

## DEFINABLE SETS IN ORDERED STRUCTURES. III

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ABSTRACT. We show that any  $o$ -minimal structure has a strongly  $o$ -minimal theory.

**0. Introduction.** In this paper we prove that an arbitrary  $o$ -minimal structure  $M$  is strongly  $o$ -minimal. This was proved in [1] in the case when the ordering on  $M$  is dense.

In §1 we show that for discrete  $M$ ,  $o$ -minimal implies strongly  $o$ -minimal. This is, of course, a result on uniform finite bounds. The proof has some interesting differences with the dense case, partly because here one has to prove uniform bound results for functions defined on *finite* intervals. In [3] it was shown that strongly  $o$ -minimal discrete structures are “trivial”, i.e. there are no definable functions other than translations in one variable. Thus, with the results of §1, discrete  $o$ -minimal structures lose their interest.

In §2 we show that the discrete and dense parts of an arbitrary  $o$ -minimal structure are “orthogonal”, from which our main result follows.

Recall that the structure  $(M, <, \dots)$  is said to be  $o$ -minimal if  $<^m$  is a linear ordering, and every definable (with parameters) subset  $X \subset M$  is a finite union of points and intervals  $(a, b)$  (where  $a \in M \cup \{-\infty\}$ ,  $b \in M \cup \{\infty\}$ ).

We use freely notation and results from previous papers on the subject [1, 2 and 3].

**1. The discrete case.** We say that the  $o$ -minimal structure  $M$  is *discrete* if every element  $a$  of  $M$  has an immediate successor  $S(a)$  and an immediate predecessor  $S^{-1}(a)$ . This is rather a strong definition, and our results here are valid for  $M$  satisfying a weaker notion of discrete, as we subsequently point out.

We now fix discrete  $o$ -minimal  $M$ .

**DEFINITION 1.1.** Let  $X \subset M$ . We say that  $X$  is *scattered* if for no  $a \in M$  does  $X$  contain both  $a$  and  $S(a)$ .

Note that by  $o$ -minimality, any definable scattered  $X \subset M$  is finite.

We are going to prove

**THEOREM 1.2.** Let  $\varphi(\bar{x}, y) \in L(M)$  be such that, for every  $\bar{a} \subset M$  ( $l(\bar{a}) = l(\bar{x})$ ),  $\varphi(\bar{a}, y)^M$  is scattered. Then there is  $N < \omega$  such that, for every  $\bar{a}$ ,  $|\varphi(\bar{a}, y)^M| < N$ .

This will be proved by induction on  $n = l(\bar{x})$ . First we need some preliminary definitions.

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DEFINITION 1.3. Let  $f$  be a definable function of one variable on the interval  $I = (a, b)$ . We call  $f$  an *order-preserving translation* on  $I$  if for all  $c$  such that  $c$  and  $S(c)$  are both in  $I$ ,  $f(S(c)) = S(f(c))$ . Similarly for  $f$  to be an *order-reversing translation*.

DEFINITION 1.4. Let  $\varphi(x, y) \in L(M)$  be such that, for all  $a \in M$ ,  $\varphi(a, y)^M$  is scattered. We say that  $a \in M$  is *good* for  $\varphi$  if for any  $b$  with  $\vDash \varphi(a, b)$ , one of the following holds.

- (i)  $\vDash \varphi(S^{-1}(a), b) \wedge \varphi(S(a), b)$ .
- (ii) for some  $m \leq 0 \leq n$ ,  $m, n \in \mathbf{Z}$ , both  $m, n$  even, we have for  $m \leq i \leq n$ ,

$$\begin{aligned} &\vDash \varphi(a, S^i(b)) && \text{iff } i \text{ is even,} \\ \vDash \varphi(S^{-1}(a), S^{i-1}(b)) && \text{iff } i \text{ is even,} \\ \vDash \varphi(S(a), S^{i+1}(b)) && \text{iff } i \text{ is even,} \end{aligned}$$

and for  $i = 1, 2$ ,

$$\begin{aligned} &\vDash \neg\varphi(a, S^{n+i}(b)) \wedge \neg\varphi(a, S^{m-1}(b)) \\ &\quad \wedge \neg\varphi(S^{-1}(a), S^{n-1+i}(b)) \wedge \neg\varphi(S^{-1}(a), S^{m-1-i}(b)) \\ &\quad \wedge \neg\varphi(S(a), S^{n+1+i}(b)) \wedge \neg\varphi(S(a), S^{m+1-i}(b)), \end{aligned}$$

- (iii) for some  $m \leq 0 \leq n$ ,  $m, n \in \mathbf{Z}$ , both even, we have for  $m \leq i \leq n$

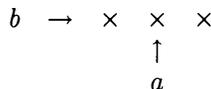
$$\begin{aligned} &\vDash \varphi(a, S^i(b)) && \text{iff } i \text{ is even,} \\ \vDash \varphi(S^{-1}(a), S^{i+1}(b)) && \text{iff } i \text{ is even,} \\ \vDash \varphi(S(a), S^{i-1}(b)) && \text{iff } i \text{ is even,} \end{aligned}$$

and for  $i = 1, 2$ ,

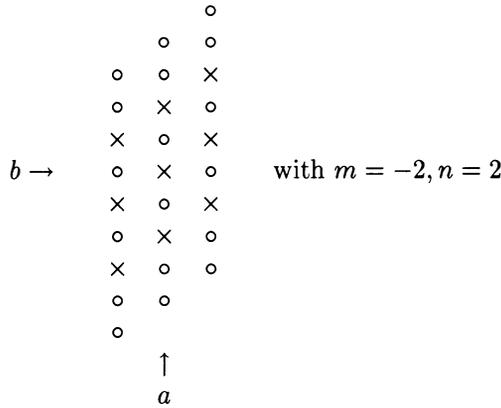
$$\begin{aligned} &\vDash \neg\varphi(a, S^{n+i}(b)) \wedge \neg\varphi(a, S^{m-i}(b)) \wedge \neg\varphi(S^{-1}(a), S^{n+1+i}(b)) \\ &\quad \wedge \neg\varphi(S^{-1}(a), S^{m+1-i}(b)) \\ &\quad \wedge \neg\varphi(S(a), S^{n-1+i}(b)) \wedge \neg\varphi(S(a), S^{m-1-i}(b)). \end{aligned}$$

We say also that  $a \in M$  is *bad* for  $\varphi$  if it is not good for  $\varphi$ . If the pair  $(a, b)$  is such that  $\vDash \varphi(a, b)$  and (i), (ii), (iii) all fail for  $(a, b)$  we say that  $(a, b)$  is a *nasty point* for  $\varphi$ .

REMARK 1.5. As  $\varphi(a, y)^m$  is finite for all  $a$  (in Definition 1.4) we see that conditions (i), (ii), (iii) are each definable conditions on the pair  $(a, b)$ . Condition (i) can be represented by the picture



Condition (ii) can be represented by



Similarly for condition (iii). (Here  $\times$  represents a point on the graph of  $\varphi$ , and  $\circ$  a point not on the graph of  $\varphi$ .)

To prove Theorem 1.2 we will prove by induction on  $n$  the following statements (1.6 being a restatement of the theorem).

(1.6) $_n$  If  $\varphi(\bar{x}, y) \in L(M)$ ,  $l(\bar{x}) = n$  and for all  $\bar{a} \subset M$   $\varphi(\bar{a}, y)^M$  is scattered then there is  $N < \omega$  such that, for all  $\bar{a} \in M$ ,  $|\varphi(\bar{a}, y)^M| < N$ .

(1.7) $_n$  Let  $f(\bar{z}, x)$  be a (partial) definable function with  $l(\bar{z}) = n$ . Then there is  $N < \omega$  such that, for any  $\bar{a}$  there are  $c_1 < \dots < c_k \in M$  with  $k \leq N$  and  $c_1 = -\infty$ ,  $c_k = +\infty$  such that, for any  $(c_i, c_{i+1}) \neq \emptyset$ ,  $f(\bar{a}, x)$  is either undefined on  $(c_i, c_{i+1})$  or  $f(\bar{a}, x) \upharpoonright (c_i, c_{i+1})$  is constant or a translation.

(1.8) $_n$  Let  $\varphi(\bar{z}, x, y) \in L(M)$  with  $l(\bar{z}) = n$  be such that, for each  $\bar{c}, a$  in  $M$ ,  $\varphi(\bar{c}, a, y)^M$  is scattered. Then there is  $N < \omega$  such that, for all  $\bar{c} \subset M$

$$|\{a \in M : a \text{ is bad for } \varphi(\bar{c}, x, y)\}| < N.$$

PROOF OF (1.6) FOR  $n = 1$ . So we have  $\varphi(x, y) \in L(M)$  such that, for all  $a \in M$ ,  $\varphi(a, y)^M$  is scattered (and so finite).

CLAIM 1.9. Only finitely many  $a \in M$  are bad for  $\varphi(x, y)$ .

PROOF OF CLAIM. We suppose not and get a contradiction. So by  $\mathcal{o}$ -minimality there is an infinite interval  $I = (a_1, a_2)$  ( $= \{x \in M : a_1 < x < a_2\}$ ) such that every  $a \in I$  is bad for  $\varphi$ . Now for  $a \in I$ , let  $f(a) =$  the first  $b$  such that the pair  $(a, b)$  is a nasty point for  $\varphi$ . By Theorem 4.2 [2] there is an infinite subinterval of  $I$  which we again call  $I$  such that  $f \upharpoonright I$  is either constant or an order-preserving or reversing bijection of  $I$  with another interval. Clearly  $f \upharpoonright I$  cannot be constant (as (i) of Definition 1.4 fails for  $(a, f(a))$ ). So let us assume  $f$  to be nonconstant and order preserving. Now define for  $a \in I$

$$\begin{aligned}
 g(a) &= \text{the greatest } b \text{ such that } b \geq f(a), \models \varphi(a, b) \\
 &\text{and } \models \neg\varphi(a, S^2(b)), \text{ and for every } c \text{ with } f(a) \leq c \leq b, \\
 &\models \varphi(a, c) \text{ or } \models \varphi(a, S(c)).
 \end{aligned}$$

(\*) Note that for every  $a \in I$ ,  $g(a)$  is defined,  $f(a) < g(a)$  and  $g(a) = S^m(f(a))$  for some even  $m \in \mathbf{Z}^+$ .

Again, for some infinite subinterval of  $I$  which we again call  $I$ ,  $g \upharpoonright I$  is constant, or an order-preserving or reversing bijection of  $I$  with another interval. If  $g$  were

order preserving, then easily, for any  $a \in I$ ,  $(a, f(a))$  satisfies (ii) of Definition 1.4 contradicting  $(a, f(a))$  being nasty for  $\varphi$ .

If  $g \upharpoonright I$  were constant, pick  $a \in I$  such that  $S^{-1}(a) \in I$ . By  $(*)$   $g(a) = S^m(f(a))$  for some even  $m \in \mathbf{Z}^+$ . But then  $g(S^{-1}(a)) = g(a) = S^m(f(a)) = S^{m+1}(S^{-1}(f(a))) = S^{m+1}(f(S^{-1}(a)))$  (as  $f$  is an order-preserving translation), which contradicts  $(*)$ ,  $m + 1$  being odd.

If  $g$  were order reversing on  $I$ , we can pick  $a \in I$  such that  $S^m(a) \in I$  for all  $m \in \mathbf{Z}^-$ . Let  $b = S^2(f(a))$ . Then clearly for  $m \in \mathbf{Z}^-$ ,  $\models \varphi(S^m(a), b)$  if and only if  $m$  is even. So the formula  $\varphi(x, b)$  cannot define a finite union of intervals and points, contradicting  $o$ -minimality.

Thus the assumption that  $f$  is order preserving leads to a contradiction. A similar argument shows that  $f$  cannot be order reversing.

This completes the proof of Claim 1.9.

CLAIM 1.10. Let  $c_1, c_2 \in M$  be such that, for all  $a \in (c_1, c_2)$ ,  $a$  is good for  $\varphi(x, y)$ . Then there is  $k < \omega$  such that, for all  $a \in (c_1, c_2)$ ,  $|\varphi(a, y)^M| = k$ .

PROOF. If not, then by the  $o$ -minimality there is  $a \in (c_1, c_2)$  such that  $S(a) \in (c_1, c_2)$  and  $|\varphi(a, y)^M| \neq |\varphi(S(a), y)^M|$ . On the other hand, it is clear from Definition 1.4 that if  $a$  is good for  $\varphi$  then  $|\varphi(S^i(a), y)^M| \geq |\varphi(a, y)^M|$  for  $i = 1, -1$ . So we get a contradiction, proving Claim 1.10.

Now, clearly from Claims 1.9 and 1.10 it follows that for some  $N < \omega$ ,  $|\varphi(a, y)^M| < N$  for all  $a \in M$ . So we have proved 1.6 for  $n = 1$ .

We now proceed with the induction steps.

PROOF OF (1.7) $_n$  ASSUMING (1.6) $_n$ . So we are given an  $M$ -definable (partial) function  $f(\bar{z}, x)$  where  $l(\bar{z}) = n$ . We first define certain sets depending on  $\bar{z}$ :

- $A_{-1}(\bar{z}) = \{x: f(\bar{z}, S^{-1}(x)) \text{ is defined and } f(\bar{z}, x) \text{ is undefined}\},$
- $A_0(\bar{z}) = \{x: f(\bar{z}, x) \text{ is undefined and } f(\bar{z}, S(x)) \text{ is defined}\},$
- $A_1(\bar{z}) = \{x: f(\bar{z}, S^{-1}(x)) = f(\bar{z}, x) = f(\bar{z}, S(x))\},$
- $A_2(\bar{z}) = \{x: f(\bar{z}, S^{-1}(x)) < f(\bar{z}, x) < f(\bar{z}, S(x))\},$
- $A_3(\bar{z}) = \{x: f(\bar{z}, S^{-1}(x)) > f(\bar{z}, x) > f(\bar{z}, S(x))\},$
- $A_4(\bar{z}) = \{x: f(\bar{z}, S^{-1}(x)) = f(\bar{z}, x) < f(\bar{z}, S(x))\},$
- $A_5(\bar{z}) = \{x: f(\bar{z}, S^{-1}(x)) = f(\bar{z}, x) > f(\bar{z}, S(x))\},$
- $A_6(\bar{z}) = \{x: f(\bar{z}, S^{-1}(x)) < f(\bar{z}, x) = f(\bar{z}, S(x))\},$
- $A_7(\bar{z}) = \{x: f(\bar{z}, S^{-1}(x)) > f(\bar{z}, x) = f(\bar{z}, S(x))\},$
- $A_8(\bar{z}) = \{x: f(\bar{z}, S^{-1}(x)) < f(\bar{z}, x) \text{ and } f(\bar{z}, x) > f(\bar{z}, S(x))\}$
- $A_9(\bar{z}) = \{x: f(\bar{z}, S^{-1}(x)) > f(\bar{z}, x) \text{ and } f(\bar{z}, x) < f(\bar{z}, S(x))\}.$

The notation in  $A_1(\bar{z})$  to  $A_9(\bar{z})$  is supposed to imply that  $f(\bar{z}, S^{-1}(x))$ ,  $f(\bar{z}, x)$  and  $f(\bar{z}, S(x))$  are well defined.

Note that each of  $A_{-1}(\bar{z})$ ,  $A_0(\bar{z})$  and  $A_4(\bar{z})$  to  $A_9(\bar{z})$  defines a scattered set. Thus by (1.6) $_n$  we obtain some  $N_1 < \omega$  such that, for all  $\bar{z}$ ,

$$\left| A_{-1}(\bar{z}) \cup A_0(\bar{z}) \cup \bigcup_{i=4}^9 A_i(\bar{z}) \right| < N_1.$$

Thus we easily obtain, for each  $\bar{z}$ , elements  $d_1(\bar{z}) < \dots < d_k(\bar{z})$  uniformly definable from  $\bar{z}$ , for some  $k < N_1$  such that for each  $1 \leq i < i + 1 \leq k$ , either  $f(\bar{z}, x) \upharpoonright (d_i(\bar{z}), d_{i+1}(\bar{z}))$  is undefined or  $(S(d_i(\bar{z})), S^{-1}(d_{i+1}(\bar{z}))) \subseteq A_j(\bar{z})$  for  $j = 1, 2$  or  $3$ . Clearly, if  $(S(d_i(\bar{z})), S^{-1}(d_{i+1}(\bar{z}))) \subseteq A_1(\bar{z})$  then  $f(\bar{z}, x) \upharpoonright (d_i(\bar{z}), d_{i+1}(\bar{z}))$  is constant.

We can also assume that if  $(S(d_i(\bar{z})), S^{-1}(d_{i+1}(\bar{z}))) \subseteq A_2(\bar{z})$  then  $f(\bar{z}, x) \upharpoonright (d_i(\bar{z}), d_{i+1}(\bar{z}))$  is order preserving. Analogously for  $A_3(\bar{z})$ . Now, fixing  $m$  with  $1 \leq m \leq N_1$ , let

$$B_2^m(\bar{z}) = \{x: (S(d_m(\bar{z})), S^{-1}(d_{m+1}(\bar{z}))) \subseteq A_2(\bar{z}) \text{ and } S(d_m(\bar{z})) < x < S^{-1}(d_{m+1}(\bar{z})) \text{ and } f(\bar{z}, S(x)) > S(f(\bar{z}, x))\}.$$

Similarly  $B_3^m(\bar{z}) = \{x: (S(d_m(\bar{z})), S^{-1}(d_{m+1}(\bar{z}))) \subseteq A_3(\bar{z}) \text{ and } S(d_m(\bar{z})) < x < S^{-1}(d_{m+1}(\bar{z})) \text{ and } f(\bar{z}, S(x)) < S^{-1}(f(\bar{z}, x))\}$ . Note that for each  $\bar{z}, m$  and  $i = 2$  or  $3$ ,  $f(\bar{z}, x) \upharpoonright B_i^m(\bar{z})$  is one-to-one and its range is a scattered set. Thus by (1.6)<sub>n</sub> there is  $N_2 < \omega$  such that, for every  $\bar{z}$ ,

$$\left| \bigcup_{\substack{i=2,3 \\ 1 \leq j \leq N_1}} B_i^j(\bar{z}) \right| < N_2.$$

Now for any  $\bar{a}$  if we enumerate

$$\{d_m(\bar{a}): m < N_1\} \cup \bigcup_{\substack{i=2,3 \\ 1 \leq m \leq N_1}} B_i^m(\bar{a}) \text{ as } c_1, \dots, c_k$$

with  $k < N_1 + N_2$  we see that on each  $(c_i, c_{i+1})$   $f(\bar{z}, x)$  is either undefined, constant, or an order-preserving or reversing translation. This proves (1.7)<sub>n</sub>.

Now (1.8)<sub>n</sub> is easily proved from (1.7)<sub>n</sub> by going through the proof of Claim 1.9 and replacing the use of Theorem 4.2 of [2] by (1.7)<sub>n</sub>.

So it remains to give

PROOF OF (1.6)<sub>n+1</sub> ASSUMING (1.6)<sub>n</sub> AND (1.8)<sub>n</sub>. Let  $\varphi(\bar{x}, y) \in L(M)$  be such that  $l(\bar{x}) = n + 1$  and for every  $\bar{a}$ ,  $\varphi(\bar{a}, y)^M$  is scattered. We will write  $\bar{x}$  as  $\bar{z} \wedge x$  where  $l(\bar{z}) = n$ . So we rewrite  $\varphi$  as  $\varphi(\bar{z}, x, y)$ . By (1.8)<sub>n</sub> there is  $N < \omega$  such that, for every  $\bar{c}$  with  $l(\bar{c}) = n$ ,  $|\{a \in M: a \text{ is bad for } \varphi(\bar{c}, x, y)\}| \leq N$ . For  $1 \leq i \leq N$ , let  $g_i(\bar{z}) =$  the  $i$ th bad point for  $\varphi(\bar{z}, x, y)$ . For  $1 \leq i \leq N$  let

$$\begin{aligned} \psi_0^i(\bar{z}, y) &\text{ be } \varphi(\bar{z}, g_i(\bar{z}), y) \\ \psi_{-1}^i(\bar{z}, y) &\text{ be } \varphi(\bar{z}, S^{-1}(g_i(\bar{z})), y), \text{ and} \\ \psi_1^i(\bar{z}, y) &\text{ be } \varphi(\bar{z}, S(g_i(\bar{z})), y). \end{aligned}$$

Note that for given  $\bar{c}$  with  $l(\bar{c}) = n$  and for  $a$  with  $g_i(\bar{c}) < a < g_{i+1}(\bar{c})$ , it follows from Claim 1.10 that

$$|\varphi(\bar{c}, a, y)^M| = |\varphi(\bar{c}, S(g_i(\bar{c})), y)^M| = |\varphi(\bar{c}, S^{-1}(g_{i+1}(\bar{c})), y)^M|.$$

Thus, for any  $\bar{c}, a$ ,

$$|\varphi(\bar{c}, a, y)^M| \leq \max\{|\psi_j^i(\bar{c}, y)^M|: 1 \leq i \leq N, -1 \leq j \leq 1\}.$$

Now, applying (1.6)<sub>n</sub> to the  $\psi_j^i(\bar{z}, y)$  we obtain our uniform bound for  $\varphi(\bar{z}, x, y)$ , completing the proof of (1.6)<sub>n+1</sub>.

This completes the proof of Theorem 1.2, and by standard arguments Theorem 1.2 implies that  $M$  is strongly  $o$ -minimal. (Namely, given  $\varphi(\bar{x}, y)$ , let  $\psi_1(\bar{x}, y)$  be  $\neg\varphi(\bar{x}, y) \wedge \varphi(\bar{x}, S(y))$ ,  $\psi_2(\bar{x}, y)$  be  $\neg\varphi(\bar{x}, y) \wedge \varphi(\bar{x}, S^{-1}(y))$  and  $\psi_3(\bar{x}, y)$  be  $\varphi(\bar{x}, y) \wedge \neg\varphi(\bar{x}, S^{-1}(y)) \wedge \neg\varphi(\bar{x}, S(y))$ . For any  $\bar{a}$  each of  $\psi_1(\bar{a}, y), \psi_2(\bar{a}, y), \psi_3(\bar{a}, y)$  defines a scattered set. So by Theorem 1.2 for some  $N < \omega$ , for every  $\bar{a}$ ,  $\varphi(\bar{a}, y)$  is a union of at most  $N$  intervals and points; thus the same is true in any  $M^1 \equiv M$ .)

Let us finally remark that trivial modifications of the above proofs show the results of this section to be valid if we consider  $o$ -minimal structures  $M$  which are discrete in the following *broad* sense: for all but finitely many  $a \in M$ ,  $a$  has an immediate successor and an immediate predecessor.

**2. Mixed  $o$ -minimal structures.** Here we will show that an arbitrary  $o$ -minimal structure  $M$  is strongly  $o$ -minimal. This will be done by breaking  $M$  into a continuous (or dense) part, and a discrete part, showing that these parts of  $M$  have no interaction with each other and then applying [1] and §1 of this paper.

We must first say some words about relativised  $o$ -minimal structures. So let  $M$  be an arbitrary structure, and  $\chi(x)$  a formula over  $\emptyset$  such that  $\chi(x)^M$  carries a  $\emptyset$ -definable linear ordering  $<$ . We will say that  $\chi(x)$  is  $o$ -minimal in  $M$  if every definable (in  $M$ ) subset  $X$  of  $\chi^M$  is a finite union of points and intervals (with endpoints). Then the proofs in [1] and §1 of this paper give:

**FACT 2.1.** Let  $\chi(x)$  be  $o$ -minimal in  $M$  with  $(\chi^M, <)$  a dense ordering without endpoints. Then for any formula  $\varphi(x, \bar{y}) \in L(M)$  there is  $N < \omega$  such that, for any  $\bar{b} \subset \chi^M$ ,  $(\varphi(x, \bar{b}) \wedge \chi(x))^M$  is a union of at most  $N$  intervals and points. Moreover the definable (in  $M$ ) subsets of  $(\chi(x)^M)^n$  satisfy all the results of [1] (i.e. in terms of decomposition into definable cells, etc.)

**FACT 2.2.** Let  $\chi(x)$  be  $o$ -minimal in  $M$  with  $(\chi^M, <)$  discrete in the broad sense of §1 of this paper. Then again for any  $\varphi(x, \bar{y}) \in L(M)$  there is  $N < \omega$  such that, for any  $\bar{b} \subset \chi^M$ ,  $(\varphi(x, \bar{b}) \wedge \chi(x))^M$  is a union of at most  $N$  intervals and points.

Now let  $(M, <, \dots)$  be an arbitrary (but fixed)  $o$ -minimal structure. Let  $c(M)$  (= the continuous part of  $M$ ) =  $\{x \in M: \exists a < x < b \text{ such that } (a, b) \text{ is dense without first or last element}\}$ . Let  $d(M)$  (= the discrete part of  $M$ ) =  $M - c(M)$ . Then by  $o$ -minimality of  $M$  it is easy to check that  $c(M)$  is either empty or the disjoint finite union of  $\emptyset$ -definable intervals  $A_i$  on each of which  $<$  is dense without endpoints and with each  $A_i$   $o$ -minimal in  $M$ . Similarly,  $d(M)$  is empty or a finite disjoint union of  $\emptyset$ -definable (possibly finite) intervals  $A_i$  such that each  $A_i$  is discrete (in the broad sense) and  $o$ -minimal in  $M$ .

*It is convenient to consider  $L$  as a 2-sorted language with variables  $x$  denoting elements in  $c(M)$  and  $y$  denoting elements in  $d(M)$ .*

We will prove

**PROPOSITION 2.3.** *For any  $L$ -formula  $\varphi(\bar{x}, \bar{y})$ , there is  $\varphi^1(\bar{x}, \bar{y})$  which is a finite Boolean combination of formulae  $\psi(\bar{x})$  and formulae  $\chi(\bar{y})$  such that  $M \models \varphi(\bar{x}, \bar{y}) \leftrightarrow \varphi^1(\bar{x}, \bar{y})$ .*

To simplify the exposition, we make the following

*Assumption.*  $c(M)$  is dense and  $o$ -minimal in  $M$ , and  $d(M)$  is discrete and  $o$ -minimal in  $M$ .

Remember in the following that  $x, x_1 \dots$  range over  $c(M)$ , and  $y, y_1 \dots$  over  $d(M)$ .

LEMMA 2.4. *Let  $\varphi(x, y_1, \dots, y_n) \in L(M)$ . Then  $X = \{a \in c(M) : \exists b_1, \dots, b_n \in d(M), a \text{ is a boundary point for } \varphi(x, b_1 \cdots b_n)^M\}$  is finite.*

PROOF. By induction on  $n$ .

$n = 1$ . Assume, by way of contradiction,  $X$  to be infinite. Clearly we can find an (infinite of course) interval  $I \subset X$  such that for each  $a \in X$  there is  $b \in d(M)$  such that  $a$  is *not* a boundary point for  $\varphi(x, b)^M$ . But then, by discreteness of  $d(M)$ , there is a definable function  $f: I \rightarrow d(M)$  such that, for all  $a \in I$ ,  $a$  is a boundary point for  $\varphi(x, f(a))^M$ . By Theorem 4.2 of [2] there is (infinite)  $I' \subset I$  such that  $f \upharpoonright I'$  is constant, say  $f(a) = c \forall a \in I'$ . But this means that  $\varphi(x, c)^M$  has an infinite number of boundary points, which is impossible.

$n \Rightarrow n + 1$ . If  $X$  were infinite then arguing as in the case  $n = 1$ , we could find infinite interval  $I \subset X$  and definable function  $f: I \rightarrow d(M)$  such that, for each  $a \in I$ ,  $\exists b_1, \dots, b_n$  [ $a$  is a boundary point of  $\varphi(x, b_1, \dots, b_n, f(a))^M$ ]. Again on some infinite  $I' \subset I$ ,  $f$  is constant, and we get a contradiction to the induction hypothesis. This proves the lemma.

LEMMA 2.5. *Let  $\varphi(x_1, \dots, x_n, \bar{y}) \in L(A)$ ,  $A \subset M$ . Let  $X$  be an  $A$ -definable open cell in  $(c(M))^n$ . Then there is a finite decomposition of  $X$  into  $A$ -definable cells  $X_i$  such that for each  $i$ , if  $\bar{a}_1, \bar{a}_2 \in X_i$  and  $\bar{b} \in d(M)$  then  $\models \varphi(\bar{a}_1, \bar{b}) \leftrightarrow \varphi(\bar{a}_2, \bar{b})$ .*

PROOF. Again by induction on  $n$ . When  $n = 1$ , the lemma follows easily from Lemma 2.4.

Now suppose the lemma is proved for  $n$ ; we prove it for  $n + 1$ . So let  $X$  be an open  $A$ -definable cell,  $X \subset c(M)^{n+1}$ . (Note that the projection of  $X$  onto the first  $n$  coordinates is an open  $A$ -definable cell in  $c(M)^n$ .) Let

$$Z_1 = \{(\bar{x}, x_{n+1}) \in X : \exists \text{ neighborhood } O \text{ of } \bar{x} \text{ in } c(M)^n \text{ such that} \\ \forall \bar{x}^1, \bar{x}^2 \in O \forall \bar{y} \in d(M), \varphi(\bar{x}^1, x_{n+1}, \bar{y}) \leftrightarrow \varphi(\bar{x}^2, x_{n+1}, \bar{y})\}.$$

Let

$$Z_2 = \{(\bar{x}, x_{n+1}) \in X : \exists \text{ open interval } I \text{ containing } x_{n+1} \text{ such that} \\ \forall x^1, x^2 \in I \forall \bar{y} \in d(M) \varphi(\bar{x}, x^1, \bar{y}) \leftrightarrow \varphi(\bar{x}, x^2, \bar{y})\}.$$

By 3.5 of [1] there is an  $A$ -definable decomposition  $P$  of  $X$  which partitions  $Z_1$  and  $Z_2$ . If  $Y \in P$  is a cell with  $\dim Y = k < n + 1$ , the  $Y$  is  $A$ -definably homeomorphic to an open cell in  $c(M)^k$  so we can use the induction hypothesis. So let us take  $Y \in P$ ,  $Y$  open.

CLAIM (I).  $Y \subset Z_1, Y \subset Z_2$ .

PROOF. To show that  $Y \subset Z_1$ , we must find a point  $(\bar{a}, b) \in Y$  with  $(\bar{a}, b) \in Z_1$ . Let  $B$  be an open box,  $B \subset Y$ . Let  $(\bar{a}, b) \in B$ . Let  $B_1 = \{\bar{a}^1 : (\bar{a}^1, b) \in B\}$ . So  $B_1$  is an open box in  $c(M)^n$ . By induction hypothesis there is an open cell  $W \subset B_1$  such that, for all  $\bar{a}_1, \bar{a}_2 \in W \forall \bar{y} \in d(M)$ ,  $\varphi(\bar{a}_1, b, \bar{y}) \leftrightarrow \varphi(\bar{a}_2, b, \bar{y})$ . So choosing  $\bar{a}^1 \in W$ , we see that  $(\bar{a}^1, b) \in Z_1$ . Similarly  $Y \subset Z_2$ .

CLAIM (II).  $\forall \bar{a}_1, \bar{a}_2$  in  $Y, \forall \bar{b}$  in  $d(M), \varphi(\bar{a}_1, \bar{b}) \leftrightarrow \varphi(\bar{a}_2, \bar{b})$ .

PROOF. If not, then for some  $\bar{b} \in d(M)$ ,  $\varphi(\bar{x}, \bar{b})$  defines a proper nonempty subset of  $Y$ . As  $Y$  is definably connected (see [1]) there is  $\bar{a}^* \in Y$  such that  $\bar{a}^*$  is a boundary point for  $\varphi(\bar{x}, \bar{b})^M \cap Y$ . Let  $B$  be an open box in  $Y$ , with  $\bar{a}^* \in B$ . Thus

(\*)  $\varphi(\bar{x}, \bar{b})^M \cap B$  is a proper nonempty subset of  $B$ .

Now let  $\bar{a}_1, \bar{a}_2 \in B$ . Write  $\bar{a}_1$  as  $\bar{c}_1 \wedge d_1$ , and  $\bar{a}_2$  as  $\bar{c}_2 \wedge d_2$ . As  $Z_1 \supset Y$  it is easy to see that  $\varphi(\bar{c}_1 \wedge d_1, \bar{b}) \leftrightarrow \varphi(\bar{c}_2 \wedge d_1, \bar{b})$ , and as  $Z_2 \supset Y$  we also have  $\varphi(\bar{c}_2 \wedge d_1, \bar{b}) \leftrightarrow \varphi(\bar{c}_2 \wedge d_2, \bar{b})$ . Thus  $\varphi(\bar{a}_1, \bar{b}) \leftrightarrow \varphi(\bar{a}_2, \bar{b})$ , which contradicts (\*), proving Claim (II).

Clearly, Claim (II) and earlier remarks suffice to prove the lemma.

PROOF OF PROPOSITION 2.3. So let  $\varphi(\bar{x}, \bar{y})$  be an  $L$ -formula, with  $l(\bar{x}) = n$  say. Take  $X$  to be  $(c(M))^n$  in Lemma 2.5. Let  $X_i$  as given by Lemma 2.5 be defined by the  $L$ -formula  $\psi_i(\bar{x})$  ( $i = 1, \dots, k$  say). Then by Lemma 2.5, we have  $M \models \varphi(\bar{x}, \bar{y}) \leftrightarrow \bigvee_{i=1}^k (\psi_i(\bar{x}) \wedge (\exists \bar{x})(\psi_i(\bar{x}) \wedge \varphi(\bar{x}, \bar{y})))$ .

The strong  $o$ -minimality of  $M$  now follows immediately from Proposition 2.3 and Facts 2.1 and 2.2.

#### REFERENCES

1. J. Knight, A. Pillay and C. Steinhorn, *Definable sets in ordered structures. II*, Trans. Amer. Math. Soc. **295** (1986), 593–605.
2. A. Pillay and C. Steinhorn, *Definable sets in ordered structures. I*, Trans. Amer. Math. Soc. **295** (1986), 565–592.
3. A. Pillay and C. Steinhorn, *Discrete  $o$ -minimal structures*, Ann. Pure Appl. Logic **34** (1987), 275–290.

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