THE SPECTRUM OF THE SCHRÖDINGER OPERATOR

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Abstract. We describe the negative spectrum of the Schrödinger operator with a singular potential. We determine the exact value of the bottom of the spectrum and estimate it from above and below. We describe the dependence of a crucial constant on the eigenvalue parameter and discuss some of its properties. We show how recent results of others are simple consequences of a theorem proved by the author in 1972.

1. Introduction

For \( V(x) \geq 0 \) in \( L^{loc}(\mathbb{R}^n) \), the smallest constant \( C_\lambda(V) \) which satisfies

\[
(Vu, u) \leq C_\lambda(V)(\|\nabla u\|^2 + \lambda^2 \|u\|^2), \quad u \in C_0^\infty,
\]

is of importance in the study of the spectrum of the Schrödinger operator

\[
H = -\Delta - V.
\]

We shall show that \(-\lambda_0^2\) is the smallest point of the spectrum of \( H \) if and only if, \( \lambda_0 \) is the smallest value of \( \lambda \geq 0 \) such that \( C_\lambda(V) \leq 1 \) (if \( C_\lambda(V) > 1 \) for all \( \lambda \geq 0 \), then the operator \( H \) is not bounded from below; the smallest point in the spectrum is \(-\infty\)). In 1972 the author obtained an expression determining the exact value of \( C_\lambda(V) \) (cf. [1, p. 498]). It is given by

\[
C_\lambda(V) = \inf_{\psi > 0} \sup_{x} \psi(x)^{-1} \int_{\mathbb{R}^n} V(y)\psi(y)G_{2,\lambda}(x-y)dy
\]

where \( G_{2,\lambda}(x) \) is the Bessel potential of order 2. It is the kernel of the operator

\[
G_{2,\lambda}f = (\lambda^2 - \Delta)^{-1}f, \quad I_2 = G_{2,0}.
\]

In (1.3) one obtains an upper bound for \( C_\lambda(V) \) by picking a particular function \( \psi(x) > 0 \), e.g., \( \psi(x) \equiv 1 \). One can improve the estimate by varying \( \psi \).

The cases \( \lambda = 0 \) and \( \lambda = 1 \) have received much attention. In 1962 Mazya [2] showed that for \( n > 2 \), \( C_0(V) \leq 1 \) if

\[
\int_\varepsilon V(x)dx \leq \frac{n-2}{4} \omega \text{cap}(\varepsilon)
\]

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holds for all compact sets $e \subset \mathbb{R}^n$. Here $\omega$ is the surface area of the unit ball in $\mathbb{R}^n$ and $\text{cap}(e)$ is the Green capacity of $e$. More recently Adams [3] showed that

$$\int \left( \int_e |x - y|^{2-n} V(y) \, dy \right)^2 \, dx \leq C \int_e V(y) \, dy, \quad e \subset \mathbb{R}^n,$$

implies a bound for $C_0(V)$. In [4], Fefferman and Phong show that

$$(1.6) \quad \int \left( \int e |x - y|^{2-n} V(y) \, dy \right)^2 \, dx \leq C \int V(y) \, dy, \quad e \subset \mathbb{R}^n,$$

implies a bound for $C_0(V)$. In [4], Fefferman and Phong show that

$$(1.7) \quad C_0(V) \leq C_p \sup_{\delta, x} \left( \delta^{2p-n} \int_{|x - y| < \delta} V(y)^p \, dy \right)^{1/p}$$

if $p > 1$. The proof of (1.7) given in [4] is rather long and involved. In the next section we shall show that it is a simple consequence of (1.3). In fact, we shall give a direct easy proof of (1.7) without involving the ideas of [4]. In [1] Kerman and Sawyer show that

$$(1.8) \quad C_\lambda(V) \sim \sup_{|Q| \leq \lambda^{-n}} \frac{\int_{\Omega} G_{1, \lambda}(x - y) V(y) \, dy}{\int_{\Omega} V(y) \, dy}$$

where the supremum is taken over all dyadic cubes $Q \subset \mathbb{R}^n$. Previous to [1], sufficient conditions for (1.1) to hold for various values of $\lambda$ were obtained by Kato, Rollnik, Schechter, Simon (cf. [14, 15] for references). Other sets of sufficient conditions were recently obtained in [12 and 16]. These authors were apparently unaware of the results of [1] where a condition which is both necessary and sufficient is obtained.

In §3 we show that there is a constant $C_p$ depending only on $n$ and $p$ such that

$$(1.9) \quad C_\lambda(V) \leq C_p \|M_{2p, 1/\lambda}[V^p]\|_{\infty}^{1/p}, \quad \lambda \geq 0,$$

where

$$M_{\alpha, \delta}[V](x) = \sup_{r \leq \delta} r^{\alpha - n} \int_{|y - x| < r} V(y) \, dy.$$

This allows us to show that the lowest point $-\mu^2$ of the spectrum of the operator (1.2) satisfies

$$\mu^2 \leq \sup_{x, \delta} \left( 2C_p \left( \delta^{-n} \int_{|y - x| < \delta} V(y)^p \, dy \right)^{1/p} - \delta^{-2} \right)$$

which is another estimate of Fefferman-Phong [4]. In our estimate only one constant appears (the one from (1.9)) and can be readily estimated. In proving (1.9) we show that there is a constant $C_{s, q}$ depending only on $s, n$ and $q$ such that

$$(1.10) \quad \|I_{s, \delta} f\|_q \leq C_{s, q} \|M_{s, \delta} f\|_q$$

where

$$I_{s, \delta} f(x) = \int_{|y - x| < \delta} |y - x|^{s-n} f(y) \, dy.$$
The estimate (1.10) is of interest in its own right. Our proof extends a method of Muckenhoupt-Wheeden [5]. As a consequence of (1.10) we obtain

\[(1.11) \quad \|G_{s, \lambda} f\|_q \leq C_s^f \|M_{s, 1/\lambda} f\|_q\]

where

\[G_{s, \lambda} f = (\lambda^2 - \Delta)^{-s/2} f.\]

In §4 we show that the constant \(C_\lambda(V)\) is continuous in \(\lambda\) in the interval \([0, \infty)\). Moreover

\[\mu^2 = \inf_{C_\lambda(V) \leq 1} \lambda^2 C_\lambda(V) = \inf_{C_\lambda(V) \leq 1} \lambda^2 \]

\[= \sup_{C_\lambda(V) > 1} \lambda^2 = \sup_{C_\lambda(V) > 1} \lambda^2 C_\lambda(V).\]

From this it follows easily that

\[\sup_\lambda \lambda^2 [C_\lambda(V) - 1] \leq \mu^2 \leq \sup_\lambda \lambda^2 [2C_\lambda(V) - 1].\]

Next we show that if \(V\) is in the Muckenhoupt-Wheeden class \(A_\infty\) (cf. [5]), then

\[(1.12) \quad C_\lambda(V) \leq N_\mu \|M_{2, 1/\lambda}\|_\infty.\]

In §5 we show that the essential spectrum of \(H\) is the same as that of \(-\Delta\), i.e.,

\[(1.13) \quad \sigma_e(H) = [0, \infty)\]

provided

(a) \(C_\lambda(V) \rightarrow 0\) as \(\lambda \rightarrow \infty\);

(b) \(C_{\lambda_0}(V^R) \rightarrow 0\) as \(R \rightarrow \infty\)

for some \(\lambda_0 \geq 0\), where

\[V^R(x) = \begin{cases} 0, & |x| \leq R, \\ V(x), & |x| > R. \end{cases}\]

2. A simple proof of the Fefferman-Phong estimate

We now show that (1.7) is a simple consequence of (1.3). Let

\[(2.1) \quad M_n[V](x) = \sup_r r^{n-n} \int_{|y-x|<r} V(y) \, dy, \quad M = M_0,\]

denote the maximal function. The right-hand side of (1.7) is equivalent to

\[K_p \|M_{2p}[V^p]\|_\infty^{1/p}.\]

By Hölder's inequality

\[(2.2) \quad M_1[V^{1/2} u] \leq M_q[V^{q/2}]^{1/q} M[|u|^{q'}]^{1/q'}.\]
holds for any \( q \geq 1 \), where \( 1/q + 1/q' = 1 \). If we take \( q = 2p > 2 \), we have

\[
\|M[V^{1/2}u]\|_2 \leq K_p^{1/2} \|M[|u|^{q'}]^{1/q'}\|_2 = K_p^{1/2} \|M[|u|^{q'}]\|_2^{1/q'} \\
\leq C K_p^{1/2} \|u|^{q'}\|_2^{1/q'} = C K_p^{1/2} \|u\|_2
\]

since \( q' < 2 \). By a theorem of Muckenhoupt and Wheeden \[5\], this implies

\[
\|I_1[V^{1/2}u]\|_2 \leq C' K_p^{1/2} \|u\|_2,
\]

where \( I_1 = G_{s,0} \). Inequality (2.4) is equivalent to

\[
\|V^{1/2} I_2[V^{1/2}u]\|_2 \leq C'' K_p \|u\|_2.
\]

If \( C > C'' K_p \) and \( h > 0 \) is in \( L^2 \), there is a \( \phi > 0 \) in \( L^2 \) such that

\[
\phi = h + C^{-1} V^{1/2} I_2[V^{1/2}\phi].
\]

This shows that the right-hand side of (1.3) is bounded by a constant times \( K_p \). Hence, (1.7) holds.

Another approach is to note that (2.4) is equivalent to

\[
\|V^{1/2} I_2\|_2 \leq C'' K_p \|u\|_2.
\]

which in turn is equivalent to

\[
(Vu, u) \leq C'' K_p \|\nabla u\|_2
\]

which shows that (1.7) holds.

**3. Estimates for arbitrary \( \lambda \)**

For \( \mu \) a locally finite Borel measure, we define

\[
I_{s,\delta} d\mu(y) = \int_{|x-y|<\delta} |x-y|^{s-n} d\mu(x), \quad 0 < s \leq n,
\]

and

\[
M_{s,\delta} d\mu(y) = \sup_{r \leq \delta} \int_{|x-y|<r} |x-y|^{s-n} d\mu(x), \quad 0 \leq s \leq n.
\]

For \( 1 \leq q < \infty \) we let

\[
\|u\|_q = \left( \int_{R^n} |u(x)|^q \, dx \right)^{1/q}
\]

by the norm in \( L^q(R^n) \). Our first result is

**Theorem 3.1.** There is a constant \( C_{s,q} \) depending only on \( s, n \) and \( q \) such that

\[
\|I_{s,\delta} d\mu\|_q \leq C_{s,q} \|M_{s,\delta} d\mu\|_q.
\]

Moreover

\[
C_{s,q} \leq 2^{n-s+1} + (\omega/s)^{s-n} (n^{n/2} 2^{2(2n+2-s)q+2s+2}.
\]

Before proving Theorem 3.1 we state some consequences.
Theorem 3.2. For each $p > 1$ there is a constant $C_p$ depending only on $n$ and $p$ such that

$$C_\lambda(V) \leq C_p \sup_x (M_{2p,1/\lambda} V^p)^{1/p}, \quad \lambda \geq 0. \quad (3.4)$$

Moreover, there is a constant $C_1$ depending only on $n$ such that

$$C_\lambda(V) \geq C_1 M_{2,1/\lambda} V. \quad (3.5)$$

Corollary 3.3. If $-\mu^2$ is the lowest point of the spectrum of the operator (1.2), then

$$\mu^2 \leq \sup_{\delta > 0} \left( 2C_p \delta^{-2} \sup_x (M_{2p,\delta} V^p)^{1/p} - \delta^{-2} \right)$$

$$= \sup_{x,\delta} \left( 2C_p \left( \delta^{-n} \int_{|y-x|<\delta} V(y)^p \, dy \right)^{1/p} - \delta^{-2} \right)$$

and

$$\mu^2 \geq \sup_{\delta} \left( C_1 \delta^{-2} \sup_x M_{2,\delta} V - \delta^{-2} \right)$$

$$= \sup_{x,\delta} \left( C_1 \delta^{-n} \int_{|y-x|<\delta} V(y) \, dy - \delta^{-2} \right).$$

Corollary 3.4. If $C_p M_{2p} V^p \leq 1$, then $\mu = 0$.

Corollaries 3.3 and 3.4 are proved by Fefferman and Phong [4]. Their proof is rather long and involved. They require two constants in (3.6) and do not provide a way of estimating them. Our proof is much shorter. They were unaware of the authors results in [1].

Proof of Theorem 3.1. Let

$$S_t = \{ x \in \mathbb{R}^n | I_{s,\delta} \, d\mu(x) < t \}$$

for each $t > 0$. If $S_t \neq \mathbb{R}^n$, then

$$S_t = \bigcup_{j=1}^{\infty} Q_j,$$

where the cubes $Q_j$ have sides parallel to the coordinate axes, have disjoint interiors and satisfy

$$d(Q_j, S^c_t) \leq 3\sqrt{n} l(Q_j) \quad (3.9)$$

where $M^c$ is the complement of $M$ in $\mathbb{R}^n$ and $l(Q)$ is the edge length of $Q$ (cf. [6, p. 10]). By subdividing $Q_j$ if necessary, we may require that

$$\rho_j \equiv 4\sqrt{n} l(Q_j) \leq \delta. \quad (3.10)$$
If (3.10) is achieved by subdivision, we lose (3.9). But in this case we can require
\[ \delta \leq 2\rho_j. \]
Thus we can make each \( Q_j \) satisfy (3.10). If it does not satisfy (3.11) as well, then it will satisfy (3.9).

Let \( b, d \) be positive numbers to be determined later. Define
\[ E_j = \{ x \in Q_j | I_{s, \delta/2} \leq d \mu(x) > tb, M_{s, \delta} \leq td \}. \]
Let \( Q \) be one of the cubes \( Q_j \), and let \( E \subset Q \) be the set given by (3.12). Assume first that \( Q \) satisfies (3.10) and (3.11). Then we have
\[
\begin{align*}
tb|E| & \leq \int_Q I_{s, \delta/2} d \mu(x) dx \\
& = \int_Q \int_{|y-x|<\delta/2} |y-x|^{s-n} d \mu(y) dx \\
& = \frac{\omega(s)(\delta/2)^s}{\omega(s)} \int_{Q+\delta} d \mu(y)
\end{align*}
\]
where \( \omega \) is the surface area of the unit sphere in \( \mathbb{R}^n \) and \( Q+\delta \) is the cube having the same center as \( Q \) but edge length equal to \( l(Q)+\delta \). Assume that \( E \) is not empty, and let \( x_0 \) be any point in \( E \). The cube \( Q+\delta \) is contained in the ball with center \( x_0 \) and radius \( \sqrt{n}l(Q) + (\delta/2) \leq (\rho/4) + (\delta/2) \leq 3\delta/4 \) by (3.10). Hence by (3.11)
\[
\begin{align*}
tb|E| & \leq (\omega/s)(\delta/2)^s (\rho/4 + \delta/2)^{n-s} M_{s, \delta} d \mu(x_0) \\
& \leq (\omega/s) \rho^s (5\rho/4)^{n-s} td \\
& \leq (\omega/s)4^s 5^{n-s} td n^{n/2} |Q|.
\end{align*}
\]
Consequently,
\[ |E| \leq (\omega/s)4^s 5^{n-s} n^{n/2} (d/b) |Q|. \]
Note that (3.13) holds if \( E \) is empty. Next assume that (3.9) and (3.10) hold. Then there is a point \( x_1 \) not in \( S \), that
\[ d(x_1, Q) \leq 3\sqrt{n}l(Q). \]
If \( x \) is in \( Q \), then
\[ |x - x_1| < \rho. \]
Consequently, if \( y \) is any point such that
\[ |y - x| > \rho. \]
then
\[(3.16) \quad |y - x_1| \leq |y - x| + |x - x_1| < 2|y - x|.
\]

Hence we have
\[
I_{s, \delta/2} \, d\mu(x) = \int_{|y - x| < \rho} \int_{\rho < |y - x| < \delta/2} |y - x|^{s-n} \, d\mu(y)
\]
\[
\leq I_{s, \rho} \, d\mu(x) + 2^{n-s} \int_{|y - x_1| < \delta} |y - x_1|^{s-n} \, d\mu(y)
\]
\[
\leq I_{s, \rho} \, d\mu(x) + 2^{n-s} I_{s, \delta} \, d\mu(x_1)
\]
\[
\leq I_{s, \rho} \, d\mu(x) + 2^{n-s} t
\]
since \( x_1 \) is not in \( S_i \). We now take \( b = 2^{n+1-s} \). This implies that if \( x \in E \), we have
\[
tb \leq I_{s, \rho} \, d\mu(x) + tb/2
\]
and consequently
\[
tb/2 \leq I_{s, \rho} \, d\mu(x).
\]
Thus \( E \) is contained in the set
\[
\{x \in Q| I_{s, \rho} \, d\mu(x) > tb/2, M_{s, \delta} \, d\mu(x) \leq td\}.
\]
Hence, if \( x \in E \)
\[
tb |E|/2 \leq \int_Q I_{s, \rho} \, d\mu(x) \, dx
\]
\[
= \int \int_{|x-y| < \rho} |x - y|^{s-n} \, dx \, d\mu(y)
\]
\[
\leq (\omega/s) \rho^{s} \int_{Q+2\rho} d\mu.
\]
Since \( 2\rho \leq \delta \) and the cube \( Q + 2\rho \) is contained in a ball of radius \( 5\rho/4 < \delta \) about any point in \( Q \), we see that
\[
tb |E|/2 \leq (\omega/s) \rho^{s} (5\rho/4)^{n-s} M_{s, \delta} \, d\mu(x_0)
\]
\[
\leq (\omega/s) r^{s} 5^{n-s} n^{n/2} td|Q|
\]
or
\[(3.17) \quad |E| \leq 2^{2s+1} (\omega/s) 5^{n-s} n^{n/2} (d/b)|Q|
\]
if we take \( x_0 \in E \). If \( E \) is empty, (3.17) holds as well. Thus we see that (3.17) holds in all cases. If we sum over all the cubes \( Q_j \), we see that
\[
|\{I_{s, \delta/2} \, d\mu(x) \geq tb, M_{s, \delta} \, d\mu(x) \leq td\}| \leq C_{n,s} d|S_i|
\]
where
\[
C_{n,s} = \omega 5^{n-s} n^{n/2} 2^{3s-n}/s.
\]
Hence
\[ |\{ I_{s,\delta/2} d\mu(x) > tb\} | \leq C_{n,s} \int |S_t| + \{|M_{s,\delta} d\mu(x) > td\}| dt^q. \]

This means that
\[
\int_0^N |\{ I_{s,\delta/2} d\mu(x) > tb\}| dt^q \\
\leq C_{n,s} d \int_0^N |S_t| dt^q + \int_0^N |\{M_{s,\delta} d\mu(x) > td\}| dt^q
\]
or
\[
b^{-q} \int_0^{Nd} |\{ I_{s,\delta/2} d\mu(x) > \tau\}| dt^q \\
\leq C_{n,s} d \int_0^N |S_t| dt^q + d^{-q} \int_0^{Nd} |\{M_{s,\delta} d\mu(x) > \tau\}| dt^q.
\]

Letting \( N \to \infty \), we have
\[
\| I_{s,\delta/2} d\mu \|_{q}^q \leq C_{n,s} d b^q \| I_{s,\delta} d\mu \|_{q}^q + (b/d)^q \| M_{s,\delta} d\mu \|_{q}^q
\]
and consequently
\[
\| I_{s,\delta/2} d\mu \|_{q} \leq C_{n,s}^{1/q} d^{1/q} b \| I_{s,\delta} d\mu \|_{q} + (b/d) \| M_{s,\delta} d\mu \|_{q}.
\]

Now
\[
I_{s,\delta} d\mu(x) = I_{s,\delta/2} d\mu(x) + \int_{\delta/2 < |y-x| < \delta} |x-y|^{s-n} d\mu(y)
\leq I_{s,\delta/2} d\mu + 2^{n-s} M_{s,\delta} d\mu.
\]

Hence
\[
\| I_{s,\delta} d\mu \|_{q} \leq C_{n,s}^{1/q} d^{1/q} b \| I_{s,\delta} d\mu \|_{q} + (b/d^{-1} + 2^{n-s}) \| M_{s,\delta} d\mu \|_{q}.
\]

Take \( 1/d = C_{n,s} 2^q b^q \). Then
\[
\| I_{s,\delta} d\mu \|_{q} \leq b(2d^{-1} + 1) \| M_{s,\delta} d\mu \|_{q}
\]
\[
= (2^{n-s+1} + (\omega/s)^{n-s} 2^{(n+2-s)q+2s+2}) \| M_{s,\delta} d\mu \|_{q}.
\]

This gives the theorem.

Next we shall prove

**Theorem 3.5.** Under the same hypothesis,
\[
\| G_{s,\lambda} d\mu \|_{q} \leq C_{s,q} \| M_{s,\lambda} d\mu \|_{q}
\]
where the constant depends only on \( s \), \( n \) and \( q \). Here
\[
G_{s,\lambda} d\mu(x) \leq \int G_{s,\lambda}(x-y) d\mu(y),
\]
\[
(\lambda^2 - \Delta)^{-s/2} f(x) = \int G_{s,\lambda}(x-y) f(y) dy.
\]
Proof. For each \( s > 0 \), the function \( G_s(x) \) has been studied extensively by Aronszajn-Smith [7]. In particular, it satisfies
\[
G_s(x) \leq \begin{cases} 
0, & \lambda|x| \leq 1, \\
\lambda^{n-s}|\lambda x|^\gamma e^{-\lambda|x|}, & \lambda|x| > 1,
\end{cases}
\]
where \( \gamma = (n-s-1)/2 \) and the \( c_j \) do not depend on \( \lambda \). Let
\[
\tilde{G}_s(x) = \begin{cases} 
0, & \lambda|x| \leq 1, \\
G_s(x), & \lambda|x| > 1.
\end{cases}
\]
It suffices to show that
\[
\|\tilde{G}_s \|_q \leq C\|M_{s,1/\lambda} \|_q.
\]
For by Theorem 3.1 and (3.20)
\[
\|G_s - \tilde{G}_s \|_q \leq c_0 \|I_{s,1/\lambda} \|_q \leq c_0 C_s \|M_{s,1/\lambda} \|_q.
\]
Now by (3.20) and (3.21)
\[
\tilde{G}_s \|_q \leq c_1 \int |\lambda|x-x'||^{n-s} \lambda^{-\lambda(x-x')} e^{-\lambda|x-x'|} d\mu(x) 
\]
\[
\leq c_1 \lambda^{n-s} \sum_{k=1}^{\infty} \int_{|x-x'|<k+1} (k+1)^{\gamma} e^{-k} d\mu(x).
\]
The set \( k < |x| < k + 1 \) can be covered by \( N(k) \) balls of radius 1 and centers \( z^{(1)}, \ldots, z^{N(k)} \) with \( N(k) \leq c_2 k^{n-1} \). Thus the set \( k < \lambda|x| < k + 1 \) can be covered by \( N(k) \) balls with centers \( z^{(1)}/\lambda, \ldots, z^{N(k)}/\lambda \) having radius \( 1/\lambda \). Hence
\[
\tilde{G}_s \|_q \leq c_1 \lambda^{n-s} \sum_{k=1}^{\infty} \sum_{j=1}^{N(k)} (k+1)^{\gamma} e^{-k} \int_{|x-x-z^{(j)}/\lambda|<1/\lambda} d\mu(x) 
\]
\[
\leq c_1 \sum_{k=1}^{\infty} \sum_{j=1}^{N(k)} (k+1)^{\gamma} e^{-k} M_{s,1/\lambda} \|\mu(x+z^{(j)}/\lambda)\|_q.
\]
Consequently
\[
\|\tilde{G}_s \|_q \leq c_1 \sum_{k=1}^{\infty} N(k)(k+1)^{\gamma} e^{-k} \|M_{s,1/\lambda} \|_q.
\]
This gives (3.22).

We can now give the

Proof of Theorem 3.2. Let \( \delta = 1/\lambda \) and put
\[
K_p = \sup_x (M_{2p,\delta} V^p)^{1/p}.
\]
If \( q = 2p > 2 \), then Hölder’s inequality gives
\[
M_{1,\delta}[V^{1/2} u] \leq M_{q,\delta}(V^{q/2})^{1/q} M_{0,\delta} |u'|^{1/q'} \leq K_p^{1/2} (M|u'|^{q'})^{1/q'}.
\]
Hence
\[ \| M_{1,\delta} [V^{1/2} u] \|_2 \leq K_p^{1/2} \| (M |u|^{q'})^{1/q'} \|_2. \]

Since \( q' < 2 \), this is bounded by
\[ K_p^{1/2} \| M |u|^{q'} \|_{2/q'}^{1/q'} \leq c' K_p^{1/2} \| |u|^{q'} \|_{2/q'}^{1/q'} = c' K_p^{1/2} \| u \|_2. \]

By Theorem 3.5, this implies
\[ \| G_{1,\lambda} [V^{1/2} u] \|_2 \leq c' C_{1,2} K_p^{1/2} \| u \|_2. \]

This implies by duality
\[ \| V^{1/2} G_{1,\lambda} \nu \|_2 \leq c' C_{1,2} K_p^{1/2} \| \nu \|_2 \]
which is equivalent to
\[ (Vu, u) \leq c'^2 C_{1,2}^2 K_p \| \nabla u \|^2 + \lambda^2 \| u \|^2. \]

Thus
\[ C_\lambda(V) \leq c'^2 C_{1,2}^2 K_p \]
which is precisely (3.4). To prove (3.5), let \( \phi(x) \) be a test function which equals 1 for \( |x| < 1 \) and 0 for \( |x| > 2 \). Put \( \phi_\lambda(x) = \phi(\lambda(x - z)) \), where \( z \in \mathbb{R}^n \) is fixed. Then
\[ (V \phi_\lambda, \phi_\lambda) \leq C_\lambda(V) \| \nabla \phi_\lambda \|^2 + \lambda^2 \| \phi_\lambda \|^2 \]
\[ = C_\lambda(V) \lambda^{2-n} \| \nabla \phi \|^2 + \| \phi \|^2 = C \lambda^{2-n} C_\lambda(V). \]

Hence
\[ \lambda^{n-2} \int_{|x - z| < 1} V(x) \, dx \leq C C_\lambda(V) \]
and consequently
\[ M_{2,1/\lambda} V(z) \leq C C_\lambda(V). \]

Remark 3.6. The constant \( C_p \) in (3.4) can be estimated readily from the proofs of Theorems 3.1, 3.2 and 3.5.

4. Properties of \( C_\lambda(V) \)

In this section we shall derive some properties of the constant \( C_\lambda(V) \).

Theorem 4.1. \( C_\lambda(V) \) is continuous in \( \lambda \) in the interval \([0, \infty)\).

Proof. Suppose
\[ C_\nu(V) \leq A, \quad \nu > \lambda. \]

Then \( C_\lambda(V) \leq A \). For we have
\[ (Vu, u) \leq A \| \nabla u \|^2 + \nu^2 \| u \|^2 \], \( u \in C_0^\infty \).

Let \( \nu \to \lambda \). Then
\[ (Vu, u) \leq A \| \nabla u \|^2 + \lambda^2 \| u \|^2 \], \( u \in C_0^\infty \).
Thus $C_\nu(V) < A$. Next, suppose $\lambda > 0$ and
$$C_\nu(V) \geq A, \quad \nu < \lambda.$$Then $C_\nu(V) \geq A$. For if $C_\lambda(V) \leq A - \epsilon$, we can find for each $\nu < \lambda$ a function $u_\nu \in C_0^\infty$ such that
\begin{equation}
\|\nabla u_\nu\|^2 + \nu^2\|u_\nu\|^2 = 1
\end{equation}
and
$$C_\nu(V) - \epsilon/2 \leq (u_\nu, u_\nu) \leq C_\nu(V)(\|\nabla u_\nu\|^2 + \lambda^2\|u_\nu\|^2).$$Thus
$$A - \epsilon/2 \leq C_\nu(V)(1 + (\lambda^2 - \nu^2)\|u_\nu\|^2) \leq C_\nu(V)\lambda^2/\nu^2$$in view of (4.1). Let $\nu \to \lambda$. We have
$$A - \epsilon/2 \leq C_\nu(V) \leq A - \epsilon$$providing a contradiction. Since $C_\lambda(V)$ is a decreasing function of $\lambda$, it must be continuous.

**Theorem 4.2.** If $-\mu^2$ is the lowest point of the spectrum of $-\Delta - V$, then
$$\mu^2 = \inf_{C_\lambda(V) \leq 1} \lambda^2 = \sup_{C_\lambda(V) > 1} \lambda^2$$
$$= \inf_{C_\lambda(V) \leq 1} \lambda^2 C_\lambda(V) = \sup_{C_\lambda(V) > 1} \lambda^2 C_\lambda(V).$$If the set $C_\lambda(V) \leq 1$ is empty, then $\mu = \infty$. If the set $C_\lambda(V) > 1$ is empty, then $\mu = 0$.
**Proof.** Let $H$ be the operator (1.2). If $C_\lambda(V) \leq 1$, then (1.1) implies
$$-C_\lambda(V)\lambda^2\|u\|^2 \leq (Hu, u).$$Thus
\begin{equation}
\mu^2 \leq C_\lambda(V)\lambda^2 \leq \lambda^2, \quad C_\lambda(V) \leq 1.
\end{equation}
If $C_\lambda(V) > 1$, then for every $\epsilon > 0$ there is a $u \in C_0^\infty$ such that
$$(u, u) \geq (C_\lambda(V) - \epsilon)(\|\nabla u\|^2 + \lambda^2\|u\|^2).$$Thus
$$(Hu, u) + \lambda^2(C_\lambda(V) - \epsilon)\|u\|^2 \leq (1 + \epsilon - C_\lambda(V))\|\nabla u\|^2.$$For $\epsilon$ sufficiently small, this is $\leq 0$. Thus
$$-\mu^2 \leq -\lambda^2(C_\lambda(V) - \epsilon) \quad \text{or} \quad \mu^2 \geq \lambda^2(C_\lambda(V) - \epsilon).$$Letting $\epsilon \to 0$, we have
\begin{equation}
\mu^2 \geq \lambda^2 C_\lambda(V) \geq \lambda^2, \quad C_\lambda(V) > 1.
\end{equation}In particular we see from this that
$$C_\mu(V) \leq 1.$$
If $\mu \neq 0$, we see by (4.2) that

$$C_\mu(V) = 1.$$  

By (4.2),

$$\mu^2 \leq \inf_{C_\lambda(V) \leq 1} \lambda^2 C_\lambda(V) \leq \inf_{C_\lambda(V) \leq 1} \lambda^2.$$  

But by (4.4) we see that equality holds. Similarly, by (4.2) we see that

$$\mu^2 \geq \sup_{C_\lambda(V)} \lambda^2 C_\lambda(V) \geq \sup_{C_\lambda(V)} \lambda^2.$$  

But there cannot be a positive $\epsilon$ such that $\mu^2 \geq \epsilon + \lambda^2$ holds for all $\lambda$ satisfying $C_\lambda(V) > 1$. For that would imply the existence of a $\nu < \mu$ such that $C_\nu(V) \leq 1$, contradicting (4.5). Thus, equality holds throughout (4.6) as well.

**Corollary 4.3.**

(4.7)  

$$\mu^2 \leq \sup_\lambda \lambda^2 [2C_\lambda(V) - 1].$$

(4.8)  

$$\mu^2 \geq \sup_\lambda \lambda^2 [C_\lambda(V) - 1].$$

**Proof.** If $C_\lambda(V) > 1$, then

$$\lambda^2 \leq \lambda^2 [2C_\lambda(V) - 1].$$

Thus $\sup_\lambda \lambda^2$ over the set $C_\lambda(V) > 1$ is bounded by the right-hand side of (4.7). Similarly, if $C_\lambda(V) > 1$, then

(4.9)  

$$\lambda^2 C_\lambda(V) \geq \lambda^2 [C_\lambda(V) - 1].$$

On the other hand, the right-hand side of (4.9) is negative if $C_\lambda(V) < 1$. Thus $\sup_\lambda \lambda^2 C_\lambda(V)$ over the set $C_\lambda(V) > 1$ is $\geq$ the right-hand side of (4.8).

Now we turn to the

**Proof of Corollary 3.3.** By (4.7) and (3.4)

$$\mu^2 \leq \sup_\lambda \lambda^2 [2C_\rho \sup_x (M_{2p, \lambda} V^p)^{1/p} - 1]$$

(4.10)  

$$= \sup_{x, \delta} [2C_\rho \delta^{-2} (M_{2p, \delta} V^p)^{1/p} - \delta^{-2}].$$

This equals the last expression in (3.6). For let $L$ be the latter expression. Then

$$\left( \delta^{-n} \int_{|y-x|<\delta} V(y)^p dy \right)^{1/p} \leq (L + \delta^{-2})/2C_\rho, \quad \delta > 0.$$  

This implies

$$(M_{2p, \delta} V^p)^{1/p} \leq (\delta^2 L + 1)/2C_\rho.$$
If we substitute this into (4.10), we obtain
\[ \mu^2 \leq \sup_{x, \delta} [\delta^{-2}(\delta^2 L + 1) - \delta^{-2}] = L. \]

The same reasoning works in reverse. The second estimate in Corollary 3.3 is proved in the same way using inequality (3.5).

Corollary 3.4 is an immediate consequence of (3.4) taking \( \lambda = 0 \).

A function \( V(x) \) is said to satisfy the \( A_\infty \) condition if there is \( p > 1 \) such that
\[
\left( \frac{1}{|Q|} \int_Q V(x)^p \, dx \right)^{1/p} \leq L_p |Q| \int_Q V(x) \, dx
\]
holds for all cubes \( Q \), where \( |Q| \) is the volume of \( Q \) (cf. [8]). We have

**Corollary 4.4.** If \( V(x) \) satisfies the \( A_\infty \) condition, then
\[
(4.11) \quad C_\lambda(V) \leq N_p \|M_{2,1/\lambda} V\|_\infty.
\]

**Proof.** From the definition we see that there is a constant \( L'_p \) such that
\[
(M_{2p,\delta} V^p)^{1/p} \leq L'_p M_{2,\delta} V.
\]

We now merely apply Theorem 3.2.

### 5. Invariance of the essential spectrum

For a closed operator \( A \) on a Banach space we define the essential spectrum of \( A \) as
\[
\sigma_e(A) = \bigcap_K \sigma(A + K)
\]
where the intersection is taken over all compact operators \( K \). We give sufficient conditions for \( H \) to have the same essential spectrum as \( -\Delta \).

**Theorem 5.1.** Assume that

(a) \( C_\lambda(V) \to 0 \) as \( \lambda \to \infty \).

(b) For some \( \lambda_0 \geq 0 \),
\[
C_{\lambda_0}(V^R) \to 0 \quad \text{as} \quad R \to \infty
\]
where
\[
V^R(x) = \begin{cases} 0, & |x| \leq R, \\ V(x), & |x| > R. \end{cases}
\]

Then
\[
(5.1) \quad \sigma_e(H) = \sigma_e(-\Delta) = [0, \infty).
\]

**Proof.** By (a) and (1.1), for each \( \epsilon > 0 \) there is a constant \( C_\epsilon \) such that
\[
(Vu, u) \leq \epsilon \|\nabla u\|^2 + C_\epsilon \|u\|^2.
\]
Moreover, if \( \phi(x) \in C_0^\infty \) is the function used in the proof of Theorem 3.2, then
\[
C_{\lambda_0}(V(1 - \phi_R)) \to 0 \quad \text{as} \quad R \to \infty
\]
by (b). These two conditions are necessary and sufficient for \( V^{1/2} \) to be compact from \( H^{1,2} \) to \( L^2 \) (cf. [14, p. 172]). This in turn is sufficient for \( H \) to have a 1/2 extension satisfying (5.1) (cf. [14, p. 149]).

**Bibliography**


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