BIFURCATION PHENOMENA ASSOCIATED TO THE p-LAPLACE OPERATOR

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ABSTRACT. We determine the structure of the set of the solutions $u$ of $-(|u_x|^{p-2}u_x)_x + f(u) = \lambda|u|^{p-2}u$ on $(0,1)$ such that $u(0) = u(1) = 0$, where $p > 1$ and $\lambda \in \mathbb{R}$. We prove that the solutions with $k$ zeros are unique when $1 < p \leq 2$ but may not be so when $p > 2$.

0. Introduction. In this article we study the structure of the set $E_\lambda$ of the solutions of the following nonlinear eigenvalue problem

$$\left\{ \begin{array}{l}
-(|u_x|^{p-2}u_x)_x + f(u) = \lambda|u|^{p-2}u \\ u(0) = u(1) = 0,
\end{array} \right. \quad (0.1)$$

where $p > 1$, $\lambda$ is a real number and $f$ is a $C^1$ real-valued odd function such that $r \mapsto g(r) = f(r)/(|r|^{p-2}r)$ is increasing on $(0, +\infty)$ with limits 0 at 0 and $+\infty$ at infinity. We first investigate the unperturbed eigenvalue problem

$$\left\{ \begin{array}{l}
-(|v_x|^{p-2}v_x)_x = \lambda|v|^{p-2}v \\ v(0) = v(1) = 0,
\end{array} \right. \quad (0.3)$$

By means of an elementary integration process we prove that (0.3) admits a non-trivial solution if and only if

$$\lambda = \lambda_k = k^p(p-1) \left[ 2 \int_0^1 \frac{dt}{(1-tp)^{1/p}} \right]^p, \quad k \in \mathbb{N}^*. \quad (0.4)$$

Moreover to each $\lambda_k$ is associated a one-dimensional eigenspace generated by a function $\omega_k$ with exactly $k - 1$ zeros in $(0,1)$. Concerning the equation (0.1) we prove that each $\lambda_k$ is a point of bifurcation as in the semilinear case ($p = 2$). More precisely we define for $k \in \mathbb{N}^*$

$$S_k = \{ \varphi \in C: \varphi \text{ has exactly } k - 1 \text{ simple zeros in } (0,1) \}, \quad (0.5)$$

where $C = \{ \varphi \in C^1([0,1]): \varphi(0) = \varphi(1) = 0 \}$ and

$$S^+_k = \{ \varphi \in S_k: \varphi_x(0) > 0 \}, \quad S^-_k = -S^+_k. \quad (0.6)$$

As $\lambda_1$ is defined as the best Poincaré constant in $W^{1,p}_0(0,1)$, that is,

$$\inf \left\{ \int_0^1 |v_x|^p \, dx : v \in W^{1,p}_0(0,1), \int_0^1 |v|^p \, dx = 1 \right\}, \quad (0.7)$$

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it is clear that $E_\lambda$ is reduced to the zero function when $\lambda \leq \lambda_1$.

When $1 < p \leq 2$ we prove that the configuration of $E_\lambda$ is exactly the same as in the case $p = 2$ [1], that is,

$$E_\lambda = \{0, \pm u_l, l = 1, \ldots, k: u_l \in S^+_l\}.$$  \hfill (0.8)

When $p > 2$ the structure of $E_\lambda$ can be quite a bit more complicated for large values of $\lambda$. Let $h$ be the inverse function of $g$ and $F(r) = \int_0^r f(s) \, ds$; we define

$$\alpha(\lambda) = \left(\frac{\lambda}{p-1} h^p(\lambda) - \frac{p}{p-1} F(h(\lambda))\right)^{1/p}$$

and

$$x(\lambda) = \int_0^{h(\lambda)} \frac{ds}{(\alpha^p(\lambda) + pF(s))/(p-1) - \lambda s^p/(p-1))^{1/p}};$$

and $\lambda \mapsto x(\lambda)$ is a decreasing positive function defined on $(0, +\infty)$. If $\lambda_k < \lambda \leq \lambda_{k+1}$ we then have

$$E_\lambda = \{0\} \cup \{\pm u_1\} \bigcup_{p=2}^{k} \{\pm E_\lambda^p\},$$  \hfill (0.11)

where $u_1 \in S^+_1$ and $E^p_\lambda \subset S^+_l$ such that

(i) $E^p_\lambda$ contains only one element if $2lx(\lambda) \geq 1$,

(ii) $E^p_\lambda$ is diffeomorphic to $[0, 1]^{l-1}$ if $0 < 2lx(\lambda) < 1$. In case (ii) the elements of $E^p_\lambda$ are constant with value $(-1)^{j+1}h(\lambda)$ on $l$ closed and disconnected subintervals $I_j \subset (0, 1)$, $j = 1, \ldots, l$, with total length $1 - 2lx(\lambda)$.

1. The eigenvalue problem. For $p > 1$ we consider the following eigenvalue problem

$$\begin{cases}
\begin{align*}
-|v_x|^{p-2}v_x &= \lambda |v|^{p-2}v \quad \text{in } (0, 1), \\
v(0) &= v(1) = 0
\end{align*}
\end{cases}$$

and let $S$ be the subset of $W^{1,p}_0(0, 1) \times \mathbb{R}$ of all the $(v, \lambda)$, $v \neq 0$, satisfying (1.1).

**Theorem 1.1.** There exists a unique sequence of functions $v_k \in S^+_k$, $k \in \mathbb{N}^*$, with maximal value 1 on $(0, 1)$ such that

$$S = \{(\mu v_k, \lambda_k): k \in \mathbb{N}^*\},$$

where $\mu$ is any nonzero real number and

$$\lambda_k = k^p \lambda_1 = k^p(p-1) \left[2 \int_0^1 \frac{dt}{(1 - t^p)^{1/p}}\right]^p.$$  \hfill (1.3)

Moreover the following holds for $m = 0, \ldots, k - 1$:

$$v_k(x) = (-1)^m v_1(kx - m), \quad m/k \leq x \leq (m + 1)/k.$$  \hfill (1.4)

Before giving the proof it must be noticed that this result is partially contained in [5], in particular formula (1.4).

**Proof.** It is clear from (1.1) and $v \in C^0([0, 1])$ and then $v \in C^1([0, 1])$ when $p > 2$ or $v \in C^2([0, 1])$ when $1 < p \leq 2$ (the complete regularity, due to Otani [5], will be given in Remark 1.1).
Step 1. If \((v, \lambda) \in S\) then \(v_x(0) \neq 0\) and \(\lambda > 0\). Multiplying (1.1) by \(v\) and integrating over \((0, 1)\) yields

\[
\int_0^1 |v_x|^p \, dx = \lambda \int_0^1 v^p \, dx.
\]

Hence necessarily \(\lambda > 0\). Multiplying (1.1) by \(v_x\) and integrating over \((0, x), 0 < x < 1\), yields the energy estimate

\[
(p - 1)|v_x(x)|^p + \lambda |v(x)|^p = (p - 1)|v_x(0)|^p + \lambda |v(0)|^p.
\]

As \(v(0) = 0\) we need \(v_x(0) \neq 0\) in order to have a nonzero \(v\).

Step 2. The explicit construction. Assume \(v\) is a nonzero solution with \(v_x(0) = \alpha > 0\) for example. Then \(v_x > 0\) on \([0, x_0)\) for some \(x_0 \in (0, 1)\) and

\[
v_x(x) = \left( \frac{\alpha^p - \frac{\lambda}{p - 1}(v(x))^p}{\alpha^p - \frac{\alpha^p - \lambda t^p}{(p - 1)^{1/p}}} \right)^{1/p}
\]

on \([0, x_0]\), from (1.6), which gives

\[
x = \int_0^{v(x)} \frac{dt}{(\alpha^p - \lambda t^p/(p - 1))^{1/p}}.
\]

Moreover this formula remains valid as long as \(v(x)\) remains smaller than the first positive zero of the function

\[
r \mapsto \varphi(\alpha, r) = \alpha^p - \lambda r^p/(p - 1)
\]

which is \(S(\alpha) = ((p - 1)/\lambda)^{1/p} \alpha\). As \(S(\alpha)\) is simple we define \(\theta(\alpha)\) by

\[
\theta(\alpha) = \int_0^{S(\alpha)} \frac{dt}{(\alpha^p - \lambda t^p/(p - 1))^{1/p}}.
\]

Moreover \(v(\theta(\alpha)) = S(\alpha)\) and \(v_x(\theta(\alpha)) = 0\). As \(\alpha^p = \lambda S^p(\alpha)/(p - 1)\) we get

\[
\theta(\alpha) = \theta_\lambda = C \left( \frac{p - 1}{\lambda} \right)^{1/p}, \quad C = \int_0^1 \frac{ds}{(1 - s^p)^{1/p}}.
\]

From (1.6) the function \(v\) is decreasing on some interval \([0, x_0]\), so we get

\[
x - \theta_\lambda = - \int_{v(x)}^{S(\alpha)} \frac{dt}{[(\lambda/(p - 1))(S^p(\alpha) - t^p)]^{1/p}},
\]

or

\[
x - \theta_\lambda = - \int_{v(x)}^{S(\alpha)} \frac{dt}{(\alpha^p - \lambda t^p/(p - 1))^{1/p}};
\]

and this formula remains valid as long as \(v\) is decreasing, in particular as long as \(v\) is positive. If \(x_1 \in (0, \theta_\lambda)\) and \(x_2 = 2\theta_\lambda - x_1\) then

\[
x_1 = \int_0^{v(x_1)} \frac{dt}{(\varphi(\alpha, t))^{1/p}}, \quad \theta_\lambda - x_1 = - \int_{v(x_2)}^{S(\alpha)} \frac{dt}{(\varphi(\alpha, t))^{1/p}}
\]

and \(v(x_1) = v(x_2)\). As a consequence \(x = \theta_\lambda\) is an axis of symmetry for the restriction of \(v\) to \([0, 2\theta_\lambda]\) and \(x = 2\theta_\lambda\) is a center of symmetry for the restriction of
v to $[0, 4\theta_\lambda]$. Hence the function $v$ is $4\theta_\lambda$-periodic on $[0, +\infty)$. The necessary and sufficient condition for the restriction of $v$ to $[0, 1]$ to be a solution of (1.1) is then
\begin{equation}
1/2\theta_\lambda \in \mathbb{N}^*,
\end{equation}
which means (1.3). As for the number of zeros of $v$ in $(0, 1)$ it is given by $1/2\theta_\lambda - 1$. Using the homogeneity of (1.1) we get the desired result as the uniqueness is a consequence of the construction of $v$.

**Remark 1.1.** Existence and uniqueness of the first positive normalized eigenfunction of $-\text{div}(D_2|D_2|^{p-2}D_2)$ in $W^{1,p}_0(\Omega)$ have been obtained by De Thelin in the radial case when $\Omega$ is a ball [7] and Guedda-Veron for general $\Omega$ with a connected $C^2$ boundary [4].

As for the regularity of $v$ we have
\begin{equation}
v \in C^{\alpha}([0,1]) \cap C^{(p)}([0,1] \setminus Z)
\end{equation}
where $Z = \{x \in (0,1): v_x(x) = 0\}$, $\alpha = \min(\{(2-p)/(p-1)\} + 1, (p))$ and $(r) = +\infty$ if $r \in 2\mathbb{N}^*$ or $(r) = \min\{n: n \in \mathbb{N}^*, n \geq r\}$ if not.

**Remark 1.2.** We have the following Poincaré type relation
\begin{equation}
\lambda_1 = \inf \left\{ \int_0^1 |u_x|^p \, dx / \int_0^1 |u|^p \, dx: u \in W^{1,p}_0(0,1) \setminus \{0\} \right\}
\end{equation}
and the infimum is achieved for $u = v_1$.

2. The bifurcation phenomena. In this section we consider the following equation

\begin{equation}
\begin{cases}
- (|u_x|^p u_{xx}) + f(u) = \lambda |u|^{p-2}u & \text{in } (0, 1), \\
u(0) = u(1) = 0,
\end{cases}
\end{equation}
where $p > 1$ and $\lambda \in \mathbb{R}$. As for $f$ we first assume that
\begin{equation}
f \text{ is a } C^1 \text{ odd function},
\end{equation}
\begin{equation}
s \mapsto f(s)/s^{p-1} \text{ is strictly increasing on } (0, +\infty) \text{ with limit } 0 \text{ at } 0,
\end{equation}
\begin{equation}
\lim_{s \to +\infty} f(s)/s^{p-1} = +\infty.
\end{equation}
We then define
\begin{equation}
h \text{ is the inverse function of the restriction of } f(s)/s^{p-1} \text{ to } (0, +\infty),
\end{equation}
\begin{equation}
H(s) = \lambda s^p - pF(s),
\end{equation}
where $F(s) = \int_0^s f(t) \, dt$. For $\lambda > 0$ we shall also consider the following hypothesis:
\begin{equation}
(p - 1)(H'(s))^2 - pH(s)H''(s) \geq 0 \text{ for any } s \in [0, h(\lambda)].
\end{equation}
Let $E_\lambda$ be the set of all the solutions of (2.1) in $W^{1,p}_0(0,1)$ and $\lambda_k$ be defined by (1.3). When $1 < p \leq 2$ the structure of $E_\lambda$ is exactly the same as in the case $p = 2$.  

THEOREM 2.1. Assume $1 < p \leq 2$ and (2.2)-(2.7). Then

(i) if $\lambda \leq \lambda_1$, $E_\lambda = \{0\}$, and

(ii) if $\lambda_k < \lambda \leq \lambda_{k+1}$ for some $k \in \mathbb{N}^*$

\[ E_\lambda = \{0, \pm u_1, \ldots, \pm u_k\}, \]

where $u_l \in S_{l+}^+$ for $l = 1, \ldots, k$.

REMARK 2.1. The assumption (2.7), which is equivalent to the fact that $s \mapsto H^{p-1}(s)/H^p(s)$ is nondecreasing on $[0, h(\lambda)]$, is essential for uniqueness but not for existence. In the particular case where $f(r) = |r|^{q-1}r$ with $q > p - 1$ then $h(\lambda) = \lambda^{1/(q+1-p)}$, $H(s) = \lambda s^p - ps^{q+1}/(q+1)$ and (2.7) is satisfied.

PROOF OF THEOREM 2.1. As in Theorem 1.1 it is clear that any solution of (2.1) in $W^{1,p}_0(0,1)$ is continuous and at least $C^2$ (remember that $1 < p \leq 2$). Multiplying the equation by $u$ yields

\[ \int_0^1 |u_x|^p dx + \int_0^1 u f(u) dx = \lambda \int_0^1 |u|^p dx. \]

From Remark 1.2 a nonzero solution of (2.1) can exist only if $\lambda > \lambda_1$, which will be assumed in the sequel.

Step 1. If $u$ is a nonzero solution of (2.1) then $u_x(0) \neq 0$. Although it is a consequence of a general result due to Franchi, Lanconelli and Serrin, we give here a direct proof which also works when $p > 2$. Multiplying (2.1) by $u_x$ yields the energy relation

\[ -\frac{p-1}{p} |u_x(x)|^p + F(u(x)) - \frac{\lambda}{p} |u(x)|^p \]

\[ = -\frac{p-1}{p} |u_x(0)| + F(u(0)) - \frac{\lambda}{p} |u(0)|^p. \]

If we assume that $u_x(0) = 0$ we get

\[ |u_x(x)|^p = \frac{1}{p-1} (pF(u(x)) - \lambda |u(x)|^p). \]

As the function $x \rightarrow pF(x) - \lambda |x|^p$ is negative on $(-\rho, \rho) \setminus \{0\}$, $u_x$ is always 0 and $u \equiv 0$.

Step 2. The explicit construction. Without any loss of generality we assume $u_x(0) = \alpha > 0$. Hence $u$ is increasing on some interval $[0, x_0]$ and from (2.10) we get

\[ u_x^p(x) = \alpha^p + \frac{p}{p-1} F(u(x)) - \frac{\lambda}{p-1} u^p(x) \]

which gives $u$ as the inverse function of a $p$-elliptic integral

\[ x = \int_0^{u(x)} \frac{dt}{(\alpha^p + pF(t)/(p-1) - \lambda t^p/(p-1))^{1/p}} \]

on $[0, x_0]$. Moreover this formula remains valid as long as $u(x)$ is smaller than the first positive zero of

\[ r \mapsto \Psi(\alpha, r) = \alpha^p + \frac{p}{p-1} F(r) - \frac{\lambda}{p-1} |r|^p. \]
But the function $\Psi(\alpha, \cdot)$ is decreasing in $[0, h(\lambda)]$ and increasing on $[h(\lambda), +\infty)$; hence there are three possibilities.

**Case 1.** $\alpha^p > \lambda h^p(\lambda)/(p-1) - p F(h(\lambda))/(p-1) = \alpha^p(\lambda)$.

In that case the function $r \mapsto \int_0^r ds/(\Psi(\alpha, s))^{1/p}$ is an increasing $C^2$ diffeomorphism from $\mathbb{R}^+$ onto $\mathbb{R}^+$ and it is the same with $u$ defined by (2.13) which cannot belong to $E_\lambda$.

**Case 2.** $\alpha^p = \alpha^p(\lambda)$.

In that case $h(\lambda)$ is a double zero for $\Psi(\alpha, \cdot)$, and as $1 < p \leq 2$.

$$\int_0^{h(\lambda)} ds/(\Psi(\alpha, s))^{1/p} = +\infty.$$ 

As in Case 1 the function $r \mapsto \int_0^r ds/(\Psi(\alpha, s))^{1/p}$ is a $C^2$ diffeomorphism from $[0, h(\lambda))$ onto $\mathbb{R}^+$ and $u$ cannot belong to $E_\lambda$.

**Case 3.** $\alpha^p < \alpha^p(\lambda)$.

In that case $\Psi(\alpha, \cdot)$ admits a simple zero $S(\alpha)$ in $(0, h(\lambda))$. As $(\partial \Psi/\partial r)(\alpha, S(\alpha)) \neq 0$, $r \mapsto (\Psi(\alpha, r))^{-1/p}$ is integrable on $(0, S(\alpha))$ and we define

$$\theta(\alpha) = \int_0^{S(\alpha)} ds/(\Psi(\alpha, s))^{1/p}.$$ 

Relation (2.13) remains valid on $[0, \theta(\alpha)]$ and we have

$$u(\theta(\alpha)) = S(\alpha), \quad u_x(\theta(\alpha)) = 0.$$ 

Using the energy relation at $\theta(\alpha)$ we have

$$\frac{p-1}{p}\|u_x(x)\|^p = \frac{\lambda}{p} S^p(\alpha) - F(S(\alpha)) - \left(\frac{\lambda}{p} u^p(x) - F(u(x))\right)$$

or

$$\|u_x(x)\|^p = \alpha^p + \frac{p}{p-1} F(u(x)) - \frac{\lambda}{p-1} u^p(x).$$

Hence $u$ is decreasing on some interval $[\theta(\alpha), \Theta]$ and we have

$$x - \theta(\alpha) = -\int_{u(x)}^{S(\alpha)} ds/(\Psi(\alpha, s))^{1/p}.$$ 

This formula remains valid as long as $u$ is decreasing, and as in §1 $x = \theta(\alpha)$ is an axis of symmetry for the restriction of $u$ to $[0, 2\theta(\alpha)]$ and $x = 2\theta(\alpha)$ is a center of symmetry for the restriction of $u$ to $[0, 4\theta(\alpha)]$; the necessary and sufficient condition for $u$ to be a solution of (2.1) is that

$$1/2\theta(\alpha) \in \mathbb{N}^*.$$ 

**Step 3.** The function $\alpha \mapsto S(\alpha)$ is convex, increasing on $[0, \alpha(\lambda))$. We have $\Psi(\alpha, S(\alpha)) = 0$ and $(\partial \Psi/\partial r)(\alpha, S(\alpha)) \neq 0$. By the implicit function theorem $\alpha \mapsto S(\alpha)$ is $C^2$. We also have

$$\frac{d}{d\alpha}(\Psi(\alpha, S(\alpha))) = \frac{\partial \Psi}{\partial \alpha}(\alpha, S(\alpha)) + \frac{\partial \Psi}{\partial r}(\alpha, S(\alpha)) \frac{dS}{d\alpha}(\alpha)$$

which gives

$$\frac{dS}{d\alpha}(\alpha) = \frac{(p-1)\alpha^{p-1}}{\lambda S^{p-1}(\alpha) - f(S(\alpha))} = \frac{p(p-1)\alpha^{p-1}}{H'(S(\alpha))}.$$
As $S(\alpha) < h(\lambda)$, $\alpha \mapsto S(\alpha)$ is increasing on $[0, \alpha(\lambda))$. Moreover
\[
\frac{d^2 S}{d\alpha^2}(\alpha) = p(p-1)(p-1)\alpha^{p-2}H'(S(\alpha)) - \alpha^{p-1}H''(S(\alpha))dS/d\alpha.
\]
Using (2.20) and the definition of $S(\alpha)$ and $H$ we get
\[
(2.21) \quad \frac{d^2 S}{d\alpha^2}(\alpha) = p(p-1)\alpha^{p-2}(p-1)(H'(S(\alpha)))^2 - pH(S(\alpha))H''(S(\alpha))
\]
\[
(H'(S(\alpha)))^3.
\]
From (2.7) we deduce $d^2 S(\alpha)/d\alpha^2 \geq 0$.

**Step 4.** The function $\alpha \mapsto \theta(\alpha)$ is continuous increasing on $[0, \alpha(\lambda))$. For $t \in [0, \alpha]$ the function $s \mapsto \Psi(t, s)$ admits a first positive zero at $S(t)$ which means
\[
t^p + \frac{p}{p-1}F(S(t)) - \frac{\lambda}{p-1}S^p(t) = 0 \quad \text{and} \quad \Psi(\alpha, S(t)) = \alpha^p - t^p.
\]
Taking $t$ as a new variable in (2.15) we get
\[
(2.22) \quad \theta(\alpha) = \int_0^\alpha \frac{dS}{dt}(t)\frac{dt}{(\alpha^p - t^p)^{1/p}}
\]
or
\[
(2.23) \quad \theta(\alpha) = \int_0^1 \frac{dS}{dt}(\alpha\sigma)\frac{d\sigma}{(1 - \sigma^p)^{1/p}}.
\]
As $ds/dt$ is increasing and $C^1$ on $[0, \alpha(\lambda))$, it is the same with $\alpha \mapsto \theta(\alpha)$.

**Step 5. End of the proof.** As $\lim_{\alpha \to 0} S(\alpha) = 0$ and $\lim_{\alpha \to 0} F(S(\alpha))/S^p(\alpha) = 0$ we get
\[
(2.24) \quad S(\alpha) \sim_{\alpha \to 0} \alpha \left(\frac{p-1}{\lambda}\right)^{1/p}
\]
which implies
\[
\lim_{\alpha \to 0} \frac{dS}{d\alpha}(\alpha) = \left(\frac{p-1}{\lambda}\right)^{1/p}
\]
and
\[
(2.25) \quad \lim_{\alpha \to 0} \theta(\alpha) = \left(\frac{p-1}{\lambda}\right)^{1/p} \int_0^1 \frac{d\sigma}{(1 - \sigma^p)^{1/p}} = \frac{1}{2} \left(\frac{\lambda_1}{\lambda}\right)^{1/p}.
\]
For the other bound we have $\lim_{\alpha \to 0} S(\alpha) = h(\lambda)$. As $h(\lambda)$ is just a double zero for $\Psi(\alpha(\lambda), r)$, there exists a continuous and bounded function $\varphi$ on $[0, \alpha(\lambda)]$ such that
\[
\Psi(\alpha(\lambda), r) = (h(\lambda) - r)^2 \varphi(r).
\]
Moreover
\[
\int_0^{S(\alpha)} \Psi(\alpha, t)^{-1/p} dt > \int_0^{S(\alpha)} \Psi(\alpha(\lambda), t)^{-1/p} dt
\]
\[
= \int_0^{S(\alpha)} (h(\lambda) - t)^{-2/p} \varphi(t)^{-1/p} dt.
\]
As $1 < p \leq 2$ we get
\[
(2.26) \quad \lim_{\alpha \to 0} \theta(\alpha) = \int_0^{h(\lambda)} (h(\lambda) - t)^{-2/p} \varphi(t)^{-1/p} dt = +\infty.
\]
As a consequence $\alpha \mapsto \theta(\alpha)$ is an increasing diffeomorphism from $(0, \alpha(\lambda))$ onto $\left(\frac{1}{2}(\lambda_1/\lambda)^{1/p}, +\infty\right)$ and $1/2\theta(\alpha)$ a decreasing diffeomorphism from $(0, \alpha(\lambda))$ onto $(0, (\lambda/\lambda_1)^{1/p})$. If we assume that $\lambda_k < \lambda \leq \lambda_{k+1}$ for some $k \in \mathbb{N}^*$ there exist exactly $k$ integers $l = 1, \ldots, k$ and $k$ positive real numbers $\alpha_l$ such that $1/2\theta(\alpha_l) = l$. If $u_l$ is the solution of the initial value problem

$$
\begin{align*}
- (|u_l|^p - 2u_l)x + f(u_l) &= \lambda |u_l|^{p-2}u_l \quad \text{on } (0, 1), \\
u_l(0) &= 0, \quad u_{lx}(0) = \alpha_l,
\end{align*}
$$

then $u_l(1) = 0$, $u_l \in S^+_l$. We get the result in considering $-u_l$, $l = 1, \ldots, k$.

**Remark 2.2.** If we represent the bifurcation diagram $(\lambda, u_\lambda)$ then there exists no secondary bifurcation along the branches of solutions in $S^\pm_k$ issuing from $\lambda_k$.

In the case $p > 2$ the main difference will come from the fact that the following integral

$$
(2.28) \quad x(\lambda) = \int_0^{h(\lambda)} \frac{ds}{\left(\alpha p(\lambda) + \frac{p}{p-1} F(s) - \frac{\lambda}{p-1} s^p\right)^{1/p}}
$$

is finite as $h(\lambda)$ is a double zero of $\Psi(\alpha(\lambda), r)$.

**Theorem 2.2.** Assume $p > 2$ and (2.2)–(2.7). Then

(i) if $\lambda \leq \lambda_1$, $E_\lambda = \{0\}$,

(ii) if $\lambda_k < \lambda \leq \lambda_{k+1}$ for some $k \in \mathbb{N}^*$

$$
(2.29) \quad E_\lambda = \{0\} \cup \{\pm u_1\} \bigcup_{l=2}^k \{\pm E^l_k\},
$$
where \( u_1 \in S_1^+ \) and \( E_1^l \subset S_l^+ \), \( l = 2, \ldots, k \), and

\( E_1^l \) is reduced to a single element if \( 2lx(\lambda) \geq 1 \),

\( E_1^l \) is diffeomorphic to \( [0, 1]^{l-1} \) if \( 0 < 2px(\lambda) < 1 \).\(^1\)

**Proof.** The idea is essentially the same as in Theorem 2.1 except that in Step 2, Case 2 (that is, if \( \alpha^p = \alpha^p(\lambda) \)) gives rise to solutions of (2.1) with maximum value \( h(\lambda) \), and in that case Serrin and Veron’s existence and uniqueness result does not apply; moreover the value \( u = h(\lambda) \) is a bifurcation value for (2.1).

**Step 1.** Assume \( 2x(\lambda) \geq 1 \). Then the construction of Theorem 2.1 works: the function \( \alpha \mapsto 1/2\theta(\alpha) \) is a decreasing diffeomorphism from \( (0, \alpha(\lambda)] \) onto \( [1/2x(\lambda), (\lambda/\lambda_1)^{1/p}] \). As \( \lambda_k < \lambda \leq \lambda_{k+1} \) there exist exactly \( k \) integers \( 1, 2, \ldots, k \) and \( k \) positive real numbers \( \alpha_1, \ldots, \alpha_k \) such that \( 1/2\theta(\alpha_l) = l \in [1/2x(\lambda), (\lambda/\lambda_1)^{1/p}] \), \( l = 1, \ldots, k \). and we get the corresponding solutions \( u_l \in S_l^+ \) by (2.26).

**Step 2.** Assume \( 4x(\lambda) \geq 1 > 2x(\lambda) \). All the elements \( u_l = 2, \ldots, k \) in \( S_l^+ \) are constructed as in Step 1. As for the element \( u_1 \in S_1^+ \) it has necessarily the following form as the initial slope must be \( \alpha(\lambda) \):

\[
\text{for } 0 \leq x \leq x(\lambda)\\
\frac{dt}{(\Psi(\alpha(\lambda), t))^{1/p}},
\]

\[
x - (1 - x(\lambda)) = - \int_{u_1(\lambda)}^{h(\lambda)} \frac{dt}{(\Psi(\alpha(\lambda), t))^{1/p}}.
\]

**Step 3.** Assume \( 0 < 2lx(\lambda) < 1 \) for some \( l \in \{2, \ldots, k\} \). We can construct all the elements of \( E_l \cap S_l^+ \) in the following way as their initial slope is necessarily \( \alpha(\lambda) \):

\[
\text{for } 0 \leq x \leq x(\lambda)\\
\frac{dt}{(\Psi(\alpha(\lambda), t))^{1/p}},
\]

\[
x = \int_0^{u_1(x)} \frac{dt}{(\Psi(\alpha(\lambda), t))^{1/p}},
\]

\[
x_1 - x(\lambda) \leq 1 - 2lx(\lambda)
\]

then \( u_1(x) = h(\lambda) \),

\[
\text{for } x_1 \leq x \leq 2x(\lambda) + x_1
\]

\[
x - x_1 = - \int_{u_1(x)}^{h(\lambda)} \frac{dt}{(\Psi(\alpha(\lambda), t))^{1/p}},
\]

\[
x + 2x(\lambda) \leq x \leq x_2 \text{ where } x_2 \in (x_1 + 2x(\lambda), 1)
\]

\[
x_2 - (x_1 + 2x(\lambda)) + x_1 - x(\lambda) \leq 1 - 2lx(\lambda)
\]

then \( u_l(x) = -h(\lambda) \).

\(^1\)And more naturally to the set \( K_t = \{x = (x^1, \ldots, x^t), x^j \geq 0, \sum_{j=1}^t x^j = 1 - 2lx(\lambda)\} \).
Continuing this procedure any solution \(u_l \in S^+_l\) is defined by the intervals \(I_j = [x_{j-1} + 2x(\lambda), x_j], \ j = 1, \ldots, l\), and \(x_0 = -x(\lambda)\) where it takes the constant value \((-1)^{j+1}h(\lambda)\) and the intervals \([x_{j-1}, x_{j-1} + 2x(\lambda)]\) where it is defined by

\[
x - x_{j-1} = - \int_{h(\lambda)}^{u_l(x)} \frac{dt}{\Psi(\alpha(\lambda), t)^{1/p}}
\]

if \(j\) is even or

\[
x - x_{j-1} = \int_{h(\lambda)}^{u_l(x)} \frac{dt}{\Psi(\alpha(\lambda), t)^{1/p}}
\]

if \(j\) is odd.

From the above construction the total length of the \(I_j\) is \(1 - 2 \lambda x(\lambda)\) and the set \(E^l_\lambda\) of the \(u_l\) is diffeomorphic to the \((l - 1)\)-dimensional cube.

**FIGURE 2. Example of construction of \(E^3_\lambda\)**

**REMARK 2.3.** It is important to notice that this type of secondary bifurcation along the branch of solutions issuing from \(\lambda_k, \ k \geq 2\), always appears if we have

\[
\lim_{\lambda \to +\infty} x(\lambda) = 0.
\]

This is in particular the case if \(f(r) \sim |r|^{-1}r\) which implies

\[
x(\lambda) \sim \lambda^{-1/p} \int_0^1 \left( \frac{q - p + 1}{(p - 1)(q + 1)} + \frac{p}{(p - 1)(q + 1)} \sigma^{q+1} - \frac{\sigma^p}{p - 1} \right)^{-1/p} d\sigma.
\]

However this is not always the case under conditions (2.2)-(2.7), for example, with \(f(r) = |r|^{p-2} \log |r| r\) for \(|r| \geq 2\), where we get

\[
\lim_{\lambda \to +\infty} x(\lambda) = \int_0^1 \left( \frac{1}{p(p-1)}(1 - \sigma^p) + \frac{1}{p - 1} \sigma^p \log \sigma \right)^{-1/p} d\sigma.
\]

We finally have the following exclusion principle.

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Theorem 2.3. Assume $p > 1$, (2.2)–(2.7), $g$ is a continuous even function increasing on $\mathbb{R}^+$ and $u_1$ and $u_2$ are two solutions of (2.1); then

(i) if $u_1$ and $u_2$ have the same number of zeros

$$
\int_0^1 g(u_1(x)) \, dx = \int_0^1 g(u_2(x)) \, dx;
$$

(ii) if $u_1$ and $u_2$ do not have the same number of zeros

$$
\int_0^1 g(u_1(x)) \, dx \neq \int_0^1 g(u_2(x)) \, dx.
$$

Proof. It is clear that for any function $\int_0^1 g(u(x)) \, dx$ is equal to $\int_0^1 g(-u(x)) \, dx$. When $p > 2$ we have only to consider two solutions of $E_\lambda$ with the same number of zeros and belonging to some $E_\lambda^l$, $l \geq 2$, in the case $2lx(\lambda) < 1$. In that case $u_1$ and $u_2$ take the value $\pm h(\lambda)$ on $l$ intervals $I_j^1$ and $I_j^2$, $j = 1, \ldots, l$, which are disconnected and have the same total length which gives

$$
\int_{\bigcup_j I_j^1} g(u_1(x)) \, dx = \int_{\bigcup_j I_j^2} g(u_2(x)) \, dx = (1 - 2lx(\lambda))g(h(\lambda)).
$$

On $(0, 1) \setminus \{\bigcup_j I_j^1\}$ or $(0, 1) \setminus \{\bigcup_j I_j^2\}$ $u_1$ and $u_2$ are defined by the same types of formula ((2.32) or (2.30)) and the integral of $g(u_i)$ over these sets is

$$
2l \int_0^{x(\lambda)} g(u_1(x)) \, dx.
$$

Hence, for $i = 1, 2$, we get

$$
\int_0^1 g(u_i(x)) \, dx = (1 - 2lx(\lambda))g(h(\lambda)) + 2l \int_0^{x(\lambda)} g(u_i(x)) \, dx
$$

which proves (i).

For proving (ii) we shall assume either $1 < p \leq 2$ or $p > 2$ but $u_1$ and $u_2$ are not constant on any subinterval of $(0, 1)$ (the other case is essentially the same). If $u_1$ and $u_2$ do not have the same number of zeros in $(0, 1)$ we can assume $u_1x(0) = \alpha$, $u_2x(0) = \beta$, $0 < \alpha < \beta$; $u_1$ is $4\theta(\alpha)$-periodic, $u_2$ is $4\theta(\beta)$-periodic and $0 < \theta(\alpha) < \theta(\beta)$. Moreover

$$
\frac{1}{2\theta(\alpha)} = k_1, \quad \frac{1}{2\theta(\beta)} = k_2, \quad k_1, k_2 \in \mathbb{N}^*, \quad k_1 > k_2.
$$

Step 1. For $0 < x < \theta(\alpha)$ we have $0 < u_1(x) < u_2(x)$. On a right neighbourhood of $0$ we have $u_1 < u_2$, and $u_1$ and $u_2$ are increasing on $[0, \theta(\alpha)]$. If we assume the existence of some $x_0 \in [0, \theta(\alpha)]$ such that $u_1(x_0) = u_2(x_0)$, we can always suppose that $u_1 < u_2$ in $(0, x_0)$ and then $u_1x(x_0) \geq u_2x(x_0)$. The energy relation implies

$$
\alpha^p + \frac{p}{p - 1} F(u_1(x_0)) - \frac{\lambda}{p - 1} u_1^p(x_0)
\geq \beta^p + \frac{p}{p - 1} F(u_2(x_0)) - \frac{\lambda}{p - 1} u_2^p(x_0)
$$

and $\alpha \geq \beta$ which is impossible.
Step 2. End of the proof. From Step 1: $0 < u_1(x) < u_2(x')$ for $0 < x < \theta(\alpha)$ and $0 < x' < \theta(\beta)$. Set $\varphi$ the lowest common multiple to $k_1$ and $k_2$. There exist $n_1$ and $n_2 \in \mathbb{N}^*$ such that $n_1 k_1 = n_2 k_2 = \varphi$ and

\begin{equation}
(2.48) \quad \frac{n_1}{\theta(\alpha)} = \frac{n_2}{\theta(\beta)}, \quad 0 < n_1 < n_2.
\end{equation}

Then

\begin{equation}
(2.49) \quad \int_0^1 g(u_1(x)) \, dx = \frac{1}{\theta(\alpha)} \int_0^{\theta(\alpha)} g(u_1(x)) \, dx,
\end{equation}

\begin{equation}
(2.50) \quad \int_0^1 g(u_2(x)) \, dx = \frac{1}{\theta(\beta)} \int_0^{\theta(\beta)} g(u_2(x)) \, dx.
\end{equation}

Setting $T = n_2 \theta(\alpha) = n_1 \theta(\beta)$, we have

\[
\frac{1}{\theta(\alpha)} \int_0^{\theta(\alpha)} g(u_1(x)) \, dx = \frac{1}{n_2 \theta(\alpha)} \int_0^{n_2 \theta(\alpha)} g \left( u_1 \left( \frac{\sigma}{n_2} \right) \right) \, d\sigma = \frac{1}{T} \int_0^{T} g \left( u_1 \left( \frac{\sigma}{n_2} \right) \right) \, d\sigma
\]

and

\[
\frac{1}{\theta(\beta)} \int_0^{\theta(\beta)} g(u_2(x)) \, dx = \frac{1}{T} \int_0^{T} g \left( u_2 \left( \frac{\sigma}{n_1} \right) \right) \, d\sigma,
\]

which implies

\begin{equation}
(2.51) \quad \int_0^1 g(u_1(x)) \, dx < \int_0^1 g(u_2(x)) \, dx.
\end{equation}

REMARK 2.4. As a consequence there exist $k + 1$ different critical values for the energy functional

\begin{equation}
(2.52) \quad J(\omega) = \frac{1}{p} \int_0^1 |\omega_x|^p \, dx + \int_0^1 F(\omega) \, dx - \frac{\lambda}{p} \int_0^1 |\omega|^p \, dx
\end{equation}

defined in $W^{1,p}_0(0,1)$, for $\lambda_k < \lambda \leq \lambda_{k+1}$; those critical values only depend on the set $S_l$, $l = 1, \ldots, k$, the critical points of (2.52) belong to. This is an immediate consequence of Theorem 2.3 and the fact that

\begin{equation}
(2.53) \quad J(u) = \int_0^1 \left( F(u) - \frac{1}{p} uf(u) \right) \, dx
\end{equation}

for $u \in E_\lambda$.

REFERENCES


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