THE CLASSIFYING TOPOS OF A CONTINUOUS GROUPOID. I

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ABSTRACT. We investigate some properties of the functor $B$ which associates to any continuous groupoid $G$ its classifying topos $BG$ of equivariant $G$-sheaves. In particular, it will be shown that the category of toposes can be obtained as a localization of a category of continuous groupoids.

If $G$ is a group, the category of $G$-sets (sets equipped with a right $G$-action) is a topos $BG$, which classifies principal $G$-bundles: for instance, if $X$ is a topological space there is an equivalence between topos morphisms $\text{Sheaves}(X) \to BG$ and principal $G$-bundles over $X$.

The construction of $BG$ also applies to the case where $G$ is a topological group, or more generally, a topological groupoid. It is a rather surprising result that this essentially exhausts the range of toposes: Joyal and Tierney (1984) have shown that any topos is equivalent to one of the form $BG$ for a topological group $G$, provided one works with a slightly generalized notion of topological space, by taking the lattice of open sets as the primitive notion, rather than the set of points (one sometimes speaks of "pointless" spaces). The continuous groupoids of this paper are the groupoid objects in this category of generalized spaces.

The aim of this paper is threefold. First, $G \to BG$ is a functor, and we wish to investigate how the properties of the topos $BG$ depend on those of the continuous groupoid $G$, and more generally how the properties of a geometric morphism $BG \to BH$ depend on those of the map of continuous groupoids $G \to H$. The second aim is to extend the Joyal-Tierney result, and not only represent toposes in terms of continuous groupoids, but also the geometric morphisms from one topos to another. There are several possible solutions to this problem. In this paper, I present one approach, and show that the category of toposes can be obtained as a category of fractions from a category of continuous groupoids. Another approach, somewhat similar in spirit to the Morita theorems for categories of modules, will be presented elsewhere. The third aim of this paper is of a more methodological nature: in presenting many arguments concerning generalized, "pointless" spaces, I have tried to convey the idea that by using change-of-base techniques and exploiting the internal logic of a Grothendieck topos, point-set arguments are perfectly suitable for dealing with pointless spaces (at least as long as one stays within the "stable" part of the theory). Although the general underlying idea is very clear (see e.g. the discussion in 5.3 below), it is a challenging open problem to express this.
as a general metatheorem which allows one to transfer (constructively valid) results concerning topological spaces immediately to the context of these generalized spaces.

Let me outline the contents of this paper in more detail. The construction of $BG$ for a continuous groupoid $G$ is a particular case of a colimit of toposes: one takes the nerve of $G$, which is a simplicial space $NG$, then one takes sheaves to obtain a simplicial topos $\text{Sh}(NG)$, and $BG$ is simply the colimit of this simplicial topos (in the appropriate bicategorical sense). Before describing this construction in more detail in §4, I will first consider the general construction of colimits of toposes, and prove the following theorem.

**Theorem 1.** All (small) indexed colimits of Grothendieck toposes exist, and are computed as indexed limits of the underlying categories and inverse image functors.

In §3 the special case of a simplicial topos is discussed.

In §5, I will take a slightly different point of view, and regard $BG$ as a category of spaces equipped with a $G$-action. The proofs in this section are also intended to serve the third, methodological, aim just mentioned. The results are of the following kind: sufficient conditions are given on homomorphisms $G \xrightarrow{\phi} H$ of continuous groupoids to imply that the induced geometric morphism $BG \xrightarrow{B\phi} BH$ is of a specific type. For instance, one can give a meaningful definition of when a map of continuous groupoids is open, full, faithful, and essentially surjective respectively (cf. 5.5), and prove (see 5.1, 5.15):

**Theorem 2.** If $G \xrightarrow{\phi} H$ is open and full then $BG \xrightarrow{B\phi} BH$ is an atomic map of toposes.

**Theorem 3.** If $G \xrightarrow{\phi} H$ is open, full and faithful, and essentially surjective, then $BG \xrightarrow{B\phi} BH$ is an equivalence of toposes.

In §6, we will show that the construction of the topos $BG$ is stable under change of base, at least when the domain and codomain maps are open. Writing $B(\mathcal{E},G)$ for the classifying $\mathcal{E}$-topos of a continuous groupoid $G$ in $\mathcal{E}$, this can be expressed as follows:

**Theorem 4.** Let $\mathcal{F} \xrightarrow{p} \mathcal{E}$ be a geometric morphism, and let $G$ be an open continuous groupoid in $\mathcal{E}$. Then there is an equivalence of toposes $B(\mathcal{F}, p^#(G)) \rightarrow \mathcal{F} \times_\mathcal{E} B(\mathcal{E},G)$.

Theorem 4 allows us to use point-set arguments in the context of these classifying toposes $BG$. A consequence of Theorem 4 is used to prove the results of §5, such as Theorems 2 and 3.

It is Theorem 3 that leads to our second aim, namely that of obtaining the category of toposes as a category of fractions from a category of continuous groupoids. Imitating the definition of an equivalence of categories, we call a map $\phi$ of continuous groupoids an essential equivalence if it satisfies the hypotheses of Theorem 3.

The following result will be proved in §7.
THEOREM 5. The class of essential equivalences admits a right calculus of fractions, and the category of toposes is equivalent to the localization of a category of continuous groupoids obtained by inverting the essential equivalences.

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1. Preliminaries. In this section I will recall some basic definitions and facts.

1.1. Spaces and locales. Our terminology concerning spaces and locales follows Joyal and Tierney (1984), in this section referred to as [JT]. So a locale is a complete Heyting algebra (a “frame”), and a morphism of locales \( A \xrightarrow{H} B \) is a function which preserves finite meets and arbitrary sups. The category of (generalized) spaces is the dual of the category of locales. It contains the category of sober topological spaces as a full subcategory. If \( X \) is a space, the corresponding locale is denoted by \( \mathcal{O}(X) \), and elements of \( \mathcal{O}(X) \) are called opens of \( X \). So a map of spaces, or a continuous map \( X \xrightarrow{f} Y \) is by definition given as a locale morphism \( f^{-1}: \mathcal{O}(Y) \to \mathcal{O}(X) \).

A point of a space is a map \( 1 \to X \), where \( 1 \) is the terminal space, \( \mathcal{O}(1) = P(\{\ast\}) = \Omega \). A neighbourhood of a point \( 1 \to X \) is an open \( U \in \mathcal{O}(X) \) such that \( \ast \notin x^{-1}(U) \), i.e. \( 1 \xrightarrow{\ast} X \) factors through the subspace \( U \subseteq X \).

We suppose that the reader is familiar with the basic properties of spaces [JT].

A presentation of a space \( X \) is a poset \( P \) equipped with a stable system of covering families, such that \( \mathcal{O}(X) \) is isomorphic to the set of downwards closed subsets of \( P \) which are closed for the system of covers, i.e. \( \mathcal{O}(X) \simeq \{S \subseteq P | (p \leq q \in S \Rightarrow p \in S) \text{ and } (T \text{ covers } p, T \subseteq S \Rightarrow p \in S)\} \); this is equivalent to saying that \( P \) is a site for the topos of sheaves on \( X \). The opens in the image of the canonical order-preserving map \( P \to \mathcal{O}(X) \) are also called basic opens of \( X \).

1.2. Open maps, étale maps (see [JT]). A map \( X \xrightarrow{f} Y \) of spaces is open if \( f^{-1} \) has a left-adjoint \( f(-): \mathcal{O}(X) \to \mathcal{O}(Y) \) such that the identity \( f(U \land f^{-1}(V)) = f(U) \land V \) holds. \( f \) is an open surjection if \( f^{-1} \) is moreover 1-1, i.e. \( ff^{-1}(U) = U \).

Open maps and open surjections are stable under composition and pullback; and if \( X \xrightarrow{f} Y \xrightarrow{g} Z \) are maps such that \( gf \) is open and \( f \) is a surjection, then \( g \) is open.

Recall that \( U \in \mathcal{O}(X) \) is called positive if every cover of \( U \) contains at least one element. \( X \to 1 \) is open iff \( X \) has a presentation consisting of positive opens. A map \( X \xrightarrow{f} Y \) is étale if it is a local homeomorphism (i.e. \( f \) is open and there is an open cover \( X = \bigvee_i U_i \) such that \( f|U_i: U_i \xrightarrow{\sim} f(U_i) \) is an isomorphism). This is equivalent to requiring that \( X \xrightarrow{f} Y \) and the diagonal \( X \to X \times_Y X \) are open. In particular, \( X \) is discrete iff \( X \to 1 \) and \( X \to X \times X \) are open. The usual equivalence between étale maps into \( X \) and sheaves on \( X \) also holds in the context of these generalized spaces.

1.3. Quotients. Colimits of spaces are computed as limits of the corresponding sets of opens. In particular, given maps \( X \xrightarrow{f} Y \) of spaces, their coequalizer \( Y \xrightarrow{q} Q \)
is described by $\mathcal{O}(Q) = \{U \in \mathcal{O}(Y) | f^{-1}(U) = g^{-1}(U)\}$, and $q^{-1}$ is the inclusion $\mathcal{O}(Q) \rightarrow \mathcal{O}(Y)$. It is easy to see that if $f$ and $g$ are open, then so is $q$. Open surjections are coequalizers of their kernel pairs (Moerdijk (1986), p. 66). It follows that such coequalizers are stable (a coequalizer $X \rightrightarrows Y \rightarrow Q$ is stable if for any space $T$, $T \times X \rightrightarrows T \times Y \rightarrow T \times Q$ is again a coequalizer). It seems to be a hard problem to describe the stable coequalizers of spaces (the corresponding problem for topological spaces is discussed in Day, Kelly (1970)). But at least we conclude that if $X \rightarrow 1$ is open and $R \subset X \times X$ is an open subspace which is an equivalence relation, then the coequalizer

$$R \rightrightarrows X \overset{q}{\rightarrow} X/R$$

is stable. (*Sketch of proof:* $q$ is an open surjection since $r_1$ and $r_2$ are, and $X/R$ is discrete because $X/R \rightarrow 1$ is open, and so is the diagonal $X/R \rightarrow X/R \times X/R$, as follows by considering the square

$$\begin{array}{ccc}
R & \xrightarrow{\text{open}} & X \times X \\
\downarrow & & \downarrow \\
X/R & \xrightarrow{\Delta} & X/R \times X/R
\end{array}$$

and using 1.2. To see that $R$ is the kernelpair of $X \rightarrow X/R$, it suffices to consider the case where $X/R = 1$, by writing $X = \bigsqcup_{t \in X/R} q^{-1}(t)$. But if $X/R = 1$, then for any two positive (cf. 1.2) opens $U$ and $V$ of $X$, $r_1 r_2^{-1}(U) = q^{-1} q(U) = q^{-1} q(V) = r_1 r_2^{-1}(V)$, from which it easily follows that $R = X \times X$.)

1. 1. Toposes. In this paper, topos means Grothendieck topos. We fix one such topos $\mathcal{S}$ as our base throughout, and work with the comma category of toposes over $\mathcal{S}$. If $\mathcal{S}$ and $\mathcal{T}$ are two such toposes, $\text{Hom}_{\mathcal{S}}(\mathcal{T}, \mathcal{S})$ is the set of geometric morphisms $\mathcal{F} \dashv \mathcal{E}$ over $\mathcal{S}$. These form a category denoted by $\text{Hom}_{\mathcal{S}}(\mathcal{F}, \mathcal{E})$, where for $f, g: \mathcal{F} \rightarrow \mathcal{E}$, the maps $\alpha: f \Rightarrow g$ are the natural transformations $f^* \rightarrow g^*$ over $\mathcal{S}$. I will often omit the subscript $\mathcal{S}$, and just write $\text{Hom}(\mathcal{F}, \mathcal{E})$, $\text{Hom}(\mathcal{F}, \mathcal{E})$. Moreover, I will often tacitly work inside $\mathcal{S}$, and abuse the language as if $\mathcal{S} = \text{Sets}$, in the usual way. We recall that (2-categorical) pullbacks of Grothendieck toposes, which are used throughout this paper, exist (see e.g. [TT, p. 131]), as well as filtered inverse limits (see Moerdijk (1986)).

1. 1.5. Change of base. We will often work with the category of internal spaces in a topos $\mathcal{E}$, $(\text{spaces})_{\mathcal{E}}$. If $\mathcal{F} \xrightarrow{p} \mathcal{E}$ is a geometric morphism, $p$ induces an adjunction

$$(\text{spaces})_{\mathcal{E}} \xleftrightarrow{p^! \dashv p^*} (\text{spaces})_{\mathcal{F}}, \quad p_! \dashv p^\#.$$ 

$p_!$ is defined by $p_!(p_! Y) = p_*(\mathcal{O}(Y))$. $p^\#$ is most easily described in terms of presentations: if $\mathcal{P}$ is a presentation of a space $X$ in $\mathcal{E}$, then the poset $p^*(\mathcal{P})$ together with the $p^*$-images of the covers in $\mathcal{P}$ give a presentation of $p^\#(X)$.

1. 6. Sheaves and spatial reflection ([JT], Johnstone (1981)). A spatial topos is a topos of the form $\text{Sh}(X) = \text{sheaves on } X$, where $X$ is a (generalized) space. A geometric morphism $\mathcal{F} \dashv \mathcal{E}$ is called *spatial* if it is equivalent to one of the canonical form $\text{Sh}_{\mathcal{E}}(X) \rightarrow \mathcal{E}$, where $X$ is a space in $\mathcal{E}$ and $\text{Sh}_{\mathcal{E}}(X)$ is the category
of $\mathcal{E}$-internal sheaves on $X$. If $\mathcal{F} \xrightarrow{f} \mathcal{E}$ is a geometric morphism, there is a reflection into spatial toposes over $\mathcal{E}$, which is stable under pullback along an arbitrary $\mathcal{E}' \rightarrow \mathcal{E}$, and which preserves products [JT, §VI.5].

1.7. Open maps. A geometric morphism $\mathcal{F} \xrightarrow{f} \mathcal{E}$ is open if $f^*$ preserves $\forall$ (i.e. $f^*(\forall_X(S)) \cong \forall_{f^*(X)}f^*(S)$ for any diagram $S \subseteq X \xrightarrow{\Delta} Y$ in $\mathcal{E}$). See the references [JT], Johnstone (1980) for equivalent characterizations. If $\mathcal{E} \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{E}$ are geometric morphisms then $f \circ g$ is open if $f$ and $g$ are, and $f$ is open if $fg$ is and $g$ is a surjection. Open maps and open surjections are stable under pullback ([JT], Johnstone (1980)); open surjections are stable under filtered inverse limits (Moerdijk (1986), Theorem 5.1(ii)).

If $\mathcal{F} \xrightarrow{f} \mathcal{E}$ is open, then the canonical maps $f^*(XY) \rightarrow f^*(X)f^*(Y)$ and $f^*(\Omega_{\mathcal{E}}) \rightarrow \Omega_{\mathcal{F}}$ are mono.

1.8. Locally connected maps. $\mathcal{F} \xrightarrow{f} \mathcal{E}$ is locally connected if $f^*$ has an $\mathcal{E}$-indexed left adjoint. This is equivalent to requiring that $f^*$ commutes with $\Pi$-functors (or that $f^*/X: \mathcal{E}/X \rightarrow \mathcal{F}/f^*(X)$ preserves exponentials, for every $X \in \mathcal{E}$). Locally connected maps are stable under composition and pullback. (See Barr and Paré (1980), and Moerdijk (1986), Appendix.)

1.9. Atomic maps. A geometric morphism $\mathcal{F} \xrightarrow{f} \mathcal{E}$ is atomic if $f^*$ preserves exponentials and the subobject classifier; see Barr-Diaconescu (1980). This is equivalent to requiring $\mathcal{F} \xrightarrow{f} \mathcal{E}$ and the diagonal $\mathcal{F} \rightarrow \mathcal{F} \times \mathcal{E}\mathcal{F}$ to be open [JT]. Atomic maps are closed under pullback and composition.

1.10. Lemma. Let $\mathcal{E} \xrightarrow{g} \mathcal{F} \xrightarrow{f} \mathcal{E}$ be geometric morphisms. If $g$ is an open surjection and $fg$ is locally connected (resp. atomic) then so is $f$.

(This follows easily from the last-mentioned property of open maps in 1.7.)

2. Colimits of toposes. The aim of this section is to show that colimits of toposes exist, and to give an explicit way to construct them.

2.1. Coequalizers. The main example will be that of a (pseudo-) coequalizer. (I will usually omit the prefix “pseudo”, following the common convention in topos theory, cf. [TT, p. 5].) Suppose we are given a parallel pair of geometric morphisms $\mathcal{F} \xrightarrow{f} \mathcal{E}$ over the base $\mathcal{S}$. Let $\mathcal{G}$ be the subcategory of $\mathcal{E}$ whose objects are pairs $(X, \theta)$, $X$ an object of $\mathcal{E}$ and $\theta: f^*(X) \rightarrow g^*(X)$ an isomorphism in $\mathcal{F}$, and whose maps $(X, \theta) \rightarrow (Y, \xi)$ are $\mathcal{E}$-morphisms $X \xrightarrow{\alpha} Y$ which are compatible with these isomorphisms, i.e. $g^*(\alpha) \circ \theta = \xi \circ f^*(\alpha)$. Write $g^*: \mathcal{G} \rightarrow \mathcal{E}$ for the forgetful functor $(X, \theta) \mapsto X; g^*$ is faithful.

Theorem. $\mathcal{D}$ is a Grothendieck topos over $\mathcal{S}$, and $g^*$ defines a geometric morphism $\mathcal{E} \xrightarrow{g} \mathcal{D}$ making $\mathcal{F} \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{D}$ into a coequalizer of toposes over $\mathcal{S}$: for any other topos $\mathcal{H}$, $q$ induces an equivalence of categories $\text{Eq}(\mathcal{H}, f, g) \cong \text{Hom}_\mathcal{S}(\mathcal{D}, \mathcal{H})$, where the left-hand side is the categories of pairs $(h, \mu)$, $\mathcal{E} \xrightarrow{h} \mathcal{H}$ a geometric morphism with $\mu: hf \cong hg$.

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PROOF. To see that \( \mathcal{D} \) is a topos and \( q^* \) defines a geometric morphism, we use Giraud's criterion [SGA 4, exposé IV; TT, p. 16]. So we have to check the following:

(a) \( \mathcal{D} \) has finite limits,
(b) \( \mathcal{D} \) has all set-indexed coproducts, and these are disjoint and universal,
(c) every equivalence relation in \( \mathcal{D} \) has a universal coequalizer,
(d) every equivalence relation in \( \mathcal{D} \) is effective, and every epimorphism in \( \mathcal{D} \) is a coequalizer,
(e) \( \mathcal{D} \) has small hom-sets,
(f) \( \mathcal{D} \) has a set of generators.

Now (a)-(e) follow from the corresponding facts for \( \mathcal{E} \), using that \( f^* \) and \( g^* \) preserve colimits and finite limits. For example in the case of (d), if \((R, \theta) \to (X, \mu) \times (X, \mu)\) is an equivalence relation in \( \mathcal{D} \), then \( R \to X \times X \) is an equivalence relation in \( \mathcal{E} \), and \( \mu \) induces an isomorphism

\[
\mu/R: f^*(X/R) \cong f^*(X)/f^*(R) \to g^*(X)/g^*(R) \cong g^*(X/R),
\]

making \( X/R \) into an object of \( \mathcal{D} \).

In particular, we see that \( q^* \) preserves finite limits and arbitrary colimits, and thus defines a geometric morphism \( \mathcal{E} \to \mathcal{D} \).

Condition (f), however, requires some argument. Suppose \( \mathcal{C} \) is a site for \( \mathcal{E} \), \( \mathcal{D} \) a site for \( \mathcal{F} \), full subcategories of \( \mathcal{E} \) resp. \( \mathcal{F} \) and both closed under finite limits, and that \( f^*, g^*: \mathcal{E} \to \mathcal{F} \) are induced by left exact continuous functors \( F, G: \mathcal{C} \to \mathcal{D} \) by left Kan extension. For \( X \in \text{Sh}(\mathcal{C}) \), \( P(X) \) is the presheaf

\[
P(X)(D) = \lim_{x \in X(C)} D(D, FC),
\]

and \( f^*(X) \) is the associated sheaf of \( P(X) \), and similarly \( g^*(X) \) is the associated sheaf of the presheaf \( Q(X) = \lim_{D \in \mathcal{D}} D(D; GC) \).

Now define an increasing sequence of (small) full subcategories of \( \mathcal{E} \), \( \langle C_n : n \in \mathbb{N} \rangle \), by

\[
C_0 = C, \\
C_{n+1} = \text{objects of the form } \prod_{j \in \mathcal{J}} C_j, \text{ where } C_j \in C_n \text{ and } \mathcal{J}
\]

is the index set of some cover of \( \mathcal{D} \).

Let \( C_\infty \) be the full subcategory whose objects are of the form \( \bigsqcup_{n \in \mathbb{N}} C_n \), \( C_n \in C_n \), and let for \( 0 \leq n \leq \infty, \hat{C}_n \) be the category whose objects are quotients of objects of \( C_n \) (as a full subcategory of \( \mathcal{E} \)). I claim that \( \mathcal{D} \) is generated by the set of objects \( (C, \mu) \in \mathcal{D} \) with \( C \in C_\infty \), \( \mu: f^*(C) \xrightarrow{\simeq} g^*(C) \).

To see this, suppose \( (X, \theta) \xrightarrow{\alpha} (Y, \xi) \) are maps in \( \mathcal{D} \) such that \( \alpha u = \beta u \) for every \( (C, \mu) \xrightarrow{\alpha} (X, \theta) \) with \( C \in \hat{C}_\infty \). To show that \( \alpha = \beta \), take \( x_0 \in X(C_0), C_0 \in C \). \( x_0 \) determines an element \( f^*(x_0) \in f^*(X)(FC_0) \), corresponding to the element of \( P(X)(FC_0) \) given by \( \text{id}_{FC_0} \) at the vertex \( (x_0, C_0) \). \( \theta_{FC_0}(x_0) \) is given by a family of elements \( y_j \in Q(X)(D_j) \), for a cover \( \{D_j \to FC\}_{j \in J} \) in \( \mathcal{D} \). Say \( y_j \) is represented by \( g_j: D_j \to G(C_j) \) at the vertex \( (x_j, C_j) \). Let \( C_1 = \bigsqcup_{j \in J} C_j \). Then we have a
map $x_1 = \{x_j\}: C_1 \to X$, with image $U_1 \subset X$ say, and $\theta \circ f^*(x_0)$ factors through $g^*(U_1)$:

$$
\begin{array}{ccc}
    f^*(C_0) & \xrightarrow{f^*(x_0)} & f^*(X) \\
    \downarrow & & \downarrow \theta \\
    g^*(C_1) & \xrightarrow{g^*(U_1)} & g^*(X).
\end{array}
$$

Now consider $x_j \in X(C_j)$. As before, $\theta(f^*(x_j))$ is given by $\{D_{ji} \to F_0C_j\}_{i \in I}$, and elements $g_{ji}: D_{ji} \to G(C_{ji})$ at the vertex $(x_{ji}, C_{ji})$. Let

$$
U_j = \text{image} \left( \coprod_{i \in I_j} C_{ji} \xrightarrow{\{x_{ji}\}} X \right),
$$

so as to get a factorization

$$
\begin{array}{ccc}
    f^*(C_j) & \xrightarrow{f^*(x_j)} & f^*(X) \\
    \downarrow & & \downarrow \theta \\
    g^*(U_j) & \xrightarrow{g^*(X)} & g^*(X).
\end{array}
$$

and let $C_2 = \coprod_{j \in J} \coprod_{i \in I_j} C_{ji} \subset C_2$, $C_2 \xrightarrow{x_2} X$ the map given by the $x_{ji}$, and $U_2 = \text{image}(C_2 \xrightarrow{x_2} X)$. Then

$$
\begin{array}{ccc}
    f^*(C_1) & \xrightarrow{x_1} & f^*(X) \\
    \downarrow & & \downarrow \\
    g^*(U_2) & \xrightarrow{g^*(X)} & g^*(X).
\end{array}
$$

Proceed by induction to produce a sequence $C_n \xrightarrow{x_n} X$, $C_n \in \mathcal{C}_n$, with image $U_n \subset X$, such that there are factorizations

$$
\begin{array}{ccc}
    f^*(U_n) & \xrightarrow{x_n} & f^*(X) \\
    \downarrow & & \downarrow \\
    g^*(U_{n+1}) & \xrightarrow{g^*(X)} & g^*(X),
\end{array}
$$

and let $U = \bigcup_n U_n \in \mathcal{C}_\infty$. Then $\theta$ restricts to a map $f^*(U) \to g^*(U)$, making $(U, \theta)$ a subobject of $X$ in $\mathcal{D}$. Since $\alpha|U = \beta|U$, also $\alpha(x_0) = \beta(x_0)$. Since $x_0 \in X(C_0)$ was arbitrary, $\alpha = \beta$.

This proves the claim. The other statements in the theorem are obvious.

2.2. Coproducts of toposes. Let $(\mathcal{E}_i)_{i \in I}$ be a set of toposes. The coproduct $\coprod_{i \in I} \mathcal{E}_i$ as a topos is the product $\prod_{i \in I} \mathcal{E}_i$ as a category. It is well known that $\prod \mathcal{E}_i$ is a topos. In fact, if $C_i$ is a site for $\mathcal{E}_i$, then the coproduct $\coprod C_i$ of the categories $C_i$, equipped with the smallest Grothendieck topology which makes all the inclusions $C_j \hookrightarrow \coprod C_i$ continuous, is a site for $\coprod \mathcal{E}_i$.

2.3. Tensors. Let $\mathcal{E}$ be a topos, and let $\mathcal{C}$ be a small category. The tensor $\mathcal{E} \otimes \mathcal{C}$ is a topos such that there is a natural equivalence

$$
\text{Hom}_{\mathcal{D}}(\mathcal{E} \otimes \mathcal{C}, \mathcal{H}) \simeq \text{Cat}(\mathcal{C}, \text{Hom}_{\mathcal{D}}(\mathcal{E}, \mathcal{H})).
$$
for any $\mathcal{E}$-topos $\mathcal{E}'$ (Cat$(-, ?)$ is the category of functors from $-$ to $?$; see Street (1976), Kelly (1982) for the general definition of this tensor). It is easily verified that we can explicitly define $\mathcal{E} \otimes C$ to be the topos $\mathcal{E}^C$ of $\mathcal{E}$-valued functors on $C$.

### 2.4. Indexed colimits

The general setting is as follows (see Street (1980)). Given a small category $K$ and two pseudofunctors

$$\mathcal{G} : K \to (\text{toposes}), \quad K^{\text{op}} \xrightarrow{w} \text{Cat},$$

we wish to construct the indexed colimit $w * \mathcal{G}$, which is to be a topos such that there is an equivalence

$$(1) \quad \text{Hom}(w * \mathcal{G}, \mathcal{E}) \simeq \text{Nat}(w, \text{Hom}(\mathcal{E}, \mathcal{F}))),$$

natural in the parameter topos $\mathcal{E}$. ($\text{Hom}$ is the category of geometric morphisms, and $\text{Nat}$ is the category of (pseudo) natural transformations between functors $K \to \text{Cat}$. If $\mathcal{G}$ and $\mathcal{E}$ are over the base $\mathcal{S}$, then so is $w * \mathcal{G}$, and (1) will hold over $\mathcal{S}$.)

For instance, in the case of 2.1 one takes $K = 0 \rightarrow 1$ (two parallel nonidentity arrows $u$ and $v$), $\mathcal{G}(0) = \mathcal{F}$, $\mathcal{G}(1) = \mathcal{E}$, $\mathcal{G}(u) = \alpha$, $\mathcal{G}(v) = \beta$, $w(1) = 1$, $w(0) = \cdot \sim \cdot$ (one nonidentity isomorphism). In the case of coproducts, cf. 2.2, one takes $K$ to be the discrete category $I$, $\mathcal{G}(i) = \mathcal{E}_i$, and $w$ the constant functor $1$. In the case of the tensor $\mathcal{E} \otimes C$, one takes $K = 1$, $\mathcal{G}$ has value $\mathcal{E}$, and $w$ has value $C$.

In the general case, $w * \mathcal{G}$ can be described as a category in the following way. The objects of $w * \mathcal{G}$ are pairs $(D_{(\cdot)}, u_{(\cdot)})$ where $D_K$ is a diagram of type $w(K)$ in the topos $\mathcal{G}(K)$, and for $K \xrightarrow{\alpha} K'$ in $K$, $u_\alpha$ is a natural isomorphism $D_K \circ w(\alpha) \sim \mathcal{G}(\alpha) * \circ D_{K'} : w(K') \to \mathcal{G}(K)$ of diagrams of type $w(K')$ in the topos $\mathcal{G}(K)$. Moreover, the $u_\alpha$ are required to be coherent, in the sense that for $K \xrightarrow{\alpha} K' \xrightarrow{\beta} K''$ in $K$, $u_{\beta \alpha} \equiv u_\beta \circ u_\alpha$ in the only way that makes sense; that is, if we suppress the isomorphisms which tell us that $w$ and $\mathcal{G}$ are pseudofunctors rather than functors, we require the natural transformation

![Diagram](https://www.ams.org/journal-terms-of-use)
to be the same as the composite

\[
\begin{array}{c}
    w(K) \\
    \downarrow^{u_\alpha} \quad \downarrow^{D_K} \\
    \mathcal{G}(K) \\
\end{array}
\]

\[
\begin{array}{c}
    w(K) \\
    \downarrow^{u_\beta} \quad \downarrow^{\mathcal{G}(\beta)^*} \\
    \mathcal{G}(K''') \\
\end{array}
\]

The morphisms \((D, u) \rightarrow (D', u')\) of \(w \ast \mathcal{G}\) are families \((\tau_K : K \in K)\) of morphisms of diagrams in \(\mathcal{G}(K)\)

\[
\begin{array}{c}
    D_K \\
    \downarrow^{\tau_K} \\
    \mathcal{G}(K) \\
\end{array}
\]

which are natural in \(K\), i.e. for \(K_1 \rightarrow K_2\),

\[
\begin{array}{c}
    \mathcal{G}(\alpha)^* \circ D_{K_1} \\
    \downarrow^{u_\alpha} \\
    D_{K_2} \circ w(\alpha) \\
\end{array}
\]

This defines a category \(w \ast \mathcal{G}\). The reader may wish to check but is advised to believe that colimits and finite limits in \(w \ast \mathcal{G}\) are computed by just taking the corresponding colimits and finite limits “pointwise” in each of the toposes \(\mathcal{G}(K)\), \(K \in K\), in the obvious way. Then the isomorphism (1) above is easily checked; in fact it is induced by the (pseudo) natural transformation

\[
\begin{array}{c}
    w \rightarrow \text{Hom}(\mathcal{G} -, w \ast \mathcal{G}),
\end{array}
\]

whose components

\[
\pi_K : w(K) \rightarrow \text{Hom}(\mathcal{G}(K), w \ast \mathcal{G})
\]

are defined as follows: taking inverse images, the functor \(\pi_K\) is the same as an inverse image functor of a geometric morphism \(\mathcal{G}(K)^w(K) \rightarrow w \ast \mathcal{G}\), and for this we can just take the projection \((D_-, u_-) \hookrightarrow D_K\).

I suppose that it is possible to show directly that \(w \ast \mathcal{G}\) has generators. But fortunately we do not have to go through this, because a result of Street (1980) says that all indexed colimits can be constructed from coequalizers (2.1), coproducts (2.2), and tensors (2.3). Thus the following theorem is proved.
2.5. Theorem. All small indexed colimits in the bicategory of Grothendieck toposes exist, and are computed as indexed (bi-)limits of the underlying categories and inverse image functors.

Remark. The reader must have noticed that given Street's result, the only work involved in the proof of 2.5 is to show the existence of coequalizers. This was proved independently by several people, among whom M. Tierney, P. Freyd, and the present author. An elegant approach to the existence of colimits is that via accessible categories, as demonstrated by recent work of Makkai and Paré (to appear).

3. Simplicial toposes and descent. In this section we consider a special type of colimit, namely that of a simplicial topos. Simplicial toposes occur naturally in a variety of circumstances; for instance, sheaves on a simplicial topological space, the étale topos of a simplicial scheme, etc. To each topos, one can associate the singular complex, which is a simplicial topos (cf. Moerdijk and Wraith (1986), and 3.10 below).

3.1. Simplicial toposes. Let $\Delta$ be the usual category of finite nonempty sets $[n] = \{0, \ldots, n\}$ ($n \geq 0$) and $\leq$-preserving maps. A simplicial topos $\mathcal{E}$ is a pseudo-functor from $\Delta^{op}$ into toposes (over the base topos $\mathcal{S}$), i.e. $\mathcal{E}$ is a sequence of toposes $\mathcal{E}_n$ ($n \geq 0$), together with geometric morphisms $\alpha: \mathcal{E}_m \to \mathcal{E}_n$ for $[n] \to [m]$, functorial up to a specified coherent isomorphism $\tau = \tau_{\alpha, \beta}: \alpha \circ \beta \cong \beta \alpha$ (i.e. $\tau$ is a natural isomorphism of inverse image functors $\beta^* \circ \alpha^* \cong (\beta \alpha)^*$). So this is the usual definition of a simplicial object in a category (see e.g. Gabriel and Zisman (1967), May (1968)), except that we have to take into account that the category of toposes and geometric morphisms can only be usefully considered as a 2-category.

Thus, a simplicial topos $\mathcal{E}$ may alternatively be described as a sequence of toposes $\mathcal{E}_n$, $n \geq 0$, together with geometric morphisms $\mathcal{E}_n \overset{d_i}{\to} \mathcal{E}_{n-1}$ ($i = 0, \ldots, n$) and $\mathcal{E}_{n-1} \overset{s_j}{\to} \mathcal{E}_n$ ($j = 1, \ldots, n - 1$) satisfying the usual simplicial identities, but only up to a coherent isomorphism $\tau$.

A (pseudo) cocone $\mathcal{E} \overset{f}{\to} \mathcal{S}$ under a given simplicial topos is a sequence of geometric morphisms $\mathcal{E}_n \overset{f_n}{\to} \mathcal{S}$ (over the base $\mathcal{S}$) into a given topos $\mathcal{S}$, together with natural isomorphisms $\sigma_\alpha: f_n \circ \alpha \Rightarrow f_m$ which are compatible with the $\tau$'s. That is, $\sigma_{id}$ is the identity for each $[n] \overset{id}{\to} [n]$, and for $[n] \overset{\alpha}{\to} [m] \overset{\beta}{\to} [k]$ the square

\[
\begin{array}{ccc}
    f_n \alpha \beta & \xrightarrow{f_n \cdot \tau} & f_n \beta \alpha \\
    \downarrow \sigma_{\alpha, \beta} & & \downarrow \sigma_{\beta \alpha} \\
    f_m \beta & \xrightarrow{\sigma_\beta} & f_k
\end{array}
\]

commutes.

The universal such cocone is the (pseudo) colimit of the simplicial topos $\mathcal{E}$, denoted by $\mathcal{E} \overset{L}{\to} L(\mathcal{E})$. The topos $L(\mathcal{E})$ can explicitly be described as the category whose objects are sequences $((X_n), \xi_\alpha)$, $X_n$ an object of $\mathcal{E}_n$ and $\xi_\alpha: X_m \overset{\sim}{\to} \alpha^* (X_n)$ an isomorphism in $\mathcal{E}_n$ for each $[n] \overset{\alpha}{\to} [m]$, compatible with the $\tau$'s. We do not need this description of the topos $L(\mathcal{E})$, but only its existence, cf. 3.3 below.
3.2. Descent. Given a simplicial topos \( \mathcal{E} \), consider that part of the data which only uses the maps

\[
\begin{array}{ccc}
\mathcal{E}_2 & \xrightarrow{d_0} & \mathcal{E}_1 \\
d_1 & \xrightarrow{d_0} & d_2 \\
\mathcal{E}_0 & \xrightarrow{s_0} & \mathcal{E}_0
\end{array}
\]

and the \( \tau \)'s between composites of these. A descent cocone into a topos \( \mathcal{F} \) is a pair \( (g, \mu) \), where \( \mathcal{E}_0 \xrightarrow{g} \mathcal{F} \) is a geometric morphism, and \( \mu: gd_1 \Rightarrow gd_0 \) is a 2-cell satisfying

(i) unit condition:

\[
\begin{array}{ccc}
gd_1s_0 & \xrightarrow{\mu \cdot s_0} & gd_0s_0 \\
g & \xrightarrow{\tau} & g \\
gd_0s_0 & \xrightarrow{\mu \cdot s_0} & gd_0s_0
\end{array}
\]

commutes, and

(ii) cocycle condition:

\[
\begin{array}{ccc}
gd_1d_2 & \xrightarrow{\mu d_2} & gd_0d_2 \\
gd_1d_1 & \xrightarrow{\mu d_1} & gd_0d_1 \\
gd_0d_1 & \xrightarrow{\tau} & gd_0d_0
\end{array}
\]

commutes.

In other words, \( \mu: gd_1 \Rightarrow gd_0 \) is a natural isomorphism such that \( s_0^*(\mu) \cong \text{id} \) and \( d_0^*(\mu) \circ d_2^*(\mu) \cong d_1^*(\mu) \), provided one plugs in enough \( \tau \)-isomorphisms for this to make sense.

The universal descent cocone is denoted by \( (p, \theta): \mathcal{E} \rightarrow \text{Desc}(\mathcal{E}) \). \text{Desc}(\mathcal{E}) exists, cf. 3.3 below, and can explicitly be described as the category whose objects are pairs \( (X, \theta) \), where \( X \) is an object of \( \mathcal{E}_0 \) and \( \theta: d_1^*(X) \xrightarrow{\sim} d_0^*(X) \) is an isomorphism such that \( s_0^*(\theta) \cong \text{id} \) and \( d_0^*(\theta) \circ d_2^*(\theta) \cong d_1^*(\theta) \), provided we plug in some \( \tau \)'s as before.

This category \( \text{Desc}(\mathcal{E}) \) plays an important role in the representation of toposes by groupoids, cf. Joyal and Tierney (1984), and below.

3.3. Existence of the toposes \( L(\mathcal{E}) \) and \( \text{Desc}(\mathcal{E}) \). These have been defined as categories, but we have to show that they are Grothendieck toposes, and that the obvious "forgetful" functors \( L(\mathcal{E}) \xrightarrow{\mathcal{E}} \mathcal{E}_n \) and \( \text{Desc}(\mathcal{E}) \xrightarrow{\mathcal{E}} \mathcal{E}_0 \) indeed define inverse images of universal geometric morphisms \( \{ \mathcal{E}_n \xrightarrow{q_n} L(\mathcal{E}) \}_n \) and \( \mathcal{E}_0 \xrightarrow{p} \text{Desc}(\mathcal{E}) \).
These are all cases of indexed colimits of toposes, however, and therefore this follows from the general result in §2.

3.4. Proposition. The toposes \( L(\mathcal{E}) \) and \( \text{Desc}(\mathcal{E}) \) are equivalent, and hence so are the 2-categories of pseudo-cocones and descent cocones, for any simplicial topos \( \mathcal{E} \).

As a sketch of proof, let me indicate how to pass from a cocone \((f, \sigma) : \mathcal{E} \to \mathcal{F}\) to a descent cocone \((g, \mu) : \mathcal{E} \to \mathcal{F}\) and vice versa, by functors \(T\) and \(U\) respectively.

Given \((f_n)_n\) and \((\sigma_\alpha)_\alpha\) as in 2.2, define

\[
T(f_n, \sigma_\alpha) = (f_0, f d_1 \overset{\alpha}{\Rightarrow} f d_0)
\]

where \(\alpha\) is the composite

\[
f_0 d_1 = f_0 \circ \overline{0} \overset{\sigma_0}{\Rightarrow} f_1 \overset{(\sigma_1)^{-1}}{\Rightarrow} f_0 \circ \overline{1} = f_0 d_0
\]

(here \(0, 1 : [0] \to [1]\) denote the maps in \(\Delta\) with the corresponding value). Given \((g, \mu)\), define \(U(g, \mu) = (g_n, \sigma_n)\), where \(g_n = g d_0 \cdots d_0 = g \overline{n}\) \((n : [0] \to [n]\) has \(n(0) = n)\), and for \([n] \overset{\alpha}{\Rightarrow} [m]\), \(\sigma_\alpha : g_n \circ \overline{\alpha} \Rightarrow g_m\) is the composite

\[
\mu(\overline{\alpha(n)}, m) g \circ \overline{1} \circ (\overline{\alpha(n)}, m) \overset{\tau}{\Rightarrow} \overline{g} \circ m,
\]

where

\[
[0] \overset{0}{\Rightarrow} [1] \overset{(\alpha(m)), m}{\Rightarrow} [m]
\]

denote the obvious maps in \(\Delta\).

A tedious argument shows that these functorial operations \(T\) and \(U\) are, up to isomorphism, mutually inverse.

3.5. Localization. Let \(\mathcal{E}\) be a truncated simplicial topos as in 3.2(1), with colimit \(\text{Desc}(\mathcal{E})\)

\[
\mathcal{E}_2 \overset{d_0}{\twoheadleftarrow} \mathcal{E}_1 \overset{d_1}{\twoheadleftarrow} \mathcal{E}_0 \to \text{Desc}(\mathcal{E}),
\]

and let \((X, \theta)\) be an object of \(\text{Desc}(\mathcal{E})\). We obtain an induced truncated simplicial topos \(\mathcal{E}./(X, \theta)\), namely

\[
\mathcal{E}_2/d_0^* d_0^*(X) \overset{d_1}{\twoheadleftarrow} \mathcal{E}_1/d_0^*(X) \overset{s}{\hookleftarrow} \mathcal{E}_0/X;
\]

the \(d_i\) and \(s\) are defined from \(d_i\) and \(s\) as follows: \(d_0^* = d_0^*/X\), i.e.

\[
d_0^*(Y \overset{\alpha}{\to} X) = d_0^*(Y) \overset{d_0^*(\alpha)}{\twoheadrightarrow} d_0^*(X),
\]

\[
d_1^*(Y \overset{\alpha}{\to} X) = d_1^*(Y) \overset{d_1^*(\alpha)}{\twoheadrightarrow} d_1^*(X) \overset{\theta}{\to} d_0^*(X),
\]

and

\[
s^*(Z \overset{\beta}{\to} d_0^*(X)) = s^*(Z) \overset{s^*(\beta)}{\to} s^*d_0^*(X) \overset{\tau}{\Rightarrow} X.
\]
The $d_i : \mathcal{E}_2/d_0 d_0(X) \to \mathcal{E}_1/d_0(X)$ are defined similarly, just by inserting enough $\tau$'s and $\theta$'s:

\[
\begin{align*}
    d_0^*(Y) &\to d_0^*(X) = d_0^*(Y) \xrightarrow{d_0^*(\alpha)} d_0^* d_0^*(X), \\
    d_1^*(Y) &\to d_0^*(X) = d_1^*(Y) \xrightarrow{d_1^*(\alpha)} d_1^* d_0^*(X) \xrightarrow{\tau} d_0^* d_0^*(X), \\
    d_2^*(Y) &\to d_0^*(X) = d_2^*(Y) \xrightarrow{d_2^*(\alpha)} d_2^* d_0^*(X) \xrightarrow{\tau} d_0^* d_1^*(X) \xrightarrow{\theta} d_0^* d_0^*(X).
\end{align*}
\]

Let $\text{Desc}(\mathcal{E}/(X, \theta))$ be the colimit of this diagram (2). Then we have the following result.

**Localization Lemma.** Let $\mathcal{E}$ be a truncated simplicial topos, and $(X, \theta)$ an object of $\text{Desc}(\mathcal{E})$, as above. Then there is a canonical equivalence of toposes

\[\text{Desc}(\mathcal{E}/(X, \theta)) \sim \text{Desc}(\mathcal{E})/(X, \theta).\]

The proof is by direct inspection.

**3.6. Theorem.** Let $\mathcal{E} = (\mathcal{E}_2 \supseteq \mathcal{E}_1 \supseteq \mathcal{E}_0)$ be a truncated simplicial topos as in (1) of 3.2, with universal morphism $\mathcal{E}_0 \to \text{Desc}(\mathcal{E})$. $p$ is obviously a surjection.

(a) If $d_0, d_1 : \mathcal{E}_1 \Rightarrow \mathcal{E}_0$ are both open, then so is $p$.

(b) If $d_0, d_1 : \mathcal{E}_1 \Rightarrow \mathcal{E}_0$ are both locally connected, and $\mathcal{E}_2 \supseteq \mathcal{E}_1$ are all open, then $p$ is locally connected.

(c) If $d_0, d_1 : \mathcal{E}_1 \Rightarrow \mathcal{E}_0$ are both atomic and $\mathcal{E}_2 \supseteq \mathcal{E}_1$ are all open, then $p$ is atomic.

**Proof.** (a) Suppose $d_0^*, d_1^*$ both preserve $\forall$ (cf. 1.7), and consider a diagram $(S, \theta) \mapsto (X, \theta) \xrightarrow{\alpha} (Y, \mu)$ in $\text{Desc}(\mathcal{E})$. To compute $\forall_\alpha(S, \theta)$ in $\text{Desc}(\mathcal{E})$, a first approximation would be to take $T = \forall_\alpha(S) \subset Y$ in $\mathcal{E}$. In general, the problem is that $T$ need not be "closed" under the action $\mu$, but if $d_0^*, d_1^*$ preserve $\forall$, then $d_1^*(Y) \xrightarrow{\Delta} d_2^*(Y)$ maps $d_2^*(T) \subset d_1^*(Y)$ into $d_0^*(T) \subset d_0^*(Y)$, as is easy to see. It is then clear that $(T, \mu) = \forall_\alpha(S, \theta)$ in $\text{Desc}(\mathcal{E})$.

(b) We have to show that $p^*$ preserves $\Pi$-functors if the assumptions of (b) hold (cf. 1.8). Since the assumptions of (b), however, are all stable under slicing by an object of $\text{Desc}(\mathcal{E})$, cf. 3.5, it is enough to check that $p^*$ preserves exponentials if $d_0^*$ and $d_1^* : \mathcal{E}_0 \Rightarrow \mathcal{E}_1$ do, and $d_0, d_1, d_2 : \mathcal{E}_2 \Rightarrow \mathcal{E}_1$ are open.

Given $(X, \theta)$ and $(Y, \mu)$ in $\text{Desc}(\mathcal{E})$, define the map $d_1^*(XY) \xrightarrow{\theta^* d_1^*(Y)} d_0^*(XY)$. We need to verify that $\theta$ satisfies the unit and cocycle conditions. The first is a straightforward diagram chase argument, using maps on test objects $T \to XY$. For the cocycle condition, we use that the $d_i : \mathcal{E}_2 \to \mathcal{E}_1$ are all open, so that for any objects $A, B \in \mathcal{E}_1$ the canonical map $d_1^*(AB) \to d_1^*(A) d_1^*(B)$ is mono (cf. 1.7). Thus, we obtain monomorphisms $d_2^* d_1^*(XY) \to d_2^* d_1^*(X) d_2^* d_1^*(Y)$, etc., and we can therefore "embed" the to-be-commutative hexagon (the inner one) into the outer hexagon, which commutes as a simple consequence of the cocycle condition for $\theta$ and $\mu$: 

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(c) To prove that \( p \) is atomic under the given assumptions, we only need to check that \( p^* \) preserves the subobject classifier, by (b). Write \( \Omega_i \) for the subobject classifier of \( \mathcal{E}_i \). Since \( \mathcal{E}_1 \models \mathcal{E}_0 \) are atomic, the canonical maps \( \sigma_i \) are isomorphisms \( d_i^*(\Omega_0) \xrightarrow{\sigma_0} \Omega_1 \xrightarrow{\sigma_1} d_i^*(\Omega_0) \), so we can define the composed map \( \sigma = \sigma_0^{-1}\sigma_1: d_i^*(\Omega_0) \to d_0^*(\Omega_0) \). We claim that \((\Omega_0, \sigma)\) is the subobject classifier of \( \text{Desc}(\mathcal{E}) \). It is rather obvious that \((\Omega_0, \sigma)\) would indeed classify subobjects in \( \text{Desc}(\mathcal{E}) \), provided we show that it is an object of \( \text{Desc}(\mathcal{E}) \) in the first place. One easily sees that \( \sigma \) satisfies the unit condition. The cocycle condition can be verified by using that the geometric morphisms \( \mathcal{E}_2 \xrightarrow{d_1^*} \mathcal{E}_1 \) are open, which implies that the canonical maps \( d_i^*(\Omega_1) \to \Omega_2 \) are mono (cf. 1.7): it is enough to show that each of the inner triangles in

\[
\begin{array}{c}
\tau \\
\downarrow \\
\Omega_2 \\
\downarrow \\
\tau \\
\end{array}
\]

commutes. For those with a \( d_i(\sigma) \) as edge this is obvious, and for those with a \( \tau \) it is straightforwardly checked.

This completes the proof of Theorem 3.6.
3.7. **Remark.** A similar but easier proof gives an analogous result for co-
equalizers \( \mathcal{F} \xrightarrow{f} \mathcal{E} \xrightarrow{p} \mathcal{D} \) as in 2.1, namely

**Theorem.** If \( f \) and \( g \) are open (resp. locally connected, atomic) then so is \( p \).

3.8. **Corollary.** Let \( \mathcal{E}_2 \supseteq \mathcal{E}_1 \supseteq \mathcal{E}_0 \to \text{Desc}(\mathcal{E}) \) be a colimit diagram, as before.

(a) If \( \mathcal{E}_0 \to \mathcal{J} \) is open, so is \( \text{Desc}(\mathcal{E}) \).

(b) If \( \mathcal{E}_0 \to \mathcal{J} \) is locally connected and \( \mathcal{E}_1 \supseteq \mathcal{E}_0 \) are open, then \( \text{Desc}(\mathcal{E}) \) is locally connected.

(c) If \( \mathcal{E}_0 \to \mathcal{J} \) is atomic and \( \mathcal{E}_1 \supseteq \mathcal{E}_0 \) are open, then so is \( \text{Desc}(\mathcal{E}) \).

**Proof.** Apply 1.10 to the triangle \( \mathcal{E}_0 \to \text{Desc}(\mathcal{E}) \to \mathcal{J} \) and use 3.6(a) for (b) and (c).

3.9. **Corollary.** Let \( \mathcal{E}_2 \supseteq \mathcal{E}_1 \supseteq \mathcal{E}_0 \to \text{Desc}(\mathcal{E}) \) be as in 3.8, and suppose that \( \mathcal{E}_0 \to \mathcal{J} \), \( \mathcal{E}_1 \xrightarrow{(d_0,d_1)} \mathcal{E}_0 \times_{\mathcal{J}} \mathcal{E}_0 \) are open (respectively open surjections). Then \( \text{Desc}(\mathcal{E}) \) is atomic (resp. atomic connected).

**Proof.** We use 1.9. \( \text{Desc}(\mathcal{E}) \to \mathcal{J} \) is open by 3.8(a); moreover by considering the diagram

\[
\begin{array}{ccc}
\mathcal{E}_1 & \xrightarrow{(d_0,d_1)} & \mathcal{E}_0 \times_{\mathcal{J}} \mathcal{E}_0 \\
\downarrow & & \downarrow \\
\text{Desc}(\mathcal{E}) & \xrightarrow{\Delta} & \text{Desc}(\mathcal{E}) \times_{\mathcal{J}} \text{Desc}(\mathcal{E})
\end{array}
\]

it follows that the diagonal \( \Delta \) is open (an open surjection) if \( \mathcal{E}_1 \to \mathcal{E}_0 \times_{\mathcal{J}} \mathcal{E}_0 \) is open (an open surjection).

3.10. **Example** (taken from Moerdijk and Wraith (1986)). Let \( \mathcal{E} \) be a topos, and let

\[1 = \Delta_0 \supseteq I = \Delta_1 \supseteq \Delta_2 \supseteq \ldots\]

be the standard cosimplicial topological space (\( \Delta_n \) is the standard \( n \)-simplex), but defined as a cosimplicial locale if \( \mathcal{J} \neq \text{Sets} \). By taking sheaves and exponentiating, we obtain the singular locale \( \mathcal{E}^{\text{Sh}(\Delta)} \), which is a simplicial topos. The colimit \( L(\mathcal{E}) \) "is" the fundamental group of \( \mathcal{E} \). For instance, if \( \mathcal{E} \) is connected and locally connected, then \( \mathcal{E}^I \to \mathcal{E} \times \mathcal{E} \) is an open surjection (Moerdijk and Wraith (1986)), and hence by 3.9 and 3.4 \( L(\mathcal{E}) \) is an atomic connected topos. A point of \( \mathcal{E} \) gives a point of \( L(\mathcal{E}) \), and it follows that \( L(\mathcal{E}) \) is continuous \( G \)-sets for a continuous group \( G \), by a result of Joyal-Tierney (1984), see also 4.3). On some other occasion, I hope to come back to this construction of a fundamental group of \( \mathcal{E} \), and to its relation to Grothendieck's construction of the fundamental group of a topos, in some more detail.
4. The classifying topos of a continuous groupoid. In this section we will describe a functor $B$ which associates a topos $BG$ to any groupoid $G$ in the category of spaces, as an indexed colimit of a simplicial topos.

4.1. Continuous groupoids. A continuous groupoid is a groupoid in the category of (generalized) spaces (cf. 1.1). So such a groupoid $G$ consists of two spaces $G_0$ (the space of objects) and $G_1$ (the space of arrows), together with domain and codomain maps $d_0$ and $d_1: G_1 \to G_0$, respectively, a unit map $G_0 \xrightarrow{s} G_1$, and a multiplication or composition map $G_1 \times_{G_0} G_1 \xrightarrow{m} G_1$ (in point-set notation: $m(g,f) = g \circ f$, i.e. $G_1 \times_{G_0} G_1$ is the pullback of $G_1 \xrightarrow{d_0} G_0$ on the left and $G_1 \xrightarrow{d_1} G_0$ on the right). These structure maps are supposed to satisfy the usual identities. The existence of an inverse $G_1 \xrightarrow{(\cdot)^{-1}} G_1$ can be expressed by requiring $G_1 \times_{G_0} G_1 \xrightarrow{(m,\pi_2)} G_1 \times_{G_0} G_1$, as well as $(\pi_1, m)$, to be isomorphisms over $G_1$.

A continuous group is a continuous groupoid with $G_0 = 1$; in this case we write $G$ for $G_1$, as usual.

A continuous homomorphism, or just a map, of continuous groupoids $G \to H$ is a pair $G_1 \xrightarrow{\phi_1} H_1$, $G_0 \xrightarrow{\phi_0} H_0$ of maps of spaces which satisfy the usual equations.

In the sequel, we will often just work with continuous groupoids $G$ having the property that $G_1 \xrightarrow{d_0} G_0$ are open maps. Notice that this implies that $m$ is open, since $m = \pi_1 \circ (m,\pi_2)$ and $(m,\pi_2)$ is an isomorphism, as just mentioned.

4.2. Definition of $BG$. Given a continuous groupoid $G$, let $N.(G)$ be the nerve of $G$, so $N.(G)$ is a simplicial space. (We number the faces and degeneracies of $N.(G)$ in such a way that $d_0$ and $d_1: G_1 \to G_0$ remain the faces $d_0$ and $d_1: N_1(G) \to N_0(G)$, respectively.)

Applying sheaves, we obtain a simplicial topos $\text{Sh}(N.(G)) \simeq N.(\text{Sh}(G))$. The topos $BG$ is by definition the (pseudo-) colimit $L(\text{Sh}(N.(G)))$ of this simplicial topos, i.e. we have a universal augmentation

$$\text{Sh}(N.G) \to BG.$$ 

By universality, $BG$ is obviously a (pseudo-)functor of $G$. If $G \xrightarrow{\phi} H$ is a map of continuous groupoids, the corresponding geometric morphism is denoted by $B\phi: BG \to BH$; sometimes we will just write $\phi^*$ for the inverse image $(B\phi)^*$.

When we wish to make the base topos explicit in the notation, we will write $B(S^\mathcal{J}, G)$ for $BG$.

By 3.4, $BG$ can alternatively be described as a descent-topos, as indicated by the diagram

$$\text{Sh}(G_1 \times_{G_0} G_1) \xrightarrow{\pi_1} \text{Sh}(G_1) \xrightarrow{d_0} \text{Sh}(G_0) \to BG.$$ 

4.3. Facts from Joyal-Tierney (1984). Recall that it is shown in loc. cit. that every topos $\mathcal{E}$ over $\mathcal{J}$ is equivalent to one of the form $BG$ for some continuous groupoid $G$. In fact, one may assume that $G_1 \xrightarrow{d_0} G_0$ are open, or even connected and locally connected. If $\mathcal{E} \to \mathcal{J}$ is open, one may take $G_0$ to be an open space (i.e. $G_0 \to 1$ is an open map). Special cases include: that every atomic connected
topos with a point is equivalent to $BG$ for a continuous group $G$ (i.e. $G_0 = 1$), and that every étendue is equivalent to $BG$ for a continuous groupoid $G$ with $G_1 \times_{G_0} G_1 \xrightarrow{m} G_1 \xrightarrow{d_0} G_0 \xrightarrow{d_1} G_0$ all étale maps (an "étale groupoid").

We list some elementary properties of the functor $G \to BG$ which follow from the results in §3.

4.4. PROPOSITION. Let $G$ be a continuous groupoid.
(a) If $d_0, d_1 : G_1 \Rightarrow G_0$ are open, $\text{Sh}(G_0) \to BG$ is open.
(b) If $d_0, d_1 : G_1 \Rightarrow G_0$ are locally connected, $\text{Sh}(G_0) \to BG$ is locally connected.
(c) If $d_0, d_1 : G_1 \Rightarrow G_0$ are étale, then $\text{Sh}(G_0) \to BG$ is atomic.

PROOF. Immediate from 3.6.

4.5. PROPOSITION. Let $G$ be a continuous groupoid.
(a) If $G_0$ is open, so is $BG$.
(b) If $G_0$ is locally connected and $G_1 \Rightarrow G_0$ are open, then $BG$ is locally connected.
(c) If $G_0$ is discrete and $G_1$ is an open space, then $BG$ is atomic. In particular, if $G$ is an open continuous group, $BG$ is an atomic topos.

PROOF. Immediate from 3.8.

4.6. PROPOSITION. Let $G \xrightarrow{f} H$ be a map of continuous groupoids, and let $BG \xrightarrow{Bf} BH$ be the induced geometric morphism.
(a) If $G_0 \xrightarrow{f_0} H_0$ is open and $H_1 \Rightarrow H_0$ are open, then $BG \xrightarrow{Bf} BH$ is open.
(b) If $G_0 \xrightarrow{f_0} H_0$ and $H_1 \Rightarrow H_0$ are locally connected, and if $G_1 \Rightarrow G_0$ are open, then $BG \to BH$ is locally connected.
(c) If $G_0 \xrightarrow{f_0} H_0$ and $H_1 \Rightarrow H_0$ are étale, and if $G_1 \Rightarrow G_0$ are open, then $BG \to BH$ is atomic.

PROOF. Use 4.5 and apply 1.10 to the square

\[
\begin{array}{ccc}
\text{Sh}(G_0) & \to & BG \\
\downarrow^{f_0} & & \downarrow^{Bf} \\
\text{Sh}(H_0) & \to & BH
\end{array}
\]

Proposition 4.5 also allows us to formulate the results of Joyal and Tierney (cf. 4.3) as if-and-only-if's; for (c), one uses 3.9.

4.7. COROLLARY. Let $\mathcal{E} \to \mathcal{S}$ be a Grothendieck topos over $\mathcal{S}$. Then
(a) $\mathcal{E} \to \mathcal{S}$ is open iff $\mathcal{E}$ is equivalent to $BG$ for a continuous groupoid $G$ with $G_0 \to 1$ all open maps;
(b) $\mathcal{E} \to \mathcal{S}$ is (connected) locally connected iff $\mathcal{E}$ is equivalent to $BG$ for a continuous groupoid $G$ with $G_1 \Rightarrow G_0 \to 1$ all (connected) locally connected maps;
(c) $\mathcal{E} \to \mathcal{S}$ is (connected) atomic iff $\mathcal{E}$ is equivalent to $BG$ for a continuous groupoid $G$ with $G_0 \to 1$ open (and surjective) and $G_1 \xrightarrow{(d_0,d_1)} G_0 \times G_0$ open (and surjective).
(d) \( \mathcal{E} \) is equivalent to an étendue if \( \mathcal{E} \) is equivalent to \( BG \) for an étale groupoid \( G \) (cf. 4.3).

5. G-spaces and étale G-spaces. In this section we will extensively analyze the properties of the functor \( B \), by viewing the topos \( BG \) as a category of spaces equipped with an action of \( G \).

5.1. Important convention. Although some of the results that follow hold for arbitrary groupoids, we will from now on assume that the structure maps \( d_0 \) and \( d_1 : G_1 \to G_0 \) of any continuous groupoid are open (this implies that \( G_1 \times_{G_0} G_1 \to G_1 \) is open). As said before, any topos \( \mathcal{E} \) is equivalent to \( BG \) for such a groupoid \( G \). For emphasis, we will sometimes call such a \( G \) an open groupoid.

5.2. G-spaces. Let \( G \) be a continuous groupoid. A G-space is a (generalized) space over \( G_0 \), \( E \to G_0 \), equipped with an action \( E \times_{G_0} G_1 \to E \) satisfying the usual axioms (the pullback here is along \( G_1 \to G_0 \)). In “point-set notation” (cf. 5.3), we have for points \( x, y, z \) in \( G_0 \), \( z \to y \) and \( y \to x \) in \( G_1 \), and \( e \) in \( E_x = p^{-1}(x) \),

\[
\begin{align*}
(1) & \quad p(e \cdot g) = y, \\
(2) & \quad e \cdot s(x) = e, \\
(3) & \quad (e \cdot g) \cdot h = e \cdot (g \cdot h).
\end{align*}
\]

A map of G-spaces from \( (E \to G_0, \cdot) \) to \( (E' \to G_0, \cdot) \) is a map of spaces \( E \to E' \) over \( G_0 \) which commutes with the action

\[
\begin{array}{ccc}
E \times_{G_0} G_1 & \to & E \\
\downarrow f & & \downarrow f \\
E' \times_{G_0} G_1 & \to & E'
\end{array}
\]

This defines a category (G-spaces).

A G-space \( E = (E \to G_0, \cdot) \) is called étale if \( p \) is an étale map (a local homeomorphism) of generalized spaces (recall that \( p \) is étale iff \( E \to G_0 \) and \( E \rightarrow E \times_{G_0} E \) are open, cf. 1.2). This gives a full subcategory (étale G-spaces) of (G-spaces).

By the equivalence between \( \text{Sh}(G_0) \) and étale spaces over \( G_0 \), we immediately conclude:

**Proposition.** The category of étale G-spaces is equivalent to the classifying topos \( BG \).

5.3. Remark on point-set notation. Of course, \( G_0 \), \( G_1 \), and \( E \) are generalized spaces, which may not have any points at all. Still, the notation in (1)–(3) of 5.2 is not merely suggestive, but can be taken to be literally the definition of an action, provided we interpret “points” in a sufficiently liberal way: it is standard practice in algebraic geometry to express conditions like (1)–(3) by means of test-spaces. So (1), for instance, becomes:

\[
(1') \quad \text{for any space } T \text{ and any two maps } T \to E \text{ and } T \to G_1 \text{ with } d_1 g = pe, \ p \cdot (e, g) = d_0 g. \ 
\]

Now a map \( T \to E \) is just a point of \( E \), if we change the base to \( \text{Sh}(T) \) and pull back our data along the map \( \text{Sh}(T) \to \mathcal{P} \) of toposes. So (1) makes sense in \( \text{Sh}(T) \), and if we interpret “point of \( E \)”, etc. as point of \( E \) in any base extension \( \text{Sh}(T) \to \mathcal{P} \) (or \( \mathcal{E} \to \mathcal{P} \) for a topos \( \mathcal{E} \), for that matter), then (1) is equivalent to \( (1') \).
Needless to say, this interpretation by change of base only gives the desired result if pulling back along \( \text{Sh}(T) \to \mathcal{S} \) also preserves constructions performed on the data (i.e. these constructions are stable under change of base). In the case of (1), for instance, this is the construction of the pullback \( E \times_{G_0} G_1 \), which is obviously preserved by change of base. One has to be very careful, however, when dealing with constructions which are not necessarily stable, such as the formation of quotients of spaces.

5.4. Functoriality of \( G \)-spaces. Let \( G \to H \) be a map of continuous groupoids. \( \phi \) induces a functor \( \phi^*: (H\text{-spaces}) \to (G\text{-spaces}) \) by pullback: given \( E \xrightarrow{\mu} H_0 \) with action \( E \times_{H_0} H_1 \xrightarrow{\lambda} E \), \( \phi^*(E) \) is the \( G_0 \)-space \( E \times_{H_0} G_0 \xrightarrow{\pi_2} G_0 \) with \( G \)-action

\[
(E \times_{H_0} G_0) \times_{G_0} G_1 \cong E \times_{H_0} G_1 \xrightarrow{(\mu \circ (E \times \phi), d_0)} E \times_{H_0} G_0;
\]

de i.e. in point-set notation, and writing \( E_y \) for the fiber \( p^{-1}(y) \) (\( y \) a point of \( H_0 \)), \( \phi^*(E)_x = E_{\phi(x)} \); and for \( x' \xrightarrow{g} x \) in \( G_1 \) and \( e \in E_{\phi(x)} \), \( e \cdot g = e \cdot \phi(g) \) defines the action of \( G \) on \( \phi^*(E) \).

Obviously, \( \phi^* \) is a functor. Moreover, \( \phi^* \) maps étale \( H \)-spaces to étale \( G \)-spaces, so as to restrict to a functor \( \phi^*: BH \to BG \). This is precisely the inverse image \( (B\phi)^* \) of the geometric morphism \( B\phi \):

\[
(H\text{-spaces}) \xrightarrow{\phi^*} (G\text{-spaces})
\]

\[
BH \xrightarrow{(B\phi)^*} BG
\]

Therefore, we will often write \( \phi^* \), rather than \( (B\phi)^* \).

5.5. Definitions. Let \( G \to H \) be a map of continuous groupoids.

(a) \( \phi \) is open if \( \phi_1: G_1 \to H_1 \) is open (it follows that \( \phi_0: G_0 \to H_0 \) is open).

(b) \( \phi \) is essentially surjective if \( G_0 \times_{H_0} H_1 \xrightarrow{d_1, \pi_2} H_0 \) ((\( x, \phi(x) \xrightarrow{h} y \) \( \mapsto y \)) is an open surjection.

(c) \( \phi \) is full if \( G_1 \times (H_0 \times H_0) \) \( (G_0 \times G_0) \) is an open surjection.

(d) \( \phi \) is fully faithful if

\[
\begin{array}{ccc}
G_1 & \xrightarrow{\phi_1} & H_1 \\
(d_0, d_1) \downarrow & & \downarrow (d_0, d_1) \\
G_0 \times G_0 & \xrightarrow{\phi_0 \times \phi_0} & H_0 \times H_0
\end{array}
\]

is a pullback of spaces.

5.6. Preview on stability. Let \( G \) be a continuous groupoid, with \( d_0 \) and \( d_1: G_1 \Rightarrow G_0 \) open (cf. 5.1). It is a consequence of the stability theorem, to be proved in the next section, that the coequalizer

\[
G_1 \Rightarrow G_0 \to \pi(G)
\]

is stable (see 6.9). That is, if \( T \) is any space, then \( T \times G_1 \Rightarrow T \times G_0 \to T \times \pi(G) \) is again a coequalizer of spaces. In particular, if the structure-maps of the groupoid are all over some base-space \( X \), \( \pi(G) \) is a space over \( X \) in a unique way which makes (1) a coequalizer of spaces over \( X \), and by interpreting what we just said in \( \text{Sh}(X) \), it follows that for any map \( T \to X \) of spaces, \( T \times_X G_1 \Rightarrow T \times_X G_0 \to T \times_X \pi(G) \) is again a coequalizer.
5.7. Theorem. Let $G \xrightarrow{\phi} H$ be an open map of continuous groupoids. Then $\phi^*$ has a left adjoint $\phi_! : (G\text{-spaces}) \to (H\text{-spaces})$; for a $G$-space $D = (D \xrightarrow{\rho} G_0, \cdot)$, $\phi_!(D)$ is defined as the coequalizer $D \otimes_G H$:

\begin{equation}
D \times_{G_0} G_1 \times_{H_0} H_1 \xrightarrow{\cdot \times H_1} D \times_{G_0} H_1 \to D \otimes_G H
\end{equation}

which is a stable coequalizer of spaces over $H_0$.

Remark. In point-set notation, the maps in (1) are $(d \in D_x, x' \xrightarrow{g} x, y \xrightarrow{h} \phi(x')) \mapsto (d \cdot g, h)$, resp. $(d, \phi(g) \circ h)$, so $D \otimes_G H$ can be thought of as the space of equivalence classes $[d, h]$ of pairs $(d \in D_x, y \xrightarrow{h} \phi(x))$. $H$ acts on $D \otimes_G H$ by composition: $[d, h] \cdot h' = [d, h \circ h']$. This point-set notation can be interpreted as a definition, using change of base, provided the coequalizer (1) is stable.

Proof. We just indicated how to define an action $(D \otimes_G H) \times_{H_0} H_1 \to (D \otimes_G H)$ making $D \otimes_G H$ into an $H$-space. However, this definition already uses the stability of (1), so as to conclude that

\begin{equation}
(D \times_{G_0} G_1 \times_{H_0} H_1) \times_{H_0} H_1 \Rightarrow (D \times_{H_0} H_1) \times_{H_0} H_1 \to (D \otimes_G H) \times_{H_0} H_1
\end{equation}

is again a coequalizer. This is indeed the case by 5.6, since the parallel pair of maps $D \times_{G_0} G_1 \times_{H_0} H_1 \Rightarrow D \times_{G_0} H_1$ are the $d_0$ and $d_1$ of a suitably defined groupoid, and are both open: $\phi_!$ is open by assumption, and hence $D \times m(\phi \times H_1)$ is open; that $\cdot \times H_1$ is open follows (because for any groupoid, $d_0$ is open iff $d_1$ is), but can also be seen directly, by observing that any action $D \times_{G_0} G_1 \to D$ must be an open map (cf. 5.11).

For a $G$-space $D = (D \xrightarrow{\rho} G_0, \cdot)$ and an $H$-space $E = (E \xrightarrow{\sigma} H_0, \cdot)$ the correspondence

$\begin{align*}
D \xrightarrow{\alpha} \phi^*(E) &= E \times_{H_0} G_0 \\
D \otimes_G H \xrightarrow{\beta} E
\end{align*}$

goes as follows. Given $\alpha$, define $\hat{\beta} : D \times_{H_0} H_1 \to E$ to be the composite

$D \times_{H_0} H_1 \xrightarrow{\alpha} (E \times_{H_0} G_0) \times_{H_0} H_1 \to E \times_{H_0} H_1 \xrightarrow{\hat{\beta}} E$

(in point-set notation: for $d \in D_x$, $y \xrightarrow{h} \phi(x)$, $\hat{\beta}_y(d, h) = \alpha_x(d) \cdot h$, where $\alpha_x(d) \in E_{\phi(x)} = \phi^*(E)_x$). Clearly $\hat{\beta}$ passes to the quotient and gives a map $D \otimes_G H \to E$ of $H$-spaces.

Conversely, given $\beta$, define $\alpha_1$ to be the composite $D \xrightarrow{D \times s \phi} D \times_{H_0} H_1 \Rightarrow D \otimes_G H \xrightarrow{\beta} E$, and let $\alpha = (\alpha_1, p) : D \to E \times_H G_0$ (in point-set notation, $\alpha_x(d) = \beta_{\phi(x)}([d, s, \phi(x)]) \in E_{\phi(x)} = \phi^*(E)_x$, for $d \in D_x$).

This proves Theorem 5.7.

In the sequel, we will often just define mappings by point-set notation, when this is justified by 5.3 (and 5.6).

5.8. Proposition. Let $G \xrightarrow{\phi} H$ be a map of continuous groupoids.

(i) If $\phi$ is essentially surjective, then $\phi^* : (H\text{-spaces}) \to (G\text{-spaces})$ is faithful.

(ii) If $\phi$ is moreover full, then $\phi^*$ is fully faithful.
PROOF. Let $E \xrightarrow{\alpha} E'$ be maps of $H$-spaces such that $\phi^*(\alpha) = \phi^*(\beta) : E \times_{H_0} G_0 \rightarrow E' \times_{H_0} G_0$. Since $d_1 \pi_2$ is a stable surjection $G_0 \times_{H_0} H_1 \rightarrow H_0$, so is $t$ in the pullback square

$$
\begin{array}{ccc}
E \times_{H_0} G_0 \times_{H_0} H_1 & \longrightarrow & E \\
\downarrow & & \downarrow p \\
G_0 \times_{H_0} H_1 & \longrightarrow & H_0
\end{array}
$$

$t(e, g, h) = e$ for $e \in E_y$, $x \in G_0$, $\phi(x) \xrightarrow{h} y$ in $H_1$). Now consider the diagram

$$
\begin{array}{c}
(E \times_{H_0} G_0) \times_{H_0} H_1 \xrightarrow{\sim} E \times_{H_0} (G_0 \times_{H_0} H_1) \xrightarrow{t} E \\
\phi^*(\alpha) \times_{H_0} H_1 \xrightarrow{\phi^*(\beta) \times_{H_0} H_1} \alpha \times_{(G_0 \times_{H_0} H_1)} \beta \times_{(G_0 \times_{H_0} H_1)} \alpha \downarrow \beta \\
(E' \times_{H_0} G_0) \times_{H_0} H_1 \xrightarrow{\sim} E' \times_{H_0} (G_0 \times_{H_0} H_1) \xrightarrow{t'} E'
\end{array}
$$

where $u$ is the map defined by $u(e, x, h) = (e \cdot h, x, h)$, for $e \in E_y$, $x \in G_0$, $(\phi(x) \xrightarrow{h} y) \in H_1$; $u'$ is defined similarly. Both right-hand squares (one with $\alpha$, one with $\beta$) commute by naturality of $t$, and both left-hand squares commute since $\alpha$ and $\beta$ preserve the action. Since $t$ is a surjection, $\phi^*(\alpha) = \phi^*(\beta)$ implies $\alpha = \beta$.

(ii) Suppose given $\phi^*(E) \rightarrow \phi^*(E')$. We define $E \rightarrow E'$ such that $\phi^*(\alpha) = \alpha$. I will give two arguments, as an illustration of the “point-set method”:

(a) In point-set language: for $y \in H_0$, there is a $\phi(x) \xrightarrow{h} y$ (because $G_0 \times_{H_0} H_1 \rightarrow H_0$ is an open surjection), and we define $\beta_y(e) = \alpha_{\phi(x)}(e \cdot h) \cdot h^{-1}$ for $e \in E_y$. This is well defined, for if $\phi(x') \xrightarrow{h'} y$ is another one, write $h^{-1}h = \phi(g)$ for an $x \xrightarrow{g} x'$ ($\phi$ is full). Then

$$
\alpha_{\phi(x')}(e \cdot h')h'^{-1} = \alpha_{\phi(x)}(e \cdot h')\alpha_{\phi(g)}(h'^{-1}) = \alpha_{\phi(x)}(e \cdot h' \cdot \phi(g))(h'\phi(g))^{-1} = \alpha_{\phi(x)}(e \cdot h') \cdot h^{-1}.
$$

This argument actually makes sense, by change of base techniques; “there exists...” is interpreted as “there exists in some open surjective base extension”, and “well defined on equivalence classes” corresponds to the fact that every open surjection is the (stable) coequalizer of its kernelpair.

(b) In the language of generalized spaces: if $G_0 \times_{H_0} H_1 \xrightarrow{d_1 \pi_2} H_0$ is an open surjection, it is the coequalizer of its kernelpair

$$(G_0 \times_{H_0} H_1) \times_{H_0} (G_0 \times_{H_0} H_1) \xrightarrow{\pi_1} G_0 \times_{H_0} H_1 \xrightarrow{d_1 \pi_2} H_0,$$

and this still holds when we pull back along $E \rightarrow H_0$, i.e.

$$
E \times_{H_0} (G_0 \times_{H_0} H_1) \times_{H_0} (G_0 \times_{H_0} H_1) \Rightarrow E \times_{H_0} (G_0 \times_{H_0} H_1) \xrightarrow{t} E
$$

is a coequalizer. To define $E \xrightarrow{\beta} E'$, it is therefore enough to define

$$
E \times_{H_0} (G_0 \times_{H_0} H_1) \xrightarrow{\beta} E'
$$
with $\hat{\beta} \circ \pi_{12} = \hat{\beta} \circ \pi_{13}$. Define $\hat{\beta}(e, x, \phi(x) \xrightarrow{h} g) = \alpha(e \cdot h)h^{-1}$ for $e \in E_y$, $x \in G_0$, $h \in H_1$. Then $\hat{\beta} \circ \pi_{12} = \hat{\beta} \circ \pi_{13}$ because $\phi$ is full, and therefore

$$H_1 \times G_0 \xrightarrow{w} (G_0 \times_{H_0} H_1) \times_{H_0} (G_0 \times_{H_0} H_1)$$

$$(\phi(x) \xrightarrow{h} y, x' \xrightarrow{\phi(g)} x) \mapsto (h \circ \phi(g))$$
is an open surjection; that $\hat{\beta} \pi_{12}w = \hat{\beta} \pi_{13}w$ is clear from the fact that $\alpha$ is a map of $G$-spaces.

The aim of this section is to investigate how properties of $BG \xrightarrow{B\phi} BH$ depend on those of $G \xrightarrow{\phi} H$. In this direction we now have the following corollary, (a) comes from 4.6(a) and 5.1, (b) and (c) follow from 5.8.

5.9. **Corollary.** Let $G \xrightarrow{\phi} H$ be a map of continuous groupoids, and $BG \xrightarrow{B\phi} BH$ the induced geometric morphism.

(a) $B\phi$ is open if $G_0 \xrightarrow{\phi_0} H_0$ is open.

(b) $B\phi$ is surjective if $\phi$ is essentially surjective.

(c) $B\phi$ is connected if $\phi$ is essentially surjective and full.

Our next aim is to see when $BG \xrightarrow{B\phi} BH$ is locally connected, i.e. when $(B\phi)^*$ has a $BH$-indexed left-adjoint. Since $B\phi^*$ is $\phi^*$ restricted to étale spaces, this comes down to asking (i) when $\phi_1$ is indexed over $(H$-spaces), and (ii) when $\phi_1$ maps étale $G$-spaces into étale $H$-spaces. We begin with the second question.

5.10. **Theorem.** Let $G \xrightarrow{\phi} H$ be an open map of continuous groupoids, with corresponding adjoint functors (cf. 5.7).

$$(G\text{-spaces}) \xrightarrow{\phi_1} (H\text{-spaces})$$

Then $\phi_1$ restricts to a functor $BG \to BH$ if $\phi$ is full (cf. 5.1.5).

**Proof.** Recall that for a $G$-space $D = (D \xrightarrow{p} G_0, \cdot)$, $\phi_1(D)$ is the (stable) coequalizer

$$D \times_{G_0} G_1 \times_{H_0} H_1 \xrightarrow{u} D \times_{H_0} H_1 \xrightarrow{r} D \otimes_G H,$$

where $u(d, g, h) = (d \cdot g, h)$, $v(d, g, h) = (d, \phi(g) \circ h)$ for $d \in D_x$, $x' \xrightarrow{\phi(g) \circ h} x$, $y \xrightarrow{h} \phi(x')$; see 5.7.

First of all, $\phi_0 p : D \to H_0$ is open since $\phi_0$ is, and hence so is its pullback $D \times_{H_0} H_1 \xrightarrow{\pi_1} H_1$ along $d_1$, and thus (cf. 5.1) $d_0 \pi_2 : D \times_{H_0} H_1 \to H_0$ is open. Since $D \times_{H_0} H_1 \to D \otimes_G H$ is a surjection, the structure map $D \otimes_G H \xrightarrow{r} H_0$ of the $H$-space $\phi_1(D)$ is open.

Second, we have to show that $q$ has an open diagonal $D \otimes_G H \to (D \otimes_G H) \times_{H_0} (D \otimes_G H)$. Since $r$ is an open surjection, so is $r \times r$, and therefore it is enough to show that $(u, v)$ is open, as in the diagram

$$
\begin{array}{ccc}
D \times_{G_0} G_1 \times_{H_0} H_1 & \xrightarrow{(u, v)} & (D \times_{H_0} H_1) \times_{H_0} (D \times_{H_0} H_1) \\
\downarrow_{ru=rv} & & \downarrow_{r \times r} \\
D \otimes_G H & \longrightarrow & (D \otimes_G H) \times_{H_0} (D \otimes_G H)
\end{array}
$$
We give a point-set argument to show that \((u, v)\) is open (i.e. we implicitly use test-spaces and change of base!).

First notice that since \(D \to G_0\) is étale, the action \(D \times_{G_0} G_1 \to D\) is not only open, but has the much stronger property that whenever \(U_{d,g}\) is a neighborhood of \(d \cdot g\) (\(d \in D_x, x' \xrightarrow{g} x\) in \(G_1\)), so small that \(U_{d,g} \overset{\phi}{\to} p(U_{d,g})\) is an isomorphism, then there are small neighborhoods \(V_g\) and \(U_d\) of \(g\) and \(d\) such that \(d_0(V) \subset p(U_{d,g})\), and for \(g' \in V_g\) and \(d' \in U_d\) with \(p(d') = d_1(g')\), \(d' \cdot g'\) is the unique point \(x\) in \(U_{d,g}\) with \(p(x) = d_0(g')\).

Now take a point \((d \in D_x, x' \xrightarrow{g} x, y \xrightarrow{h} \phi(x'))\) of \(D \times_{G_0} G_1 \times_{H_0} H_1\), and take sufficiently small neighborhoods \(V_g \subset G_1\) and \(W_h, W_{\phi(g)oh} \subset H_1\). In fact, since \(\phi\) is full, the map \(G_1 \times_{G_0} H_1 \overset{M}{\to} (G_0 \times_{H_0} H_1) \times_{H_0} (G_0 \times_{H_0} H_1)\) is an open surjection, where in point-set notation: \(G_1 \times_{G_0} H_1\) has points \((x \xrightarrow{g} x', y \xrightarrow{h} \phi(x))\), \((G_0 \times_{H_0} H_1) \times_{H_0} (G_0 \times_{H_0} H_1)\) has points \((x_1, z \xrightarrow{h_1} \phi(x_1), x_2, z \xrightarrow{h_2} \phi(x_2))\), and \(M(g, h) = (x, h, x', \phi(g) \circ h)\). (\(M\) comes from pulling back the open surjection \(G_1 \overset{\phi}{\to} H_1 \times_{(H_0 \times H_0)} (G_0 \times G_0)\) of 5.5 along the map \((G_0 \times_{H_0} H_1) \times_{H_0} (G_0 \times_{H_0} H_1) \to H_1 \times_{(H_0 \times H_0)} (G_0 \times G_0)\) given by \((z \xrightarrow{h_1} \phi(x_1), z \xrightarrow{h_2} \phi(x_2)) \mapsto h_2 h_1^{-1}\).)

Moreover, choose small neighborhoods \(U_d, U_{d,g} \subset D\) (on which \(p\) restricts to an isomorphism) such that \(U_d \cdot V_g = U_{d,g}\) (this can be done if we take \(V_g\) small enough, by the observation just made about the action \(D \times_{G_0} G_1 \to D\).

We claim that
\[
(u, v)(U_d \times_{G_0} V_g \times_{H_0} W_h) \geq (U_{d,g} \times_{H_0} W_h) \times_{H_0} (U_d \times_{H_0} W_{\phi(g)oh}).
\]
Indeed, take a point \(((d_1, h_1), (d_2, h_2))\) of the right-hand space, say \(z \xrightarrow{h_1} \phi(x_1) \in W_h, z \xrightarrow{h_2} \phi(x_2) \in W_{\phi(g)oh}, d_1 \in U_{d,g} \cap D_x, d_2 \in U_d \cap D_{x'}.\) Since \(\phi\) is full, we can find (after extending the base by some open surjection of toposes!) a point \(x_1 \xrightarrow{k} x_2 \in V_g\) such that \(\phi(k) \circ h_1 = h_2\) (cf. (1)). But then
\[
(u, v)(d_2, k, h_1) = ((d_2 \cdot k, h_1), (d_2, \phi(k) \circ h_1)) = ((d_1, h_1), (d_2, h_2)),
\]
because \(\phi(k) \circ h_1 = h_2\) by choice of \(k\), and \(d_2 \cdot k\) is an element of \(U_{d,g}\) over \(x_1\), and hence cannot be anything else but \(d_1\), by the remark above on the action \(D \times_{G_0} G_1 \to D\).

This completes the proof of the theorem.

**5.11. Localization.** Let \(G \overset{\phi}{\to} H\) be a map of continuous groupoids, inducing \(\phi^*: (H\text{-spaces}) \to (G\text{-spaces}),\) and let \(E \overset{p}{\to} H_0\) be an \(H\)-space. \(E\) gives rise to a groupoid \(\hat{E} = E \times_{H_0} H_1,\) i.e.
\[
\hat{E}_0 = E, \quad \hat{E}_1 = E \times_{H_0} H_1 \overset{\pi_1}{\to} E
\]
where \(\cdot\) is the \(d_0\) of \(\hat{E}\) and \(\pi_2\) the \(d_1\). (So this is the “diagram” of \(E\): the objects of \(\hat{E}\) are elements of \(E\), and the arrows from \(e'\) to \(e\) are \(H\)-maps \(p(e') \overset{h}{\to} p(e)\) such
that \( e \cdot h = e' \). \( \hat{E} \) is again an open groupoid (cf. 5.1) if \( H \) is an open groupoid (in particular, the action \( E \times_{H_0} H_1 \rightarrow E \) is open).

There is a projection \( \pi_E : \hat{E} \rightarrow H \) which is a map of continuous groupoids. \( E \hookrightarrow \hat{E} \) is a functor from \( H \)-spaces into continuous groupoids over \( H \).

**Proposition.** (a) There is a canonical equivalence

\[
(H\text{-spaces})/E \xrightarrow{\sim} (\hat{E}\text{-spaces}),
\]

and the functor \((H\text{-spaces}) \rightarrow (H\text{-spaces})/E\) given by taking the fibered product over \( H_0 \) with \( E \rightarrow H_0 \) coincides—modulo the equivalence (1)—with \( \pi_E^* \).

(b) \( G \rightarrow H \) induces a map of continuous groupoids \( \phi_E : \hat{\phi^*(E)} \rightarrow \hat{E} \) such that

\[
\xymatrix{ \phi^*(E) \ar[r]^{\phi_E} \ar[d]_{\pi_{\phi^*(E)}} & \hat{E} \ar[d]^{\pi_E} \\
G \ar[r]^{\phi} & H }
\]

commutes, and the functor \( \phi^*/E \) corresponds to \( \phi_E^* \) under the equivalence (1), i.e. the diagram

\[
(H\text{-spaces})/E \xrightarrow{\phi^*/E} (G\text{-spaces})/\phi^*(E)
\]

remains commutative, up to canonical isomorphism.

**Proof.** Straightforward and omitted.

**Remark.** Everything in the preceding proposition can be restricted to étale spaces, so as to get corresponding statements about localization of \( BG \) and \( BH \): for an étale \( H \)-space \( E \) we have

\[
BH/E \xrightarrow{\sim} B\hat{E};
\]

and the square

\[
B(\phi^*(E)) \xrightarrow{B\phi_E} B\hat{E}
\]

commutes, and is moreover a pullback of toposes; and the square

\[
BG/\phi^*(E) \xrightarrow{B\phi^*/E} BH/E
\]

of geometric morphisms commutes (compare also 3.5).

Notice that if \( G \rightarrow H \) is open (respectively full), then so is \( \phi_E : \hat{\phi^*(E)} \rightarrow \hat{E} \) for any \( H \)-space \( E \). In particular, \( \phi_E \) restricts to a functor \( B(\hat{\phi^*(E)}) \rightarrow B(\hat{E}) \) if \( \phi \) is open and full, by Theorem 5.10.
5.12. THEOREM. Let $G 	o H$ be open and full, as in 5.10. Then for any map $E \to E'$ of étale $H$-spaces, with corresponding map $\hat{E} \to \hat{E}'$ of continuous groupoids over $H$, the diagram

\[
\begin{array}{ccc}
B(\phi^*(E)) & \xrightarrow{(\phi_E)_!} & B\hat{E} \\
\uparrow (\phi^*(\alpha))^* & & \uparrow \alpha^* \\
B(\phi^*(E')) & \xrightarrow{(\phi_{E'})_!} & B\hat{E}'
\end{array}
\]

commutes (up to canonical isomorphism).

PROOF. By replacing $G \to H$ by $\phi^*(E) \to \hat{E}'$, we may without loss assume that $E' = 1$; so it is enough to show that

\[
\begin{array}{ccc}
BG & \xrightarrow{\phi^!} & BH \\
\uparrow \phi^*(\alpha)^* & & \uparrow \alpha^*
\end{array}
\]

commutes. (In what follows, I will indicate what the points of spaces are and what the maps do to them, just to remind you which pullback is along which map, etc.)

Take a $G$-space $D = (D \to G_0, \cdot)$. $\phi_1(D)$ is the coequalizer

\[
D \times_{G_0} G_1 \times_{H_0} H_1 \xrightarrow{u} D \times_{H_0} H_1 \xrightarrow{r} D \otimes_G H
\]

(recall that the points of the left-hand space are triples $(d \in D_x, x' \xrightarrow{g} x, y \xrightarrow{h} \phi(x))$, and $u(d, g, h) = (d \cdot g, h)$, $v(d, g, h) = (d, \phi(g) \circ h)$). So by stability (cf. 5.7), $\alpha^*\phi_1(D)$ is the coequalizer

\[
(1) \quad D \times_{G_0} G_1 \times_{G_0} H_1 \times_{H_0} E \xrightarrow{u \times E} (D \times_{H_0} H_1) \times_{H_0} H_1 \xrightarrow{r \times E} (D \otimes_G H) \times_{H_0} E
\]

(points of the left-hand space are of the form $(d, g, h, e \in E_y)$, with $(d, g, h)$ as above). Notice that the action of $H$ on $(D \otimes_G H) \times_{H_0} E$ is defined by acting on both the $H$- and the $E$-coordinate, i.e. for $d \in D_x$, $y \xrightarrow{h} \phi(x)$, $e \in E_y$, $y' \xrightarrow{k} y$, we have $[(d, h), e] \cdot k = [(d, h'k), e \cdot k]$.

The other way round, $\phi^*(\alpha)^*(D)$ is the $\phi^*(E)$-space $D \times_{G_0} (G_0 \times_{H_0} E) \cong D \times_{H_0} E$ (points are pairs $(d \in D_x, e \in E_{\phi(x)})$), and hence $(\phi_E)_!(\phi^*(\alpha)^*(D))$ is the coequalizer $\phi^*(\alpha)^*(D) \otimes_{\phi^*(E)} \hat{E}$ of

\[
(2) \quad (D \times_{G_0} G_0 \times_{H_0} E) \xrightarrow{\phi^*(E)_0} \phi^*(E)_1 \times_{\hat{E}_0} \hat{E}_1 \xrightarrow{u'} D \times_{G_0} (G_0 \times_{H_0} E) \times_{\hat{E}_0} \hat{E}_1.
\]

Unwinding the definitions, it is not difficult to check that the coequalizer of (2) is isomorphic to the coequalizer (3)

\[
(3) \quad D \times_{G_0} E \times_{G_0} G_1 \times_{H_0} H_1 \xrightarrow{\phi^!} D \times_{H_0} E \times_{H_0} H_1 \xrightarrow{\phi^*(\alpha)^*(D) \otimes_{\phi^*(E)} \hat{E}}
\]
(to remind you of where we are: points of the left-hand space in (3) are quadruples 
\( (d \in D_x, e \in E_{\phi(x)}, x' \xrightarrow{\theta} x, y \xrightarrow{h} \phi(x')) \), and \( \tilde{u} \) sends this to \( (d \cdot g, e \cdot \phi(g), h) \) while \( \tilde{v} \) sends it to \( (d, e, \phi(g) \circ h) \).

But (1) and (3) are the same under the isomorphisms \( \alpha \) and \( \beta \),

\[
\begin{align*}
\text{(1)}: & \quad (D \times_{G_0} G_1 \times_{G_0} H_1) \times_{H_0} E \xrightarrow{u \times E} D \times_{H_0} H_1 \times_{H_0} E \\
& \quad \uparrow \alpha \quad \uparrow \beta \\
\text{(3)}: & \quad D \times_{G_0} E \times_{G_0} G_1 \times_{G_0} H_1 \xrightarrow{\tilde{u}} D \times_{H_0} E \times_{H_0} H_1
\end{align*}
\]

where \( \alpha \) and \( \beta \) are defined by \( \alpha(d, e, g, h) = (d, g, e \cdot \phi(g) \cdot h) \), \( \beta(d, e, h) = (d, h, e \cdot h) \). Then \( \beta \circ \tilde{u} = (u \times E) \circ \alpha \), \( \beta \circ \tilde{v} = (v \times E) \circ \alpha \), so (1) and (3) have isomorphic coequalizers.

This proves the theorem.

5.13. PROPOSITION. Let \( G \xrightarrow{\phi} H \) be a map of continuous groupoids, and suppose \( \phi \) is open and fully faithful (5.5). Then the composite \( BG \xrightarrow{\phi} BH \xrightarrow{\phi^*} BG \) is canonically isomorphic to the identity on \( BG \).

PROOF. Take an étale \( G \)-space \((D \to G_0, \cdot)\). \( \phi_!(D) \) is the coequalizer

\[
D \times_{G_0} G_1 \times_{G_0} H_1 \xrightarrow{u} D \times_{H_0} H_1 \xrightarrow{\tau} D \otimes_G H
\]

over \( H_0 \). This coequalizer is stable under pullback along \( G_0 \to H_0 \) (5.7), so \( \phi^* \phi_!(D) \) is the coequalizer

\[
(D \times_{G_0} G_1 \times_{H_0} H_1) \times_{H_0} G_0 \xrightarrow{u \times G_0} D \times_{H_0} H_1 \times_{H_0} G_0 \to (D \otimes_G H) \times_{H_0} G_0
\]

(so points of the left-hand space are quadruples \( (d \in D_x, x' \xrightarrow{\theta} x, \phi(x'') \xrightarrow{h} \phi(x'), x'') \)). If \( \phi \) is fully faithful, this is isomorphic to the coequalizer of

\[
D \times_{G_0} G_1 \times_{G_0} G_1 \xrightarrow{\times G_1} D \times_{G_0} G_1.
\]

But

\[
D \times_{G_0} G_1 \times_{G_0} G_1 \xrightarrow{\times G_1} D \times_{G_0} G_1 \xrightarrow{D \times m} D
\]

is a split coequalizer, for any \( G \)-space \( D \).

5.14. PROPOSITION. Let \( G \xrightarrow{\phi} H \) be a map of continuous groupoids, and suppose \( \phi \) is open and full. Then \( \phi_! \) induces an isomorphism of subobject lattices \( \text{Sub}_{BG}(E) \xrightarrow{\sim} \text{Sub}_{BH}(\phi_! E) \) for each étale \( G \)-space \( E \).

PROOF. (a) Let us first note that \( \phi_! \) maps monos to monos: Take \( S \xrightarrow{u} E \) in \( BG \), and consider the diagram

\[
\begin{array}{ccc}
S \times_{G_0} G_1 \times_{H_0} H_1 & \xrightarrow{q} & S \times_{H_0} H_1 \\
\downarrow & & \downarrow \\
E \times_{G_0} G_1 \times_{H_0} H_1 & \xrightarrow{p} & E \times_{H_0} H_1 \\
& \downarrow \phi_!(u) & \\
& \phi_!(S) & \xrightarrow{\phi_!(u)} \phi_!(E)
\end{array}
\]

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the rows of which are stable coequalizers (cf. 5.7). We use change of base and “point-set language”. Suppose given points \(x, x'\) of the space \(S \otimes_G H\) (in any extension of the base topos \(S \rightarrow \mathcal{P}\)) such that \(\phi(u)(x) = \phi(u)(x')\). Since \(q\) is an open surjection, we can find representing points \((s, h), (s', h') \in S \times_{H_0} H_1\) with \([s, h] = q(s, h) = x, [s', h'] = x'\) (in some further base extension \(\mathcal{P}'\) where \(\mathcal{P}' \rightarrow \mathcal{P}\) is an open surjection). Then \(p(u(s), h) = p(u(s'), h')\), so (by going to an \(\mathcal{P}''\) where \(\mathcal{P}'' \rightarrow \mathcal{P}'\) is an open surjection) there is an \(x \stackrel{u}{\rightarrow} x'\) such that \(u(s') \cdot g = u(s)\) and \(\phi(g) \circ h = h'\) (here we use that \(E \times G_0 \times_{H_0} H_1 \rightarrow (E \times_{H_0} H_1) \times_{H_0} (E \times_{H_0} H_1)\) is open, cf. 5.10). Since \(u\) is mono, \(s'g = s\), hence \(x = [s, h] = [s', h'] = x'\) in \(\mathcal{P} \otimes_G H\). (This latter identity holds in \(\mathcal{P}''\), but since \(\mathcal{P}'' \rightarrow \mathcal{P}'\) is an open surjection, it follows that it must hold in \(\mathcal{P}\), the base over which \(x\) and \(x'\) were originally defined). This proves that \(\phi(u)(x)\) is a monomorphism of spaces.

(b) Next, we verify that \(\text{Sub}_{BG}(E) \xrightarrow{\phi_1} \text{Sub}_{BH}(\phi_1 E)\) are mutually inverse functors, where for \(A \subseteq \phi_1(E)\), \(\phi_1(A)\) is defined as the pullback along the unit of the adjunction \(\phi_1 \dashv \phi^*:\)

\[
\begin{array}{ccc}
\phi(A) & \longrightarrow & E \\
\downarrow & & \downarrow \eta \\
A & \longrightarrow & \phi^* \phi_1(E).
\end{array}
\]

For \(S \subseteq E\) and \(A \subseteq \phi_1(E)\), the inclusions \(S \subseteq \phi_1\phi(A)\) and \(\phi_1\phi(A) \subseteq A\) are obvious. To prove the inclusions \(\phi_1\phi_1(S) \subseteq S\) and \(A \subseteq \phi_1\phi(A)\) we use point-set arguments as in (a). This can be done since everything in sight is stable. Leaving the subsequent base extensions of the base topos \(\mathcal{P}\) implicit, the proof is as follows. For \(\phi_1(S) \subseteq S\), take \(e \in \phi_1\phi_1(S)_x \subseteq E_x\), i.e. \([e, s(\phi(x))] \in \phi_1(S)_{\phi(x)}, \phi_1(S)_{\phi(x)} \subseteq \phi_1(E)_{\phi(x)} = (E \otimes_G H)_{\phi(x)}\) (recall that \(G_0 \xrightarrow{\sigma} G_1\) “sends \(x\) to the identity at \(x'\)”). This means that there is a pair \([t, h] \in S \otimes_G H\) with \([e, s(\phi(x))] = [t, h]\), say with \(\phi(x) \xrightarrow{h} \phi(x')\) and \(t \in S_x\). Then there is an \(x' \xrightarrow{g} x\) such that \(t = e \cdot g\) and \(\phi(g) \circ h = s(\phi(x))\). So \(e = t \cdot g^{-1} \in S_x\) since \(t \in S_x\). Thus \(\phi_1(S) \subseteq S\).

For \(A \subseteq \phi_1\phi(A), \) take \(a \in A_y \subseteq \phi_1(E)_y\) over a point \(y\) of \(H_0\). So \(a\) is a class \(a = [e_0, h_0], \) where \(e_0 \in E_x\) and \(y \xrightarrow{h_0} \phi(x)\). Now \(\phi(A)_{x}\) consists of points \(e \in E_x\) such that \([e, s(\phi(x))] \in A_{\phi(x)}, \) so \(e_0 \in \phi(A)_n\) because \([e_0, s(\phi(x))] = a \cdot h_0^{-1} \in A_{\phi(x)}\). Then \(\phi(e_0) = [e_0, s(\phi(x))] \in \phi_1\phi(A)_{\phi(x)}, \) i.e. \(a \cdot h_0^{-1} \in \phi_1\phi(A)_{\phi(x)}\). So \(a \in \phi_1\phi(A)_y\).

We now reformulate the preceding results in terms of properties of the geometric morphism \(BG \xrightarrow{\phi} BH\).

5.15. SUMMARY. Let \(G \xrightarrow{\phi} H\) be a continuous map of groupoids, with induced geometric morphism \(BG \xrightarrow{B\phi} BH\).

(i) If \(\phi_0 : G_0 \rightarrow H_0\) is open, then \(BG \rightarrow BH\) is open.

(ii) If \(\phi\) is essentially surjective then \(B\phi\) is surjective.

(iii) If \(\phi\) is essentially surjective and full then \(B\phi\) is connected.

(iv) If \(\phi\) is open and full, then \(BG \xrightarrow{B\phi} BH\) is atomic.
(v) If $\phi$ is open, fully faithful and essentially surjective, then $BG \overset{B\phi}{\longrightarrow} BH$ is an equivalence of toposes.

**Proof.** For (i)–(iii) see 5.9. For (iv), notice that $B\phi^*: BH \to BG$ has a BH-indexed left adjoint by 5.11 (so $B\phi$ is locally connected). $B\phi^*$ also preserves the subobject classifier: writing $\Omega_G, \Omega_H$ for the subobject classifiers of $BG$ and $BH$, we have for $E \in BG$,

\[
\begin{align*}
E & \to \Omega_G \\
A & \subseteq E \text{ in } BG \\
S & \subseteq \phi_!(E) \text{ in } BH \\
\phi_!(E) & \to \Omega_H \\
E & \to (B\phi)^*(\Omega_H)
\end{align*}
\]

(by 5.14)

so $\Omega_G \cong (B\phi)^*(\Omega_H)$. Finally, (v) follows from (iii) and 5.13.

6. **The Stability Theorem.** Recall that if $G$ is a continuous groupoid in a topos $\mathcal{E}$, $B(\mathcal{E}, G)$ denotes the $\mathcal{E}$-topos of étale $G$-spaces over (or “in”) $\mathcal{E}$. If $\mathcal{F} \overset{p}{\to} \mathcal{E}$ is a geometric morphism, we obtain a continuous groupoid $p^#(G)$ in $\mathcal{F}$ (see 1.5). The aim of this section is to show there is a canonical equivalence of toposes $B(\mathcal{F}, p^#(G)) \overset{\sim}{\longrightarrow} \mathcal{F} \times _{\mathcal{E}} B(\mathcal{E}, G)$ for any (open!, cf. 5.1) continuous groupoid $G$ in $\mathcal{E}$.

6.1. **Generators for $BG$.** Let $G$ be a continuous groupoid (we still tacitly assume that $d_0$ and $d_1: G_1 \to G_0$ are open maps). The aim is to find a more manageable set of generators for $BG$ than the one coming from the proof of the existence of colimits (cf. 2.1).

Let $E = (E \overset{p}{\to} G_0, \cdot)$ be an étale $G$-space, and let $U \overset{t}{\to} E$ be a section of $p$ over some $U \in \mathcal{E}(G_0)$. Let $N_t \subseteq G_1$ be the subspace defined by the pullback

\[
\begin{align*}
G_1 \cap d_1^{-1}(U) & \overset{i}{\longrightarrow} E \\
\downarrow & \downarrow \\
N_t & \overset{t}{\longrightarrow} U
\end{align*}
\]

where $i$ is the composition $G_1 \cap d_1^{-1}(U) \overset{td_1 \times 1}{\longrightarrow} E \times_{G_0} G_1 \overset{\pi}{\longrightarrow} E$ (i.e. “$N_t = \{ x \overset{g}{\to} y \in G_1 | y \in U \text{ and } t(y) \cdot g = t(x) \}$”). Then

(i) $N_t$ is an open subspace of $G_1$, and $d_0(N_t), d_1(N_t) \subseteq U$,

(ii) $N_t$ contains all identities, and is closed under inverse and composition

(in point-set notation: $x \in U \Rightarrow s(x) \in N_t$, $g \in N_t \Rightarrow g^{-1} \in N_t$, and $g, h \in N_t \Rightarrow g \circ h \in N_t$ when $d_0 g = d_1 h$).

Now consider the pullback of spaces

\[
\begin{align*}
G_1 \times_{G_0} G_1 & \overset{m(1 \times r)}{\longrightarrow} G_1 \\
\downarrow & \downarrow \\
R_t & \longrightarrow N_t
\end{align*}
\]

where $G_1 \overset{r}{\to} G_1$ is the inverse, and $G_1 \times_{G_0} G_1$ is the pullback of $G_1 \overset{d_0}{\to} G_0 \overset{d_0}{\leftarrow} G_1$ (so $R_t = \{(x \overset{g}{\to} y, x \overset{h}{\to} z | gh^{-1} \in N_t \}$). $R_t$ defines an equivalence relation on
$G_1 \cap d_{-1}^{-1}(U)$ over $G_0$ (via $G_1 \xrightarrow{d_0} G_0$), and we write $G_1 \cap d_{-1}^{-1}(U)/N_t$ for the quotient, i.e. there is a coequalizer of spaces over $G_0$:

\[
\begin{array}{ccc}
R_t & \longrightarrow & G_1 \cap d_{-1}^{-1}(U) \\
\downarrow & & \downarrow \text{do} \\
G_1 \cap d_{-1}^{-1}(U)/N_t & \longrightarrow & G_0
\end{array}
\]

(1)

Notice that this coequalizer (1) is stable, because $R_t \to G_1 \times_G G_1$ is open, and $G_1 \cap d_{-1}^{-1}(U)$ is an open space over $G_0$ (so we apply the last fact of 1.3 to spaces over $G_0$).

We claim that $G_1 \cap d_{-1}^{-1}(U)/N_t$ is an étale $G$-space. Composition $G_1 \times_G G_1 \to G_1$ induces an action $(G_1 \cap d_{-1}^{-1}(U)/N_t) \times_{G_0} G_1 \to G_1 \cap d_{-1}^{-1}(U)/N_t$ (in point-set notation: $[g] \cdot h = [gh]$ for a class $[g] \in G_1 \cap d_{-1}^{-1}(U)/N_t$, $g \in G_1, d_1(g) \in U$). Moreover, $G_1 \cap d_{-1}^{-1}(U)/N_t \xrightarrow{\text{do}} G_0$ is open because $G_1 \xrightarrow{\text{do}} G_0$ is open (by assumption 5.1), while the diagonal over $G_0$

\[
G_1 \cap d_{-1}^{-1}(U)/N_t \to (G_1 \cap d_{-1}^{-1}(U)/N_t) \times_{G_0} (G_1 \cap d_{-1}^{-1}(U)/N_t)
\]

is open, as follows by considering the square

\[
\begin{array}{ccc}
R_t & \longrightarrow & G_1 \cap d_{-1}^{-1}(U) \\
\downarrow & & \downarrow \\
G_1 \cap d_{-1}^{-1}(U)/N_t & \longrightarrow & (G_1 \cap d_{-1}^{-1}(U)/N_t) \times_{G_0} (G_1 \cap d_{-1}^{-1}(U)/N_t).
\end{array}
\]

Thus $G_1 \cap d_{-1}^{-1}(U)/N_t$ is an object of $BG$.

Furthermore, the section $t: U \to E$ induces a map of étale $G$-spaces $\tilde{t}$, defined by factoring the map $G_1 \cap d_{-1}^{-1}(U) \xrightarrow{i} E$, $(g \mapsto t(d_1g) \cdot g)$ through the coequalizer (1):

\[
\begin{array}{ccc}
G_1 \cap d_{-1}^{-1}(U)/N_t & \longrightarrow & E \\
\downarrow & & \downarrow \text{t} \\
G_0 & \longrightarrow & E
\end{array}
\]

(2)

$\tilde{t}$ contains the section $t$, in the sense that there is a commutative diagram

\[
\begin{array}{ccc}
G_1 \cap d_{-1}^{-1}(U)/N_t & \longrightarrow & E \\
\downarrow \text{[s]} & & \downarrow \text{t} \\
U & \longrightarrow & E
\end{array}
\]

([s] stands for the composite $U \xrightarrow{s} G_1 \cap d_{-1}^{-1}(U) \to G_1 \cap d_{-1}^{-1}(U)/N_t$.)

We conclude that the étale $G$-spaces of the form $G_1 \cap d_{-1}^{-1}(U)/N_t \xrightarrow{\text{do}} G_0$ (with $G$-action defined by composition) generate $BG$.

**Definition.** Let $S_C$ be the full subcategory of $BG$ whose objects are étale $G$-spaces of the form $G_1 \cap d_{-1}^{-1}(U)/N$, where $U \subset G_0$ and $N \subset G_1$ are open subspaces such that $d_0(N), d_1(N) \subset U$, and $s(U) \subset N$, $m(N \times_{G_0} N) \subset N$, $N^{-1} \subset N$ (cf. (i),

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If we define the covering families of \( S_G \) to be the epimorphic families of \( BG \), then \( S_G \) is a site for \( BG \), i.e. there is a canonical equivalence
\[
BG \simeq \text{Sh}(S_G).
\]

**Notation.** We write \([G, U, N]\) for the \( G \)-space \((G_1 \cap d^{-1}_1(U)/N \overset{d_0}{\to} G_0, \cdot)\), where \( U \) and \( N \) are as in the preceding definition.

For later use, we note the following lemma.

**6.2. LEMMA.** Let \( G \to H \) be a map of continuous groupoids, with corresponding geometric morphism \( B\phi: BG \to BH \), and let \( S_G, S_H \) be the sites for \( BG \) and \( BH \) as just defined. If \( \phi \) is a fibration in the sense that \( G_1 \overset{(d_0, \phi_1)}{\to} G_0 \times_{H_0} H_1 \) is an open surjection, then \((B\phi)^*\) maps \( S_H \) into \( S_G \); in fact for \([H, V, N] \in S_H\),
\[
(B\phi)^*[H, V, N] = [G, \phi^{-1}_0(V), \phi^{-1}_1(N)].
\]

**PROOF.** Take \( V \subset G_0, N \subset G_1 \) such that \([H, V, N] = H_1 \cap d^{-1}_1(V)/N \) is an object of the site \( S_H \), and write \( R_N \subset (H_1 \cap d^{-1}_1(V)/N) \times_{H_0} (H_1 \cap d^{-1}_1(V)/N) \) for the equivalence relation corresponding to \( N \). Note that \( R_N \) is an open sublocale, and that \( R_{\phi^{-1}_1(N)} = (\phi \times \phi)^{-1}(R_N) \). Since quotients by an open equivalence relation are stable (1.3), we find that
\[
(B\phi)^*[H, V, N] = (G_0 \times_{H_0} (H_1 \cap d^{-1}_1(V)))/G_0 \times_{H_0} R_N.
\]

Clearly, \( \phi \) induces a map \( G_1 \cap d^{-1}_1\phi^{-1}_0(V) \overset{(d_0, \phi_1)}{\to} G_0 \times_{H_0} (H_1 \cap d^{-1}_1(V)) \), which passes to the quotient to give a map
\[
G_1 \cap d^{-1}_1\phi^{-1}_0(V)/R_{\phi^{-1}_1(N)} \xrightarrow{\alpha} G_0 \times_{H_0} (H_1 \cap d^{-1}_1(V))/(G_0 \times_{H_0} R_N).
\]

Also, since by assumption
\[
G_1 \cap d^{-1}_1\phi^{-1}_0(V) \overset{(d_0, \phi_1)}{\to} G_0 \times_{H_0} (H_1 \cap d^{-1}_1(V))
\]
is an open surjection, it is a coequalizer of its kernelpair \( K \rightrightarrows (G_1 \cap d^{-1}_1\phi^{-1}_0(V)) \times_{G_0} (G_1 \cap d^{-1}_1\phi^{-1}_0(V)) \). But \( K \subset R_{\phi^{-1}_1(N)} \), so we obtain a factorization
\[
K \rightrightarrows G_1 \cap d^{-1}_1\phi^{-1}_0(V) \overset{(d_0, \phi_1)}{\to} G_0 \times_{H_0} (H_1 \cap d^{-1}_1(V))
\]
where \( \pi \) is the projection. It is easy to see that \( \beta \) passes to the quotient to give a map
\[
G_0 \times_{H_0} H_1 \cap d^{-1}_1(V)/(G_0 \times_{H_0} R_N) \overset{\beta}{\to} G_1 \cap d^{-1}_1\phi^{-1}_0(V)/\phi^{-1}_1(R_N),
\]
and that \( \alpha \) and \( \beta \) are mutually inverse maps of \( G \)-spaces.

**6.3. Morphisms in \( S_G \).** It is clear that for any étale \( G \)-space \( E \) and any object \([G, U, N] \) of \( S_G \), there is a bijective correspondence between \( G \)-maps \([G, U, N] \to E \) and sections \( U \overset{t}{\to} E \) with \( N_t \subset N \)
\[
(1) \quad \frac{[G, U, N] \overset{t}{\to} E}{U \overset{t}{\to} E, N \subset N_t},
\]

given $\tilde{t}, t = \tilde{t} \circ [s]$, where $U \xrightarrow{[s]} G_1 \cap d_1^{-1}(U)/N$ is as in 6.1; and given $t$, the map $\tilde{t}$ is defined as the projection $G_1 \cap d_1^{-1}(U)/N \to G_1 \cap d_1^{-1}(U)/N_t$ followed by the map $\tilde{t}$ as described in 6.1(2).

In particular, the maps $[G, U, N] \xrightarrow{i} [G, V, M]$ of $S_G$ correspond to sections $t$ of $[G, V, M]$ over $U$ such that (in point-set notation:) for each $x \xrightarrow{h} y \in N$, $t(y) \cdot h = t(x)$ in $[G, V, M]$.

6.4. Covers in $S_G$. Let $[G, U, N] \xrightarrow{f} [G, V, M]$ be a map in $S_G$; i.e. $f$ is an étale map $G_1 \cap d_1^{-1}(U)/N \to G_1 \cap d_1^{-1}(V)/M$ of $G$-spaces, induced by a section $U \xrightarrow{s} G_1 \cap d_1^{-1}(V)/M$. Then $N \subset N_t$, and there is a factorization

$$G_1 \cap d_1^{-1}(U)/N \xrightarrow{\pi} G_1 \cap d_1^{-1}(U)/N_t \xrightarrow{\iota} G_1 \cap d_1^{-1}(V)/N_t$$

$\pi$ is the projection onto the quotient, and $\iota$ is mono. We also write $\text{Ker}(f)$ for $N_t$.

Now suppose $\{[G, U_i, N_i] \xrightarrow{f_i} [G, V, M]\}_i$ is a cover in $S_G$, i.e. an epimorphic family of étale $G$-spaces. Factor each $f_i$ as

$$[G, U_i, N_i] \xrightarrow{\pi_i} [G, U_i, \text{Ker}(f_i)] \xrightarrow{u_i} [G, V, M].$$

Each $\pi_i$ is a singleton covering family in $S_G$.

Consider the pullback

$$\begin{array}{ccc}
G_1 \cap d_1^{-1}(U)/\text{Ker}(f_i) & \xrightarrow{u_i} & G_1 \cap d_1^{-1}(V)/M \\
\downarrow{\pi_i} & & \downarrow{[s]} \\
V_i & \subset & V
\end{array}$$

(1) The $V_i$ form an open cover of $V$, inducing a cover

$$\{G_1 \cap d_1^{-1}(V_i)/(M|V_i) \xrightarrow{\eta_i} G_1 \cap d_1^{-1}(V)/M\}$$

in $S_G$, where $M|V_i$ is the restriction of $M$ to $V_i$ ("$M|V_i = \{x \xrightarrow{g} y| g \in M, x, y \in V_i\}$"), and $\eta_i$ is induced by the identity-section $[s] \colon V_i \to G_1 \cap d_1^{-1}(V)/M$. For each $i$, the triangle

$$\begin{array}{ccc}
G_1 \cap d_1^{-1}(V_i)/\text{Ker}(f_i) & \xrightarrow{u_i} & G_1 \cap d_1^{-1}(V)/M \\
\downarrow{\pi_i} & & \downarrow{[s]} \\
G_1 \cap d_1^{-1}(V_i)/M|V_i & \subset & G_1 \cap d_1^{-1}(V)/M|V_i
\end{array}$$

commutes, i.e. the cover $\{\eta_i\}$ refines the cover $\{u_i\}$.

We conclude that the Grothendieck topology on $S_G$ is generated by (is the smallest one containing) the covers of the following two types

(1) projections $[G, U, N] \to [G, U, M], N \subset M$,

(2) covers $\{[G, U_i, M|U_i] \xrightarrow{\eta_i} [G, U, M]\}_i$ coming from open covers $\{U_i\}$ of $U$ in $G_0$.  

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Notice that this is a stable system, i.e. the pullback of a cover of type (1) or (2) along a map $[G,V,L] \xrightarrow{f} [V,U,M]$ in $\mathcal{S}_G$ is again of type (1) or (2).

6.5. Bases for $G$. Let $\mathcal{B}_0$ be a basis for $G_0$, $\mathcal{B}_1$ a basis for $G_1$. We say that the pair $\mathcal{B} = (\mathcal{B}_0, \mathcal{B}_1)$ is a basis for $G$ if both are closed under finite meets and

(i) if $B_0 \in \mathcal{B}_0$ then $d_0^{-1}(B_0), d_1^{-1}(B_0) \in \mathcal{B}_1,$
(ii) if $B_1 \in \mathcal{B}_1$ then $s^{-1}(B_1) \in \mathcal{B}_0,$

(iii) $\mathcal{B}_1$ is closed under "closure for composition", i.e. if $B \in \mathcal{B}_1$ then $mB \in \mathcal{B}_1$ (here for $U \in \mathcal{C}(G_1)$, $U^+ = m(U \times_{G_0} U)$—recall that $G_1 \times_{G_0} G_1 \xrightarrow{m} G_1$ is open, cf. 5.1—and $U^{(0)} = U, U^{(n+1)} = U^{(n)} + mU = \bigcup_{n} U^{(n)}$). Then clearly, given any object $[G,U,N]$ of $\mathcal{S}_G$, there is a canonical map $[G, B_0, B_1] \to [G, U, N]$ whenever $B_0 \subset U$ and $B_1 \subset N$ (the map induced by the section $B_0 \to U \xrightarrow{s} [G, U, N]$), and these form a cover in $\mathcal{S}_G$, $\{[G, B_0, B_1] \to [G, U, N]\}_{B_0B_1}$, indexed by all $B_0 \in \mathcal{B}_0$ with $B_0 \subset U$ and $B_1 \in \mathcal{B}_1$ with $B_1 \subset N$.

Consequently by the comparison lemma (SGA 4, III.4), the full subcategory of $\mathcal{S}_G$ consisting of only those objects $[G, B_0, B_1], B_0 \in \mathcal{B}_0, B_1 \in \mathcal{B}_1$, for some basis $(\mathcal{B}_0, \mathcal{B}_1)$ for $G$, equipped with the Grothendieck topology induced from $\mathcal{S}_G$, still form a site for $BG$.

It is clear that if $\mathcal{B} = (\mathcal{B}_0, \mathcal{B}_1)$ is a basis for $G$, then for any geometric morphism $\mathcal{E} \xrightarrow{\mathcal{P}} \mathcal{S}$, the bases (presentations, cf. 1.1) $p^*(\mathcal{B}_0)$ for $p^#(G_0)$ and $p^*(\mathcal{B}_1)$ for $p^#(G_1)$ still satisfy conditions (i)–(iii), and thus define a basis $p^*(\mathcal{B})$ for $p^#(G)$.

6.6. Lemma. (a) Let $E \xrightarrow{p} X$ and $F \xrightarrow{q} Y$ be étale maps of (generalized) spaces and let $U \xrightarrow{t} E \times F$ be a section of $p \times q$ over an open $U \subset X \times Y$. Then there exists a cover $U = \bigcup V_i (V_i \times W_i)$ such that $t|V_i \times W_i$ is of the form $r_i \times s_i$ for sections $V_i \xrightarrow{r_i} E, W_i \xrightarrow{s_i} F$.

(b) In particular, taking $q = \text{identity}$, sections of $E \times Y \xrightarrow{p \times Y} X \times Y$ locally do not depend on the $Y$-variable.

Proof. Obvious.

6.7. Stability Theorem. Let $\mathcal{F} \xrightarrow{p} \mathcal{E}$ be a geometric morphism, and let $G$ be an open continuous groupoid in $\mathcal{E}$. Then the canonical geometric morphism $B(\mathcal{F}, p^#(G)) \xrightarrow{\sim} \mathcal{F} \times_{\mathcal{E}} B(\mathcal{E}, G)$ is an equivalence of toposes.

Proof. By arguing constructively in $\mathcal{E}$, it is enough to consider the case $\mathcal{E} = \text{Sets}$. We may also assume that $\mathcal{F}$ is sheaves on a space $Y$, because for any $\mathcal{F}$ there is an open surjection $\text{Sh}(Y) \xrightarrow{q} \mathcal{F}$, and if we prove the theorem for the composite $p \circ q$, then it will follow for $p$ since equivalences of toposes are reflected down open surjections; i.e. if

$$
\begin{array}{ccc}
\mathcal{Y} & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
\mathcal{F} & \longrightarrow & \mathcal{E}
\end{array}
$$

is a pullback, $\mathcal{Y} \to \mathcal{F}$ is an open surjection and $\mathcal{X} \to \mathcal{Y}$ is an equivalence, then $\mathcal{X} \to \mathcal{F}$ is also an equivalence (see Moerdijk (to appear), lemma in 2.4).

Let $\mathcal{B} = (\mathcal{B}_0, \mathcal{B}_1)$ be the maximal basis for $\mathcal{F}$, $\mathcal{B}_i = \mathcal{C}(G_i)$, with corresponding site $\mathcal{S}_G$ for $BG = B(\mathcal{E}, G)$. By applying $p^*$ to the category $\mathcal{S}_G$ and to the covers
of $S_G$ we obtain a site $S = p^#(S_G)$ in $\mathcal{F}$ (or rather, a category with a stable generating system for the topology) for $\mathcal{F} \times \mathcal{G} B(\mathcal{G}, G)$.

On the other hand, $p^*(\mathcal{B}_0)$, $p^*(\mathcal{B}_1)$ give a basis for $p^#(G)$ as noted in 6.5, with a corresponding site $T = S_{p^*(\mathcal{B})} \subset S_{p^#(G)}$ for $B(\mathcal{F}, p^#G)$ in $\mathcal{F}$.

We will compare these two sites $S$ and $T$ in $\mathcal{F}$ by an obvious functor $S \to T$ induced by $p^*$.

(i) The objects of $S \in \mathcal{F}$ are (at least locally) the same as those of $S_G \in \mathcal{G}$ (i.e. $S_0 = p^*(S_{G0})$, the constant sheaf), and $P$ sends an object $G_1 \cap d^{-1}_1(V)/N$ of $S_G$ to the corresponding object $p^#(G_1) \cap d^{-1}_1(p^#(V))/p^#(N)$. But observe that the quotient $G_1 \cap d^{-1}_1(V)/N$ is stable, i.e. we have

$$p^#(G_1) \cap d^{-1}_1(p^#(V))/p^#(N) \cong p^#(G_1 \cap d^{-1}_1(V)/N),$$

so $S$ and $T$ have essentially the same objects.

(ii) A map $G_1 \cap d^{-1}_1(V)/N \to G_1 \cap d^{-1}_1(W)/M$ in $S$ comes (locally) from a section $V \to G_1 \cap d^{-1}_1(W)/M$ (cf. 6.3), and this gives a section

$$p^#(V) \to p^#(G_1 \cap d^{-1}_1(W)/M) \cong p^#(G_1) \cap d^{-1}_1p^#(W)/p^#(M),$$

and hence a map

$$p^#(G_1) \cap d^{-1}_1(p^#V)/p^#(M) \to p^#(G_1) \cap d^{-1}_1p^#(W)/p^#(M)$$

(this describes $P$ on maps). There are many more maps $p^#(G_1 \cap d^{-1}_1(V)/N) \to p^#(G_1 \cap d^{-1}_1(W)/M)$ in $T$ than there are maps $G_1 \cap d^{-1}_1(V)/M \to G_1 \cap d^{-1}_1(W)/M$ in $S$ (i.e. essentially, in $S_G$). However, if $u$ is such a map in $T$ corresponding to a section $p^#(V) \to G_1 \cap d^{-1}_1(W)/M$ of $p^#(d_0)$, then it is true internally (in $\mathcal{F}$) that there exists an open cover $\{U_i\}_{i \in I}$ of $p^#(V)$ such that $U_i$ is of the form $p^#(r_i)$ for a section $r_i$ of $G_1 \cap d^{-1}_1(V)/M \to G_0$ in $\mathcal{G}$—this follows from Lemma 6.6(b). Moreover $p^*(\mathcal{B}_0)$ gives a presentation of the locale $p^#(G_0)$, so the covers of $p^#(V)$ are generated by families of the form $\{p^#(V_j)\}_{j \in p^#(J)}$, where $\{V_j\}_{j \in J}$ is a cover of $V$ by elements of $\mathcal{B}_0$ in $\mathcal{G}$. But any such cover $V = \bigvee V_j$ in $G_0$ gives a cover of $G_1 \cap d^{-1}_1(V)/M$ in the site $S_G$ in $\mathcal{G}$: $\{G_1 \cap d^{-1}_1(V_i)/(M_i)\}_i \to G_1 \cap d^{-1}_1(V)/M \to G_0$ (cf. 6.4). So what we conclude is that the composite to $p^#(r_j)$ is of the form $p^#(r_j)$.

In other words, $S \to T$ is locally full in the sense that for any map $PS \to PS'$ in $T$ there is a cover $\{S_j \to S\}_j$ in $S$ such that $u \circ P(\alpha_j) = P(v_j)$ for some $v_j : S_j \to S'$ in $S$.

(iii) It remains to compare the covers in $S$ and in $T$. Clearly $P$ preserves covers (cf. 6.4). $P$ also reflects covers, since every cover $\{PS_i \overset{u_i} \to PS\}_i \in I$ in $T$ has a refinement of the form $\{PS_j \overset{Pw_j} \to PS\}_j \in J$ where $\{w_j\}$ is a cover in $S$. To see this, we may first assume that $v_i = P(f_i)$, by (ii). Now factor $f_i = u_i \circ \pi_i$ as in 6.4, and construct the pullback 6.4(1). Clearly $\pi_i$ is a cover in $S$. Moreover the $u_i$ give a family $\{V_i \to V\}_i \in I$ such that $\{p^#(V_i)\}_i$ is a cover of $p^#(V)$ in $p^#(G_0)$. This does not necessarily mean that $\{V_i\}$ is a cover of $V$ in $\mathcal{G}$, but it certainly implies that the induced family $\{[p^#G_1,p^#(V_i),p^#(N|V_i)] \to [p^#G_1,p^#(V),p^#(N)]\}$ is a cover in $S$, since the cover $\{p^#(V_i)\}$ of $p^#(V)$ must at least be generated from open covers in $G_0$ in $\mathcal{G}$, essentially by definition of the space $p^#(G_0)$ (cf. 1.5).

It now follows by the Comparison Lemma (SGA 4, III) that $S$ and $T$ give equivalent toposes of sheaves, and the proof of Theorem 6.7 is complete.

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6.8. THEOREM. Let \( \{G^i\}_i \) be a family of open continuous groupoids in a topos \( \mathcal{E} \). Then there is an equivalence of \( \mathcal{E} \)-toposes \( \prod_{\mathcal{E}} B(\mathcal{E}, G^i) \cong B(\mathcal{E}, \prod G^i) \) (on the left, \( \prod_{\mathcal{E}} \) is the (possibly infinite) fibered product over \( \mathcal{E} \); on the right, \( \prod \) is the obvious direct product of continuous groupoids).

PROOF. The proof is similar to that of 5.7, in fact easier. I will only say a few words about the case of a binary product \( G^1 \times G^2 \). If \( \mathcal{B}^i = (\mathcal{P}^i_0, \mathcal{P}^i_1) \) is a basis for \( G^i \) (\( i = 1, 2 \)), then \( \mathcal{B} = (\mathcal{P}^1_0 \times \mathcal{P}^2_0, \mathcal{P}^1_1 \times \mathcal{P}^2_1) \) is a basis for \( G^1 \times G^2 \) (where \( \mathcal{P}^1_1 \times \mathcal{P}^2_1 = \{B^1 \times B^2 | B^j \in \mathcal{P}^j_1 \} \)), and there is a comparison functor \( S_{\mathcal{B}^1} \times S_{\mathcal{B}^2} \rightarrow S_{\mathcal{B}} \) which is essentially the identity on objects, locally full by 6.6(a), and preserves and reflects covers because any cover of a product of spaces \( B^1 \times B^2 \) is generated from covers in each coordinate separately (by the very definition of the product of spaces).

As a corollary of 6.7 and 6.8, we obtain some stable coequalizers of spaces (see also 5.6).

6.9. COROLLARY. (1) Let \( G \) be an open groupoid over a space \( T \), and let

\[
G_1 \xrightarrow{d_0} G_0 \xrightarrow{p} \pi G
\]

be a coequalizer of spaces (over \( T \)). Then \( Y \times_T G_1 \rightrightarrows Y \times_T G_0 \rightarrow Y \times_T \pi(G) \) is again a coequalizer, for any space \( Y \) over \( T \).

(2) Let \( G \) and \( H \) be continuous groupoids over a space \( T \), and let

\[
G_1 \xrightarrow{d_0} G_0 \xrightarrow{p} \pi G \quad \text{and} \quad H_1 \xrightarrow{d_0} H_0 \xrightarrow{q} \pi H
\]

be coequalizers of spaces over \( T \). Then

\[
G_1 \times_T H_1 \rightrightarrows G_0 \times_T H_0 \rightarrow \pi(G) \times_T \pi(H)
\]

is again a coequalizer.

PROOF. \( G \) is a continuous groupoid in the topos \( \text{Sh}(T) \), and \( \pi(G) \) is the reflection of \( BG \rightarrow \text{Sh}(T) \) into spaces over \( T \) (i.e. spaces in \( \text{Sh}(T) \)). So (1) follows from 6.7 and the fact that the spatial reflection is preserved by pullback (cf. 1.6). (2) follows similarly from 6.8 and the fact that the spatial reflection commutes with products (1.6).

Just like 6.8, 6.9(2) holds for arbitrary (small) products, and not just for binary ones.

7. Toposes as a localization of continuous groupoids. The aim of this section is to obtain the category of toposes (over a given base topos \( \mathcal{S} \)) as a category of fractions from the category of continuous groupoids. The category of toposes here could mean the 2-category with geometric morphisms as 1-cells and natural isomorphisms as 2-cells. This 2-categorical version requires a calculus of fractions for 2-categories. Here, however, one looses oneself in an orgy of coherence conditions involved in pseudofunctors, pseudo-natural transformations, etc. Therefore, I will only present a version for ordinary categories, namely the category [toposes] of toposes and isomorphism classes of geometric morphisms (under natural isomorphism of inverse image functors), and a category of continuous groupoids and isomorphism classes of continuous homomorphisms, as in the following definition.
7.1. Definition. Let $G \xrightarrow{\phi} H$ be maps of continuous groupoids. A map
of (2-cell) $\alpha: \phi \Rightarrow \psi$ is a continuous map of spaces $G_0 \xrightarrow{\alpha} H_1$ such that $d_0\alpha = \phi_0$,
d$1\alpha = \psi_0$, and the diagram (1) commutes:

$$
\begin{array}{c}
G_1 \xrightarrow{(\psi, \alpha d_0)} H_1 \times_{H_0} H_1 \xrightarrow{m} H_1,
\end{array}
$$

Any 2-cell is an isomorphism; $\phi$ and $\psi: G \to H$ are isomorphic if there exists a
2-cell $\phi \Rightarrow \psi$. $\text{Hom}(G, H)$ denotes the category of maps $G \to H$ and 2-cells between
them.

7.2. Étale-complete groupoids. A continuous groupoid $G$ is étale-complete if

$$
\begin{array}{c}
\text{Sh}(G_i) \xrightarrow{p} \text{Sh}(G_0)
\end{array}
$$

is a pullback of toposes. Here $p$ is the canonical map ($p^*$ is the forgetful functor
$(E \to G, \cdot) \mapsto E$), and $\mu_G: d_1^*p^* \Rightarrow d_0^*p^*$ is the natural transformation correspond-
ing to the action of $G$ on étale-spaces.

If $x$ is a point of $G_0$, let $\text{ev}_x: BG \to \mathcal{S}$ denote the functor “evaluate at $x$”,
which takes the fiber at $x$: $\text{ev}_x(E) = E_x$. Using the Stability Theorem (6.7),
one can describe étale-completeness as follows: $G$ is étale-complete iff for any two
points $x, y \in G_0$, any natural isomorphism $\alpha: \text{ev}_x \Rightarrow \text{ev}_y$, is induced by a unique
point $y \xrightarrow{\alpha} x$ of $G_1$ (here point means: point in any base extension).

If $G$ is an arbitrary continuous groupoid, there is an étale-completion $\hat{G}$
associated to it, defined by $\hat{G}_0 = G_0$ and $\hat{G}_1$ is given by the pullback
$\text{Sh}(G_1) = \text{Sh}(G_0) \times_{BG} \text{Sh}(G_0)$. (So in point-set language, for $x, y \in G_0$ the maps $x \to y$ in
$\hat{G}_1$ are the natural isomorphisms $\text{ev}_y \Rightarrow \text{ev}_x: BG \to \mathcal{S}$.) $G$ is étale-complete iff $G$
is isomorphic to $\hat{G}$. Note that $\hat{G}$ is open if $G$ is (cf. 5.1).

There is a canonical map $G \xrightarrow{\eta} \hat{G}$ of continuous groupoids, which is universal
in the sense that for any étale-complete groupoid $H$, any $G \xrightarrow{\phi} H$ has a unique
extension $\hat{\phi}$ to $\hat{G}$

Moreover, if $G \xrightarrow{\psi} H$ is another map with unique extension $\hat{\psi}$, there is a natural
1-1 correspondence between 2-cells $\phi \Rightarrow \psi$ and 2-cells $\hat{\phi} \Rightarrow \hat{\psi}$. (More precisely, if
$\alpha: G_0 \to H_1$ is a 2-cell from $\phi$ to $\psi$, i.e. the diagram 7.1(1) commutes, then the
same $\alpha$ also makes (1) commute with $G_1$ replaced by $\hat{G}_1$). Thus, $\eta$ induces an
isomorphism of categories $\text{Hom}(\hat{G}, H) \to \text{Hom}(G, H)$.

7.3. Lemma. Let $\mathcal{F} \xrightarrow{p} \mathcal{E}$ be a geometric morphism, and let $G$ be a continuous
groupoid in $\mathcal{E}$. If $G$ is étale-complete, then so is $p^\#(G)$.

Proof. Obvious from the Definition and the Stability Theorem (6.7).
7.4. Étale-complete groups. At least in the case of a continuous group $G$ (i.e. $G_0 = 1$, $G_1 = G$), the étale completion is easy to describe explicitly. Let $\mathcal{S}$ be the system of open subgroups of $G$, partially ordered by inclusion. If $U \subset V$ in $\mathcal{S}$, there is a canonical projection map of discrete spaces (of right $G$-sets) $G/U \to G/V$. Here $G/U$ is defined as the coequalizer $R_U \rightrightarrows G \xrightarrow{q_U} G/U$, where $R_U \subset G \times G$ is given by the pullback

$$
\begin{array}{ccc}
R_U & \longrightarrow & G \times G \\
\downarrow & & \downarrow \chi y^{-1} \\
U & \longrightarrow & G
\end{array}
$$

(so this construction of $G/U$ is a special case of the construction of $G_1 \cap d_1^{-1}(V)/N$ from §6). Taking the inverse limit of this system of projections, one obtains a prodiscrete space

$$M_G = \varprojlim_{U \in \mathcal{S}} G/U.$$

We claim that $M_G$ is a monoid object in the category of spaces. The neutral element $1 \in M_G$ is defined by $\pi_U \circ e = 1 \in G \xrightarrow{q_U} G/U$, where $M_G \xrightarrow{\pi_U} G/U$ is the projection. Multiplication $\mu: M_G \times M_G \to M_G$ (also denoted by $\cdot$), is defined as follows: Write $G/U \times M_G = \bigsqcup_{K \in \mathcal{S}/U} \{K\} \times M_G$ ($G/U$ is discrete), and let $m^U: G/U \times M_G \to G/U$ be the map with $m^U|\{K\} \times M_G$ equal to the composite $\{K\} \times M_G \xrightarrow{\pi_K} G/\bar{K} \xrightarrow{\bar{l}} G/U$, where $\bar{K} = q_U^{-1}(K)^{-1} \cdot q_U(K) \subset G$, and $\bar{l}$ is the function (of discrete sets!) defined by the commutative diagram

$$
\begin{array}{ccc}
q_U^{-1}(K) \times G & \longrightarrow & G \\
\downarrow q_K \circ \pi_2 & & \downarrow \\
G/\bar{K} & \xrightarrow{\bar{l}} & G/U.
\end{array}
$$

A more intelligible definition of $\mu_G$ can be given in point-set language: points of $M_G$ “are” systems $x = (U x_U)_{U \in \mathcal{S}}$ of cosets, and $\mu$ is simply defined by

$$
(1) \quad \mu(x, y)_U = (x \cdot y)_U = U x_U y_{(x_U^{-1}U x_U)}.
$$

REMARK. It is important to note that (1) may actually be taken as a definition, by the usual techniques of change of base: We have to define a natural (in $X$) function $\mu_X: \text{Cts}(X, M_G \times M_G) \to \text{Cts}(X, M_G)$. By working in Sh($X$), it is enough to give an explicit definition for the case $X = 1$. So take two points $\bar{x}$ and $\bar{y}$ of $M_G$. These are sequences of elements of $G/U$, $U \in \mathcal{S}$. By changing the base along an open surjection (pulling back along $G \to G/U$), every point $1 \to G/U$ can be represented as an actual coset $U x_U$ for a point $x_U$ of $G$. (Going to a base extension does not affect $M_G$, since (i) the quotient $G/U$ is stable, and (ii) it is enough to consider a cofinal system of open subgroups.) By taking the filtered inverse limit over $\mathcal{S}$ of all these base extensions, we obtain another open surjection $\text{Sh}(A) \to \mathcal{S}$ (Moerdijk (1986), Theorem 5.1(ii)), such that in $\text{Sh}(A)$, $\bar{x}$ is given as a sequence of cosets $(U x_U)_{U \in \mathcal{S}}$ for points $x_U \in G$; and similarly we may choose $\bar{y}$ to be represented as $(U y_U)_{U \in \mathcal{S}}$. Then $(\bar{x} \cdot \bar{y})_U = U x_U y_{(x_U^{-1}U x_U)}$ as in (1) defines a point of $M_G$ inside $\text{Sh}(A)$, i.e. a map $A \to M_G$. We have to show that it factors
through \( \text{Sh}(A) \), i.e. a map \( A \to \mathcal{S} \) and gives an actual point \( 1 \to M_G \) in \( \mathcal{S} \). But since \( \text{Sh}(A) \to \mathcal{S} \) is an open surjection, and open surjections are coequalizers of their kernel pairs, this precisely means that the definition (1) is independent of the points \( x_U, y_U \) chosen to represent the cosets \( Ux_U, Uy_U \), which is obvious.

Using such change of base techniques, it is easy to check that \( M_G \) is a well-defined monoid, and that the map \( G \to M_G \) given by \( \pi \circ \pi = q_U : G \to G/U \), is a continuous homomorphism. Now let \( \mathcal{J}(M_G) \) be the subspace of invertible elements of \( M_G \). Then this is precisely the étale completion of \( G : \hat{G} \cong \mathcal{J}(M_G) \).

7.5. Essential equivalences. Call a map \( G \xrightarrow{\phi} H \) an essential equivalence if \( \phi \) is open, essentially surjective, and fully faithful (cf. 5.5; note that if \( \phi \) is fully faithful, \( \phi_1 : G_1 \to H_1 \) is open when \( \phi_0 : G_0 \to H_0 \) is). Let \( E \) denote the class of essential equivalences.

Clearly \( E \) is closed under composition. Moreover, if

\[
\begin{array}{ccc}
P & \xrightarrow{\mu} & H \\
\downarrow \nu & & \downarrow \phi \\
K & \xrightarrow{\psi} & G
\end{array}
\]

is a lax pullback-square of continuous groupoids and \( \phi \) is an essential equivalence, then so is \( \nu \). The laxpullback (1) is the groupoid \( P \) defined by \( P_0 = H_0 \times_{G_0} G_1 \times_{G_0} K_0 \) (i.e. \( \text{"the space of triples (y, g, z), y} \in H_0, g : \phi(y) \to \psi(z) \text{ in } G_1 \" \)), and \( P_1 \) is the equalizer

\[
P_1 \to H_1 \times_{(H_0 \times H_0)} (P_0 \times P_0) \times_{(K_0 \times K_0)} K_1 \xrightarrow{m(\psi_1, \pi_1)} G_1
\]

(i.e. \( \text{"the maps (y, g, z) } \to (y', g', z') \text{ in } P \text{ are pairs } y \xrightarrow{h} y', z \xrightarrow{k} z' \text{ such that } \psi(k) \circ g = g' \circ \phi(h) \" \)). Finally, observe that if \( G \xrightarrow{\phi} H \) are maps of continuous groupoids, and \( H \xrightarrow{\epsilon} K \) is an essential equivalence, then any 2-cell \( \alpha : \epsilon \phi \Rightarrow \epsilon \psi \) (cf. 7.2) factors through \( \epsilon \) — i.e. there is a 2-cell \( \beta : \phi \Rightarrow \psi \) with \( \epsilon \cdot \beta = \alpha \).

Let \( CG \) be the category of open (5.1) continuous groupoids and isomorphism classes of maps (cf. 7.2). Let \( E \) also stand for the family of morphisms in \( CG \) which come from essential equivalences. The properties of \( E \) as just pointed out show that \( E \subset CG \) admits a calculus of right fractions (see Gabriel and Zisman (1967)).

Let \( ECG \) be the full subcategory of \( CG \) whose objects are étale-complete continuous groupoids. Clearly, \( E \) also admits a calculus of right fractions on \( ECG \) (by 7.3, the inclusion \( ECG \hookrightarrow CG \) has a left adjoint).

7.6. Lemma. Let \( C \xrightarrow{F} D \) be a functor, and \( \Sigma \subset C \) a class of morphisms admitting a right calculus of fractions. Suppose

(i) \( F \) is surjective on objects, and faithful,

(ii) \( F \) sends morphisms from \( \Sigma \) to isomorphisms,
(iii) for any map FC \xrightarrow{q} FC' there is a commutative diagram

\[
\begin{array}{ccc}
FC & \xrightarrow{q} & FC' \\
\uparrow^{F\sigma} & & \nearrow^{Ff} \\
FC_0 & & \\
\end{array}
\]

with \(\sigma \in \Sigma\).

Then the functor \(C[\Sigma^{-1}] \to D\) induced by \(F\) (by (ii)) is an equivalence of categories.

**Proof.** Since \(\Sigma\) admits a calculus of right fractions, we can explicitly construct \(C[\Sigma^{-1}]\) as in Gabriel and Zisman (1967). \(F\) then induces a functor \(\mathcal{C}[\Sigma^{-1}] \to \mathcal{D}\) and the conditions of the lemma simply state that \(F'\) is surjective on objects, and fully faithful.

Note that "\(F\) is faithful" can be replaced by the weaker condition that whenever \(f, g\) are parallel arrows in \(C\) with \(Ff = Fg\), then there is a \(\sigma \in \Sigma\) such that \(f\sigma = g\sigma\).

7.7. **Theorem.** The functor \((\text{continuous groupoids}) \xrightarrow{B} (\text{toposes})\) induces an equivalence of categories \(ECG[\Sigma^{-1}] \xrightarrow{\sim} [\text{toposes}]\).

**Proof.** The descent theorem of Joyal and Tierney (Joyal-Tierney (1984); see also Moerdijk (1985)) implies that the restriction of \(B\) to étale complete groupoids is essentially surjective. Moreover, \(B\) restricts to a faithful functor \(ECG \to [\text{toposes}]\). In fact, if \(G \xrightarrow{\phi} H\) are maps of continuous groupoids and \(H\) is étale complete, then any 2-isomorphism \((B\phi)^* \xrightarrow{\alpha} (B\psi)^* : BH \to BG\) comes from a 2-cell \(\beta : \psi \Rightarrow \phi\) as in 7.2, as is immediate from étale completeness. By 7.6 it therefore suffices to show that for any geometric morphism \(BG \xrightarrow{f} BH\) there exists a diagram \(G \xrightarrow{\epsilon} K \xrightarrow{\phi} H\) of continuous groupoids such that \(\epsilon\) is an essential equivalence, and

\[
\begin{array}{ccc}
BG & \xrightarrow{f} & BH \\
\downarrow^{B\epsilon} & & \downarrow^{B\phi} \\
BK & & \\
\end{array}
\]

commutes (up to natural isomorphism). Construct the pullback square

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\phi_0} & Sh(H_0) \\
\downarrow^{\epsilon_0} & & \downarrow^{q} \\
Sh(G_0) & \xrightarrow{P} & BG & \xrightarrow{f} & BH \\
\end{array}
\]

of toposes. \(\mathcal{F}\) must be of the form \(Sh(K_0)\) for a unique space \(K_0\). The points of \(K_0\) can be thought of as triples \((x, \alpha, y)\) where \(x \in G_0, y \in H_0\) are points, and \(\alpha : ev_y \xrightarrow{\sim} ev_x \circ f^*\) is a natural isomorphism. Notice that \(\epsilon_0 : K_0 \to G_0\) is an open surjection since \(Sh(H_0) \to BH\) is.

Write \(\alpha : q\phi_0 \xrightarrow{\sim} f\epsilon_0\) for the 2-isomorphism up to which (2) commutes.
The space $K_1$ is similarly defined as the pullback of toposes

\[
\begin{array}{ccc}
\text{Sh}(K_1) & \overset{\phi_1}{\longrightarrow} & \text{Sh}(H_1) \\
\downarrow^\epsilon_1 & & \downarrow^{q_0} \\
\text{Sh}(G_1) & \overset{p_0}{\longrightarrow} & BG \\
\end{array}
\]

with a natural isomorphism $\beta: q_0 \pi_H \cong f p_0 \pi_G$. So the points of $K_1$ can be thought of as triples $(g, h, \beta)$, where $x \xleftarrow{\partial} x'$ is a point of $G_1$, $y \xrightarrow{h} y'$ a point of $H_1$, and $\beta: \text{ev}_y \to \text{ev}_x f^*$ is a natural isomorphism.

There are canonical maps $K_1 \xrightarrow{d_0} K_0$ resulting from the universal properties of (2) and (3). In fact, $d_0$ and $d_1$ are most easily described by using test spaces and stability (6.7): Given a map $T \xrightarrow{t} K_1$, this can be thought of (inside $\text{Sh}(T)$) as a triple $t = (g, h, \beta)$ of points as above, and $d_0 t = (x, y, \beta)$ defines a point of $K_0$ in $\text{Sh}(T)$, i.e. a map $T \xrightarrow{d_0} K_0$. Similarly, $d_1 (g, h, \beta) = (x', y', g^{-1} \circ \beta \circ h)$. In other words, if we allow change of base, then given two points $(x, y, \alpha)$, $(x', y', \alpha')$ of $K_0$, the maps $(x, y, \alpha) \to (x', y', \alpha')$ in $K_1$ are pairs $(g, h)$ such that $x \xleftarrow{\partial} x' \in G$, $y \xrightarrow{h} y' \in H$, and the diagram of natural transformations

\[
\begin{array}{ccc}
\text{ev}_y & \overset{\alpha}{\longrightarrow} & \text{ev}_x f^* \\
\uparrow^h & & \uparrow^g \\
\text{ev}_{y'} & \xrightarrow{\alpha'} & \text{ev}_{x'} f^* \\
\end{array}
\]

commutes. $K$ is a groupoid in the obvious way, and $\epsilon_1, \phi_1$ give maps of continuous groupoids $K \xrightarrow{\epsilon} G$, $K \xrightarrow{\phi} H$.

If $G$ is étale complete, it easily follows from the construction that

\[
\begin{array}{ccc}
K_1 & \xrightarrow{(d_0,d_1)} & K_0 \times K_0 \\
\downarrow^{\epsilon_1} & & \downarrow^{\epsilon_0 \times \epsilon_0} \\
G_1 & \xrightarrow{(d_0,d_1)} & G_0 \times G_0
\end{array}
\]

is a pullback. Thus $K \xrightarrow{\epsilon} G$ is an essential equivalence.

Notice that $K$ is indeed an object of $ECG$ if $G$ and $H$ are. First of all, the pullback (3) can be constructed stepwise

\[
\begin{array}{ccc}
K_1 = G_1 \times_{G_0} K_0 \times_{H_0} H_1 & \longrightarrow & K_0 \times_{H_0} H_1 \\
\downarrow & & \downarrow \\
K_0 & \longrightarrow & H_1
\end{array}
\]

and all the vertical maps are open surjections. It follows that $d_0$ and $d_1: K_1 \xrightarrow{\epsilon} K_0$ are open maps. Moreover, since $K \xrightarrow{\epsilon} G$ is fully faithful and $G$ is étale complete, so is $K$. 
Finally, it remains to show that the diagram

\[
\begin{array}{ccc}
BK & \xrightarrow{B\phi} & BH \\
\downarrow{Be} & & \downarrow{f} \\
BG & \xrightarrow{\beta} & BH
\end{array}
\]

commutes, up to isomorphism. Write \( \text{Sh}(K_0) \xrightarrow{\iota} BK \) for the canonical geometric morphism. Then \( \alpha : q\phi_0 = f\pi_0 \) gives a natural isomorphism \( \nu : B\phi \circ \alpha \xrightarrow{\simeq} f \circ B\pi \circ \alpha \). For \( E \in BH \), the component \( \nu_E \) is a priori just a map of spaces \( \phi^*(E) \rightarrow e^*f^*(E) \) over \( K_0 \). However, it easily follows from the construction of \( K_1 \) (from the naturality of \( \beta \)) that it is in fact a map of \( K \)-spaces, i.e. \( \nu_H \cong t^*\mu_H \) for some (unique) \( \mu_H \).

This completes the proof of 7.7.

There are several variants, such as the following analogue of 7.7 for open \( \mathcal{S} \)-toposes and open maps.

7.8. Open toposes. Let \( \mathcal{O} \)-toposes denote the category of open \( \mathcal{S} \)-toposes and open geometric morphisms (over \( \mathcal{S} \)), and \( \mathcal{O} \)-toposes the corresponding category with isomorphism classes of open geometric morphisms as maps. Let \( \mathcal{O} \)-ECG be the subcategory of ECG given by open maps of continuous groupoids (5.5). It can be shown that \( B : \mathcal{O} \)-ECG \( \rightarrow \mathcal{O} \)-toposes induces an equivalence \( \mathcal{O} \)-ECG \( \left\{ E \right\} \cong \mathcal{O} \)-toposes. The proof is completely analogous to that of 7.7.

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