BIFURCATION PHENOMENA ASSOCIATED TO THE $p$-LAPLACE OPERATOR

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ABSTRACT. We determine the structure of the set of the solutions $u$ of
$$
-(|u_x|^{p-2}u_x)_x + f(u) = \lambda|u|^{p-2}u \quad \text{on} \ (0,1)
$$
such that $u(0) = u(1) = 0$, where $p > 1$ and $\lambda \in \mathbb{R}$. We prove that the solutions with $k$ zeros are unique when $1 < p \leq 2$ but may not be so when $p > 2$.

0. Introduction. In this article we study the structure of the set $E_\lambda$ of the solutions of the following nonlinear eigenvalue problem

$$
\begin{cases}
-(|u_x|^{p-2}u_x)_x + f(u) = \lambda|u|^{p-2}u & \text{in} \ (0,1), \\
u(0) = u(1) = 0,
\end{cases}
$$

where $p > 1$, $\lambda$ is a real number and $f$ is a $C^1$ real-valued odd function such that
$$
r \mapsto g(r) = f(r)/(|r|^{p-2}r)
$$
is increasing on $(0, +\infty)$ with limits 0 at 0 and $+\infty$ at infinity. We first investigate the unperturbed eigenvalue problem

$$
\begin{cases}
-(|v_x|^{p-2}v_x)_x = \lambda|v|^{p-2}v & \text{in} \ (0,1), \\
v(0) = v(1) = 0.
\end{cases}
$$

By means of an elementary integration process we prove that (0.3) admits a non-trivial solution if and only if

$$
\lambda = \lambda_k = k^p(p-1) \left[ 2 \int_0^1 \frac{dt}{(1-tp)^{1/p}} \right]^p, \quad k \in \mathbb{N}^*.
$$

Moreover to each $\lambda_k$ is associated a one-dimensional eigenspace generated by a function $\omega_k$ with exactly $k - 1$ zeros in $(0,1)$. Concerning the equation (0.1) we prove that each $\lambda_k$ is a point of bifurcation as in the semilinear case ($p = 2$). More precisely we define for $k \in \mathbb{N}^*$

$$
S_k = \{ \varphi \in C : \varphi \text{ has exactly } k - 1 \text{ simple zeros in } (0,1) \},
$$

where $C = \{ \varphi \in C^1([0,1]) : \varphi(0) = \varphi(1) = 0 \}$ and

$$
S_k^+ = \{ \varphi \in S_k : \varphi_x(0) > 0 \}, \quad S_k^- = -S_k^+.
$$

As $\lambda_1$ is defined as the best Poincaré constant in $W_0^{1,p}(0,1)$, that is,

$$
\inf \left\{ \int_0^1 |v_x|^p \, dx : v \in W_0^{1,p}(0,1), \int_0^1 |v|^p \, dx = 1 \right\},
$$

Received by the editors July 4, 1987 and, in revised form, October 12, 1987.
1980 Mathematics Subject Classification (1985 Revision). Primary 35J60; Secondary 34C30.
it is clear that $E_\lambda$ is reduced to the zero function when $\lambda \leq \lambda_1$.

When $1 < p \leq 2$ we prove that the configuration of $E_\lambda$ is exactly the same as in the case $p = 2$ [1], that is,

$$E_\lambda = \{0, \pm u_l, l = 1, \ldots, k: u_l \in S^+_l\}. $$

(0.8)

When $p > 2$ the structure of $E_\lambda$ can be quite a bit more complicated for large values of $\lambda$. Let $h$ be the inverse function of $g$ and $F(r) = \int_0^r f(s) \, ds$; we define

$$\alpha(\lambda) = \left(\frac{\lambda}{p-1} h^p(\lambda) - \frac{p}{p-1} F(h(\lambda))\right)^{1/p} $$

(0.9)

and

$$x(\lambda) = \int_0^{h(\lambda)} \frac{ds}{(\alpha^p(\lambda) + p F(s)/(p-1) - \lambda s^p/(p-1))^{1/p}};$$

(0.10)

and $\lambda \mapsto x(\lambda)$ is a decreasing positive function defined on $(0, +\infty)$. If $\lambda_k < \lambda \leq \lambda_{k+1}$ we then have

$$E_\lambda = \{0\} \cup \{\pm u_l\} \bigcup_{p=2}^k \{\pm E^l_\lambda\},$$

(0.11)

where $u_1 \in S^+_1$ and $E^l_\lambda \subset S^+_l$ such that

(i) $E^l_\lambda$ contains only one element if $2lx(\lambda) \geq 1$,

(ii) $E^l_\lambda$ is diffeomorphic to $[0, 1]^{l-1}$ if $0 < 2lx(\lambda) < 1$. In case (ii) the elements of $E^l_\lambda$ are constant with value $(-1)^{j+1} h(\lambda)$ on $l$ closed and disconnected subintervals $I_j \subset (0, 1), j = 1, \ldots, l$, with total length $1 - 2lx(\lambda)$.

1. The eigenvalue problem. For $p > 1$ we consider the following eigenvalue problem

$$- (|v_x|^{p-2} v_x)_x = \lambda |v|^{p-2} v \quad \text{in } (0, 1),$$

$$v(0) = v(1) = 0$$

and let $S$ be the subset of $W^{1,p}_0(0, 1) \times \mathbb{R}$ of all the $(v, \lambda), v \neq 0$, satisfying (1.1).

**Theorem 1.1.** There exists a unique sequence of functions $v_k \in S^+_k, k \in \mathbb{N}^*$, with maximal value 1 on $(0, 1)$ such that

$$S = \{(\mu v_k, \lambda_k): k \in \mathbb{N}^*\},$$

(1.2)

where $\mu$ is any nonzero real number and

$$\lambda_k = k^p \lambda_1 = k^p(p-1) \left[ 2 \int_0^1 \frac{dt}{(1 - t^p)^{1/p}} \right]^p.$$  

(1.3)

Moreover the following holds for $m = 0, \ldots, k - 1$:

$$v_k(x) = (-1)^m v_1(kx - m), \quad m/k \leq x \leq (m + 1)/k.$$  

(1.4)

Before giving the proof it must be noticed that this result is partially contained in [5], in particular formula (1.4).

**Proof.** It is clear from (1.1) and $v \in C^0([0, 1])$ and then $v \in C^1([0, 1])$ when $p > 2$ or $v \in C^2([0, 1])$ when $1 < p \leq 2$ (the complete regularity, due to Otani [5], will be given in Remark 1.1).
Step 1. If $(v, \lambda) \in S$ then $v_x(0) \neq 0$ and $\lambda > 0$. Multiplying (1.1) by $v$ and integrating over $(0, 1)$ yields
\begin{equation}
\int_0^1 |v_x|^p \, dx = \lambda \int_0^1 v^p \, dx.
\end{equation}
Hence necessarily $\lambda > 0$. Multiplying (1.1) by $v_x$ and integrating over $(0, x)$, $0 < x < 1$, yields the energy estimate
\begin{equation}
(p - 1)|v_x(x)|^p + \lambda |v(x)|^p = (p - 1)|v_x(0)|^p + \lambda |v(0)|^p.
\end{equation}
As $v(0) = 0$ we need $v_x(0) \neq 0$ in order to have a nonzero $v$.

Step 2. The explicit construction. Assume $v$ is a nonzero solution with $v_x(0) = \alpha > 0$ for example. Then $v_x > 0$ on $[0, x_0)$ for some $x_0 \in (0, 1)$ and
\begin{equation}
v_x(x) = \left( \frac{\alpha^p - \lambda}{p - 1} v(x) \right)^{1/p}
\end{equation}
on $[0, x_0]$, from (1.6), which gives
\begin{equation}
x = \int_0^{v(x)} \frac{dt}{(\alpha^p - \frac{\lambda t^p}{(p - 1)})^{1/p}}.
\end{equation}
Moreover this formula remains valid as long as $v(x)$ remains smaller than the first positive zero of the function
\begin{equation}
r \mapsto \varphi(\alpha, r) = \alpha^p - \lambda r^p/(p - 1)
\end{equation}
which is $S(\alpha) = ((p - 1)/\lambda)^{1/p} \alpha$. As $S(\alpha)$ is simple we define $\theta(\alpha)$ by
\begin{equation}
\theta(\alpha) = \int_0^{S(\alpha)} \frac{dt}{(\alpha^p - \frac{\lambda t^p}{(p - 1)})^{1/p}}.
\end{equation}
Moreover $v(\theta(\alpha)) = S(\alpha)$ and $v_x(\theta(\alpha)) = 0$. As $\alpha^p = \lambda S^p(\alpha)/(p - 1)$ we get
\begin{equation}
\theta(\alpha) = \theta_\lambda = C \left( \frac{p - 1}{\lambda} \right)^{1/p}, \quad C = \int_0^1 \frac{ds}{(1 - s^p)^{1/p}}.
\end{equation}
From (1.6) the function $v$ is decreasing on some interval $[0, x_0)$, so we get
\begin{equation}
-x - \theta_\lambda = - \int_{v(x)}^{S(\alpha)} \frac{dt}{[\lambda/(p - 1)](S^p(\alpha) - t^p)]^{1/p}},
\end{equation}
or
\begin{equation}
-x - \theta_\lambda = - \int_{v(x)}^{S(\alpha)} \frac{dt}{(\alpha^p - \frac{\lambda t^p}{(p - 1)})^{1/p}},
\end{equation}
and this formula remains valid as long as $v$ is decreasing, in particular as long as $v$ is positive. If $x_1 \in (0, \theta_\lambda)$ and $x_2 = 2\theta_\lambda - x_1$ then
\begin{equation}
x_1 = \int_0^{v(x_1)} \frac{dt}{(\varphi(\alpha, t))^{1/p}}, \quad \theta_\lambda - x_1 = - \int_{v(x_2)}^{S(\alpha)} \frac{dt}{(\varphi(\alpha, t))^{1/p}}
\end{equation}
and $v(x_1) = v(x_2)$. As a consequence $x = \theta_\lambda$ is an axis of symmetry for the restriction of $v$ to $[0, 2\theta_\lambda]$ and $x = 2\theta_\lambda$ is a center of symmetry for the restriction of
v to $[0, 4\theta_\lambda]$. Hence the function $v$ is $4\theta_\lambda$-periodic on $[0, +\infty)$. The necessary and sufficient condition for the restriction of $v$ to $[0, 1]$ to be a solution of (1.1) is then

$$1/2\theta_\lambda \in \mathbb{N}^*,$$

which means (1.3). As for the number of zeros of $v$ in $(0, 1)$ it is given by $1/2\theta_\lambda - 1$.

Using the homogeneity of (1.1) we get the desired result as the uniqueness is a consequence of the construction of $v$.

**Remark 1.1.** Existence and uniqueness of the first positive normalized eigenfunction of $-\text{div}(|D|^p - 2D_p)$ in $W^{1,p}_0(\Omega)$ have been obtained by De Thelin in the radial case when $\Omega$ is a ball [7] and Guedda-Veron for general $\Omega$ with a connected $C^2$ boundary [4].

As for the regularity of $v$ we have

$$v \in C^\alpha([0, 1]) \cap C^{(p)}([0, 1]\setminus Z)$$

where $Z = \{x \in (0, 1): v_x(x) = 0\}$, $\alpha = \min((2-p)/(p-1)) + 1$, $\langle p \rangle$ and $\langle r \rangle = +\infty$ if $r \in 2\mathbb{N}^*$ or $\langle r \rangle = \min\{n: n \in \mathbb{N}^*, n \geq r\}$ if not.

**Remark 1.2.** We have the following Poincaré type relation

$$\lambda_1 = \inf \left\{ \int_0^1 |u_x|^p dx/ \int_0^1 |u|^p dx: u \in W^{1,p}_0(0, 1) \setminus \{0\} \right\}$$

and the infimum is achieved for $u = v_1$.

**2. The bifurcation phenomena.** In this section we consider the following equation

$$\begin{cases}
-|u_x|^{p-2}u_x + f(u) = \lambda|u|^{p-2}u \\ u(0) = u(1) = 0,
\end{cases}$$

where $p > 1$ and $\lambda \in \mathbb{R}$. As for $f$ we first assume that

$$\text{(2.2)} \quad f \text{ is a } C^1 \text{ odd function},$$

$$\text{(2.3)} \quad s \mapsto f(s)/s^{p-1} \text{ is strictly increasing on } (0, +\infty) \text{ with limit 0 at 0},$$

$$\text{(2.4)} \quad \lim_{s \to +\infty} f(s)/s^{p-1} = +\infty.$$  

We then define

$$\text{(2.5)} \quad h \text{ is the inverse function of the restriction of } f(s)/s^{p-1} \text{ to } (0, +\infty),$$

$$\text{(2.6)} \quad H(s) = \lambda s^p - pF(s),$$

where $F(s) = \int_0^s f(t) dt$. For $\lambda > 0$ we shall also consider the following hypothesis:

$$\text{(2.7)} \quad (p-1)(H'(s))^2 - pH(s)H''(s) \geq 0 \quad \text{for any } s \in [0, h(\lambda)].$$

Let $E_\lambda$ be the set of all the solutions of (2.1) in $W^{1,p}_0(0, 1)$ and $\lambda_k$ be defined by (1.3). When $1 < p \leq 2$ the structure of $E_\lambda$ is exactly the same as in the case $p = 2$.  

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THEOREM 2.1. Assume $1 < p \leq 2$ and (2.2)–(2.7). Then

(i) if $\lambda \leq \lambda_1$, $E_\lambda = \{0\}$, and

(ii) if $\lambda_k < \lambda \leq \lambda_{k+1}$ for some $k \in \mathbb{N}^*$

$$E_\lambda = \{0, \pm u_1, \ldots, \pm u_k\},$$

where $u_l \in S_{l+}^+$ for $l = 1, \ldots, k$.

REMARK 2.1. The assumption (2.7), which is equivalent to the fact that $s \mapsto H^{p-1}(s)/H^p(s)$ is nondecreasing on $[0, h(\lambda)]$, is essential for uniqueness but not for existence. In the particular case where $f(r) = |r|^{q-1}r$ with $q > p - 1$ then $h(\lambda) = \lambda^{1/(q+1-p)}$, $H(s) = \lambda s^p - ps^{q+1}/(q + 1)$ and (2.7) is satisfied.

PROOF OF THEOREM 2.1. As in Theorem 1.1 it is clear that any solution of (2.1) in $W^{1,p}_0(0,1)$ is continuous and at least $C^2$ (remember that $1 < p \leq 2$). Multiplying the equation by $u$ yields

$$\int_0^1 |u_x|^p \, dx + \int_0^1 uf(u) \, dx = \lambda \int_0^1 |u|^p \, dx.$$  \hspace{1cm} (2.9)

From Remark 1.2 a nonzero solution of (2.1) can exist only if $\lambda > \lambda_1$, which will be assumed in the sequel.

Step 1. If $u$ is a nonzero solution of (2.1) then $u_x(0) \neq 0$. Although it is a consequence of a general result due to Franchi, Lanconelli and Serrin, we give here a direct proof which also works when $p > 2$. Multiplying (2.1) by $u_x$ yields the energy relation

$$-\frac{p-1}{p} |u_x(x)|^p + F(u(x)) - \frac{\lambda}{p} |u(x)|^p$$

$$= -\frac{p-1}{p} |u_x(0)| + F(u(0)) - \frac{\lambda}{p} |u(0)|^p.$$  \hspace{1cm} (2.10)

If we assume that $u_x(0) = 0$ we get

$$|u_x(x)|^p = \frac{1}{p-1} (pF(u(x)) - \lambda |u(x)|^p).$$  \hspace{1cm} (2.11)

As the function $x \mapsto pF(x) - \lambda |x|^p$ is negative on $(-\rho, \rho) \setminus \{0\}$, $u_x$ is always 0 and $u \equiv 0$.

Step 2. The explicit construction. Without any loss of generality we assume $u_x(0) = \alpha > 0$. Hence $u$ is increasing on some interval $[0, x_0]$ and from (2.10) we get

$$u_x^p(x) = \alpha^p + \frac{p}{p-1} F(u(x)) - \frac{\lambda}{p-1} u^p(x)$$

which gives $u$ as the inverse function of a $p$-elliptic integral

$$x = \int_0^{u(x)} \frac{dt}{\left(\alpha^p + pF(t)/(p-1) - \lambda t^p/(p-1)\right)^{1/p}}$$  \hspace{1cm} (2.13)

on $[0, x_0]$. Moreover this formula remains valid as long as $u(x)$ is smaller than the first positive zero of

$$r \mapsto \Psi(\alpha, r) = \alpha^p + \frac{p}{p-1} F(r) - \frac{\lambda}{p-1} |r|^p.$$  \hspace{1cm} (2.14)
But the function \( \Psi(\alpha, \cdot) \) is decreasing in \([0, h(\lambda)]\) and increasing on \([h(\lambda), +\infty)\); hence there are three possibilities.

Case 1. \( \alpha^p > \lambda h^p(\lambda)/(p - 1) - p F(h(\lambda))/(p - 1) = \alpha^p(\lambda) \). In that case the function \( r \mapsto \int_0^r ds/(\Psi(\alpha, s))^{1/p} \) is an increasing \( C^2 \) diffeomorphism from \( R^+ \) onto \( R^+ \) and it is the same with \( u \) defined by (2.13) which cannot belong to \( E_{\lambda} \).

Case 2. \( \alpha^p = \alpha^p(\lambda) \). In that case \( h(\lambda) \) is a double zero for \( \Psi(\alpha, \cdot) \), and as \( 1 < p < 2 \)

\[ \int_0^{h(\lambda)} ds/(\Psi(\alpha, s))^{1/p} = +\infty. \]

As in Case 1 the function \( r \mapsto \int_0^r ds/(\Psi(\alpha, s))^{1/p} \) is a \( C^2 \) diffeomorphism from \([0, h(\lambda)]\) onto \( R^+ \) and \( u \) cannot belong to \( E_{\lambda} \).

Case 3. \( \alpha^p < \alpha^p(\lambda) \). In that case \( \Psi(\alpha, \cdot) \) admits a simple zero \( S(\alpha) \) in \((0, h(\lambda))\). As \( (\partial \Psi/\partial r)(\alpha, S(\alpha)) \neq 0 \), \( r \mapsto (\Psi(\alpha, r))^{-1/p} \) is integrable on \((0, S(\alpha))\) and we define

\[ (2.15) \theta(\alpha) = \int_0^{S(\alpha)} ds/(\Psi(\alpha, s))^{1/p}. \]

Relation (2.13) remains valid on \([0, \theta(\alpha)]\) and we have

\[ u(\theta(\alpha)) = S(\alpha), \quad u_x(\theta(\alpha)) = 0. \]

Using the energy relation at \( \theta(\alpha) \) we have

\[ (2.17) \frac{p - 1}{p} |u_x(x)|^p = \frac{\lambda}{p} S^p(\alpha) - F(S(\alpha)) - \left( \frac{\lambda}{p} u^p(x) - F(u(x)) \right) \]

or

\[ |u_x(x)|^p = \alpha^p + \frac{p}{p - 1} F(u(x)) - \frac{\lambda}{p - 1} u^p(x). \]

Hence \( u \) is decreasing on some interval \([\theta(\alpha), \Theta]\) and we have

\[ (2.18) x - \theta(\alpha) = -\int_{S(\alpha)}^{u(x)} ds/(\Psi(\alpha, s))^{1/p}. \]

This formula remains valid as long as \( u \) is decreasing, and as in \( \S 1 \) \( x = \theta(\alpha) \) is an axis of symmetry for the restriction of \( u \) to \([0, 2\theta(\alpha)]\) and \( x = 2\theta(\alpha) \) is a center of symmetry for the restriction of \( u \) to \([0, 4\theta(\alpha)]\); the necessary and sufficient condition for \( u \) to be a solution of (2.1) is that

\[ (2.19) 1/2\theta(\alpha) \in N^*. \]

Step 3. The function \( \alpha \mapsto S(\alpha) \) is convex, increasing on \([0, \alpha(\lambda)]\). We have \( \Psi(\alpha, S(\alpha)) = 0 \) and \( (\partial \Psi/\partial r)(\alpha, S(\alpha)) \neq 0 \). By the implicit function theorem \( \alpha \mapsto S(\alpha) \) is \( C^2 \). We also have

\[ \frac{d}{d\alpha}(\Psi(\alpha, S(\alpha))) = \frac{\partial \Psi}{\partial \alpha}(\alpha, S(\alpha)) + \frac{\partial \Psi}{\partial r}(\alpha, S(\alpha)) \frac{dS}{d\alpha}(\alpha) \]

which gives

\[ (2.20) \frac{dS}{d\alpha}(\alpha) = \frac{(p - 1)\alpha^{p-1}}{\lambda S^{p-1}(\alpha) - f(S(\alpha))} = \frac{p(p - 1)\alpha^{p-1}}{H'(S(\alpha))}. \]
As \( S(\alpha) < h(\lambda) \), \( \alpha \mapsto S(\alpha) \) is increasing on \([0, \alpha(\lambda))\). Moreover
\[
d^2S\over d\alpha^2(\alpha) = p(p - 1)(p - 1)\alpha^{p-2}H'(S(\alpha)) - \alpha^{p-1}H''(S(\alpha))dS/d\alpha \over (H'(S(\alpha)))^2.
\]
Using (2.20) and the definition of \( S(\alpha) \) and \( H \) we get
\[
d^2S\over d\alpha^2(\alpha) = p(p - 1)\alpha^{p-2}(p - 1)(H'(S(\alpha)))^2 - pH(S(\alpha))H''(S(\alpha)) \over (H'(S(\alpha)))^3.
\]
From (2.7) we deduce \( d^2S(\alpha)/d\alpha^2 \geq 0 \).

**Step 4.** The function \( \alpha \mapsto \theta(\alpha) \) is continuous increasing on \([0, \alpha(\lambda))\). For \( t \in [0, \alpha] \) the function \( s \mapsto \Psi(t, s) \) admits a first positive zero at \( S(t) \) which means
\[
 t^p + {p \over p - 1} F(S(t)) - {\lambda \over p - 1} S^p(t) = 0 \quad \text{and} \quad \Psi(\alpha, S(t)) = \alpha^p - t^p.
\]
Taking \( t \) as a new variable in (2.15) we get
\[
 \theta(\alpha) = \int_0^\alpha {dS\over dt}(t) dt \over (\alpha^p - t^p)^{1/p}
\]
or
\[
 \theta(\alpha) = \int_0^1 {dS\over d\alpha}(\alpha \sigma) d\sigma \over (1 - \sigma^p)^{1/p}
\]
As \( ds/dt \) is increasing and \( C^1 \) on \([0, \alpha(\lambda))\), it is the same with \( \alpha \mapsto \theta(\alpha) \).

**Step 5. End of the proof.** As \( \lim_{\alpha \to 0} S(\alpha) = 0 \) and \( \lim_{\alpha \to 0} F(S(\alpha))/S^p(\alpha) = 0 \) we get
\[
 S(\alpha) \sim_{\alpha \to 0} \alpha^{1/p} \left( {p - 1 \over \lambda} \right)
\]
which implies
\[
 \lim_{\alpha \to 0} {dS\over d\alpha}(\alpha) = \left( {p - 1 \over \lambda} \right)^{1/p}
\]
and
\[
 \lim_{\alpha \to 0} \theta(\alpha) = \left( {p - 1 \over \lambda} \right)^{1/p} \int_0^1 {d\sigma\over (1 - \sigma^p)^{1/p}} = {1 \over 2} \left( {\lambda_1 \over \lambda} \right)^{1/p}
\]
For the other bound we have \( \lim_{\alpha \to 1, \alpha(\lambda)} S(\alpha) = h(\lambda) \). As \( h(\lambda) \) is just a double zero for \( \Psi(\alpha(\lambda), r) \), there exists a continuous and bounded function \( \varphi \) on \([0, \alpha(\lambda)]\) such that
\[
 \Psi(\alpha(\lambda), r) = (h(\lambda) - r)^2 \varphi(r).
\]
Moreover
\[
 \int_0^{S(\alpha)} (\Psi(\alpha, t))^{-1/p} dt > \int_0^{S(\alpha)} (\Psi(\alpha(\lambda), t))^{-1/p} dt
\]
\[
 = \int_0^{S(\alpha)} (h(\lambda) - t)^{-2/p}(\varphi(t))^{-1/p} dt.
\]
As \( 1 < p \leq 2 \) we get
\[
 \lim_{\alpha \to 1, \alpha(\lambda)} \theta(\alpha) = \int_0^{h(\lambda)} (h(\lambda) - t)^{-2/p}(\varphi(t))^{-1/p} dt = +\infty.
\]
As a consequence \( \alpha \mapsto \theta(\alpha) \) is an increasing diffeomorphism from \( (0, \alpha(\lambda)) \) onto \( (\frac{1}{2}(\lambda_1/\lambda)^{1/p}, +\infty) \) and \( 1/2\theta(\alpha) \) a decreasing diffeomorphism from \( (0, \alpha(\lambda)) \) onto \( (0, (\lambda/\lambda_1)^{1/p}) \). If we assume that \( \lambda_k < \lambda \leq \lambda_{k+1} \) for some \( k \in \mathbb{N}^* \) there exist exactly \( k \) integers \( l = 1, \ldots, k \) and \( k \) positive real numbers \( \alpha_l \) such that \( 1/2\theta(\alpha_l) = l \). If \( u_l \) is the solution of the initial value problem

\[
\begin{cases}
-|u_{lx}|^{p-2}u_{lx} + f(u_l) = \lambda|u_l|^{p-2}u_l & \text{on } (0,1), \\
u_l(0) = 0, \quad u_{lx}(0) = \alpha_l,
\end{cases}
\]

then \( u_l(1) = 0, \ u_l \in S_l^+ \). We get the result in considering \(-u_l, \ l = 1, \ldots, k\).

**REMARK 2.2.** If we represent the bifurcation diagram \((\lambda, u_\lambda)\) then there exists no secondary bifurcation along the branches of solutions in \( S_k^\pm \) issuing from \( \lambda_k \).

In the case \( p > 2 \) the main difference will come from the fact that the following integral

\[
x(\lambda) = \int_0^{h(\lambda)} \frac{ds}{(\alpha^p(\lambda) + \frac{p}{p-1}F(s) - \frac{\lambda}{p-1}sp)^{1/p}}
\]

is finite as \( h(\lambda) \) is a double zero of \( \Psi(\alpha(\lambda), r) \).

**THEOREM 2.2.** Assume \( p > 2 \) and (2.2)-(2.7). Then

(i) if \( \lambda \leq \lambda_1 \) \( E_\lambda = \{0\} \),

(ii) if \( \lambda_k < \lambda \leq \lambda_{k+1} \) for some \( k \in \mathbb{N}^* \)

\[
E_\lambda = \{0\} \cup \{\pm u_1\} \bigcup_{l=k}^k \{\pm E^{l}_l\},
\]
where \( u_1 \in S^+_1 \) and \( E^l_\lambda \subset S^+_1, \ l = 2, \ldots, k, \) and

\( E^l_\lambda \) is reduced to a single element if \( 2l \alpha(x) \geq 1, \)

\( E^l_\lambda \) is diffeomorphic to \([0, 1]^{l-1}\) if \( 0 < 2l \alpha(x) < 1. \)

**Proof.** The idea is essentially the same as in Theorem 2.1 except that in Step 2, Case 2 (that is, if \( \alpha^p = \alpha^p(\lambda) \)) gives rise to solutions of (2.1) with maximum value \( h(\lambda), \) and in that case Serrin and Veron’s existence and uniqueness result does not apply; moreover the value \( u = h(\lambda) \) is a bifurcation value for (2.1).

**Step 1.** Assume \( 2 \alpha(x) \geq 1. \) Then the construction of Theorem 2.1 works: the function \( \alpha \mapsto 1/2\theta(\alpha) \) is a decreasing diffeomorphism from \((0, \alpha(\lambda)] \) onto \((1/2 \alpha(x)), (\lambda/\lambda_1)^{1/p}) \). As \( \alpha_k < \lambda \leq \lambda_{k+1} \) there exist exactly \( k \) integers \( 1, 2, \ldots, k \) and \( k \) positive real numbers \( \alpha_1, \ldots, \alpha_k \) such that \( 1/2 \theta(\alpha_l) = l \in [1/2 \alpha(x)), (\lambda/\lambda_1)^{1/p}), \ l = 1, \ldots, k. \) and we get the corresponding solutions \( u_l \in S^+_1 \) by (2.26).

**Step 2.** Assume \( 4 \alpha(x) > 1 > 2 \alpha(x) \). All the elements \( u_l = 2, \ldots, k \) in \( S^+_1 \) are constructed as in Step 1. As for the element \( u_1 \in S^+_1 \) it has necessarily the following form as the initial slope must be \( \alpha(\lambda): \)

\[
\begin{align*}
\text{for } 0 \leq x \leq x(\lambda) & \\
\frac{dt}{(\Psi(\alpha(\lambda), t))^{1/p}}, & \\
\text{for } x(\lambda) \leq x \leq 1 - x(\lambda) & = h(\lambda), \\
\text{for } 1 - x(\lambda) \leq x \leq 1 \\
\end{align*}
\]

(2.30)

**Step 3.** Assume \( 0 < 2 \alpha(x) < 1 \) for some \( l \in \{2, \ldots, k\}. \) We can construct all the elements of \( E_\lambda \cap S^+_1 \) in the following way as their initial slope is necessarily \( \alpha(\lambda): \)

\[
\begin{align*}
\text{for } 0 \leq x \leq x(\lambda) & \\
\frac{dt}{(\Psi(\alpha(\lambda), t))^{1/p}}, & \\
\text{for } x(\lambda) \leq x \leq x_1 \text{ where } x_1 \in (x(\lambda), 1) \text{ and} & \\
x_1 - x(\lambda) \leq 1 - 2l \alpha(x) & \quad \text{then } u_1(x) = h(\lambda), \\
\text{for } x_1 \leq x \leq 2x(\lambda) + x_1 & \\
h(\lambda) & \quad \text{then } u_1(x) = -h(\lambda).
\end{align*}
\]

(2.31)

(2.32)

(2.33)

(2.34)

(2.35)

(2.36)

(And more naturally to the set \( K_1 = \{x = (x^1, \ldots, x^t), x^j \geq 0, \sum_{j=1}^t x^j = 1 - 2l \alpha(x)\}. \)
Continuing this procedure any solution \( u_1 \in S^+_l \) is defined by the intervals \( I_j = [x_{j-1} + 2x(\lambda), x_j], \) \( j = 1, \ldots, l \), and \( x_0 = -x(\lambda) \) where it takes the constant value \( (-1)^{j+1} h(\lambda) \) and the intervals \([x_{j-1}, x_{j-1} + 2x(\lambda)]\) where it is defined by

\[
x - x_{j-1} = -\int_{x_j}^{h(\lambda)} \frac{dt}{(\Psi(\alpha(\lambda), t))^{1/p}}
\]

if \( j \) is even or

\[
x - x_{j-1} = \int_{h(\lambda)}^{u_j(x)} \frac{dt}{(\Psi(\alpha(\lambda), t))^{1/p}}
\]

if \( j \) is odd.

From the above construction the total length of the \( I_j \) is \( 1 - 2l x(\lambda) \) and the set \( E^l_\lambda \) of the \( u_j \) is diffeomorphic to the \( (l - 1) \)-dimensional cube.

![Figure 2. Example of construction of \( E^3_\lambda \)](image)

**Remark 2.3.** It is important to notice that this type of secondary bifurcation along the branch of solutions issuing from \( \lambda_k, k \geq 2 \), always appears if we have

\[
\lim_{\lambda \to +\infty} x(\lambda) = 0.
\]

This is in particular the case if \( f(r) \sim |r|^{q-1}r \) which implies

\[
x(\lambda) \sim_{\lambda \to +\infty} \lambda^{-1/p} \int_0^1 \left( \frac{q + 1 - p}{(p - 1)(q + 1)} + \frac{p}{(p - 1)(q + 1)} \sigma^{q+1} - \frac{(q - p)}{p - 1} \right)^{-1/p} d\sigma.
\]

However this is not always the case under conditions (2.2)–(2.7), for example, with \( f(r) = (|r|^{p-2} \log |r|)r \) for \( |r| \geq 2 \), where we get

\[
\lim_{\lambda \to +\infty} x(\lambda) = \int_0^1 \left( \frac{1}{p(p-1)} (1 - \sigma^p) + \frac{1}{p - 1} \sigma^p \log \sigma \right)^{-1/p} d\sigma.
\]

We finally have the following exclusion principle.
THEOREM 2.3. Assume $p > 1$, (2.2)–(2.7), $g$ is a continuous even function increasing on $\mathbb{R}^+$ and $u_1$ and $u_2$ are two solutions of (2.1); then

(i) if $u_1$ and $u_2$ have the same number of zeros

\[
\int_0^1 g(u_1(x)) \, dx = \int_0^1 g(u_2(x)) \, dx;
\]

(ii) if $u_1$ and $u_2$ do not have the same number of zeros

\[
\int_0^1 g(u_1(x)) \, dx \neq \int_0^1 g(u_2(x)) \, dx.
\]

PROOF. It is clear that for any function $\int_0^1 g(u(x)) \, dx$ is equal to $\int_0^1 g(-u(x)) \, dx$. When $p > 2$ we have only to consider two solutions of $E_\lambda$ with the same number of zeros and belonging to some $E_{\lambda,l}$, $l \geq 2$, in the case $2l x(\lambda) < 1$. In that case $u_1$ and $u_2$ take the value $\pm h(\lambda)$ on $l$ intervals $I_j^1$ and $I_j^2$, $j = 1, \ldots, l$, which are disconnected and have the same total length which gives

\[
\int_{\bigcup_j I_j^1} g(u_1(x)) \, dx = \int_{\bigcup_j I_j^2} g(u_2(x)) \, dx = (1-2l x(\lambda))g(h(\lambda)).
\]

On $(0,1)\setminus\{\bigcup_j I_j^1\}$ or $(0,1)\setminus\{\bigcup_j I_j^2\}$ $u_1$ and $u_2$ are defined by the same types of formula ((2.32) or (2.30)) and the integral of $g(u_i)$ over these sets is

\[
2l \int_0^{x(\lambda)} g(u_1(x)) \, dx.
\]

Hence, for $i = 1, 2$, we get

\[
\int_0^1 g(u_i(x)) \, dx = (1-2l x(\lambda))g(h(\lambda)) + 2l \int_0^{x(\lambda)} g(u_i(x)) \, dx
\]

which proves (i).

For proving (ii) we shall assume either $1 < p \leq 2$ or $p > 2$ but $u_1$ and $u_2$ are not constant on any subinterval of $(0,1)$ (the other case is essentially the same). If $u_1$ and $u_2$ do not have the same number of zeros in $(0,1)$ we can assume $u_{1x}(0) = \alpha$, $u_{2x}(0) = \beta$, $0 < \alpha < \beta$; $u_1$ is $4\theta(\alpha)$-periodic, $u_2$ is $4\theta(\beta)$-periodic and $0 < \theta(\alpha) < \theta(\beta)$. Moreover

\[
\frac{1}{2\theta(\alpha)} = k_1, \quad \frac{1}{2\theta(\beta)} = k_2, \quad k_1, k_2 \in \mathbb{N}^*, \quad k_1 > k_2.
\]

Step 1. For $0 < x < \theta(\alpha)$ we have $0 < u_1(x) < u_2(x)$. On a right neighbourhood of $0$ we have $u_1 < u_2$, and $u_1$ and $u_2$ are increasing on $[0,\theta(\alpha)]$. If we assume the existence of some $x_0 \in [0,\theta(\alpha)]$ such that $u_1(x_0) = u_2(x_0)$, we can always suppose that $u_1 < u_2$ in $(0, x_0)$ and then $u_{1x}(x_0) \geq u_{2x}(x_0)$. The energy relation implies

\[
\alpha^p + \frac{p}{p-1} F(u_1(x_0)) - \frac{\lambda}{p-1} u_1^p(x_0)
\]

\[
\geq \beta^p + \frac{p}{p-1} F(u_2(x_0)) - \frac{\lambda}{p-1} u_2^p(x_0)
\]

and $\alpha \geq \beta$ which is impossible.
Step 2. End of the proof. From Step 1: \(0 < u_1(x) < u_2(x')\) for \(0 < x < \theta(\alpha)\) and \(0 < x' < \theta(\beta)\). Set \(\varphi\) the lowest common multiple to \(k_1\) and \(k_2\). There exist \(n_1\) and \(n_2 \in \mathbb{N}^*\) such that \(n_1k_1 = n_2k_2 = \varphi\) and

\[
n_1/\theta(\alpha) = n_2/\theta(\beta), \quad 0 < n_1 < n_2.
\]

Then

\[
\int_0^1 g(u_1(x)) \, dx = \frac{1}{\theta(\alpha)} \int_0^{\theta(\alpha)} g(u_1(x)) \, dx,
\]

(2.49)

\[
\int_0^1 g(u_2(x)) \, dx = \frac{1}{\theta(\beta)} \int_0^{\theta(\beta)} g(u_2(x)) \, dx.
\]

(2.50)

Setting \(T = n_2\theta(\alpha) = n_1\theta(\beta)\), we have

\[
\frac{1}{\theta(\alpha)} \int_0^{\theta(\alpha)} g(u_1(x)) \, dx = \frac{1}{n_2\theta(\alpha)} \int_0^{n_2\theta(\alpha)} g\left(u_1\left(\frac{\sigma}{n_2}\right)\right) \, d\sigma
\]

\[
= \frac{1}{T} \int_0^T g\left(u_1\left(\frac{\sigma}{n_2}\right)\right) \, d\sigma,
\]

and

\[
\frac{1}{\theta(\beta)} \int_0^{\theta(\beta)} g(u_2(x)) \, dx = \frac{1}{T} \int_0^T g\left(u_2\left(\frac{\sigma}{n_1}\right)\right) \, d\sigma,
\]

(2.48) which implies

\[
\int_0^1 g(u_1(x)) \, dx < \int_0^1 g(u_2(x)) \, dx.
\]

REMARK 2.4. As a consequence there exist \(k + 1\) different critical values for the energy functional

\[
J(\omega) = \frac{1}{p} \int_0^1 |\omega_x|^p \, dx + \int_0^1 F(\omega) \, dx - \lambda \int_0^p |\omega|^p \, dx
\]

(2.52) defined in \(W^{1,p}_0(0,1)\), for \(\lambda_k < \lambda < \lambda_{k+1}\); those critical values only depend on the set \(S_l, l = 1, \ldots, k\), the critical points of (2.52) belong to. This is an immediate consequence of Theorem 2.3 and the fact that

\[
J(u) = \int_0^1 \left(F(u) - \frac{1}{p} uf(u)\right) \, dx
\]

for \(u \in E_\lambda\).

REFERENCES


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