_n}^{s,t}(Y)$ for $t - 3s < 2^{n+1} - 3$. (Of all new elements in all subsequent KSS's, the one with smallest $t - 3(s + \sigma)$ is $h_{1,n+1}$.) The elements in the asserted differentials in the KSS converging to $v_1^{-1} \text{Ext}_{A^*_n}(Y)$ all satisfy $t - 3(s + \sigma) < 2^{n+1} - 11$, and by 2.9(ii) these elements cannot be present in $v_1^{-1} \text{Ext}_{A^*_n}(Y)$. Thus the asserted
differentials must be as claimed. The following diagram of polynomial generators may help.

$$\begin{align*}
\sigma = 0 & \quad h_{2,1}, \ldots, h_{n-1,1}, v_2^{2n-2}, \ldots, v_{n-1}^2 \\
\sigma = 1 & \quad h_{1,n}, \ldots, h_{n-2,3}, h_{n,1} \\
\sigma = 2 & \quad v_n^2
\end{align*}$$

Step (iii). To see that there are no other nonzero differentials, we use that $d_r$ increases $\sigma$ by $r$ but decreases $t - 3(\sigma + s)$ by $3$. Using the multiplicative structure, it suffices to show

$$
\begin{align*}
& d_r(h_{n,1}) = 0 \quad \text{for } r \geq 1, \\
& d_r(v_n^2) = 0 \quad \text{for } r \geq 1, \text{ and} \\
& d_r(v_{j+1}^{2n+1-j}) = 0 \quad \text{for } r \geq 2.
\end{align*}$$

After elimination of $h_{1,n}, \ldots, h_{n-2,3}$, $E^2_{t,*}$ begins with $h_{n,1}^2$ in $t - 3(\sigma + s) = 2n+2 - 10$. This $t - 3(\sigma + s)$-degree is at most $2n+2 - 16$ for the $v_j^{2n+1-j}$, and for $h_{n,1}$ it is much less. This establishes the first two parts of (5.4), and the third part follows similarly.

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