REALIZING ROTATION VECTORS FOR TORUS HOMEOMORPHISMS

BY JOHN FRANKS

ABSTRACT. We consider the rotation set $\rho(F)$ for a lift $F$ of a homeomorphism $f: T^2 \to T^2$, which is homotopic to the identity. Our main result is that if a vector $v$ lies in the interior of $\rho(F)$ and has both coordinates rational, then there is a periodic point $x \in T^2$ with the property that

$$\frac{F^q(x_0) - x_0}{q} = v$$

where $x_0 \in R^2$ is any lift of $x$ and $q$ is the least period of $x$.

In this article we consider the rotation set $\rho(F)$ as defined in [MZ], for a lift $F$ of a homeomorphism $f: T^2 \to T^2$, which is homotopic to the identity. Our main result is that if a vector $v$ lies in the interior of $\rho(F)$ and has both coordinates rational, then there is a periodic point $x \in T^2$ with the property that

$$\frac{F^q(x_0) - x_0}{q} = v$$

where $x_0 \in R^2$ is any lift of $x$ and $q$ is the least period of $x$. This should be compared with the well-known fact that if a homeomorphism of the circle has rational rotation number $p/q$ then it has a periodic point (with rotation number $p/q$).

R. MacKay and J. Llibre [ML] have proved a similar result using the ideas our Proposition (2.4) below. They require the stronger hypothesis that $v$ is in the interior of the convex hull of vectors in $\rho(F)$ which represent periodic orbits of $f$.

1. BACKGROUND AND DEFINITIONS

Suppose $f: T^2 \to T^2$ is a homeomorphism homotopic to the identity map, and let $F: R^2 \to R^2$ be a lift.

(1.1) Definition. Let $\rho(F)$ denote the set of accumulation points of the subset of $R^2$

$$\left\{ \frac{F^n(x) - x}{n} \mid x \in R^2, n \in Z^+ \right\},$$

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thus \( \nu \in \rho(F) \) if there are sequences \( x_i \in \mathbb{R}^2 \) and \( n_i \in \mathbb{Z}^+ \) with \( \lim n_i = \infty \) such that
\[
\lim_{i \to \infty} \frac{F^{n_i}(x_i) - x_i}{n_i} = \nu.
\]
In [MZ] the rotation set is defined for a map homotopic to the identity (rather than a homeomorphism) \( f : T^n \to T^n \). However, we shall be concerned only with homeomorphisms of \( T^2 \). In [MZ] it is shown that for homeomorphisms of \( T^2 \), \( \rho(F) \) is convex.

We now briefly review the elementary theory of attractor-repeller pairs and chain recurrence developed by Charles Conley in [C]. In the following \( f : X \to X \) will denote a homeomorphism of a compact metric space \( X \).

**Definition.** An \( \varepsilon \)-chain for \( f \) is a sequence \( x_1, x_2, \ldots, x_n \) of points in \( X \) such that
\[
d(f(x_i), x_{i+1}) < \varepsilon \quad \text{for } 1 \leq i \leq n - 1.
\]
If \( x_i = x_n \) it is called a periodic \( \varepsilon \)-chain.

A point \( x \in X \) is called chain recurrent if for every \( \varepsilon > 0 \) there is an \( n \) (depending on \( \varepsilon \) ) and an \( \varepsilon \)-chain \( x_1, x_2, \ldots, x_n \) with \( x_1 = x_n = x \). The set \( R \) of chain recurrent points is called the chain recurrent set of \( f \).

It is easily seen that \( R \) is compact and invariant under \( f \).

If \( A \subset X \) is a compact subset and there is an open neighborhood \( U \) of \( A \) such that \( f(cl(U)) \subset U \) and \( \bigcap_{n \geq 0} f^n(cl(U)) = A \), then \( A \) is called an attractor and \( U \) is its isolating neighborhood. It is easy to see that if \( V = X - cl(U) \) and \( A^* = \bigcap_{n \geq 0} f^{-n}(cl(V)) \), then \( A^* \) is an attractor for \( f^{-1} \) with isolating neighborhood \( V \). The set \( A^* \) is called the repeller dual to \( A \). It is clear that \( A^* \) is independent of the choice of isolating neighborhood \( U \) for \( A \). Obviously \( f(A) = A \) and \( f(A^*) = A^* \).

If we define a relation \( \sim \) on \( R \) by \( x \sim y \) if for every \( \varepsilon > 0 \) there is an \( \varepsilon \)-chain from \( x \) to \( y \) and another from \( y \) to \( x \), then it is clear that \( \sim \) is an equivalence relation.

The equivalence classes in \( R(f) \) for the equivalence relation \( \sim \) above are called the chain transitive components of \( R(f) \).

**Definition.** A complete Lyapounov function for \( f : X \to X \) is a continuous function \( g : X \to \mathbb{R} \) satisfying:

1. If \( x \notin R(f) \), then \( g(f(x)) < g(x) \).
2. If \( x, y \in R(f) \), then \( g(x) = g(y) \) iff \( x \sim y \) (i.e., \( x \) and \( y \) are in the same chain transitive component.
3. \( g(R(f)) \) is a compact nowhere dense subset of \( \mathbb{R} \).

By analogy with the smooth setting, elements of \( g(R(f)) \) are called critical values of \( g \).

A theorem of C. Conley [C] asserts that a complete Lyapounov function exists for any flow or homeomorphism of a compact space. The proof in [C] is given for flows; for an exposition in the case of homeomorphisms see [F2].
In general the number of chain transitive components for a homeomorphism can be infinite (even uncountable). However, if we specify a fixed $\delta > 0$ and work with $\delta$-chains we can decompose $R(f)$ into a finite number of pieces.

(1.4) Definition. For a fixed $\delta > 0$ we say that $x, y \in R(f)$ are $\delta$-equivalent if there is a $\delta$-chain from $x$ to $y$ and one from $y$ to $x$. This is an equivalence relation and the equivalence classes will be called $\delta$-transitive components of $R(f)$. We will say a compact $f$-invariant set $\Lambda \subset R(f)$ is $\delta$-transitive if for every $x, y \in \Lambda$, $x$ is $\delta$-equivalent to $y$.

(1.5) Lemma. Given $\delta > 0$ and a homeomorphism $f : X \to X$ of a compact space, then there are finitely many $\delta$-transitive components.

Proof. A $\delta$-transitive component is a union of chain transitive components. Two chain transitive components which are in different $\delta$-transitive components must be at least distance $\delta$ apart. Hence if there were infinitely many $\delta$-transitive components, there would be infinitely many subsets each at least distance $\delta$ from the others. This is impossible since $X$ is compact.

(1.6) Theorem. Given $\delta > 0$ and a homeomorphism of a compact space $f : X \to X$, there is a complete Lyapounov function $g : X \to \mathbb{R}$ for $f$, and regular values for $g$, $c_0 < c_1 < c_2 < \cdots < c_n$ such that if $\Lambda_i = R(f) \cap g^{-1}([c_{i-1}, c_i])$, then $\{\Lambda_i\}$, $1 \leq i \leq n$, are the $\delta$-transitive components of $f$.

Proof. Let $\Lambda_1, \ldots, \Lambda_n$ be the $\delta$-transitive components for $f$. We order them in such a way that if $i < j$ there is no $\delta$-chain from $\Lambda_i$ to $\Lambda_j$. This is possible since there can be no "cycle" of $\Lambda_i$'s with each one having a $\delta$-chain to the next and the last having a $\delta$-chain to the first.

Let $U_i$ denote the set of all $z \in X$ such that there is a $\delta$-chain from $\Lambda_i$ to $z$. $U_i$ is an open set. Moreover, $f(\text{cl}(U_i)) \subset U_i$, because if $z \in \text{cl}(U_i)$, there is $z_0 \in U_i$ such that $d(f(z), f(z_0)) < \delta$ and consequently a $\delta$-chain from $x$ to $z_0$ gives a $\delta$-chain $x = x_1, x_2, \ldots, x_k, z_0, f(z)$ from $x$ to $f(z)$.

Thus if $A_i = \bigcap_{n \geq 0} f^n(\text{cl}(U_i))$ and $A_i^* = \bigcap_{n \geq 0} f^{-n}(X-U_i)$, then $A_i, A_i^*$ are an attractor repeller pair and $\Lambda_i \subset A_i$. A result of Conley (see Lemma (1.7) of [F2] for a proof) asserts there is a continuous function $g_i : X \to [0, 1]$ such that $A_i = g_i^{-1}(0)$, $A_i^* = g_i^{-1}(1)$ and $g_i(f(x)) < g_i(x)$ for all $x \in X-(A_i \cup A_i^*)$. If $i < j$, then $\Lambda_j \subset A_j^*$ so $g_i(\Lambda_j) = \{1\}$.

Let $h(x) = \sum_{i=1}^{n} 2^i g_i(x)$ and note that $h(f(x)) \leq h(x)$ for all $x \in X$. For $x \in R(f) = \bigcup \Lambda_i$, $h(x)$ is an even integer between 0 and $2^n+1$. Also note if $x, y \in R(f)$, then $h(x) = h(y)$ if and only if $g_i(x) = g_i(y)$ for all $i$. Hence if $x \in \Lambda_i$, $y \in \Lambda_j$, $i < j$, then $h(x) \neq h(y)$ since $g_i(x) \neq g_i(y)$. Now if $g_0 : X \to [0, 1]$ is a complete Lyapounov function, then $g(x) = g_0(x) + h(x)$ is the desired function.
2. THE $\delta$-TRANSITIVE CASE

We begin with a sequence of results leading to our main theorem. Assume throughout that $f : T^2 \to T^2$ is a homeomorphism homotopic to the identity and $F : R^2 \to R^2$ is a lift, i.e., if $\pi : R^2 \to T^2$ is the covering projection then $\pi \circ F = f \circ \pi$.

(2.1) **Lemma.** If $F$ has no fixed points, then there is an $\varepsilon > 0$ such that no periodic $\varepsilon$-chain for $F$ exists.

**Proof.** This result and its proof are quite similar to (2.1) of [F1] and (2.2) of [F2]. Let

$$\delta = \min_{x \in R^2} |F(x) - x|.$$  

Note this minimum is assumed since it suffices to consider only $x$ in a compact fundamental domain for $\pi$. Hence $\delta > 0$.

A result of Oxtoby [Ox] says that there is a $\gamma > 0$ such that for any finite set of pairs $\{(x_i, y_i)\}$ of elements in $R^2$ with $\|x_i - y_i\| < \gamma$ there is a pairwise disjoint set of piecewise linear arcs $\alpha_i$ from $x_i$ to $y_i$ with the diameter of each $< \delta$.

Let $\varepsilon = \gamma$; we will show there is no periodic $\varepsilon$-chain for $F$. Suppose to the contrary that $z_1 = z, z_2, z_3, \ldots, z_n = z$ is a periodic $\varepsilon$-chain. Letting $y_i = z_i, x_i = F(z_{i-1})$, we see that there are pairwise disjoint arcs $\alpha_i$ from $F(z_{i-1})$ to $z_i$, with diameter $< \delta$. By isotoping in a neighborhood of these arcs we can produce a perturbation $G$ of $F$ satisfying

\begin{enumerate}
  \item $\|F(x) - G(x)\| < \delta$ for all $x \in R^2$, and
  \item $G(z_{i-1}) = z_i$.
\end{enumerate}

Now $G$ has a periodic point, namely $z$. Hence by results of [Br or Fa] $G$ has a fixed point $p$. Thus $\|F(p) - p\| \leq \|F(p) - G(p)\| + \|G(p) - p\| < \delta$ which is a contradiction. $\square$

(2.2) **Lemma.** Suppose $\Lambda$ is a $\delta$-transitive compact invariant subset of $R(f)$ for a homeomorphism $f : T^2 \to T^2$ and $F$ is a lift of $f$. There is a constant $K > 0$, such that for any $x_0, y_0 \in \Lambda$, $x \in \pi^{-1}(x_0)$ there is a $\delta$-chain for $F$ from $x$ to a point $y \in \pi^{-1}(y_0)$ with $\|y - x\| < K$.

**Proof.** Fix $\omega \in \pi^{-1}(\Lambda)$ and let $Q_n$ denote the set of $z \in \Lambda$ such that there is a $\delta$-chain for $f$ from $\pi(\omega)$ to $z$ of length less than $n$. $Q_n$ is open by definition and $\Lambda = \bigcup_{n \geq 1} Q_n$ so compactness of $\Lambda$ implies $Q_N = \Lambda$ for some $N > 0$. Hence given $y_0 \in \Lambda$ there is a $\delta$-chain from $\pi(\omega)$ to $y_0$ of length less than $N$. Lifting this to $R^2$, starting at $w$, we obtain a $\delta$-chain from $w$ to some $y' \in \pi^{-1}(y_0)$. If $P = \sup \|F(\nu) - \nu\|$, then since this $\delta$-chain from $w$ to $y'$ has length less than $N$, it follows that $\|y' - w\| < C_1 = N(P + \delta)$.

A similar argument shows that given $x_0 \in \Lambda$ there is an $x' \in \pi^{-1}(x_0)$ with a $\delta$-chain from $x'$ to $w$ and $\|x' - w\| < C_2$ for some constant $C_2$ independent of
Piecing these together we obtain a δ-chain from \( x' \) to \( y' \) with \( \| y' - x' \| < K = C_1 + C_2 \). Now given any \( x \in \pi^{-1}(x_0) \) translate this δ-chain by the integer vector \( x - x' \) to obtain a δ-chain from \( x \) to \( y \), where \( y = y' + (x - x') \) satisfies \( \pi(y) = y_0 \) and \( \|y - x\| = \|y' - x'\| < K \). □

\[(2.3) \text{Definition. If } \Lambda \subset T^2 \text{ is a compact invariant set for } f: T^2 \to T^2, \text{ and } F \text{ is a lift of } f, \text{ we denote by } \rho(f, \Lambda), \text{ the accumulation points of the set} \]

\[
\left\{ \frac{F^n(x) - x}{n} \mid \pi(x) \in \Lambda \text{ and } n > 0 \right\}.
\]

\[(2.4) \text{Proposition. Suppose } \Lambda \subset T^2 \text{ is a compact invariant subset of } \mathcal{R}(f) \text{ for } f: T^2 \to T^2 \text{ and for some } \delta > 0, \Lambda \text{ is } \delta\text{-transitive. If } 0 \text{ is in the interior of the convex hull of } \rho(F, \Lambda), \text{ then there is a periodic } \delta\text{-chain for } F.\]

\[\text{Proof. The hypothesis guarantees that there are vectors } \nu_1, \nu_2, \nu_3, \nu_4 \in \rho(F, \Lambda) \text{ such that } 0 \text{ is in the interior of their convex hull (see Steinitz's theorem in [HDK]). Choose neighborhoods } U_i \text{ of } \nu_i \text{ in } R^2 \text{ so small that whenever } \nu_i' \in U_i, 0 \text{ is also in the interior of the convex hull of } \nu_1', \nu_2', \nu_3' \text{ and } \nu_4'. \text{ Fix } z_0 \in \Lambda \text{ and } z \in \pi^{-1}(z_0). \text{ Now by (2.2) and the fact that } \nu_1 \in \rho(F, \Lambda) \text{ we can find } x_i \in R^2 \text{ and } n_i > i \text{ such that} \]

\begin{enumerate}
    \item \( \lim_{n \to \infty} \frac{F^n(x_i) - x_i}{n_i} = \nu_1. \)
    \item There is a δ-chain from \( z \) to \( x_i \) and \( \|x_i - z\| < K. \)
    \item There is a δ-chain from \( F^n(x_i) \) to \( z_i' \in \pi^{-1}(z_0) \) and \( \|F^n(x_i) - z_i'\| < K. \)
\end{enumerate}

Notice that piecing together the δ-chain from \( z \) to \( x_i \), the orbit segment from \( x_i \) to \( F^n(x_i) \) and the δ-chain from \( F^n(x_i) \) to \( z_i' \) we obtain a δ-chain from \( z \) to \( z_i' \). Also (1), (2), and (3) imply

\[\lim_{n \to \infty} \frac{z_i' - z}{n_i} = \nu_1.\]

Choose \( i \) sufficiently large that

\[\frac{z_i' - z}{n_i} \in U_1\]

and set \( w_1 = z_i' - z, m_1 = n_i \) so that there is a δ-chain from \( z \) to \( z + w_1 \) and \( w_1/m_1 \in U_1 \). Note that \( \pi(z_i') = \pi(z) = z_0 \) implies \( w_1 \) is an integer vector.

Now in a similar fashion construct \( w_2, m_2, w_3, m_3, \) and \( w_4, m_4 \), with the analogous properties.

Since \( 0 \) is in the convex hull of \( w_1/m_1, w_2/m_2, w_3/m_3, w_4/m_4 \) and the vectors \( w_1, w_2, w_3, w_4 \) are integers, it is possible to solve

\[Aw_1 + Bw_2 + Cw_3 + Dw_4 = 0\]
for positive integers $A$, $B$, $C$, $D$. Any translate of a $\delta$-chain by an integer vector is another $\delta$-chain. Hence piecing together $A$ translates of the $\delta$-chain from $z$ to $z + w$, with $B$ translates of the $\delta$-chain from $z$ to $z + w_2$, $C$ translates of the $\delta$-chain from $z$ to $z + w_3$, etc., we obtain a $\delta$-chain from $z$ to $z + Aw_1 + Bw_2 + Cw_3 + Dw_4 = z$ as desired. □

3. THE GENERAL CASE

As before we assume $f : T^2 \to T^2$ is a homeomorphism and $F : R^2 \to R^2$ is a lift.

(3.1) Proposition. Suppose $\nu_1$, $\nu_2$, $\nu_3$ and $\nu_4$ are extreme points of the convex set $\rho(F)$ and 0 is in the interior of their convex hull. Then $F$ possesses a fixed point.

Proof. In [MZ] it is shown that since $\nu_i$ is an extreme point of $\rho(F)$ there is an ergodic Borel measure realizing $\nu_i$ and hence a nonwandering point $x_i \in T^2$ such that if $x \in \pi^{-1}(x_i)$

\[
\lim_{n \to \infty} \frac{F^n(x) - x}{n} = \nu_i.
\]

We will need only the fact that such an $x_i$ exists with $x_i \in R(f)$.

To show that $F$ has a fixed point it suffices by (2.1) to show that for every $\delta > 0$ there is a periodic $\delta$-chain for $F$. Given $\delta > 0$, let $R(f) = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_m$ be a decomposition of the chain recurrent set into $\delta$-transitive pieces as given in (1.6) and let $g : T^2 \to R$ be a complete Lyapounov function compatible with this decomposition. We will show that there exists a piece $\Lambda_i$ of this decomposition and points $y_i \in \Lambda_i$, $i = 1, 2, 3, 4$, such that whenever $y \in \pi^{-1}(y_i)$,

\[
\nu_i = \lim_{n \to \infty} \frac{F^n(y) - y}{n}.
\]

It then follows by (2.4) that $F$ has a $\delta$-chain. Since this holds for all $\delta > 0$ we conclude by (2.1) that $F$ has a fixed point.

Choose a smooth approximation $g_0 : T^2 \to R$ to $g$ and regular values $c_1, c_2, \ldots, c_m$ such that the manifolds with boundary $M_i = g_0^{-1}((-\infty, c_i])$ satisfy

1. $f(M_i) \subset \text{int } M_i$, and
2. $\Lambda_i \subset M_i - M_{i-1}$.

Let $N_i$ be the manifold $\text{cl}(M_i - M_{i-1})$, so $T^2 = \bigcup N_i$ and $N_i \cap N_k$ consists of a finite set of circles if $k = i \pm 1$ and otherwise is empty if $i \neq k$.

These circles are the components of $g_0^{-1}([c_1, c_2, \ldots, c_m])$. We first observe that none of these circles is essential in $T^2$. If there were such a circle, say
\( \gamma \), then it would be in the boundary of \( M_j \) for some \( j \) and \( M_j \) would have to have another boundary component which is isotopic to \( \gamma \). (There might also be some inessential circles in the boundary of \( M_j \).) It follows that \( M_j \) is an essential annulus (perhaps with some disks removed) in \( T^2 \). Let \( \tilde{M}_j \) be a component of \( \pi^{-1}(M_j) \) and choose a lift \( F_0 \) of \( f \) so that \( F_0(\tilde{M}_j) \subset \tilde{M}_j \). Now \( \tilde{M}_j \) is an infinite strip (perhaps with holes) which has a rational slope. It follows since \( F_0(\tilde{M}_j) \subset \tilde{M}_j \) that for any \( x \in R^2 \), if \( \lim_{n \to \infty} \frac{F^n(x) - x}{n} \) exists, then it must lie on a line with this slope, since \( F_0^n(x) \) is constrained between parallel translates of \( \tilde{M}_j \). From this and the fact that \( F(x) = F_0(x) + w \) for some integer vector \( w \), it follows that the convex hull of the vectors \( \nu_i \) given in our hypothesis is a line segment. This contradicts the assumption that 0 is in the interior of the convex hull; so none of the boundary components of the \( N_i \) can be essential in \( T^2 \).

Since each of these boundary circles is inessential, each of them bounds a unique smooth disk in \( T^2 \). The complement of the union of these disks consists of the interior of a single one of the \( N_i \)'s, say \( N_j \). The complement of \( \text{int}(N_j) \) in \( T^2 \) consists of a finite set of disks, say \( D_1, D_2, \ldots, D_r \). Number these disks so that

\[
D_i \subset M_j \quad \text{for} \quad 1 \leq i \leq s
\]

and

\[
D_i \subset \text{cl}(T^2 - M_j) \quad \text{for} \quad s < i \leq r.
\]

Then

\[
f(D_i) \subset \bigcup_{k=1}^{s} D_k \quad \text{for} \quad 1 \leq i \leq s
\]

and

\[
f^{-1}(D_i) \subset \bigcup_{k=s+1}^{r} D_k \quad \text{if} \quad s < i \leq r.
\]

Consider now a point \( x \in \pi^{-1}(x_1) \) such that

\[
\nu_1 = \lim_{n \to \infty} \frac{F^n(x) - x}{n}.
\]

We will show that if \( x_1 \) is not in \( \Lambda_j \), there is another point \( y_1 \in \Lambda_j \) so that whenever \( y \in \pi^{-1}(y_1) \),

\[
\nu_1 = \lim_{n \to \infty} \frac{F^n(y) - y}{n}.
\]

Since the same is true for \( \nu_2, \nu_3, \) and \( \nu_4 \), we will have completed the proof by the remarks above.

Suppose now that \( x_1 \in D_p \) for \( 1 \leq p \leq s \). There exists \( q > 0 \) such that \( f^q(D_p) \subset D_p \) (recall that \( x_1 \) is recurrent). Hence if \( D \subset R^2 \) is the lift of
If \( x \in D_p \) and \( s < p \), then a similar argument applied to \( f^{-1} \) leads to a fixed point \( z_0 \) of \( G \) with the same properties.

We want to find a fixed point for \( G \) which is in \( \pi^{-1}(N_j) \). To do this we consider fixed points of \( f^q \) on \( T^2 \). We will use the fact that \( f^q \) is homotopic to a map with no fixed points so the index sum of the set of fixed points in any Nielsen class for \( f^q \) is zero (see [B, Theorem 3, p. 94]). Recall that two fixed points \( p_1 \) and \( p_2 \) are in the same Nielsen class for \( f^q \) provided any lift of \( f^q \) to \( R^2 \) which pointwise fixes \( \pi^{-1}(p_1) \) also pointwise fixes \( \pi^{-1}(p_2) \).

We will consider points in the Nielsen class of the point \( \pi(z_0) \) where \( z_0 \) is the fixed point of \( G \) mentioned above. Any such points which are not in \( N_j \) will lie in a \( D_i \) with a lift \( \tilde{D}_i \) for which \( G(\tilde{D}_i) \subset \tilde{D}_i \) or with \( G^{-1}(\tilde{D}_i) \subset \tilde{D}_i \). Hence the contribution to the index of the points in \( D_i \) will be +1. Thus the index of the set of fixed points in the Nielsen class of \( \pi(z_0) \) which are not in \( N_j \) is positive (the disk \( D_p \) contributes at least one +1). It follows there must be a fixed point \( y_1 \in N_j \) of \( f^q \) in the Nielsen class of \( \pi(z_0) \). Since \( y_1 \) is in the Nielsen class of \( \pi(z_0) \), if \( y \in \pi^{-1}(y_1) \), then \( G(y) = y \). Hence

\[
\nu_1 = \lim_{n \to \infty} \frac{F^n(y) - y}{n}.
\]

Also \( y_1 \) is a periodic point of \( f \) in \( N_j \) so \( y_1 \in \Lambda_j \). The same argument implies the existence of \( y_2, y_3, y_4 \in \Lambda_j \), so this completes the proof.  

(3.2) Theorem. Suppose \( f : T^2 \to T^2 \) is a homeomorphism homotopic to the identity and \( F : R^2 \to R^2 \) is a lift. If \( \nu \) is a vector with rational coordinates in the interior of \( \rho(F) \), then there is a point \( p \in R^2 \) such that \( \pi(p) \in T^2 \) is a periodic point for \( f \) and

\[
\nu = \lim_{n \to \infty} \frac{F^n(p) - p}{n}.
\]

Proof. Suppose \( \nu = (r/q, s/q) \) with the greatest common divisor of \( r, s \), and \( q \) equal to 1. If \( G(x) = F^q(x) - (r, s) \), then a fixed point \( p \) of \( G \) will satisfy \( F^q(p) = p + (r, s) \) and hence be the desired point.

It is easy to check (see [MZ]) that \( \rho(G) = q\rho(F) - (r, s) \). Thus since \( (r/q, s/q) \) is in the interior of \( \rho(F) \), it follows that 0 is in the interior of \( \rho(G) \). Since \( \rho(G) \) is closed and convex there exist extreme points \( \nu_1, \nu_2, \nu_3, \nu_4 \in \rho(G) \) such that 0 is in their convex hull (see Steinitz's theorem in [HDK]). It now follows from (3.1) that \( G \) possesses a fixed point \( p \).  

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Department of Mathematics, Northwestern University, Evanston, Illinois 60201