REGULAR COVERINGS OF HOMOLOGY 3 SPHERES
BY HOMOLOGY 3 SPHERES

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ABSTRACT. We study 3-manifolds that are homology 3-spheres and which admit nontrivial regular coverings by homology 3-spheres. Our main theorem establishes a relationship between such coverings and the canonical covering of the 3-sphere \( S^3 \) onto the dodecahedral space \( D^3 \). We also give methods for constructing irreducible sufficiently large homology 3-spheres \( \tilde{M}, M \) together with a degree 1 map \( h: M \to D^3 \) such that \( \tilde{M} \) is the covering space of \( M \) induced from the universal covering \( S^3 \to D^3 \) by means of the degree 1 map \( h: M \to D^3 \). Finally, we show that if \( p: \tilde{M} \to M \) is a nontrivial regular covering and \( \tilde{M}, M \) are homology spheres with \( M \) Seifert fibered, then \( \tilde{M} = S^3 \) and \( M = D^3 \).

1. INTRODUCTION

The dodecahedral space \( D^3 \) is the only known irreducible 3-manifold with finite (nontrivial) fundamental group, that is also a homology 3-sphere. It is covered by the 3-sphere. The fundamental group \( \pi_1(D^3) \) of \( D^3 \) is the binary icosahedral group, denoted by \( I^* \).

In this paper we investigate those 3-manifolds that are homology 3-spheres and which admit a nontrivial regular covering by a homology 3-sphere. Our main result is the following.

**Main Theorem.** Let \( M, \tilde{M} \) be homology 3-spheres and \( p: \tilde{M} \to M \) a nontrivial regular covering. Then the following hold:

1. The group of covering transformations of \( p: \tilde{M} \to M \) is the binary icosahedral group \( I^* \).
2. The mapping cone \( C_p \) of \( p: \tilde{M} \to M \) is homotopy equivalent to the mapping cone \( C_q \) of the universal covering \( q: S^3 \to D^3 \).
3. There is a map \( f: M \to D^3 \) with \( f_*(\pi_1(M)) = \pi_1(D^3) \), such that the degree of \( f \) is relatively prime to 120 and
   \[ p_*(\pi_1(\tilde{M})) = \ker(f_*: \pi_1(M) \to \pi_1(D^3)), \]
   that is, the covering \( p: \tilde{M} \to M \) is the pullback of the covering \( q: S^3 \to D^3 \).
If the homology 3-sphere $M$ is not irreducible it can be decomposed into a connected sum of irreducible homology 3-spheres, and this will induce a corresponding decomposition of $\widetilde{M}$ and the covering $p: \widetilde{M} \to M$ (Theorem (3.6)).

If the homology 3-sphere $M$ admits a Seifert fibration and if it also admits a nontrivial regular covering by a homology 3-sphere $\widetilde{M}$, then necessarily $M = D^3$ and $\widetilde{M} = S^3$ (Theorem (4.1)).

There is an abundance of irreducible homology 3-spheres that admit nontrivial regular coverings by homology 3-spheres. In §5 we construct examples by utilizing the dodecahedral space $D^3$. The irreducible homology 3-spheres in these examples are all sufficiently large. This raises the following question:

**Question.** Is there an example of an irreducible homology 3-sphere (with infinite fundamental group) that is not sufficiently large or is hyperbolic and that is regularly covered by a homology 3-sphere?

It is a well-known conjecture that the fundamental group of a compact 3-manifold $M$ is residually finite, i.e. there is a sequence $\{G_i\}_{i=1,2, \ldots}$ of subgroups of finite index in $\pi_1 M$ with $\bigcap_i G_i = 1$. Applying statement 1 of our Main Theorem we obtain that if $M$ is a homology 3-sphere such that $\pi_1 M$ has a subgroup of finite index which is not a divisor of 120, then there are infinitely many distinct subgroups of finite index in $\pi_1 M$ (Corollary 3.2).

2. Preliminaries

In this section we collect the background material we need in order to prove our theorems. We will work throughout in the PL category. A PL homeomorphism we simply call an isomorphism. Our reference for 3-manifold concepts is [He].

By the term surface we will mean a compact, connected 2-manifold. A closed surface $F$ in a 3-manifold $M$ is said to be incompressible if it is not a 2-sphere and if for each 2-cell $B \subset M$ with $B \cap F = \partial B$, there is a 2-cell $B' \subset F$ with $\partial B = \partial B'$.

A 3-manifold $M$ is said to be irreducible if each 2-sphere in $M$ bounds a 3-cell in $M$. If $p: \widetilde{M} \to M$ is a covering onto the orientable 3-manifold $M$, then $\widetilde{M}$ is irreducible if and only if $M$ is irreducible [MSY].

A closed orientable connected 3-manifold is sufficiently large if it is irreducible and if it contains a 2-sided incompressible closed surface.

The following will be used in §5.

**Lemma (2.1).** Let $M$ be a 3-manifold and $M_1, M_2$ submanifolds such that $M = M_1 \cup M_2$ and $M_1 \cap M_2 = \partial M_1 \cap \partial M_2 = F$ is a component of $\partial M_1$ and $\partial M_2$. If $M_1$ and $M_2$ are irreducible and if $F$ is incompressible in $M_1$ and $M_2$, then $M$ is irreducible.
There are various descriptions of the dodecahedral space, for example, see [Ro]. We will use the following presentation as a Seifert fibered space. For basic definitions regarding Seifert fibrations of 3-manifolds we refer to [O] or [S].

Let $S^2$ be the 2-sphere and let $B_0$, $B_1$, $B_2$, $B_3 \subset S^2$ be four disjoint 2-cells. Then $S^2 = S^2 - (B_0 \cup B_1 \cup B_2 \cup B_3)$ is a 2-sphere with 4 holes. Let $S^1$ be the 1-sphere. $\partial (S^2 \times S^1)$ consists of the 4 tori $\partial B_0 \times S^1$, $\ldots$, $\partial B_3 \times S^1$. We give $S^2$ and $S^1$ fixed orientations. These define a unique orientation on $S^2 \times S^1$.

Then the dodecahedral space $D^3$ is obtained from $S^2 \times S^1$ by attaching 4 solid tori $B'_0 \times S^1$, $\ldots$, $B'_3 \times S^1$ to the 4 boundary tori of $S^2 \times S^1$ via isomorphisms $h_i : \partial B'_i \times S^1 \to \partial B_i \times S^1$, $i = 0, 1, 2, 3$, that satisfy

$$h_0 \circ [\partial B'_0] = [\partial B_0] - [S^1] \text{ in } H_1(\partial B_0 \times S^1),$$

$$h_i \circ [\partial B'_i] = \alpha_i [\partial B_i] + [S^1] \text{ in } H_1(\partial B_i \times S^1), \quad 1 \leq i \leq 3,$$

where $\alpha_i = 5, 2, 3$, respectively. That is,

$$D^3 = (S^2 \times S^1) \cup_{h_0} (B'_0 \times S^1) \cup_{h_1} (B'_1 \times S^1) \cup_{h_2} (B'_2 \times S^1) \cup_{h_3} (B'_3 \times S^1).$$

Thus the dodecahedral space $D^3$ is a Seifert fibered space having 3 singular fibers with Seifert invariants $(5,1)$, $(2,1)$, $(3,1)$ determined by the solid tori $B'_1 \times S^1$, $B'_2 \times S^1$, $B'_3 \times S^1$, respectively, and with Seifert surface a 2-sphere. The solid torus $B'_0 \times S^1$ determines a regular fiber. In the terminology of [S], $D^3$ has the description

$$(0, o; o| - 1; 5, 1; 2, 1; 3, 1).$$

The fundamental group $\pi_1(D^3)$ is the binary icosahedral group $I^*$. It has order 120 and its center is a cyclic group of order 2. Each regular fiber $S^1_0 \subset D^3$ defines a generator $[S^1_0] \in \pi_1(D^3)$ of the center. In §5 we will give a more detailed description of the universal covering $q : S^3 \to D^3$ of the 3-sphere $S^3$ onto the dodecahedral space $D^3$.

For any group $G$ let $\varepsilon : Z[G] \to Z$ denote the augmentation homomorphism of the integral group ring $Z[G]$ and $A[G] = \ker \varepsilon$ the augmentation ideal. If $G$ is a finite group then let $N$ denote the norm element, $N = \sum_{x \in G} x$ in $Z[G]$.

For any integer $r$ the left ideal generated by $r$ and $N$ in $Z[G]$ is denoted by $(r, N)$. If $r$ is relatively prime to the order of $G$ then the ideal $(r, N)$ is a finitely generated projective $Z[G]$-module [Sw, Proposition 7.1, p. 570]. Therefore $(r, N)$ determines an element, denoted by $[r, N]$, of the reduced Grothendieck group $\tilde{K}_0(Z[G])$ of finitely generated projective $Z[G]$-modules.

A $(G, m)$-complex is a finite connected $m$-dimensional CW complex $X$ such that $\pi_1(X) \cong G$ and the universal covering space $\tilde{X}$ is $(m-1)$-connected. To any $(G, m)$-complex $X$ there is associated its algebraic $m$-type, that is the triple $T(X) = (\pi_1(X), \pi_m(X), k(X))$ where $k = k(X) \in H^{m+1}(G, \pi_m(X))$ is the $k$-invariant (see [D, p. 249]).
An abstract $m$-type is a triple $T = (G, \pi_m, k)$, where $G$ is a group, $\pi_m$ is a $\mathbb{Z}[G]$-module, and $k \in H^{m+1}(G, \pi_m)$. There are notions of homomorphism and isomorphism for abstract $m$-types (see [D, p. 250] for details).

**Theorem (2.2)** (see [D]). Two $(G, m)$-complexes $X, Y$ are homotopically equivalent if, and only if, $T(X)$ and $T(Y)$ are isomorphic as abstract $m$-types.

Let $Z_n$ denote the ring of integers mod $n$ and let $Z_n^* \subset Z_n$ denote its group of units. Now let $X$ be a $(G, m)$-complex, where $G$ is a group of order $n$. Then $H^{m+1}(G, \pi_m(X)) \cong Z_n$ and the only $k$-invariants $r$ which can possibly arise from $(G, m)$-complexes with algebraic $m$-type $(G, \pi_m(X), r)$ must be in $Z_n^* \subset Z_n$ (see [D]). By [D, Theorem 2.5], the map $\nu : Z_n^* \to K_0(\mathbb{Z}[G]), \nu(r) = [r, N]$ is a homomorphism.

**Theorem (2.3)** [D, Theorem 3.5]. Suppose $m \geq 3$. The abstract $m$-type $(G, \pi_m(X), r)$ is the algebraic $m$-type of some $(G, m)$-complex if, and only if, $\nu(r) = [r, N] = 0$, that is $[r, N]$ is stably free.

Consequently, the $k$-invariants $r$ which can arise from $(G, m)$-complexes with algebraic $m$-type $(G, \pi_m(X), r)$ form a subgroup of $Z_n^*$. In particular $(G, \pi_m(X), 1)$ is the algebraic $m$-type of a $(G, m)$-complex (namely $X$).

Let $G$ be a finite group of order $n$ such that there is a $(G, m)$-complex $X$. As in [D] we make the definitions

$$Q_m(\pi_m(X)) = \{ r \in Z_n^* \subset H^{m+1}(G, \pi_m(X))|(G, \pi_m(X), 1) \cong (G, \pi_m(X), r) \} ,$$

where $\cong$ is isomorphism as abstract $m$-types, and

$$F(G) = \{ r \in Z_n^*|(r, N) \text{ is free} \} .$$

Suppose now that $G$ has periodic cohomology and minimal free period $k$. The following is a consequence of [D, Corollary (8.4), (a), p. 275].

**Theorem (2.4).** $F(G) \subset Q_k(A[G])$.

3. Proof of the main theorem

**Theorem (3.1).** Suppose $p : \tilde{M} \to M$ is a nontrivial regular covering with $M$ and $\tilde{M}$ homology 3-spheres. Then the group of covering transformations is the binary icosahedral group $I^*$.  

**Proof.** Let $G$ be the group of covering transformations of $p : \tilde{M} \to M$. Then $G$ has periodic cohomology and its period is either 2 or 4. From the exact sequence $1 \to \pi_1(\tilde{M}) \to \pi_1(M) \to \pi_1(M)/p_*(\pi_1(\tilde{M})) \cong G \to 1$ we see that $G$ is perfect. Therefore $G \cong I^*$ (see [Sj]). Q.E.D.

**Corollary (3.2).** Let $M$ be a homology 3-sphere such that $\pi_1(M)$ has a subgroup of finite index which is not a divisor of 120. Then there is a sequence $\{G_i\}_{i=1,2,...}$ of subgroups of finite index in $\pi_1(M)$ with $G_{i+1} \subset G_i$ , $i = 1, 2, \ldots$.

**Proof.** Let $G_1 \subset \pi_1(M)$ be a subgroup of finite index which is not a divisor of 120. Let $\pi_1(M) = g_0G_1 \cup \cdots \cup g_kG_1$ be the left coset decomposition of
Figure 1

$\pi_1(M)$. Then $G_1 = \cap_{i=0}^{k} g_i G_i g_i^{-1}$ is a normal subgroup of $\pi_1(M)$ with index $(G_1 : \pi_1(M)) = \text{index}(G_1 : G_1) \cdot \text{index}(G_1 : \pi_1(M))$. Hence index $(G_1 : \pi_1(M))$ does not divide 120. Let $p : M_1 \to M$ be the covering with $p_* \pi_1(M_1) = G_1$. By Theorem 3.1, $M_1$ cannot be a homology 3-sphere. Let

$$G_2 = p_* \ker(\pi_1(M_1) \to H_1(M_1) \to \text{onto finite abelian group } \neq 0).$$

Then index $(G_2 : \pi_1(M_1))$ does not divide 120 and the construction can be continued. Q.E.D.

If $X$ is a space and $f : X \to Y$ is a map let $CX$, $SX$ and $C_f$ denote the unreduced cone, suspension and mapping cone, respectively.

Now let $M$, $\tilde{M}$ be homology 3-spheres and let $p : \tilde{M} \to M$ be a regular covering with $I^*$ as group of covering transformations. Define $W = I^* \times C\tilde{M} / (g, \tilde{x}, 0) \sim (h, \tilde{x}, 0)$. See Figure 1.

Note that $W$ is 3-connected since collapsing one of the cones to a point gives a homotopy equivalence

$$W \simeq \bigvee_{119} S\tilde{M} \vee \bigvee_{119} S\tilde{M} \simeq S^4 \vee \cdots \vee S^4$$

Also note that there is a natural action $I^* \times W \to W$, $g \cdot (h, \tilde{x}, t) = (gh, g\tilde{x}, t)$ and that $W / I^* = C\tilde{M} / (\tilde{x}, 0) \sim (g\tilde{x}, 0) = C_p$. Since this action is fixed point free this implies that $W$ is the universal covering space of $C_p$.

Lemma (3.3). $C_p$ is an $(I^*, 4)$-complex whose algebraic 4-type is $(I^*, A[I^*], r)$ for some $r \in Z_{120}^*$. Proof. The only part requiring proof is that $\pi_4(C_p) \cong A[I^*]$ as (left) $Z[I^*]$-modules. Thus consider the following portion of the homology exact sequence of the pair $(W, \tilde{M})$:

$$0 \to H_4(W) \to H_4(W, \tilde{M}) \to H_3(\tilde{M}) \to 0.$$

This is an exact sequence of $Z[I^*]$-modules with $H_4(W) \cong \pi_4(C_p)$ as $Z[I^*]$-modules and $H_3(\tilde{M}) \cong Z$ as a trivial $Z[I^*]$-module.

Let $U = \{(g, \tilde{x}, t) \in W | t \leq \frac{1}{2}\}$. See Figure 2.
Then we have the following isomorphism of \( \mathbb{Z}[I^*] \)-modules:
\[
H_4(W, \tilde{M}) \cong H_4(W, U) \cong H_4(W - \text{int } U, U - \text{int } U) \cong H_4(I^* \times (C\tilde{M}, \tilde{M})) \\
\cong \mathbb{Z}[I^*] \quad \text{since } H_4(C\tilde{M}, \tilde{M}) \cong \mathbb{Z}.
\]

With respect to these isomorphisms the boundary homomorphism
\[
\partial : H_4(W, \tilde{M}) \to H_3(\tilde{M})
\]
is just the augmentation homomorphism and therefore \( \pi_4(C_p) \cong A[I^*] \) as \( \mathbb{Z}[I^*] \)-modules. Q.E.D.

**Theorem (3.4).** Up to homotopy there is only one \((I^*, 4)\)-complex \( X \) such that \( \pi_4(X) \cong A[I^*] \). In particular, if \( p : \tilde{M} \to M \) is a nontrivial regular covering of a homology 3-sphere \( \tilde{M} \) onto the homology 3-sphere \( M \), then \( C_p \) is homotopy equivalent to \( C_q \), where \( q : S^3 \to D^3 \) is the universal covering.

**Proof.** According to (3.3), \( C_p \) and \( C_q \) are \((I^*, 4)\) complexes with \( \pi_4 \cong A[I^*] \). If \( X \) is any \((I^*, 4)\) complex with algebraic 4-type \((I^*, A[I^*], r)\), then \([r, N] = 0 \) in \( K_0(\mathbb{Z}[I^*]) \) (see (2.3)). A result of Swan (see [SW2, Theorem I]) is that \( r \in F(I^*) \). Then from (2.4) we see that there is only one isomorphism class of algebraic 4-types \((I^*, A[I^*], r)\). Using (2.2) we now have that \( C_p \) is homotopy equivalent to \( C_q \). Q.E.D.

**Theorem (3.5).** Let \( M, \tilde{M} \) be homology 3-spheres and \( p : \tilde{M} \to M \) a nontrivial regular covering. Then there is a map \( f : M \to D^3 \) onto the dodecahedral space \( D^3 \) such that \( f_* (\pi_1(M)) = \pi_1(D^3) \), the degree of \( f \) is relatively prime to 120, and \( p_*(\pi_1(\tilde{M})) = \ker(f_* : \pi_1(M) \to \pi_1(D^3)) \).

**Proof.** By (3.4) there is a homotopy equivalence \( h : C_p \to C_q \). Let \( i : M \to C_p \) and \( j : D^3 \to C_q \) be the inclusions. Then we can alter \( h \) by a homotopy, if necessary, so that \( hi(M) \subset D^3 \). Let \( f = hi : M \to D^3 \). Thus we have the commutative diagram
\[
\begin{array}{ccc}
M & \xrightarrow{f} & D^3 \\
\downarrow i & & \downarrow j \\
C_p & \xrightarrow{h} & C_q
\end{array}
\]
The map \( f \) has the desired properties. Q.E.D.
Theorems (3.1), (3.4) and (3.5) prove the Main Theorem. It should be pointed out that one can construct such a map \( f : M \to D^3 \) by elementary obstruction theory. In fact the regular covering \( p : \tilde{M} \to M \) induces an epimorphism \( \theta : \pi_1(M) \to I^* \) and \( f \) can be chosen so that \( \theta \) corresponds to \( f_* : \pi_1(M) \to \pi_1(D^3) \). The map \( f : M \to D^3 \) lifts to a map \( \tilde{f} : \tilde{M} \to S^3 \), and this will then produce a map \( h : C_p \to C_q \) by coning. However, \( h \) will not in general be a homotopy equivalence.

Theorem (3.5) raises the following open question.

**Question.** Suppose \( M, \tilde{M} \) are homology 3-spheres and \( p : \tilde{M} \to M \) is a non-trivial regular covering. Then is there a degree 1 map \( f : M \to D^3 \) such that \( p_* (\pi_1(M)) = \ker(f_* : \pi_1(M) \to \pi_1(D^3)) \)?

It follows from [Ol] that for each integer \( m \) there is a map \( h_m : D^3 \to D^3 \) that induces the identity on \( \pi_1(D^3) \) and that has degree \( 1 + 120m \). The binary icosahedral group \( I^* \) has only one nontrivial outer automorphism \( \alpha : I^* \to I^* \). There is a map \( h_\alpha : D^3 \to D^3 \) that induces \( \alpha \) on \( \pi_1(D^3) \) and that has degree 49 [Pl]. Composing the map \( f : M \to D^3 \) with maps of the types \( h_m, h_\alpha \), we see that we can alter the degree of the map \( f \) to \( \deg f + 120m \), or to 49 \( \deg f + 120m \).

Suppose that \( p : \tilde{M} \to M \) is a (regular) covering of the homology 3-sphere \( \tilde{M} \) onto the homology 3-sphere \( M \) (e.g. \( q : S^3 \to D^3 \)). Let \( X \) be an arbitrary homology 3-sphere and let \( n \) be the number of points in a fiber \( p^{-1}(x), x \in M \) (\( n = 120 \) if the covering is regular and nontrivial). Then \( p : \tilde{M} \to M \) extends to a (regular) covering \( \tilde{p} : \tilde{M} \# nX \to M \# X \) of connected sums, where \( M \# X \) is the connected sum defined by removing a 3-cell \( E^3 \subset M \) from \( M \), a 3-cell from \( X \), and identifying their boundaries, and \( \tilde{M} \# nX \) is defined by removing the \( n \) 3-cells \( p^{-1}(E^3) \) from \( \tilde{M} \) and sewing in \( n \) copies of \( \tilde{X} \)(3-cell). See Figure 3. The connected sums \( M \# X \) and \( \tilde{M} \# nX \) are homology 3-spheres.

**Theorem (3.6).** Let \( \tilde{M}, M \) be homology 3-spheres and \( p : \tilde{M} \to M \) a nontrivial regular covering. Suppose that \( M \) is not irreducible. Then there are irreducible homology 3-spheres \( \tilde{M}_0, M_0 \); a nontrivial regular covering \( p_0 : \tilde{M}_0 \to M_0 \); and a homology 3-sphere \( X \) so that \( M = M_0 \# X, \tilde{M} = \tilde{M}_0 \# 120X \), and \( p = p_0 \).

**Proof.** Since \( M \) is not irreducible we have \( M = M_0 \# M_1 \# \cdots \# M_k \) with \( M_0, \ldots, M_k \) irreducible homology 3-spheres. The covering \( p : \tilde{M} \to M \) defines canonical coverings \( p_i : \tilde{M}_i \to M_i, i = 0, \ldots, k \). The components of \( \tilde{M}_i \) must be homology 3-spheres. If \( M'_i \subset \tilde{M}_i \) is a component, then \( p_i : M'_i \to M_i \) is a covering. There must exist at least one \( i \) and one component \( M'_i \subset \tilde{M}_i \) such that \( p_i : M'_i \to M_i \) is nontrivial (otherwise, replacing each \( M'_i \) by a 3-sphere we can construct a nontrivial covering \( S^3 \to S^3 = S^3_0 \# \cdots \# S^3_k \), a contradiction).

Suppose \( p_0 : M'_0 \to M_0 \) is nontrivial. Define \( X \) to be \( M_1 \# \cdots \# M_k \). Then
Figure 3

\[ M = M_0 \# X \] Note that \( p_0^* : M'_0 \to M_0 \) is a regular covering. By Theorem 3.1, both \( p : \tilde{M} \to M \) and \( p_0 : M'_0 \to M_0 \) are 120-sheeted. Therefore \( M'_0 = \tilde{M}_0 \). Since \( \tilde{M} \) is a homology 3-sphere, \( p^{-1}(X) \) consists of 120 copies of \( X \). Q.E.D.

4. Regular coverings of Seifert fibered homology 3-spheres by homology 3-spheres

We have the following uniqueness result.

**Theorem (4.1).** Let \( M \) be a homology 3-sphere that admits a Seifert fibration and a nontrivial regular covering \( p : \tilde{M} \to M \) by a homology 3-sphere. Then necessarily \( M = D^3 \) and \( \tilde{M} = S^3 \).

**Proof.** Let \((\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)\) be the Seifert invariants of the Seifert fibration of \( M \). By Satz 12 of [S] we must have \( r \geq 3 \) and \( \alpha_1, \ldots, \alpha_r \) relatively prime in pairs. We give \( \tilde{M} \) the Seifert fibration induced by the covering \( p : \tilde{M} \to M \).

If \( S^1 \subset M \) is a regular fiber, then the components of \( p^{-1}(S^1) \) are all regular fibers. If \( S^1 \subset M \) is a singular fiber, we claim that the components of \( p^{-1}(S^1) \) must also be regular fibers. To prove this suppose \( \tilde{S}_1 \subset p^{-1}(S^1) \) is a singular fiber. First we show that \( \tilde{S}_1 = p^{-1}(S^1) \). Otherwise there is another component \( \tilde{S}_1 \subset p^{-1}(S^1) \). Then, since the group of covering transformations of the regular covering \( p : \tilde{M} \to M \) acts transitively on \( p^{-1}(S^1) \), \( \tilde{S}_1 \) and \( \tilde{S}_1 \) must have the same Seifert invariants. Since \( \tilde{M} \) is a homology 3-sphere this contradicts Satz 12 of [S]. Therefore \( \tilde{S}_1 = p^{-1}(S^1) \). This now contradicts the fact that the group of covering transformations is the noncyclic group \( I^* \).

Thus the Seifert fibration of \( \tilde{M} \) has no singular fibers. By the remark preceding Satz 12 of [S], \( \tilde{M} \) must be the 3-sphere. Therefore the fundamental group
of \( M \) must be finite. Again by Satz 12 of [S], \( M \) must be the dodecahedral space \( D^3 \). Q.E.D.

5. EXAMPLES OF REGULAR COVERINGS OF IRREDUCIBLE HOMOLOGY 3-SHEPES BY HOMOLOGY 3-SHEPES

We present two methods of constructing regular coverings \( p: \widetilde{M} \to M \) such that \( M, \widetilde{M} \) are irreducible homology 3-spheres.

**Theorem (5.1).** Let \( p_0: \widetilde{M}_0 \to M_0 \) be a regular covering of the irreducible homology 3-sphere \( M_0 \) by the homology 3-sphere \( \widetilde{M}_0 \). Then there is a sufficiently large homology 3-sphere \( M \) containing an incompressible torus, \( M \) and \( M_0 \) not homotopy equivalent, a regular covering \( p: \widetilde{M} \to M \) of \( M \) by a homology 3-sphere, and there are degree 1 maps \( h: M \to M_0 \), \( \tilde{h}: \widetilde{M} \to \widetilde{M}_0 \) such that the following diagram

\[
\begin{array}{ccc}
\widetilde{M} & \xrightarrow{\tilde{h}} & \widetilde{M}_0 \\
\downarrow p & & \downarrow p_0 \\
M & \xrightarrow{h} & M_0
\end{array}
\]

commutes.

**Proof.** Let \( W \) be an irreducible orientable compact 3-manifold with \( \partial W \) a torus, \( H_1(W) = \mathbb{Z} \), and \( W \) not a solid torus (e.g. let \( X \) be any irreducible homology 3-sphere with \( \pi_1(X) \neq 1 \), and \( S^1 \subset X \) a 1-sphere which is not nullhomotopic in \( X \); or \( X = S^3 \) and \( S^1 \subset S^3 \) a nontrivial knot. Then \( W = X - N(S^1) \), where \( N(S^1) \) is a regular neighbourhood of \( S^1 \) in \( X \), is an irreducible orientable compact 3-manifold with \( \partial W \) a torus, \( H_1(W) = \mathbb{Z} \), and \( W \) is not a solid torus). Note that \( \partial W \) is incompressible in \( W \). By a standard argument there is a proper surface \( F \subset W \) with \( F \cap \partial W = \partial F \) a 1-sphere. Let \( \partial W = S^1 \times \partial F \) be a representation such that \([S^1]\) is a generator of \( H_1(W) = \mathbb{Z} \).

By a result of [Ha] there is a 1-sphere \( S^1_0 \subset M_0 \) which is nullhomotopic in \( M_0 \) and such that \( C = M_0 - N(S^1_0) \) is a fiber bundle over a 1-sphere with fiber a surface \( F_0 \), where \( N(S^1_0) = S^1_0 \times D^2_0 \) is a regular neighbourhood of \( S^1_0 \) in \( M_0 \). Applying the exact Mayer-Vietoris sequence of the pair \((C , N(S^1_0))\), we may assume that \([\partial F_0] = [S^1]\) in \( H_1(\partial N(S^1_0))\). Note that \( C \) is irreducible and that the torus \( \partial C \) is incompressible in \( C \). Let \( g: (W, \partial W) \to (N(S^1_0), \partial N(S^1_0)) \) be a map such that \( g|: \partial W \to \partial N(S^1_0) \) is an isomorphism with \( g(F) = D^2_0 \) and \( g_*[S^1] = [S^1_0] \) in \( H_1(\partial N(S^1_0))\). Define

\[
M = C \cup W/x = g(x) , \quad x \in \partial C ,
\]

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and the map $h: M \to M_0$ by

$$h(x) = \begin{cases} 
  x, & x \in C, \\
  g(x), & x \in W.
\end{cases}$$

The closed 3-manifold $M$ is orientable, irreducible and the torus $\partial W = \partial C$ is incompressible in $M$. From the exact Mayer-Vietoris sequence of the pair $(C, W)$ it follows that $M$ is a homology 3-sphere. The map $h: M \to M_0$ has degree 1. By construction $N = N(S^1) \subset M_0$ is nullhomotopic and therefore $p_0^{-1}(N)$ consists of 120 copies $\tilde{N}_1, \ldots, \tilde{N}_{120}$ of $N$. Now take 120 copies $W_i$ of $W$, $i = 1, \ldots, 120$. Define: $\tilde{M} = \bigcup_{i=1}^{120} W_i \cup p^{-1}_0(C)$. See Figure 4. The map $h: M \to M_0$ lifts to a degree 1 map $\tilde{h}: \tilde{M} \to \tilde{M}_0$. A Mayer-Vietoris sequence applied to the map $h: (\tilde{M}, p^{-1}(W), p^{-1}(C)) \to (M_0, p_0^{-1}(N), p_0^{-1}(C))$ proves that $\tilde{M}$ is a homology 3-sphere. Finally, $M$ and $M_0$ cannot be homotopy equivalent since $\pi_1(M)$ and $\pi_1(M_0)$ cannot be isomorphic. Namely, $h_*: \pi_1(M) \to \pi_1(M_0)$ is an epimorphism with $\ker(h_*) \neq 1$. If $\pi_1(M) \cong \pi_1(M_0)$, then $\pi_1(M) \cong \pi_1(M)/\ker(h_*)$ and $\pi_1(M)$ is not Hopfian. But $M$ is sufficiently large and therefore $\pi_1(M)$ is residually finite and hence Hopfian, a contradiction. Q.E.D.

Starting with the regular covering $q: S^3 \to D^3$ we can thus construct an abundance of sufficiently large homology 3-spheres containing incompressible tori that admit regular coverings by homology 3-spheres.
For the second construction we utilize the Seifert fibration of \( D^3 \). Let \( q: S^3 \to D^3 \) be the universal covering of the dodecahedral space \( D^3 \). Then \( q \) lifts the Seifert fibration of \( D^3 \) to a Seifert fibration of \( S^3 \). The group of covering transformations acts equivariantly on the fibers of the induced Seifert fibration.

Recall that the binary icosahedral group \( \pi_1(D^3) \) has order 120 and that its center is a cyclic group of order 2. Since each regular fiber \( S^1_0 \subset D^3 \) defines a generator \([S^1_0] \in \pi_1(D^3)\) of the center, \( q^{-1}(S^1_0) \) has 60 components. If \( \tilde{S}^1_0 \subset q^{-1}(S^1_0) \) is a component, it is a regular fiber with \( q|: \tilde{S}^1_0 \to S^1_0 \) a 2-sheeted covering.

We complete the description of \( q: S^3 \to D^3 \) as follows. Let \( S^1_a \subset D^3 \) be a singular fiber with Seifert invariant \((\alpha,1)\), \( \alpha = 2,3,5 \). Let \( \tilde{S}^1 \subset q^{-1}(S^1_a) \) be a component and suppose that it has Seifert invariant \((\tilde{\alpha},\tilde{\beta})\). Assume that \( q|: \tilde{S}^1 \to S^1_a \) is a \( \sigma \)-sheeted covering. Then by the remarks on p. 196 of [S] we have \( \tilde{\alpha} = \alpha/(\alpha,\sigma) \). Now \((\alpha,\sigma) = 1\) is not possible, since otherwise \( \tilde{\alpha} = \alpha \). But then, since \( \pi_1(D^3) \) acts transitively on \( q^{-1}(S^1_0) \), the induced Seifert fibration of \( S^3 \) has more than one fiber with the same Seifert invariants. A contradiction to Satz 12 of [S]. Therefore \((\alpha,\sigma) = \alpha \) and hence \( \tilde{\alpha} = 1 \). Thus each fiber \( \tilde{S}^1 \subset q^{-1}(S^1_a) \) is regular. The Seifert fibration induced on \( S^3 \) has no singular fibers; therefore it is the Hopf fibration [S].

Now let \( F \) be a closed orientable surface, \( B \subset F \) a 2-cell, and \( \phi: F \to F \) an orientation preserving isomorphism such that \( \phi(x) = x \) for all \( x \in B \). Define

\[
M_\phi = F \times [0,1]/(x,0) \sim (\phi(x),1),
\]

\[
\pi: M_\phi \to S^1 = [0,1]/0 \sim 1 \text{ by } \pi(x,t) = t.
\]

Then \( M_\phi \) is a bundle over \( S^1 \) with fiber \( F \) and bundle map \( \pi \). An application of a Mayer-Vietoris sequence gives the following exact sequence in homology.

\[
0 \to H_1(F)/(\phi_* - \text{id})H_1(F) \xrightarrow{i_*} H_1(M_\phi) \xrightarrow{\pi_*} H_1(S^1) = Z \to 0.
\]

Here \( i_* \) is the map induced by the inclusion \( i: F \to M_\phi, \ i(x) = (x,0) \). Thus \( H_1(M_\phi) \cong Z \oplus \text{coker}(\phi_* - \text{id}) \). In a similar fashion we define

\[
W_\phi = (F-B) \times [0,1]/(x,0) \sim (\phi(x),1) = M_\phi - B \times S^1.
\]

Again we have \( H_1(W_\phi) \cong Z \oplus \text{coker}(\phi_* - \text{id}) \). In particular, if \( \phi_* - \text{id} \) is invertible it follows that \( H_1(W_\phi) \cong Z \) with generator \([S^1]\).

In the dodecahedral space \( D^3 = S^2_4 \times S^1 \cup_{h_0} B_0' \times S^1 \cup \cdots \cup_{h_3} B_3' \times S^1 \) let \( B \subset \text{int} S^2_4 \) be a 2-cell and let \( D_0^3 \) be a neighborhood of \( B \) in \( D^3 \). Then \( H_1(D_0^3) \cong Z \) with generator \([\partial B]\).

Define \( M(\phi) = W_\phi \cup_{\partial} D_0^3 \) by identifying the boundary tori \( \partial B \times S^1 \) of \( W \) and \( D_0^3 \) as suggested by the notation. Notice that \( M(\phi) \) is irreducible (by (2.1)) and contains the incompressible torus \( \partial B \times S^1 \). We have \( H_1(M(\phi)) \cong \text{coker}(\phi_* - \text{id}) \).
since \([\partial B] = 0\) in \(H_1(W_\phi)\) and \([S^1] = 0\) in \(H_1(D^3_0)\). Therefore, if \(\phi_* - \text{id}\) is invertible it follows that \(M(\phi)\) is a homology 3-sphere.

Next we construct a degree 1 map \(h: M(\phi) \to D^3\). Let \(h|D^3_0 = \text{id}\). The isomorphism \(\text{id}: \partial W_\phi = \partial B \times S^1 \to \partial D^3_0 = \partial B \times S^1\) extends to a map \(W_\phi \to B \times S^1\) by mapping a collar \(F - B \times [-\epsilon, \epsilon]\) onto a collar \(B \times [-\epsilon, \epsilon]\) and then extending this map to a map of \((W_\phi - F - B) \times [-\epsilon, \epsilon]\) onto the 3-cell \(B \times S^1 - B \times [-\epsilon, \epsilon]\).

Summarizing, we have for each orientation preserving isomorphism \(\phi: (F, B) \to (F, B)\), where \(\phi = \text{id}\) on the 2-cell \(B\), constructed a 3-manifold \(M(\phi)\) and a degree 1 map \(h: M(\phi) \to D^3\). Moreover, \(M(\phi)\) is irreducible, contains an incompressible torus, and is a homology 3-sphere if, and only if, \(\phi_* - \text{id}: H_1(F) \to H_1(F)\) is invertible.

Our goal is to find conditions on \(\phi\) which will ensure that the covering of \(M(\phi)\) induced by \(h\) from the universal covering \(q: S^3 \to D^3\) will also be a homology 3-sphere. Let \(\tilde{M}(\phi)\) denote this covering.

Let \(d: S^1 \to S^1\) be the 2-sheeted covering of \(S^1\) and let \(d: M_{\phi^2} \to M_{\phi}\) be the corresponding 2-sheeted fiber preserving covering. If we use the notation \([x, t]\) for a typical point then the coordinate description of \(d\) is

\[
d[x, t] = \begin{cases} 
[\phi(x), 2t] & \text{if } 0 \leq t \leq 1/2, \\
[x, 2t - 1] & \text{if } 1/2 \leq t \leq 1.
\end{cases}
\]

Then \(d^{-1}(B \times S^1) = B \times \tilde{S}^1\) and \(d: B \times \tilde{S}^1 \to B \times S^1\) is a 2-sheeted covering with \(d_*[\partial B] = [\partial B]\) and \(d_*[\tilde{S}^1] = 2[S^1]\) in \(H_1(\partial B \times S^1)\). Therefore we can induce a 2-sheeted covering \(d: W_{\phi^2} \to W_\phi\).

From the description of \(q: S^3 \to D^3\) we see that \(q^{-1}(B \times S^1)\) consists of 60 distinct solid tori \(B \times \tilde{S}^1_i, 1 \leq i \leq 60\), and that \(q: B \times \tilde{S}^1_i \to B \times S^1\) is a 2-sheeted covering satisfying \(q_*[\partial B] = [\partial B]\), \(q_*[\tilde{S}^1_i] = 2[S^1]\) in \(H_1(\partial B \times S^1)\). Now take 60 copies \(\tilde{W}_i, 1 \leq i \leq 60\), of \(W_{\phi^2}\) and define \(\tilde{M} = (\bigcup_{i=1}^{60} \tilde{W}_i) \cup_{\partial_{\tilde{W}_i}} q^{-1}(D^3_0)\), where we identify the boundary torus \(\partial \tilde{W}_i = \partial B \times \tilde{S}^1_i\) of \(\tilde{W}_i\) with the boundary torus \(\partial B \times \tilde{S}^1_i\) of \(q^{-1}(D^3_0)\) as suggested by the notation, \(1 \leq i \leq 60\).

Now define a covering projection \(p: \tilde{M} \to M(\phi)\) by the formulas: \(p|q^{-1}(D^3_0) = q|q^{-1}(D^3_0), p|\tilde{W}_i = d|\tilde{W}_i, 1 \leq i \leq 60\). This definition is valid since on \(q^{-1}(D^3_0) \cap \tilde{W}_i = \partial B \times \tilde{S}^1_i\) the maps \(q\) and \(d\) agree.

The map \(\tilde{h}: (W_\phi, \partial W_\phi) \to (B \times S^1, \partial B \times S^1)\) lifts to a map \(\tilde{h}: (\tilde{W}_i, \partial \tilde{W}_i) \to (B \times \tilde{S}^1_i, \partial B \times \tilde{S}^1_i)\) which is the identity on the boundary torus \(\partial \tilde{W}_i, 1 \leq i \leq 60\), and which makes the following diagram commute:

\[
\begin{array}{ccc}
\tilde{W}_i & \xrightarrow{\tilde{h}_i} & B \times \tilde{S}^1_i \\
\downarrow d & & \downarrow q \\
W_\phi & \xrightarrow{h} & B \times S^1
\end{array}
\]
Thus we can define \( \widetilde{h} : \widetilde{M} \rightarrow S^3 \) by
\[
\widetilde{h}|q^{-1}(D_0^3) = \text{id}, \quad \widetilde{h}|\overline{W}_i = \widetilde{h}_i, \quad 1 \leq i \leq 60.
\]
Then \( \widetilde{h} \) has degree 1 and is a lift of \( h : M(\phi) \rightarrow D^3 \).

It follows that \( p : \widetilde{M} \rightarrow M(\phi) \) is the covering \( \widetilde{M}(\phi) \rightarrow M(\phi) \) induced from \( q : S^3 \rightarrow D^3 \) by \( h : M(\phi) \rightarrow D^3 \).

Finally we compute \( H_1(M(\phi)) \). To do this we apply a Mayer-Vietoris sequence to \( M(\phi) = (\bigcup_{i=1}^{60} \overline{W}_i) \cup_0 q^{-1}(D_0^3) \). Note that \( H_1(q^{-1}(D_0^3)) \cong 60\mathbb{Z} \) with generators \([\partial B_i], \quad 1 \leq i \leq 60\), and \( H_1((\bigcup_{i=1}^{60} \overline{W}_i) \cong 60H_1(W_{\phi^i}) \cong 60\mathbb{Z} \oplus 60\text{coker}(\phi_*^2 - \text{id}), \) with generators \([\overline{S}^1_i], \quad 1 \leq i \leq 60\), for the free summand. It follows that \( H_1(\widetilde{M}(\phi)) \cong 60\text{coker}(\phi_*^2 - \text{id}) \).

The following theorem summarizes the results of the above construction.

**Theorem (5.2).** Suppose \( F \) is a closed orientable surface and \( \phi : F \rightarrow F \) is an orientation preserving isomorphism which is the identity on some 2-cell \( B \subset F \). Let \( M(\phi) = \{(F^\phi \backslash B) \times [0,1] / (x,0) \sim (\phi(x),1) \} \cup_0 D_0^3 \).

(a) There exists a degree 1 map \( h : M(\phi) \rightarrow D^3 \) which is the identity on \( D_0^3 \).
(b) \( H_1(M(\phi)) \cong \text{coker}(\phi_*^2 - \text{id} : H_1(F) \rightarrow H_1(F)) \).
(c) If \( p : M(\phi) \rightarrow M(\phi) \) is the covering induced from \( q : S^3 \rightarrow D^3 \) by \( h : M(\phi) \rightarrow D^3 \), then \( H_1(M(\phi)) \cong 60\text{coker}(\phi_*^2 - \text{id} : H_1(F) \rightarrow H_1(F)) \).
(d) \( M(\phi), \widetilde{M}(\phi) \) are both irreducible and both contain incompressible tori.

**Corollary (5.3).** If \( \phi_*^2 - \text{id} : H_1(F) \rightarrow H_1(F) \) is invertible then \( M(\phi), \widetilde{M}(\phi) \) are homology 3-spheres and \( p : M(\phi) \rightarrow M(\phi) \) is the regular covering induced from \( q : S^3 \rightarrow D^3 \) by \( h : M(\phi) \rightarrow D^3 \) means of the degree 1 map \( h : M(\phi) \rightarrow D^3 \).

**Question.** Is there a homology 3-sphere \( M \) (with \( \pi_1(M) \) infinite) that is not sufficiently large and such that there is a degree 1 map \( h : M \rightarrow D^3 \) with the corresponding regular covering \( \widetilde{M} \) not a homology 3-sphere?

We conclude with some examples.

**Example.** If \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is an invertible \( 2 \times 2 \) matrix over the integers such that \( A^2 - I \) is invertible (over the integers) then \( \det A = -1 \) and trace \( A = \pm 1 \). Conversely, if \( A \) has determinant \(-1\) and trace \( \pm 1 \) then \( A, A - I \) and \( A^2 - I \) will all be invertible over the integers. It follows that there are no orientation preserving isomorphisms \( \phi : S^1 \times S^1 \rightarrow S^1 \times S^1 \) such that \( M(\phi) \) and \( \widetilde{M}(\phi) \) are homology 3-spheres (see (5.3)).

**Example.** Suppose \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) has determinant 1. Then \( A - I \) is invertible (over the integers) if, and only if, trace \( A = 1 \) or 3. If \( A \) is any such matrix and \( \phi : S^1 \times S^1 \rightarrow S^1 \times S^1 \) is the corresponding orientation preserving isomorphism then \( M(\phi) \) is a homology 3-sphere, but \( \widetilde{M}(\phi) \) will not be a homology 3-sphere. In fact, \( H_1(M(\phi)) \cong 60\text{coker}(A^2 - I) \cong 60\mathbb{Z}_3 \) (resp. \( 60\mathbb{Z}_5 \) since...
det(\(A^2 - I\)) = 3 (resp. 5) if trace \(A = 1\) (resp. 3). As a particular example consider \(A = [0 -1]\). Then
\[
A^2 = \begin{bmatrix}
-1 & -1 \\
1 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
1 & -1
\end{bmatrix}^{-1},
\]
and therefore \(M_\phi, M_{\phi^2}\) are orientable spherical space forms (\(M_5, M_3\) resp. in the notation of [LS]). \(M(\phi)\) is a homology 3-sphere, but \(H_1(M(\phi)) \cong 60\ Z_3\).

**Example.** A matrix of the form \(A = [\begin{smallmatrix} P & I \\ -I & 0 \end{smallmatrix}]\), where all blocks are \(g \times g\), is symplectic if, and only if, \(A \pm I\) will be invertible over the integers if, and only if, \(\det(P + 2I) = \pm 1\) and \(\det(P - 2I) = \pm 1\). If \(g = 2\) one can show that \(P\) must have the form \(P = [\begin{smallmatrix} x & y \\ y & -x \end{smallmatrix}]\), where \(x^2 + y^2 = 5\), i.e., \((x, y)\) must be one of \(\pm(1, 2), \pm(1, -2), \pm(2, 1), \pm(2, -1)\). A particular example when \(g = 3\) is given by
\[
P = \begin{bmatrix}
-2 & 1 & 1 \\
1 & -1 & 2 \\
1 & 2 & 0
\end{bmatrix}.
\]
By taking direct sums of copies of these matrices for \(g = 2\) and \(g = 3\), we can find \(g \times g\) matrices \(P\), any \(g \geq 2\), so that \(\det(P + 2I) = \pm 1\) and \(\det(P - 2I) = \pm 1\). It follows that the \(2g \times 2g\) matrix \(A = [\begin{smallmatrix} P & I \\ -I & 0 \end{smallmatrix}]\) will be symplectic and satisfy \(\det(A + I) = \pm 1\), \(\det(A - I) = \pm 1\). Therefore, if \(F\) is a closed orientable surface of genus \(g \geq 2\) there are orientation preserving isomorphisms \(\phi: F \to F\) so that \(\phi_* \pm \text{id}: H_1(F) \to H_1(F)\) are isomorphisms. According to (5.2) this means that \(M(\phi), M(\phi^2)\) are homology 3-spheres.

**References**


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