RIGIDITY FOR COMPLETE WEINGARTEN HYPERSURFACES

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ABSTRACT. We classify, locally and globally, the ruled Weingarten hypersurfaces of the Euclidean space. As a consequence of the local classification and a rigidity theorem of Dajczer and Gromoll, it follows that a complete Weingarten hypersurface which does not contain an open subset of the form $L^3 \times \mathbb{R}^{n-3}$, where $L^3$ is unbounded and $n \geq 3$, is rigid.

Introduction

Recently Dajczer and Gromoll [DG] showed that a complete hypersurface $M^n$, $n \geq 4$, of the euclidean space $\mathbb{R}^{n+1}$ is rigid, unless it contains an open subset $U$ such that either $U = L^3 \times \mathbb{R}^{n-3}$ with $L^3$ unbounded or $U$ is completely ruled. We recall that a completely ruled submanifold is a ruled submanifold with complete rulings. It is not known if there exists a nowhere ruled three-dimensional irreducible hypersurface which is not rigid (see [DG2]).

We observe that there is an abundance of hypersurfaces of the euclidean space which admit local isometric deformations. A classification of such hypersurfaces was obtained by Sbrana [S] and Cartan [C]. A special case is given by the minimal hypersurfaces of rank two discussed in [DG].

In this paper we consider the rigidity question for complete hypersurfaces $M^n$ which satisfy the additional condition of being Weingarten, i.e. there exists a differentiable function relating the mean curvature and the scalar curvature of $M$. Our main result is the following.

Theorem A. Let $M^n$, $n \geq 4$, be a complete Weingarten immersed hypersurface of $\mathbb{R}^{n+1}$, which does not contain an open subset $U = L^3 \times \mathbb{R}^{n-3}$ with $L^3$ unbounded. Then $M$ is rigid.

The above result is an immediate consequence of the rigidity theorem of Dajczer and Gromoll and the following local classification of ruled Weingarten hypersurfaces.

Theorem B. Let $M^n$, $n \geq 3$, be a connected ruled Weingarten hypersurface of $\mathbb{R}^{n+1}$. Then $M^n$ is either

(i) flat;

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or it is an open subset of one of the following:

(ii) $Q^3 \times \mathbb{R}^{n-3}$, where $Q^3 \subset \mathbb{R}^4$ is a cone over a product of circles in $S^3$, or over a minimal ruled surface in $S^3$;

(iii) $Q^2 \times \mathbb{R}^{n-2}$, where $Q^2 \subset \mathbb{R}^3$ is a ruled helicoidal surface or a hyperboloid of revolution.

The classification for $n = 2$ was obtained in 1865 by Beltrami [B] and Dini [D], see (2.29). We observe that the classification of Theorem B is complete since the minimal ruled surfaces in $S^3$ are given in [L], see (2.16).

Now if we assume $M$ to be complete, we have

**Corollary C.** Let $M^n$, $n \geq 3$, be a complete connected ruled Weingarten hypersurface in $\mathbb{R}^{n+1}$. Then, $M$ is either

(i) a product $Q^2 \times \mathbb{R}^{n-2}$, where $Q^2$ is a complete ruled helicoidal surface of a hyperboloid of revolution; or

(ii) a cylinder over a complete curve.

1. Preliminaries

Let $M^n \subset \mathbb{R}^{n+1}$ be a connected orientable immersed hypersurface endowed with the induced metric. The *relative nullity* of the immersion at a point $p \in M$, is $\ker A(p)$, where $A$ denotes the second fundamental form of the hypersurface. Suppose that the relative nullity has constant dimension $\nu = n - k$. Then the Gauss map $\phi: M^n \to S^n \subset \mathbb{R}^{n+1}$ is parallel along each leaf of the relative nullity foliation, and provides (locally) a Gauss parametrization of $M$ as it was defined in [DG]. More precisely, there exists an isometric immersion $g: L^k \to S^n$, which is a local parametrization of the image of the Gauss map $\phi$, and a differentiable function $\gamma: L^k \to \mathbb{R}$ (support function) such that

$$X: U \subset \Lambda \to M^n \subset \mathbb{R}^{n+1},$$

$$(x, v) \mapsto X(x, v) = \gamma(x)g(x) + \text{grad} \gamma(x) + v$$

is a local parametrization of $M^n$, where $\Lambda$ is the normal bundle of the immersion $g$. $X$ is the so-called Gauss parametrization of $M$.

For each $(x, v) \in U \subset \Lambda$, let $\text{Hess} \gamma(x)$ denote the hessian of $\gamma$ and $B_v$ the second fundamental form of the immersion $g$ at $x \in L^k$, relative to the normal vector $v$. Then the selfadjoint operator defined on the tangent space of $L^k$ at $x$,

$$P_{(x, v)} = \gamma(x)I + \text{Hess} \gamma(x) - B_v$$

is nonsingular. Moreover, the second fundamental form $A_{(x, v)}$ of $X$ at $(x, v)$ is given by $-P^{-1}$, when restricted to the orthogonal complement of the relative nullity distribution. We refer to [DG] for the above results.
For each vector field \( e: L \to \mathbb{R}^{n+1} \), we may consider an associated vector field \( \overline{e}: U \subset \Lambda \to \mathbb{R}^{n+1} \) defined by
\[
\overline{e}(x, v) = e(x), \quad \forall (x, v) \in U,
\]
i.e. \( \overline{e} \) is the euclidean parallel transport of \( e(x) \) along the leaves of the relative nullity foliation of \( M \). Therefore, if \( e \) is a vector field normal (resp. tangent) to the immersion \( g \), then the associated vector field \( \overline{e} \) belongs (resp. is orthogonal) to the relative nullity distribution.

In what follows we consider hypersurfaces \( M^n \subset \mathbb{R}^{n+1} \) with constant index of relative nullity \( \nu = n-2 \), locally parametrized as in (1.1). Moreover, we choose orthonormal vector fields \( e_1, \ldots, e_n \), locally defined on \( L^2 \), such that \( e_1(x), e_2(x) \) are tangent to the immersion \( g \) at \( x \) and \( e_3(x), \ldots, e_n(x) \) generate the normal space of the immersion in \( S^n \). Let \( \overline{e}_i(x, v) = e_i(x), \ 1 \leq i \leq n, (x, v) \in U \subset \Lambda, \) be the associated vector fields on \( M \). With respect to this frame the second fundamental form of \( X \) at \( (x, v) \) is given by
\[
A = \begin{pmatrix}
-P_{x,v}^{-1} & 0 \\
0 & 0
\end{pmatrix},
\]
where \( P \) is defined by (1.2).

It follows that the mean curvature \( \overline{H} \) and the scalar curvature \( \overline{S} \) of \( M \) at \((x, v)\) are given respectively by
\[
\overline{H}(x, v) = -\operatorname{tr} A = \frac{\operatorname{tr} P}{\det P},
\]
\[
\overline{S}(x, v) = \frac{1}{\det P}.
\]

**Lemma 1.6.** Let \( M^n \subset \mathbb{R}^{n-1} \) be a ruled immersed hypersurface with constant index of relative nullity \( \nu = n-2 \). Then the immersion \( g \) is a ruled surface in \( S^n \).

**Proof.** Let
\[
X(s, \lambda, \mu_j) = c(s) + \lambda \xi(s) + \sum_{j=1}^{n-2} \mu_j \eta_j(s)
\]
be a local parameterization of \( M \), where \( c(s) \) is a curve orthogonal to the ruling, \( \eta_j, 1 \leq j \leq n-2 \), generate the relative nullity and \( \{\xi, \eta_j\} \) generate the ruling of \( M^n \). Then the Gauss map depends only on the parameters \( s, \lambda, \) since \( \eta_j \) generate the relative nullity distribution. Moreover, for \( s = s_0 \), the Gauss map describes a curve which is orthogonal to the subspace generated by \( \xi(s_0), \eta_j(s_0), 1 \leq j \leq n-2 \). Therefore it is contained in a great circle of \( S^n \). Q.E.D.

**Fact 1.7.** It follows from the above lemma that if \( M \) is a ruled hypersurface then the frame considered earlier may be chosen such that \( e_1(x) \) is tangent to the ruling of the immersion \( g \). Thus the second fundamental form \( \theta \) of \( g \) with
values in the normal bundle satisfies $\theta(e_i, e_i) = 0$. Therefore, the associated frame tangent to $M$, $\overline{e}_i(x, v) = e_i(x)$, is such that $\overline{e}_i$, $3 \leq i \leq n$, generate the relative nullity, $\overline{e}_i$, $2 \leq i \leq n$, generate the ruling and $\langle A\overline{e}_2, \overline{e}_2 \rangle = 0$.

For such a frame, the second fundamental form of the immersion $g$, with respect to $e_i$, $3 \leq i \leq n$, will be denoted by

$$(1.8) \quad B_i(x) = \begin{pmatrix} 0 & \beta_i \\ \beta_i & \lambda_i \end{pmatrix}, \quad 3 \leq i \leq n,$$

and the operator $\gamma(x)I + \text{Hess}\gamma(x)$ will be denoted by

$$(1.9) \quad \gamma(x)I + \text{Hess}\gamma(x) = \begin{pmatrix} 0 & h \\ h & \alpha \end{pmatrix}.$$

Now we assume that the submanifold $M^n \subset \mathbb{R}^{n+1}$ is Weingarten, i.e. there exists a differentiable function $F(\overline{H}, \overline{S}) = 0$. Taking exterior derivatives we obtain

$$\frac{\partial F}{\partial \overline{H}}d\overline{H} + \frac{\partial F}{\partial \overline{S}}d\overline{S} = 0.$$

Therefore, applying to vector fields tangent to $M$, we conclude that

$$(1.10) \quad d\overline{H} \wedge d\overline{S} = 0,$$

since the partial derivatives of $F$ are not simultaneously zero.

**Fact 1.11.** Let $M^n \subset \mathbb{R}^{n+1}$ be a ruled Weingarten hypersurface with constant index of relative nullity $\overline{\nu} = n - 2$. Then it follows from (1.4) to (1.10) that

$$(1.12) \quad d \left( \alpha(x) - \sum_{i=3}^{n} t_i \lambda_i(x) \right) \wedge d \left( h(x) - \sum_{j=3}^{n} t_j \beta_j(x) \right) = 0$$

for $t_i \in \mathbb{R}$.

### 2. Proofs of the theorems

For the proof of Theorem B we will need the following three propositions.

**Proposition 2.1.** Let $M^n \subset \mathbb{R}^{n+1}$ be a connected ruled Weingarten hypersurface without flat points. Suppose that the dimension of the first normal space of $g$ is constant equal to 1. Then, there exists a totally geodesic submanifold $S^3 \subset S^n$ such that $g(L^2) \subset S^3$ is a ruled Weingarten surface which satisfies

$$H^2 + c^2(K - 1) = 0,$$

where $H$ and $K$ are the mean and Gaussian curvature and $c$ is a constant. Moreover, $M^n$ is contained in a euclidean product $Q^3 \times \mathbb{R}^{n-3}$, where $Q^3 \subset \mathbb{R}^4$ is a ruled Weingarten surface with index of relative nullity $\nu = 1$.

**Proposition 2.2.** Let $g: L^2 \rightarrow S^3$ be a connected ruled surface in $S^3$ such that

$$H^2 + c^2(K - 1) = 0.$$
Then either $H = 0$ or $H = c \neq 0$ and $K = 0$. In the latter case the immersed surface is contained in the product of two circles.

**Proposition 2.3.** Let $M^3 \subset \mathbb{R}^4$ be a connected ruled Weingarten hypersurface, with index of relative nullity $\nu = 1$. Suppose that the image of the Gauss map $g(L^2)$ is either

(i) a minimal surface in $S^3$; or
(ii) it is contained in the product of two circles.

Then $M^3$ is an open subset of a cone over $g(L^2)$.

We need the following result. Recall that the first normal space of an immersion is the subspace generated by the second fundamental form.

**Lemma 2.4.** Let $M^n$ be a ruled Weingarten hypersurface without flat points. Then, for each $x \in L^2$, the dimension of the first normal space $N_1$ of $g$ is less than or equal to 1.

**Proof.** The ruled hypersurface $M$ has no flat points if and only if the index of relative nullity is constant $\nu = n - 2$.

Since $M^n$ is a ruled hypersurface, it follows from Lemma 1.6 that $g(L^2)$ is a ruled surface. Let $e_1(x), e_2(x)$ be a locally defined tangent frame to the immersion $g$ such that $e_1(x)$ is tangent to the ruling. Let $N_1(x)$ be the first normal space of $g$ at $x$. Since $N_1$ is generated by $\theta(e_1, e_2)$, $\theta(e_2, e_2)$, it follows that $\dim N_1 \leq 2$.

Suppose $\dim N_1(x) = 2$. We choose $e_3(x), e_4(x)$ generating $N_1$ such that $e_4$ is orthogonal to $\theta(e_1, e_2)$. Then the second fundamental form with respect to $e_3$ and $e_4$ in the tangent basis $e_1, e_2$ is given respectively by

$$B_3 = \begin{pmatrix} 0 & \beta_3 \\ \beta_3 & \lambda_3 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_4 \end{pmatrix}.$$ 

Since $M^n$ is a Weingarten hypersurface it follows from (1.13) that $\lambda_4 \beta_3 = 0$. If $\lambda_4 = 0$, then $B_4 = 0$. If $\beta_3 = 0$, then $\theta(e_1, e_2) = 0$. In both cases we have a contradiction, since we assumed that $\dim N_1 = 2$. Q.E.D.

**Proof of Proposition 2.1.** Since $M$ is a ruled hypersurface without flat points, the index of relative nullity is constant $\nu = n - 2$. Let $e_1(x), \ldots, e_n(x)$ be an orthonormal frame defined locally on $L^2$ as in Fact 1.7. Moreover, we can choose $e_3(x)$ to generate the first normal space $N_1$ of $g$. For such a frame, the second fundamental form of $g$ (1.8) reduces to

$$B_3 = \begin{pmatrix} 0 & \beta \\ \beta & \lambda \end{pmatrix}, \quad B_i = 0, \quad 4 \leq i \leq n. \tag{2.5}$$

Since $M$ is Weingarten, it follows from (1.12) that

$$d(\alpha - t\lambda) \wedge d(h - t\beta) = 0, \quad t \in \mathbb{R}. \tag{2.6}$$

Applying (2.6) to the pair $(e_i, \partial/\partial t)$, $i = 1, 2$, we obtain

$$-\beta d\alpha + \lambda dh = 0. \tag{2.7}$$
It follows from (2.8) that there exist constants $c_1, c_2$, not simultaneously zero, such that

$$c_1 \beta + c_2 \lambda = 0. \tag{2.9}$$

Observe that $c_2 \neq 0$. In fact if $c_2 = 0$, then from (2.9) we have $\beta = 0$. Now, $\dim N_1 = 1$ implies that $\lambda \neq 0$ and (2.7) implies that $h$ is a constant. Therefore, it follows from (1.5) that $\bar{S} = -1/h^2$ is constant. However, Theorem 3.4 in [DG] implies that $\dim N_1 = 0$, which is a contradiction.

Therefore, we have

$$\lambda = c\beta \tag{2.10}$$

and $\beta \neq 0$ in $L^2$. Moreover, it follows from (2.7) that $\alpha = ch + \bar{c}$, where $c$ and $\bar{c}$ are constants.

Now we prove that the first normal space of the immersion $g: L^2 \to S^n$ is parallel. In fact, let $\eta$ be any vector field generated by $e_4, \ldots, e_n$. Then it follows from the Codazzi equation that

$$\langle \nabla_{e_1} \eta, e_3 \rangle e_2 = \langle \nabla_{e_2} \eta, e_3 \rangle e_1.$$

Using (2.5) we get

$$\langle \nabla_{e_1} \eta, e_3 \rangle \beta e_1 + \left[\langle \nabla_{e_1} \eta, e_3 \rangle \lambda - \langle \nabla_{e_2} \eta, e_3 \rangle \beta \right] e_2 = 0.$$

Since $\beta \neq 0$ we conclude that

$$\langle \nabla_{e_1} \eta, e_3 \rangle = \langle \nabla_{e_2} \eta, e_3 \rangle = 0.$$

Hence the first normal space of the immersion $g$ is parallel. It follows that there exists a totally geodesic submanifold $S^3 \subset S^n$ which contains the image of $g$. Therefore, the normal bundle $\Lambda$ of $g$ splits into $\Lambda = \Lambda_1 + \Lambda_{n-3}$, where $\Lambda_1$ is the normal bundle in $S^3$ and the orthogonal complement $\Lambda_{n-3}$ is parallel in $\mathbb{R}^{n+1}$. Hence, $M^n$ splits as a consequence of the Gauss parametrization.

Finally, from (2.5) we obtain that the mean curvature $H$ and the Gaussian curvature $K$ satisfies $H = \lambda$ and $K - 1 = \beta^2$. Therefore, it follows from (2.10) that $H^2 + c^2(K - 1) = 0$. Q.E.D.

**Fact 2.11.** It follows from the preceding proof that if $M^n$ satisfies the hypothesis of Proposition 1 then there is a frame locally defined on $L^2$ for which

$$B_3 = \begin{pmatrix} 0 & \beta \\ \beta & \lambda \end{pmatrix}, \quad B_i = 0, \quad 4 \leq i \leq n,$$

$$\gamma I + \text{Hess} \gamma = \begin{pmatrix} 0 & h \\ h & \alpha \end{pmatrix},$$

where $\lambda = c\beta$, $\alpha = ch + \bar{c}$, and $c$, $\bar{c}$ are constants.
Proof of Proposition 2.2. If the constant \( c \) is zero, then \( H = 0 \). Otherwise, we will show that \( K = 0 \) and hence \( H = c \).

Let \( g: L^2 \to S^3 \) be a parametrized ruled surface in \( S^3 \). We may consider

\[ g(s, t) = \cos t \sigma(s) + \sin t e(s), \]

where \( \sigma(s) \) and \( e(s) \) are vectors in \( \mathbb{R}^4 \) such that

\[ |\sigma| = 1 = |e|, \quad \langle e, \sigma \rangle = 0, \quad \langle e', \sigma' \rangle = 0. \]

Moreover, we may choose the parameter \( s \) such that \( |e'| = 1 \). We introduce the following notation

\[ p(s) = (e', \sigma' \times e \times \sigma) = (e' \sigma' e \sigma), \]

\[ A(s) = |\sigma'|^2 - \langle \sigma', e \rangle^2, \]

\[ B(s) = 1 - \langle \sigma', e \rangle^2, \]

\[ G(s, t) = A \cos^2 t + B \sin^2 t. \]

We observe that \( p = \sqrt{AB} \). Moreover, it follows by a straightforward computation that the mean and Gaussian curvature of the surface are given by

\[ H = \frac{l - 2p(\sigma', e)}{2G^{3/2}}, \quad K - 1 = -\frac{p^2}{G^2}, \]

where

\[ l(s, t) = \cos^2 t (\sigma'' \sigma' e \sigma) + \sin^2 t (e'' e' e \sigma) \]

\[ + \sin t \cos t [(\sigma'' e' e \sigma) + (e'' \sigma' e \sigma)]. \]

By hypothesis \( H^2 + c^2(K - 1) = 0 \), therefore, without loss of generality, we have

\[ l - 2p(\sigma', e) - 2cpG^{1/2} = 0. \]

Taking a derivative with respect to \( t \) we get

\[ \frac{\partial l}{\partial t} - cpG^{-1/2} \frac{\partial G}{\partial t} = 0. \]

In particular for \( t = 0 \), it follows from (2.14) and (2.15) that

\[ (\sigma'' e' e \sigma) + (e'' \sigma' e \sigma) = 0. \]

Hence (2.15) reduces to

\[ 2 \sin t \cos t [(\sigma'' \sigma' e \sigma) + (e'' e' e \sigma)] - cpG^{-1/2} \frac{\partial G}{\partial t} = 0, \]

which is equivalent to

\[ \left[ \frac{1}{AB} \left( \frac{d}{ds} (\sigma' \times e \times \sigma) e' e \sigma \right) - cG^{1/2} \right] \frac{\partial G}{\partial t} = 0, \quad \forall s, t. \]

Since \( c \) is a nonzero constant, it follows that

\[ \frac{\partial G}{\partial t} = 0, \quad \forall s, t. \]
Therefore, using (2.12) we get $A = B = p = A = G$. From (2.13) we get $K = 0$ and hence $H = c$.

In order to show that the surface is contained in a product of two circles, we consider a local orthonormal frame field such that the second fundamental form is given by

$$B = \begin{pmatrix} 0 & \beta \\ \beta & \lambda \end{pmatrix}.$$ 

From $K = 0$ and $H = c$, we have $\det B = -1$ and $\lambda = 2c$, so that

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 2c \end{pmatrix}.$$ 

We conclude the proof by using the uniqueness part of the fundamental theorem for surfaces in the sphere, see [S]. Q.E.D.

**Proof of Proposition 2.3. Part (i).** Since $\nu = 1$, there exists an immersion

$$g: L^2 \to G \subset S^3$$

and a local Gauss parametrization of $M^3$ given by

$$X: \Lambda \to M^3 \subset R^4, \quad (x, v) \mapsto y(x)g(x) + \grad y(x) + v,$$

where $\Lambda$ is the normal bundle of the immersion $g$ and $y: L^2 \to R$ is a differentiable function.

$M^3$ is a ruled hypersurface, therefore it follows from Lemma 1.6 that $g(L^2)$ is a ruled surface in $S^3$. Since $g(L^2)$ is also minimal, we have $g$ locally given by

$$g(x_1, x_2) = \cos x_1 (\cos kx_2, \sin kx_2, 0, 0)$$

$$+ \sin x_1 (0, 0, \cos x_2, \sin x_2),$$

where $k$ is a positive constant, see [L or BDJ]. Let us consider the orthonormal tangent frame

$$e_1 = \frac{\partial g}{\partial x_1}, \quad e_2 = \frac{1}{\sqrt{E}} \frac{\partial g}{\partial x_2},$$

where $E = k^2 \cos^2 x_1 + \sin^2 x_1$. Let $e_3$ be a unitary normal vector field for the immersion $g$. Then the second fundamental form with respect to this frame is given by

$$B(x) = \begin{pmatrix} 0 & -k/E \\ -k/E & 0 \end{pmatrix}, \quad x = (x_1, x_2).$$

Let $\overline{e}_i$ be the associated frame defined on $M^3$, i.e.

$$\overline{e}_i(x, v) = e_i(x), \quad 1 \leq i \leq 3, \quad (x, v) \in \Lambda.$$ 

Then the second fundamental form for $M^3 \subset R^4$, with respect to this frame, is given by

$$A(x, v) = \begin{pmatrix} -P_{(x,v)}^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad (x, v) \in \Lambda.$$
where
\[ P(x, v) = \gamma(x) + \text{Hess} \gamma(x) - (v, e_3)B(x). \]
Moreover, it follows from Fact 2.11 that \( \gamma I + \text{Hess} \gamma \) is of the form
\[ \gamma I + \text{Hess} \gamma = \begin{pmatrix} 0 & h \\ h & \alpha \end{pmatrix}, \]
where
\[ 0 = -ck/E, \quad \alpha = ch + \bar{c}. \]
Hence, \( c = 0 \) and \( \alpha = \bar{c}. \)

Now, we want to determine \( \gamma: L^2 \to R \) such that
\[ (2.19) \quad \gamma I + \text{Hess} \gamma = \begin{pmatrix} 0 & * \\ * & \bar{c} \end{pmatrix}. \]
It follows from (2.17) and (2.19) that \( \gamma \) must satisfy
\[ (2.20) \quad \gamma + \frac{2}{x_1^2} = 0, \quad \gamma + \frac{1}{E} \left( \frac{\partial^2 \gamma}{\partial x_2^2} + \sin x_1 \cos x_1 (1 - k^2) \frac{\partial \gamma}{\partial x_1} \right) = \bar{c}. \]
From the first equation we get
\[ \gamma = f(x_2) \cos x_1 + h(x_2) \sin x_1. \]
Substituting into (2.20) we get \( \bar{c} = 0. \) Therefore, the trace of \( P \) and hence the trace of \( A \) is zero, i.e. \( M^3 \) is a minimal surface in \( R^4. \)

To conclude the proof in this case we use [BDJ].

**Part (ii).** By hypothesis \( g(L^2) \) is contained in the product of two circles, therefore the immersion \( g \) is locally given by
\[ (2.21) \quad g(x_1, x_2) = r_1 \left( \sin \frac{x_1}{r_1}, \cos \frac{x_1}{r_1}, 0, 0 \right) + r_2 \left( 0, 0, \sin \frac{x_2}{r_2}, \cos \frac{x_2}{r_2} \right), \]
where \( r_1^2 + r_2^2 = 1. \)

Let us consider the orthonormal frame field defined by
\[ (2.22) \quad e_1 = r_1 \frac{\partial g}{\partial x_1} - r_2 \frac{\partial g}{\partial x_2}, \quad e_2 = r_2 \frac{\partial g}{\partial x_1} + r_1 \frac{\partial g}{\partial x_2}. \]
Then the second fundamental form of the immersion \( g \) with respect to \( e_1, e_2 \) is given by
\[ B = \begin{pmatrix} 0 & 1 \\ r_2^2 - r_1^2 & r_1 \frac{\partial g}{\partial x_2} \end{pmatrix}. \]
Let \( e_3 \) be a unitary normal vector field for the immersion \( g \) and
\[ \bar{e}_i(x, v) = e_i(x), \quad 1 \leq i \leq 3, \quad (x, v) \in \Lambda, \]
the associated frame defined on \( M^3. \) Then the second fundamental form for \( M^3 \subset R^4, \) with respect to this frame, is given by
\[ A(x, v) = \begin{pmatrix} -P^{-1}(x,v) & 0 \\ 0 & 0 \end{pmatrix}, \]
where \( P(x, v) = \gamma(x) + \text{Hess} \gamma(x) - (v, e_3)B. \)
Moreover, it follows from Fact 2.11 that with respect to this frame

\[ \gamma I + \text{Hess} \gamma = \begin{pmatrix} 0 & h \\ h & \alpha \end{pmatrix}, \]

where

\[ c = \frac{r_2^2 - r_1^2}{r_1 r_2}, \quad \alpha = ch + \bar{c}. \]

We want to determine \( \gamma: L^2 \to \mathbb{R} \) which satisfies the above conditions. It follows from (2.22) that \( \gamma \) must satisfy

\[
\begin{align*}
\gamma + r_1^2 \frac{\partial^2 \gamma}{\partial x_1^2} - 2r_1 r_2 \frac{\partial^2 \gamma}{\partial x_1 \partial x_2} + r_2^2 \frac{\partial^2 \gamma}{\partial x_2^2} &= 0, \\
\gamma + r_1^2 \frac{\partial^2 \gamma}{\partial x_1^2} + r_4 r_1 \frac{\partial^2 \gamma}{\partial x_1 \partial x_2} + r_2^2 \frac{\partial^2 \gamma}{\partial x_2^2} &= \bar{c}.
\end{align*}
\]

(2.23)

Subtracting the above equation we get

\[
\frac{1}{r_1 r_2} \frac{\partial^2 \gamma}{\partial x_1 \partial x_2} = \bar{c}.
\]

Therefore

\[ \gamma(x_1, x_2) = \bar{c} r_1 r_2 x_1 x_2 + \gamma_1(x_1) + \gamma_2(x_2), \]

where \( \gamma_1 \) and \( \gamma_2 \) are functions which depend only on \( x_1 \) and \( x_2 \) respectively. Substituting (2.24) into (2.23), we obtain

\[ \gamma_1 + r_1^2 \frac{d^2 \gamma_1}{dx_1^2} + \gamma_2 + r_2^2 \frac{d^2 \gamma_2}{dx_2^2} + \bar{c} r_1 r_2 x_1 x_2 = 2r_1^2 r_2^2 \bar{c}. \]

(2.25)

Taking derivatives with respect to \( x_1 \), and then with respect to \( x_2 \) we conclude that \( \bar{c} = 0 \). Therefore, (2.24) reduces to

\[ \gamma(x_1, x_2) = \gamma_1(x_1) + \gamma_2(x_2), \]

where \( \gamma_1 \) and \( \gamma_2 \) satisfy the following equations

\[ \gamma_1 + r_1^2 \frac{\partial^2 \gamma_1}{\partial x_1^2} = a, \quad \gamma_2 + r_2^2 \frac{\partial^2 \gamma_2}{\partial x_2^2} = a, \]

(2.27)

where \( a \) is a constant.

Now we want to show that the Gauss parametrization of \( M^3 \) describes a cone over \( G \). In fact

\[ X(x_1, x_2, s) = \gamma(r_1 u_1 + r_2 v_1) + \frac{\partial \gamma_1}{\partial x_1} v_1 + \frac{\partial \gamma_2}{\partial x_2} v_2 + s(-r_2 u_1 + r_1 u_2), \]

where

\[
\begin{align*}
u_1 &= \left( \sin \frac{x_1}{r_1}, \cos \frac{x_1}{r_1}, 0, 0 \right), \\
u_2 &= \left( 0, 0, \sin \frac{x_2}{r_2}, \cos \frac{x_2}{r_2} \right), \\
v_1 &= \partial X/\partial x_1, \quad v_2 = \partial X/\partial x_2.
\end{align*}
\]
It follows from (2.26) and (2.27) that $X(x_1, x_2, s(x_1, x_2))$ is constant for

$$s(x_1, x_2) = \frac{a}{r_1 r_2} - \frac{r_2}{r_1} \gamma_1 + \frac{1}{r_2} \gamma_2,$$

which concludes the proof of case (ii). Q.E.D.

Finally, we prove Theorem B using the preceding results.

Proof of Theorem B. Let $\overline{M} = \{ p \in M; \overline{S}(p) \neq 0 \}$. Since $M$ is a ruled hypersurface, the sectional curvature $K$ at points of $\overline{M}$ is not identically zero. It follows from Lemma 2.4 applied to $\overline{M}$ that at each point of the image of the Gauss map the first normal space $N_1$ has dimension $\leq 1$. We have $\overline{M} = \overline{M}_0 \cup \overline{M}_1$, where at $\overline{M}_0$ the Gauss map is totally geodesic in $S^n$ and $\overline{M}_1$ is the open subset of points where $N_1$ has dimension 1.

Let $V_1$ be a connected component of $\overline{M}_1$, let $X: U \subset \Lambda \rightarrow V_1 \subset \mathbb{R}^{n+1}$ be a Gauss parametrization and let $g: L^2 \rightarrow S^n$ be the associated local parametrization of the Gauss map of $V_1$. It follows from Proposition 2.1 that there exists a totally geodesic submanifold $S^3 \subset S^n$ such that $g(L^2) \subset S^3$ is a ruled Weingarten surface which satisfies $H^2 + c^2(K - 1) = 0$. Moreover, $V_1$ is contained in a euclidean product $Q^3 \times \mathbb{R}^{n-3}$, where $Q^3 \subset \mathbb{R}^4$ is a ruled Weingarten surface with constant index of relative nullity $\nu = 1$.

Using Proposition 2.2, we obtain that either $g$ is a minimal immersion in $S^3$ or $K = 0, H = c$, and the image of $g$ is contained in the product of two circles of $S^3$. It follows from Proposition 2.3, that $Q^3$ is an open subset of a cone over the image of $g$, i.e. $V_1$ satisfies (ii).

Let $V_0$ be a connected open subset of $\overline{M}_0$. We have a Gauss parametrization for $V_0$ and $g$ the associated local parametrization of the image of the Gauss map of $V_0$. Since $g$ is totally geodesic in $S^n$, the normal bundle $\Lambda$ of the immersion $g$ is parallel in $\mathbb{R}^{n+1}$. Hence, using the Gauss parametrization we obtain that $V_0$ is an open subset of $Q^2 \times \mathbb{R}^{n-2}$, where $Q^2 \subset \mathbb{R}^3$ is a ruled Weingarten surface. It follows from the classical result of Beltrami [B] and Dini [D] that $Q^2$ is a ruled helicoidal surface or a hyperboloid of revolution, i.e. $V_0$ satisfies (iii).

We now observe that the boundary of $V_0$ does not intersect the boundary of $V_1$, since the determinant of the second fundamental form of the image of the Gauss map of $V_1$ in $S^3$ is bounded away from zero. Moreover, the boundaries of $V_0$ and of $V_1$ do not contain points where the scalar curvature $\overline{S}$ is zero. Since $\overline{M}$ is connected, this concludes the proof of the theorem. Q.E.D.

Remark 2.28. The ruled Weingarten surfaces $Q^2 \subset \mathbb{R}^3$ classified by Beltrami and Dini are given by

$$X(s, t) = (a \cos s + ct \sin s, a \sin s - ct \cos s, bs + \sqrt{1 - c^2}t)$$

where $a, b, c$ are constants.
Proof of Corollary C. We use Theorem B. If $M$ is complete, then it cannot be a cone. If $M$ splits as in (iii), then $M = Q^2 \times \mathbb{R}^2$, where $Q^2$ is a complete ruled helicoidal surface or a hyperboloid of revolution. If $M$ is flat, it follows from [HN] that $M$ is a cylinder over a complete curve. Q.E.D.

Proof of Theorem A. If $M^n \subset \mathbb{R}^{n+1}$, $n \geq 4$, is a complete hypersurface, it follows from [DG] that $M$ is rigid, unless it contains an open subset $U$ which is completely ruled.

We will show that the existence of such a subset $U$ contradicts the hypothesis of Theorem A. In fact, if we apply Theorem B to each connected component $U_0$ of $U$, we conclude that $U_0$ is completely ruled and flat. We consider a connected component of $U_0$ where the nullity is $n-1$. Then the ruling coincides with the nullity and therefore the nullity is complete. The argument used in [HN] implies that this component of $U_0$ is a cylinder over a curve (not necessarily complete). Moreover, each connected component of $U_0$ where the nullity is $n$ is totally geodesic. Hence, in both cases we obtain open subsets of type $L^3 \times \mathbb{R}^{n-3}$, with $L^3$ unbounded, which is a contradiction. Therefore, $M$ is rigid. Q.E.D.

References


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