SOME FAMILIES OF ISOPARAMETRIC HYPERSURFACES AND RIGIDITY IN A COMPLEX HYPERBOLIC SPACE

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ABSTRACT. The geometric notion of equivalence for submanifolds in a chosen ambient space is that of congruence. In this study, a certain type of isoparametric hypersurface of a complex hyperbolic space form is shown to have a rigid immersion by utilizing the congruences of a Lorentzian hyperbolic space form that lies as an S^1-fiber bundle over the complex hyperbolic space. Several families of isoparametric hypersurfaces (namely tubes and horospheres) are constructed whose immersions are rigid.

0. INTRODUCTION

Analogous to the Hopf fibration $S^1 \to S^{2m+1} \to CP^m$ of an odd-dimensional sphere over complex projective space, we have a Lorentzian S^1-fiber bundle, called anti-De Sitter spacetime (denoted by H_1^{2n+1}), that lies over a complex hyperbolic space form $CH^n(-4)$. The rigidity of an immersion of a hypersurface into CP^m is usually obtained by considering the S^1-fiber bundle over the hypersurface as a hypersurface of S^{2m+1} and then applying isometries of $\mathbf{R}^{2m+1}$ that restrict to isometries of S^{2m+1}. We shall use the same approach to establish the rigidity of certain isoparametrical real hypersurfaces of $CH^n(-4)$.

Let M^{2n-1} be a complete, connected real hypersurface of $CH^n(-4)$ with immersion i and $\pi : H_1^{2n+1} \to CH^n$ be the Riemannian submersion with timelike, totally geodesic fibers. If $\widetilde{M}_1^{2n} = \pi^{-1}(M)$ is the S^1-fiber bundle over M, there is an induced immersion $\tilde{i} : \widetilde{M} \to H_1^{2n+1}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\widetilde{M}_1^{2n} & \xrightarrow{\ \tilde{i}\ } & H_1^{2n+1} \\
\Big\downarrow{\scriptstyle \pi|_{\widetilde{M}}} & & \Big\downarrow{\scriptstyle \pi} \\
M^{2n-1} & \xrightarrow{\ i\ } & CH^n
\end{array}
$$

Now by considering the natural imbedding of H_1^{2n+1} into the complex Lorentzian euclidean space $\mathbf{C}_1^{n+1}$, we can view $\widetilde{M}$ as a Lorentzian submanifold of $\mathbf{R}_2^{2n+2}$, of real codimension 2, and thereby use known congruence results for submanifolds of semi-Riemannian euclidean spaces to obtain a congruence result for hypersurfaces of CH^n:

Theorem 2. *Let M and N be isoparametric orientable real hypersurfaces of CH^n that each have the distinguished direction U as a principal direction. If M and N have the same second fundamental form, i.e., the corresponding principal curvatures are the same, then M and N are congruent.*

This theorem establishes the rigidity of several families of isoparametric hypersurfaces of CH^n that are constructed in this study. These families form the model spaces for classifications of isoparametric hypersurfaces that will appear in other papers. Along the way, we will construct a horosphere and explicitly display the convergence of a family of geodesic hyperspheres to a horosphere using Theorem 2. The notion of a certain distinguished direction on a real hypersurface being a principal direction is central to the study.

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1. Real hypersurfaces of $CH^n(-4)$

Let $CH^n(-4)$, $n \geq 2$, denote a complex hyperbolic space equipped with the Bergman metric tensor, i.e., a complex space form of constant holomorphic sectional curvature -4. Let M^{2n-1} be an orientable real hypersurface of CH^n, ∇ and $\overline{\nabla}$ be the metric connections on M and CH^n, respectively, so that the Gauss and Weingarten formulae can be written as

(1.1) $\overline{\nabla}_X Y = \nabla_X Y + \langle HX, Y \rangle \xi$ and $\overline{\nabla}_X \xi = -HX$ for all $X, Y \in T(M)$,

where ξ is a unit normal field on M in CH^n and H denotes the second fundamental form (in this case the Weingarten map of ξ in $\text{End}[T(M)]$). We shall refer to the eigenvalues and eigenvectors of H in $\mathbf{R}$ and $T(M)$, respectively, as principal curvatures and principal directions.

Let J denote the complex structure of the ambient complex space form. It induces a bundle endomorphism ϕ of rank $2n - 2$ and a linear functional f on $T(M)$ given by setting at each point p of M

(1.2) $JX = \phi X + f(X)\xi$ for all X in $T_p(M)$.

Let $U = -J\xi$. As M is of codimension one we have $U \in T(M)$. The following equations now hold for all X, Y in $T(M)$:

(1.3) $f(X) = \langle X, U \rangle$,

(1.4) $f(\phi X) = 0$,

$$(1.5) \qquad\qquad \phi U = 0\,,$$

$$(1.6) \qquad\qquad \phi^2 X = -X + f(X)U\,,$$

$$(1.7) \qquad\qquad \langle X\,,\phi Y\rangle = -\langle \phi X\,,Y\rangle$$
$$\langle \phi X\,,\phi Y\rangle = \langle X\,,Y\rangle - f(X)f(Y).$$

$(\phi\,,f\,,U)$ is an example of what is called an almost contact structure on M. The tensor fields ϕ and U have the following derivatives:

$$(1.9) \qquad\qquad \nabla_X U = \phi H X\,,$$

$$(1.10) \qquad\qquad (\nabla_X \phi)Y = f(Y)HX - \langle HX\,,Y\rangle U.$$

We also have the usual Gauss and Codazzi equations for a real hypersurface of a complex space form (of holomorphic sectional curvature -4) in terms of ϕ and H:

$$(1.11) \quad R(X\,,Y)Z = \langle X\,,Z\rangle Y - \langle Y\,,Z\rangle X + \langle \phi X\,,Z\rangle \phi Y - \langle \phi Y\,,Z\rangle \phi X$$
$$- 2\langle X\,,\phi Y\rangle \phi Z + \langle HY\,,Z\rangle HX - \langle HX\,,Z\rangle HY\,,$$

$$(1.12) \qquad (\nabla_X H)Y - (\nabla_Y H)X = -f(X)\phi Y + f(Y)\phi X - 2\langle X\,,\phi Y\rangle U$$

for all $X\,,Y\,,Z \in T(M)$, where R is the curvature tensor on M.

An important special case will be when U is a principal direction on M. Under this assumption, more information can be gained concerning the structure of M. For example, it is straightforward to show that the direction U is principal on M if and only if the trajectories of U on M are geodesic [12]. In particular, the assumption that U is a principal direction will force a strong relationship to hold between H and ϕ:

Lemma 1 [12]. *Suppose that U is a principal direction on M with principal curvature α. Then*

$$(1.13) \qquad\qquad 2(H\phi H + \phi) = \alpha(\phi H + H\phi) \quad in\ T(M).$$

Since $\ker(f)$ is ϕ-invariant (formulae (1.3)–(1.7)), it is natural to investigate the action of ϕ on $\ker(f)$.

Lemma 2 [12]. *If λ is a principal curvature on M, let D_λ denote the distribution of principal directions on M with principal curvature λ. If $\lambda^2 - 1 \neq 0$ on a neighborhood in M and $X \in D_\lambda \cap \ker(f)$ on that neighborhood, then ϕX is principal on that neighborhood.*

Some information can be gleaned from the case that remains in the wake of Lemma 2:

Let $p \in M$ be a point at which $\lambda^2 = 1$ and λ is a principal curvature with a direction $X \in (\ker(f))_p$. We may choose $\alpha = 2$ at p by selecting a suitable orientation of M. Assume that $\lambda = -1$ at p. Then, for $X \in (D_\lambda)_p$, (1.13) yields ϕX principal with curvature 1. However, if $\alpha = 2$ and $\lambda = 1$ at p, then

for $X \in (D_\lambda)_p$, (1.13) is an identity and no information is gained concerning ϕX. This case must be treated separately.

We also note that assuming (1.13), $\lambda^2 \neq 1$, $\alpha \neq \pm 2$ and $\alpha \neq 2\lambda$ are equivalent statements.

The hypersurfaces studied in the sequel sections will have U principal and constant principal curvatures. If M is such a hypersurface with $\alpha \neq \pm 2$, we can choose a frame $\{X_1, \dots, X_{n-1}, \phi X_1, \dots, \phi X_{n-1}, U\}$ on M that consists entirely of principal directions, with the property that ϕ interchanges the distributions $\mathrm{span}\{X_1, \dots, X_{n-1}\}$ and $\mathrm{span}\{\phi X_1, \dots, \phi X_{n-1}\}$. In this case ϕ acts on $\ker(f)$ as a complex structure and, from (1.13), the principal curvatures of a ϕ-invariant plane $\mathrm{span}\{X_i, \phi X_i\}$ will be related by the equation

$$(1.14) \qquad \gamma_i = (\alpha \lambda_i - 2)/(2\lambda_i - \alpha)$$

where $HX_i = \lambda_i X_i$ and $H\phi X_i = \gamma_i \phi X_i$.

2. Some families of isoparametric hypersurfaces

In this section, families of isoparametric hypersurfaces in $CH^n(-4)$ are constructed that have the distinguished direction U as a principal direction. Our initial discussion will be of a more general nature: tubes in Riemannian manifolds. (For more detail, see [1, 4, 7, 9, 18].)

Recall first the notions of cut point and cut locus. (A detailed and analytic discussion of cut loci can be found in Vol. II of [10] and in [11].) A *cut point* of a point p in a Riemannian manifold M is a point $c = \gamma(t)$, where γ is a geodesic emanating from $p = \gamma(0)$ with the property that for $s > t$, the length of the curve $\gamma(J)$, $J = [0, s]$, is greater than the distance $d_M(p, \gamma(s))$. For instance, if $p \in S^2(r)$, its only cut point is its antipodal point.

The *cut locus* of a point $p \in M$, written $\mathrm{Cut}(p)$, is the set of all cut points of p. The cut locus of a point on a sphere is a singleton, whereas for a point p on a cylinder over S^1 in $\mathbf{R}^3$, $\mathrm{Cut}(p)$ is the axial line opposite p. Define $c(p) = \min\{d(p, q) \mid q \in \mathrm{Cut}(p)\}$.

Let N^m be an immersed submanifold of a Riemannian manifold M^n. Define the unit normal sphere bundle of N by

$$S^\perp(N) = \{X \in T(N)^\perp : \|X\| = 1\}.$$

Set $c(N) = \inf\{c(p) \mid p \in N\}$. Now for each $r \in (0, c(N))$, define the *tube of radius r about N in M* to be the hypersurface given by

$$N_r = \{\exp_q(rX): q \in N, \ X \in S^1(N)\}.$$

Let $\tau(t, X_q)$ be parallel translation of vector fields along the geodesic $(\gamma_X)_q$: $t \to \exp_q(tX)$. For $p = \exp_q(rX) \in N_r$, $\tau(t, X_q): T_q(M) \to T_p(M)$ is a linear isometry [15, p. 66]. (See Figure 1.) By the generalized Gauss lemma [11, p. 121], we have

$$(2.1) \qquad T_p(N_r) = \tau(r, X_q)(\{X_q\}^\perp) = \{\tau(r, X_q)X_q\}^\perp$$

and

(2.2) $T_p(N_r) \cong T_q(N) \oplus [\{X_q\}^\perp \cap T_q(N)^\perp]$,

where $\cong$ denotes the isomorphism of parallel translation.

For $X \in S^\perp(N)$ and $q \in N$, define $\widetilde{R}_X(t) \in \text{End}[T_q(N)]$, for each $t > 0$, by

$$\widetilde{R}_X(t)Y_q = \tau(t,X_q)^{-1}\{R(\tau(t,X_q)Y_q,\tau(t,X_q)X_q)\tau(t,X_q)X_q\}$$

where R is the curvature tensor of M. As we are primarily interested in the tangent space of the tube N_r, set

$$R_X(t) = \widetilde{R}_X(t)|_{\{X\}^\perp}.$$

Finally, define $F(t,X) \in \text{End}(\{X\}^\perp)$ for each $X \in S^\perp(N)$ to be the solution of the initial value problem

(2.3) $(d^2/dt^2)[F(t,X_q)] + R_X(t) \circ F(t,X_q) = 0$,

$$F(0,X_q) = P, \qquad (d/dt)[F(t,X_q)]|_{t=0} = -A_X \circ P + P^\perp,$$

for each $q \in N$, where $P: \{X\}^\perp \to T(N)$ and $P^\perp: \{X\}^\perp \to T(N)^\perp \cap \{X\}^\perp$ are orthogonal projections of the vector bundle $\{X\}^\perp = T(N) \oplus [T(N)^\perp \cap \{X\}^\perp]$ onto the indicated component distributions, and A_X is the Weingarten map of X on N in M. To simplify notation we shall write $F'(s,X_q)$ for $(d/dt)[F(t,X_q)]|_{t=s}$ and $F''(s,X_q)$ for $(d^2/dt^2)[F(t,X_q)]|_{t=s}$.

Theorem 1 [4, 9, 16]. *The second fundamental form of N_r at $p = \exp_q(rX)$ is given by*

(2.4) $H_r = \tau(t,X_q) \circ F'(r,X_q) \circ F(r,X_q)^{-1} \circ \tau(r,X_q)^{-1}$.
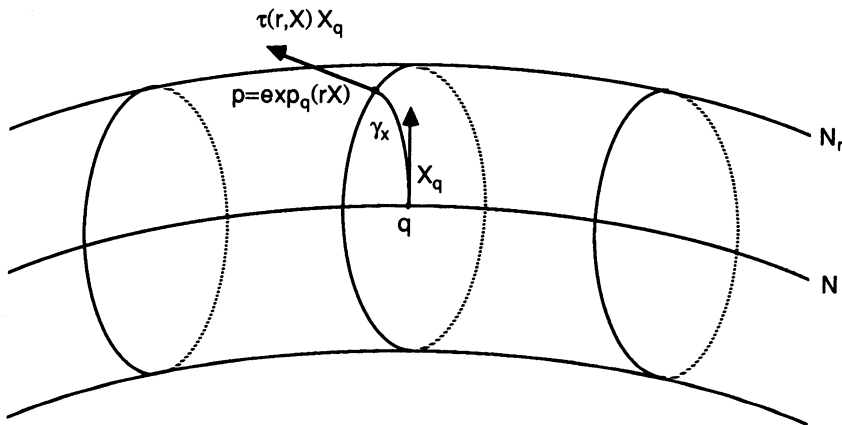

FIGURE 1

Hence, in order to find an explicit representation of the second fundamental form of a tube, we need merely select a suitable basis of $T(N_r)$ using (2.1) and (2.2), solve (2.3), and then compute (2.4). Of course (2.4) says that $H_r \in$ End$[T(N_r)]$ at $p = \exp_q(rX)$ is nothing more than parallel displacement of the endomorphism

$$F'(r, X_q) \circ F(r, X_q)^{-1} \in \text{End}[\{X_q\}^{\perp}]$$

along the geodesic γ_X emanating from q and passing through p.

Now we will assume that M is not an arbitrary Riemannian manifold but the ambient space discussed in the previous section, namely a complex hyperbolic space. Let N be an immersed orientable submanifold of $\mathbf{CH}^n(-4)$. As $\mathbf{CH}^n$ is a symmetric space, once a suitable basis of $\{X_q\}^{\perp}$ is selected (where $q \in N$ and $X \in S^{\perp}(N)$), parallel displacement along the geodesic γ_X will preserve the basis and the respective orthogonality relations between its elements. Thus, H_r will have the same matrix representation with respect to the displaced basis as $F'(r, X_q) \circ F(r, X_q)^{-1}$ has with respect to the chosen basis of $\{X_q\}^{\perp}$. This simplifies the calculation of (2.4) considerably.

An additional feature of $\mathbf{CH}^n$ is that $R_X(r)$ is of a particularly simple form. Let $X \in S^{\perp}(N)$ and $Y \in \{X, JX\}^{\perp}$. Direct computations using the Gauss equation show that $(R_X(t)Y)_q = -Y_q$ and $(R_X(t)JX)_q = -4JX_q$ for all $t > 0$.

As N_r is orientable, let ξ be a local unit normal field on N_r. From the earlier discussion on tubes, at any point $p = \exp_q(rX) \in N_r$, we can write $\xi_p = \tau(r, X_q)X_q$. In this way a unique point q and a unique direction in $S^{\perp}(N_q)$ can be associated to each point $p \in N_r$.

In order to simplify notation, set $Y^* = \tau(r, X_q)Y \in T_p(N_r)$ for any $Y \in \{X_q\}^{\perp}$. In particular, we shall write ξ^* for a normal field on N_r, and ξ will refer to the associated direction in $S^{\perp}(N)$; that is, $\xi_p^* = \exp_q(r\xi_q)$. In terms of §1, $U_p^* = -J\xi_p^* = \tau(r, X_q)J\xi_q$ and

$$\{U_p^*\}^{\perp} \cap T_p(N_r) = \tau(r, \xi_q)[(T_q(N) \cap (J\xi_q)^{\perp}) \oplus (\{J\xi_q, \xi_q\}^{\perp} \cap T_q(N)^{\perp})]$$

is the ϕ-invariant subspace, $(\ker(f))_p$, of $T_p(N_r)$.

Also, $\mathbf{CH}^n(-4)$, as a space of constant negative holomorphic sectional curvature, is a space of negative sectional curvature. Since $\mathbf{CH}^n(-4)$ is simply connected, by Theorem 8.1, Chapter VIII, Vol. 2 of [10], all cut loci will be empty. This means that tubes of radius less than $\min\{d(p, N): p$ a focal point of $N\}$ may be constructed about N.

The following are examples of tubes that form families of isoparametric hypersurfaces of the type discussed in §1. In each, it will be crucial that $U_q = \tau(r, X_q)^{-1}U_p^*$ is tangent or normal to the core at q. (Also see [1].)

Example 1. Let $N = H^n(-1)$ be a real space form of constant sectional curvature -1 immersed in $\mathbf{CH}^n$ as a totally geodesic and totally real submanifold. (See the proof of Theorem 1 in [3].) As N has no focal points, for
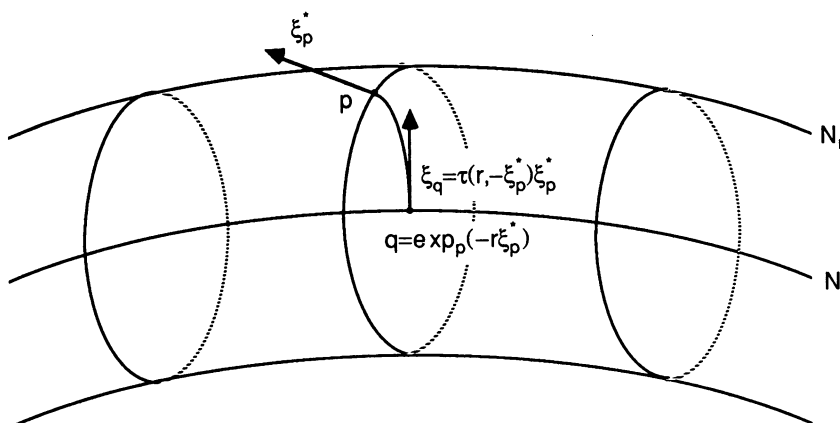
$$\xi_p^*$$

$$p$$

$$\xi_q = \tau(r, -\xi_p^*)\xi_p^*$$

$$q = \exp_p(-r\xi_p^*)$$

$$N_r$$

$$N$$

<div align="center">FIGURE 2</div>

any $r > 0$ let N_r be the tube of radius r about N in $\mathbf{C}H^n$. Since N is a totally real submanifold of dimension n, if ξ^* is a unit normal to N_r, at each point $p \in \exp_q(r\xi) \in N_r$ we have $U_q = -J\xi_q \in T_q(N)$. So let $\{X_1, X_2, \dots, X_{n-1}, U_q\}$ be an orthonormal basis of $T_q(N)$. If (ϕ, f) is the almost contact structure induced by J on N_r, then

$$B_p = \{X_1^*, \dots, X_{n-1}^*, \phi(X_1^*), \dots, \phi(X_{n-1}^*), U_p^*\}$$

forms an orthonormal basis of $T_p(N_r)$. Setting $\phi X_i = \tau(r, \xi_q)^{-1}[\phi(X_i^*)]$ for each $i = 1, \dots, n-1$ allows us to write

$$B_q = \{X_1, \dots, X_{n-1}, \phi X_1, \dots, \phi X_{n-1}, U_q\}$$

for an orthonormal basis of $\{\xi_q\}^\perp$. With respect to this basis, (2.5) has the matrix solution

$$F(t, \xi_q) = \operatorname{diag}(\cosh(t)I_{n-1}, \sinh(t)I_{n-1}, \cosh(2t)).$$

Now, from (2.4), the second fundamental form, H_r, of the tube will have the following matrix representation with respect to the basis B_p:

$$(2.6) \qquad H_r = \operatorname{diag}(\tanh(r)I_{n-1}, \coth(r)I_{n-1}, 2\tanh(2r)).$$

We see that the representation of H_r depends only on r and is hence constant on N_r. As a consequence N_r is isoparametric. Also notice that U is principal on N_r.

Example 2. Let $N = \mathbf{C}H^k$, $k = 0, 1, \dots, n-1$, be a complex space form immersed in $\mathbf{C}H^n(-4)$ as a totally geodesic submanifold (see [3]). In case $k = 0$, we are regarding a point to be a trivial complex space form. Otherwise, from [3], the $\mathbf{C}H^k$ will have constant holomorphic sectional curvature -4. Again, as N has no focal points, for any $r > 0$ let N_r be the tube of radius r about N in $\mathbf{C}H^n(-4)$ and ξ^* be a global unit normal to N_r. As N is

a complex submanifold of CH^n, its tangent space is invariant under J. So $\tau(r,\xi_q)[T_q(N)]$ is a ϕ-invariant subspace of $T_p(N_r)$ and

$$U_q = -\tau(r,\xi_q)^{-1}J\xi^* \in T_q^\perp(N).$$

Let $\{X_1, \ldots, X_k, JX_1, \ldots, JX_k\}$ be an orthonormal basis of $T_q(N)$ and extend to an orthonormal basis

$$B_q = \{X_1, \ldots, X_k, JX_1, \ldots, JX_k, X_{k+1}, \ldots, X_{n-1}, JX_{k+1}, \ldots, JX_{n-1}, U_q\}$$

of $\{\xi_q\}^\perp$. The last $2n-1-2k$ elements of B_q form a basis of $T_q(N)^\perp \cap \{\xi_q\}^\perp$. The radially symmetric matrix solution to (2.5), with respect to B_q, has the matrix representation

$$F(t,\xi_q) = \mathrm{diag}(\cosh(t)I_{2k}, \sinh(r)I_{2n-2k-2}, \sinh(2r)).$$

Computing (2.4) yields the following matrix representation for the second fundamental form on N_r with respect to the translation of B_q:

(2.7) $H_r = \mathrm{diag}(\tanh(r)I_{2k}, \coth(r)I_{2n-2k-2}, 2\coth(2r)).$

Again, $U^* = -J\xi^*$ is principal on N_r and N_r is isoparametric.

Example 3. The previous two examples, despite their obvious differences as tubes with different cores, do have one thing in common. Each core has no focal points so that tubes of any radii may be constructed. Hence, if the basis of $T_p(N_r)$, $p = \exp_q(r\xi_q)$, is chosen for each $r > 0$ to be compatible (via τ) with B_q, then we can discuss the limit of the tensor H_r, viewed as acting on $\mathrm{span}(B_q)$, as $r \to \infty$. Clearly, this matrix limit is given by

(2.8) $\displaystyle\lim_{r\to\infty} H_r = \mathrm{diag}(I_{2n-2}, 2).$

The geometric significance of this is obscure from our view of these tensors as acting on the tangent space of the core of the tube. However, in §4 (originally in [13]), a hypersurface is constructed in an analytic way that has a second fundamental form of the form (2.8). So we know that such a hypersurface exists; the question is whether we can obtain a geometric characterization.

Choose a point $p \in CH^n$ and any direction $\xi \in T(CH^n)$. For each $r > 0$, let $q(r) = \exp_p(r\xi_p)$ and γ_ξ be geodesic with initial direction ξ_p that joins p to $q(r)$. Then p is on each geodesic hypersphere, $G_r(q(r))$, centered at $q(r)$ with radius r.
It is known that as $q(r)$ recedes from p $(r \to \infty)$ the $G_r(q(r))$ approach a limiting hypersurface, M^∞, called a horosphere (see [5, 8]).

The horosphere will have an extrinsic geometry that is obtained as a limiting hypersurface of these expanding geodesic hyperspheres. That is, M^∞ will have a second fundamental form with a representation (2.8) with respect to a suitable basis of $T_p(M^\infty)$. We shall establish this analytically in §4 when, subsequent to more theory, we will be able to show the convergence of the geodesic
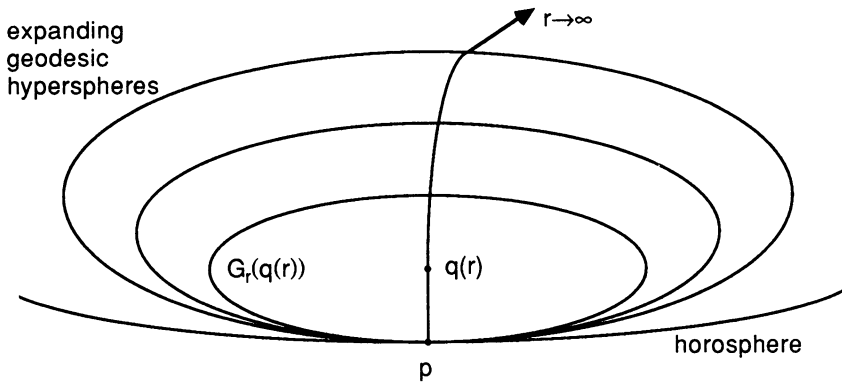
FIGURE 3

hyperspheres to a hypersurface with a second fundamental form of the type (2.8).

The hypersurfaces of Examples 1, 2, and 3 will yield satisfying geometric characterizations for isoparametric hypersurfaces of CH^n that satisfy a number of conditions. However, to make further progress in this direction we must show that hypersurfaces of this type are completely determined (up to congruences of CH^n) by their second fundamental forms. In order to do this, CH^n must be thought of as the base manifold of a certain Lorentzian S^1-fiber bundle.

3. THE LORENTZIAN CIRCLE BUNDLE OVER $CH^n(-4)$ AND RIGIDITY OF CERTAIN ISOPARAMETRIC HYPERSURFACES OF $CH^n(-4)$

The best understood of all non-Euclidean complex space forms, complex projective space (usually written CP^n), is constructed using a natural equivalence on an odd-dimensional sphere, S^{2n+1}, itself immersed in C^{n+1}. A complex hyperbolic space can be constructed in a similar way (see [2; 10, vol. II]). In this case CH^n is formed by taking the equivalence on a real Lorentzian hyperbolic space form in C^{n+1} instead of on a real Riemannian space form.

Define a hermitian bilinear form F on C^{n+1} by

$$F(z, w) = -z_0 \overline{w}_0 + \sum_{j=1}^{n} z_j \overline{w}_j$$

for all $z = (z_0, z_1, \ldots, z_n)$ and $w = (w_0, w_1, \ldots, w_n)$ in C^{n+1}. F forms a complex Lorentzian metric, so that C^{n+1} equipped with the metric F forms a complex Lorentzian space, C_1^{n+1}, where at the origin $(z_0, 0, \ldots, 0)$ is a timelike vector if $z_0 \neq 0$. C^{n+1} can also be regarded as a real semi-Riemannian euclidean space, R_2^{2n+2}, if it is equipped with the metric $\text{Re}(F)$. (At the origin $(1, 0, \ldots, 0)$ and $(i, 0, \ldots, 0)$ form a basis of the negative definite subspace of $T_0(R_2^{2n+2})$, with respect to the metric $\text{Re}(F)$.) Anti-De Sitter spacetime is

the hyperquadric defined by

$$H_1^{2n+1} = \{z \in \mathbf{C}^{n+1} \mid F(z, z) = -1\}.$$

The tangent space of anti-De Sitter space is determined by its immersion into the ambient complex euclidean space; explicitly:

$$T_z(H_1^{2n+1}) = \{w \in \mathbf{C}^{n+1} \mid \operatorname{Re}[F(z, w)] = 0\}$$

for any $z \in H_1^{2n+1}$. As a real hypersurface of $\mathbf{R}_2^{2n+2}$, H_1^{2n+1} has $\operatorname{Re}(F)|_{H_1^{2n+1}}$ as a natural Lorentzian metric that is of constant sectional curvature -1.

An S^1-action can be defined on H_1^{2n+1} (in fact on $\mathbf{C}_1^{n+1}$) by $z \to \lambda z$ for any $z \in H_1^{2n+1}$ and $\lambda \in \mathbf{C}$ with $|\lambda| = 1$. At each point $z \in H_1^{2n+1}$, the vector $V = iz$ is tangent to the flow and has length -1. Given a point $z \in H_1^{2n+1}$, the flow of V through z in H_1^{2n+1} will be given by the orbit

$$O_z = \{x_t = e^{it}z \mid t \in \mathbf{R}\}$$

that satisfies the differential equation $dx_t/dt = ix_t$. This in turn shows that O_z lies in the intersection of the negative definite plane $\operatorname{span}\{z, V\}$ with H_1^{2n+1}. Let $\sim$ be the equivalence given by the orbits of the action, that is, $w \sim z$ if $w \in O_z$. Then the natural projection $\pi : H_1^{2n+1} \to H_1^{2n+1}/\sim\ = CH^n$ is a Riemannian submersion with fundamental tensor the natural complex structure J on $\mathbf{C}_1^{n+1}$ (see [13, 16]) and with timelike totally geodesic fibers, each of which is a trajectory of the vertical vector $V = iz$ at any point $z \in H_1^{2n+1}$. The complex Riemannian space, CH^n, obtained in this way has its complex structure induced from that on $\mathbf{C}_1^{n+1}$ and has constant holomorphic sectional curvature -4 with the metric induced from $\operatorname{Re}(F)$.

The differential of the submersion, π_*, is a linear isometry [16]; i.e., it preserves the metric tensor on the horizontal distribution as it projects onto $T(CH^n)$. So we shall make no distinction between the metric on H_1^{2n+1} and that on CH^n. We can now write

$$T_z(H_1^{2n+1}) \cong T_{\pi(z)}(CH^n) \oplus \operatorname{span}\{V\}$$

where the isomorphism is given by $\pi_* \otimes I|_{\operatorname{span}\{V\}}$.

π_*, as a linear isometry of the distribution $\mathscr{H}$ of horizontal vector fields on H_1^{2n+1} onto $T(CH^n)$, induces relations between the connections $\dot{\nabla}$ and $\overline{\nabla}$ of H_1^{2n+1} and CH^n, respectively. (See [6, 16].) These are

$$(3.1) \qquad \dot{\nabla}_{(X^\sim)}Y^\sim = (\overline{\nabla}_X Y)^\sim + \langle J(X^\sim), Y^\sim \rangle V,$$

$$(3.2) \qquad \dot{\nabla}_V(X^\sim) = \dot{\nabla}_{(X^\sim)}V = (JX)^\sim = J(X^\sim)$$

for all $X, Y \in T(CH^n) \cong \mathscr{H}$, and where $X^\sim$ denotes the unique horizontal lift of a vector field $X \in T(CH^n)$.

If M^{2n-1} is an orientable real hypersurface of $\mathbf{C}H^n(-4)$ then the hypersurface $\widetilde{M}_1^{2n} = \pi^{-1}(M)$ of H_1^{2n+1} is orientable and invariant under the S^1-action, and $\pi|_{\widetilde{M}}: \widetilde{M} \to M$ is a Riemannian submersion with timelike totally geodesic fibers. Conversely, if $\widetilde{M}_1^{2n}$ is an orientable S^1-invariant hypersurface of H_1^{2n+1}, then $\pi|_{\widetilde{M}}$ is a Riemannian submersion of $\widetilde{M}$ onto $M^{2n-1} = \pi(\widetilde{M})$ with timelike totally geodesic fibers. Hence, we have the following commutative diagram:

$$
\begin{array}{ccc}
\widetilde{M}_1^{2n} & \xrightarrow{\ \tilde{j}\ } & H_1^{2n+1}(-1) \\[6pt]
\Big\downarrow {\scriptstyle \pi|_{\widetilde{M}}} & & \Big\downarrow {\scriptstyle \pi} \\[6pt]
M^{2n-1} & \xrightarrow{\ j\ } & \mathbf{C}H^n(-4)
\end{array}
$$

where $j: M^{2n-1} \to \mathbf{C}H^n(-4)$ and $\tilde{j}: \widetilde{M}_1^{2n} \to H_1^{2n+1}(-1)$ are immersions compatible with the fibration. If ξ is a normal field on M, its unique lift, $\xi^{\sim}$, is horizontal and forms a normal field on $\widetilde{M}$. Henceforth, we will drop this distinction between ξ and $\xi^{\sim}$.

Let $\widetilde{\nabla}$ be the metric connection of $\widetilde{M}$ as a hypersurface in H_1^{2n+1} and $\widetilde{H}$ be its second fundamental form. The Gauss and Weingarten formulae in this case are given by

$$
(3.3) \qquad \dot{\nabla}_X Y = \widetilde{\nabla}_X Y + \langle \widetilde{H}X, Y\rangle \xi \quad \text{and} \quad \dot{\nabla}_X \xi = -\widetilde{H}X
$$

for all $X, Y \in T(\widetilde{M})$. Now combining (1.1), (3.1), (3.2), and (3.3) we have the following relations between $\widetilde{H}$ and H:

$$
(3.4) \qquad \widetilde{H}(X^{\sim}) = (HX)^{\sim} - f(X)V
$$

for all $X \in T(M) \cong \{V\}^{\perp} \cap T(\widetilde{M})$ and

$$
(3.5) \qquad \widetilde{H}V = U^{\sim}.
$$

Let $\widetilde{R}$ be the curvature tensor of $\widetilde{M}$. We shall have the opportunity to use the Gauss and Codazzi equations for $\widetilde{M}$ in H_1^{2n+1}:

$$
(3.6) \quad \widetilde{R}(X, Y)Z = \langle X, Z\rangle Y - \langle Y, Z\rangle X + \langle \widetilde{H}Y, Z\rangle \widetilde{H}X - \langle \widetilde{H}X, Z\rangle \widetilde{H}Y,
$$

$$
(3.7) \qquad (\widetilde{\nabla}_X \widetilde{H})Y = (\widetilde{\nabla}_Y \widetilde{H})X
$$

for all $X, Y \in T(\widetilde{M})$. These and the preceding formulae can be used to prove the following useful identities:

$$
(3.8) \quad
\begin{aligned}
(\widetilde{\nabla}_W \widetilde{R})(X, Y)Z &= \langle (\widetilde{\nabla}_W \widetilde{H})Y, Z\rangle \widetilde{H}X - \langle (\widetilde{\nabla}_W \widetilde{H})X, Z\rangle \widetilde{H}Y \\
&\quad + \langle \widetilde{H}Y, Z\rangle (\widetilde{\nabla}_W \widetilde{H})X - \langle \widetilde{H}X, Z\rangle (\widetilde{\nabla}_W \widetilde{H})Y
\end{aligned}
$$

for all $X, Y, Z, W \in T(\widetilde{M})$, and

$$
(3.9) \qquad (\widetilde{\nabla}_{(X^{\sim})} \widetilde{H})V = [(\phi H - H\phi)X]^{\sim}
$$

for any $X \in T(M)$.

An immediate consequence of the preceding formulae is that M and $\widetilde{M}$ have the same mean curvatures. Thus, $\widetilde{M}$ is isoparametric when M is isoparametric.

If $\{X_1, \ldots, X_{2n-1}\}$ is a local frame on M consisting of principal directions in $T(M)$ with corresponding principal curvatures $\{\lambda_1, \ldots, \lambda_{2n-1}\}$, then $\{X_1^{\sim}, \ldots, X_{2n-1}^{\sim}, V\}$ forms a local frame on $\widetilde{M}$ with respect to which $\widetilde{H}$ is represented by the matrix

$$\begin{bmatrix} \lambda_1 & & & f(X_1) \\ & & 0 & \vdots \\ & 0 & \lambda_{2n-1} & f(X_{2n-1}) \\ -f(X_1) & \cdots & -f(X_{2n-1}) & 0 \end{bmatrix},$$

all of whose entries are functions on $\widetilde{M}$. Hence, even the stipulation that $\widetilde{M}$ be S^1-invariant does not guarantee an easy analysis of the structure of $\widetilde{M}$ via an investigation of those subbundles of $T(\widetilde{M})$ held invariant by $\widetilde{H}$, for we are not even assured the existence of principal directions that have real principal curvatures.

However, in case U is a principal direction on M, we can choose $U = X_{2n-1}$ so that $\widetilde{H}$ is represented by the matrix

$$(3.10) \qquad \text{diag}\left(\lambda_1, \ldots, \lambda_{2n-2}, \begin{bmatrix} \alpha & 1 \\ -1 & 0 \end{bmatrix}\right).$$

We shall see that if M is also isoparametric, then M is rigid; that is, M is unique up to congruence.

In order to characterize congruence in $CH^n(-4)$, the principal S^1-fiber bundle over CH^n must be taken into account as well as the fiber bundle's own imbedding in $\mathbf{C}_1^{n+1}$. From the preceding discussion we see that rigid motions of $\mathbf{C}_1^{n+1}$ that leave H_1^{2n+1} invariant will induce rigid motions of CH^n.

The isometries of $\mathbf{C}_1^{n+1}$ under F,

$$U(1, n) = \{A \in GL(n+1; \mathbf{C}): F(Az, Aw) = F(z, w), \forall z, w \in \mathbf{C}_1^{n+1}\},$$

are precisely the rigid motions that leave H_1^{2n+1} invariant and act transitively on H_1^{2n+1}. Hence, the elements of $U(1, n)$ will induce isometries of CH^n via π.

Now let M and N be isoparametric hypersurfaces of CH^n that each has the distinguished direction of the induced almost contact structure as a principal direction. Suppose that M and N have second fundamental forms with the same matrix representation with respect to suitably chosen orientations of M and N and local orthonormal bases of principal directions of $T(M)$ and $T(N)$. Let $\widehat{M}$, ρ and $\widehat{N}$, σ be the simply connected covering spaces of the lifts $\widetilde{M} = \pi^{-1}(M)$ and $\widetilde{N} = n^{-1}(N)$, respectively. Since $\widetilde{N}$ and $\widetilde{M}$ are isoparametric, following an argument in [17], combine (3.10) with the fact that

the ambient space of $\widetilde{N}$ and $\widetilde{M}$ is of constant sectional curvature to show that the principal curvatures α and λ_i, $i = 1, \ldots, 2n - 1$, are constant.

If $i \colon M \to CH^n$ and $j \colon N \to CH^n$ are the isometric immersions in complex hyperbolic space and $\tilde{i} \colon \widetilde{M} \to H_1^{2n+1}$ and $\tilde{j} \colon \widetilde{N} \to H_1^{2n+1}$ are the induced immersions of the respective lifts, we have the following commutative diagram of immersions, submersions, and covering maps:

$$
\begin{array}{ccccc}
\widehat{M} & & & & \widehat{N} \\
\downarrow \rho & & & & \downarrow \sigma \\
\widetilde{M} & \overset{\tilde{i}}{\to} & H_1^{2n+1} & \overset{\tilde{j}}{\leftarrow} & \widetilde{N} \\
\downarrow \pi|_{\widetilde{M}} & & \downarrow \pi & & \downarrow \pi|_{\widetilde{N}} \\
M & \overset{i}{\to} & CH^n & \overset{j}{\leftarrow} & N
\end{array}
$$

The maps $\tilde{i} \circ \rho \colon \widehat{M} \to H_1^{2n+1}$ and $\tilde{j} \circ \sigma \colon \widehat{N} \to H_1^{2n+1}$ are now isometric immersions of simply connected Lorentzian spaces of codimension 1 into the Lorentzian symmetric space H_1^{2n+1} that have the same constant matrix representation for their second fundamental forms with respect to canonically chosen orthonormal bases of $T(\widehat{M})$ and $T(\widehat{N})$.

Lemma 3. *Let N and M be isoparametric oriented real hypersurfaces of $CH^n(-4)$ that have*

(i) U as a principal direction, and

(ii) the same corresponding principal curvatures, i.e., the same matrix representation for the second fundamental form with respect to suitably chosen bases of corresponding principal directions.

If $\widehat{N}$ and $\widehat{M}$ are the simply connected covering spaces of $\widetilde{N} = \pi^{-1}(N)$ and $\widetilde{M} = \pi^{-1}(M)$, respectively, then $\widehat{N}$ and $\widehat{M}$ are isometric.

Sketch of Proof. Let $x \in \widehat{N}$ and $y \in \widehat{M}$. By hypothesis and using the ideas of §1 we can choose orthonormal bases of $T_{\pi(\sigma(x))}(N)$ and $T_{\pi(\rho(y))}(M)$ that consist entirely of principal directions, with respect to which the second fundamental forms have the same representations. Now pull back and extend to orthonormal bases of $T_x(\widehat{N})$ and $T_y(\widehat{M})$. (These constructions rely primarily on the distinguished direction U being principal on both M and N.)

Define a linear isometry $\psi \colon T_x(\widehat{N}) \to T_y(\widehat{M})$ in the "obvious" way; that is, ψ maps basis elements of $T_x(\widehat{N})$ onto the corresponding basis elements of $T_y(\widehat{M})$. Let $\widehat{H}$ and $\widehat{H}'$ denote the second fundamental forms of the isometric immersions $\tilde{j} \circ \sigma$ and $\tilde{i} \circ \rho$, and similarly let $\widetilde{H}$ and $\widetilde{H}'$ denote the second fundamental forms of the immersions $\tilde{j}$ and $\tilde{i}$, respectively. As σ and ρ are local isometries, $\widehat{H}_x$ and $\widehat{H}'_y$ will agree with $\widetilde{H}_{\sigma(x)}$ and $\widetilde{H}'_{\rho(y)}$, respectively. By hypothesis and applying (3.4) and (3.5) we have that $\psi(\widehat{H}_x(Z)) = \widehat{H}'_y(\psi(Z))$ for any $Z \in T_x(\widehat{N})$. Let $\widehat{R}$ and $\widehat{R}'$ denote the curvature tensors of $\widehat{N}$ and $\widehat{M}$, respectively. By (3.6), $\psi(\widehat{R}_x(X, Y)Z) = \widehat{R}'_y(\psi X, \psi Y)\psi Z$ for all $X, Y, Z \in T_x(\widehat{N})$.

Let $\widehat{\nabla}$ and $\widehat{\nabla}'$ denote the connections on $\widehat{N}$ and $\widehat{M}$. Using (3.7) and (3.9), it follows that ψ maps the tensor $(\widehat{\nabla}\widehat{H})_x$ to the tensor $(\widehat{\nabla}'\widehat{H}')_y$. Hence by (3.8), ψ maps $(\widehat{\nabla}\widehat{R})_x$ to $(\widehat{\nabla}'\widehat{R}')_y$. The fact that $\widehat{H}$ and $\widehat{H}'$ have corresponding constant matrix representations provides the inductive step to show that ψ maps $(\widehat{\nabla}^m\widehat{R})_x$ to $(\widehat{\nabla}'^m\widehat{R}')_y$ for any $m > 0$.

Now by Corollary 7.3, Chapter VI, Vol. I of [10], there is a unique isometry $\Omega: \widehat{N} \to \widehat{M}$ such that $(d\Omega)_x = \psi$.

The point of Lemma 3 is to get into position to establish a general congruence theorem for real hypersurfaces of complex space forms. We shall establish this by using known congruence results for submanifolds of semi-Riemannian euclidean spaces.

Theorem 2. *Let M and N be isoparametric orientable real hypersurfaces of CH^n that each have the distinguished direction U as a principal direction. If M and N have the same second fundamental form, i.e., the corresponding principal curvatures are the same, then M and N are congruent.*

Sketch of the Proof. Regard the simply connected covering spaces, $\widehat{M}$ and $\widehat{N}$ of the lifts of M and N in H_1^{2n+1}, as Lorentzian submanifolds of the semi-Riemannian manifold $\mathbf{R}_2^{2n+2}$. Define $\psi^\perp : T_x(\widehat{N})^\perp \subset T_x(\mathbf{R}_2^{2n+2}) \to T_y(\widehat{M})^\perp \subset T_x(\mathbf{R}_2^{2n+2})$ by $\psi^\perp(x) = y$ and $\psi^\perp(\xi_x) = \xi'_y$, where ξ and ξ' are the normals to $\widehat{N}$ and $\widehat{M}$ in H_1^{2n+1}, and then extend linearly to obtain a linear isometry of the two-dimensional normal spaces. Utilize the hypotheses that M and N are isoparametric and have the direction U principal and the fact that H_1^{2n+1} is a totally umbilic (and isoparametric) hyperquadric in $\mathbf{R}_2^{2n+2}$ to show that for any curve α through x in $\widehat{N}$, the linear isometry of normal spaces

$$\psi_{\alpha(s)}^\perp = P_{\Omega(\alpha(s))} \circ \psi^\perp \circ P_{\alpha(s)}^{-1} : T_{\alpha(s)}(\widehat{N})^\perp \to T_{\Omega(\alpha(s))}(\widehat{M})^\perp,$$

where P is normal parallel translation along smooth curves, maps the second fundamental form of $\widehat{N}$ at $\alpha(s)$ onto that of $\widehat{M}$ at $\Omega(\alpha(s))$, regarded as submanifolds of $\mathbf{R}_2^{2n+2} \approx \mathbf{C}_1^{n+1}$.

Finally, use Lemma 3 and Theorem 41, Chapter 4 of [15], to show that there exists an isometry $\overline{\psi}$ of $\mathbf{C}_1^{n+1}$ such that $\overline{\psi}|_{\widehat{N}} = \Omega$. Hence, $\overline{\psi}$ induces a rigid motion of CH^n that maps N isometrically onto M.

Theorem 2 will be used in the following section to establish the existence of a horosphere in CH^n independent of [5, 8, 18]. Theorem 2 will also be useful in the future for classifying real hypersurfaces of CH^n that satisfy certain tensor equations.

4. Application: An analytic construction of a horosphere

Using the congruence results of §3, we can now place the model spaces used in [13] into the context of the hypersurfaces constructed in §2. First, recall

an elegant and well-known (e.g., [13, 14]), analytic method of determining the extrinsic geometry of a level hypersurface of a C^∞ function on a space form imbedded as a hypersurface in a euclidean space, modified to fit the particular needs of this section.

Let $f: \mathbf{R}_2^{n+1} \to \mathbf{R}$ be a C^∞ function and $M_1^n(c)$ be an imbedded Lorentzian space form in $\mathbf{R}_2^{n+1}$ of sectional curvature c. Let ∇f denote the gradient of f in $T(M)$ as a function on M, and $\widetilde{\nabla} f$ denote the gradient of f as a function on $\mathbf{R}_2^{n+1}$. Let S be the set of all $s \in \mathbf{R}$ such that $\widetilde{M}_s = f^{-1}(s)$ is a hypersurface of $\mathbf{R}_2^{n+1}$. Then for any $s \in S$, $\widetilde{M}_s$ and $\widetilde{\nabla} f/\|\widetilde{\nabla} f\|$ as a unit normal field in $T(\mathbf{R}_2^{n+1})$. Similarly, let T be the set of all $s \in S$ such that $M_s = \widetilde{M}_s \cap M_1^n(c)$ is a hypersurface of $M_1^n(c)$. Then M_s will have $\nabla f/\|\nabla f\|$ as a unit normal in $T(M_1^n(c))$, for each $s \in T$.

For a given f, $\widetilde{\nabla} f$ is usually easy to calculate. Once this is done,

$$(4.1) \qquad \nabla f = \widetilde{\nabla} f + \langle \widetilde{\nabla} f, \zeta \rangle \zeta$$

where $\langle \, , \rangle$ is the standard metric of $\mathbf{R}_2^{n+1}$ and ζ is a unit timelike normal field to $M_1^n(c)$ in $\mathbf{R}_2^{n+1}$. (Notice that the choice of positive coefficient of ζ is necessitated by the causal nature of ζ.)

Let $\mathrm{Hess}(f; \mathbf{R}_2^{n+1})$ denote the hessian of f as an operator on $\mathbf{R}_2^{n+1}$. For each $s \in S$, the second fundamental form $\widetilde{H}$ of $\widetilde{M}_s$ in $\mathbf{R}_2^{n+1}$ is given by

$$(4.2) \qquad \langle \widetilde{H} X, Y \rangle = \mathrm{Hess}(f; \mathbf{R}_2^{n+1})(X, Y)/\|\widetilde{\nabla} f\|$$

for all $X, Y \in T(\widetilde{M}_s)$, which for a given f is usually easy to compute. For each $s \in T$, the second fundamental form H of M_s in $M_1^n(c)$ can be expressed similarly:

$$(4.3) \qquad \langle H X, Y \rangle = \mathrm{Hess}(f; M_1^n(c))(X, Y)/\|\nabla f\|$$

for all $X, Y \in T(M_s)$ and where $\mathrm{Hess}(f; M_1^n(c))$ denotes the hessian of f as an operator on $T(M_1^n(c))$. Since $\mathrm{Hess}(f; M_1^n(c))$ is defined to be the second covariant derivative of f on $M_1^n(c)$ [15]:

$$\mathrm{Hess}(f; M_1^n(c))(X, Y) = XYf - (\nabla_X Y)f = \langle \nabla_X (\nabla f), Y \rangle,$$

where ∇ denotes the connection on $M_1^n(c)$; once (4.2) has been computed, (4.1) can be used to obtain a representation of (4.3), thereby yielding an explicit calculation of the second fundamental form of M_s in $M_1^n(c)$.

Example 4. Consider the function $G_p: \mathbf{C}_1^{n+1} \to \mathbf{R}$, for each $p = 0, 1, \ldots, n$, defined by

$$G_p(z) = -|z_0|^2 + \sum_{i=1}^{p} |z_i|^2$$

where $z = (z_0, z_1, \ldots, z_n)$. For $r > 0$, define a level hypersurface of $\mathbf{C}_1^{n+1}$ by

$$\widetilde{M}_p(r) = \{z \in \mathbf{C}_1^{n+1} \mid G_p(z) = -\cosh^2(r)\}.$$

The gradient of G_p in $\mathbf{C}_1^{n+1}$ is computed to be

$$\tilde{\nabla} G_p(z) = 2(z_0, z_1, \ldots, z_p, 0, \ldots, 0)$$

for all $z \in \mathbf{C}_1^{n+1}$, so that

$$\tilde{\nabla} G_p(z)/2\cosh(r) = (\operatorname{sech}(r) z_0, \operatorname{sech}(r) z_1, \ldots, \operatorname{sech}(r) z_p, 0, \ldots, 0)$$

is a unit (timelike) normal to $\widetilde{M}_p(r)$ in $\mathbf{C}_1^{n+1}$, for all $z \in \widetilde{M}_p(r)$.

The level hypersurface $M_p(r) = \widetilde{M}_p(r) \cap H_1^{2n+1}$ is nothing more than the model $M_{2p+1,2q+1}(\tanh^2(r))$ of example 4.1 in [13]. Notice that $M_p(r)$ is isometric to the product $H_1^{2p+1}(-\cosh^2(r)) x S^{2(n-p)-1}(\sinh^2(r))$. The gradient of G_p on $M_p(r)$ in $T(H_1^{2n+1})$ is given by (4.1)

$$\nabla G_p(z) = -2\cosh^2(r)(\tanh^2(r) z_0, \ldots, \tanh^2(r) z_p, z_{p+1}, \ldots, z_n)$$

for all $z \in M_p(r)$. Thus,

$$\xi(z) = -(\tanh(r) z_0, \ldots, \tanh(z) z_p, \coth(r) z_{p+1}, \ldots, \coth(r) z_n)$$

is a unit normal to $M_p(r)$ in H_1^{2n+1}, for all $z \in M_p(r)$.

At this point we can see that the second fundamental form of $M_p(r)$ in H_1^{2n+1} is diagonalizable with respect to a real basis of $T_z(M_p(r)) = \{z, \xi(z)\}^\perp$ and has constant principal curvatures $\tanh(r)$ and $\coth(r)$ of real multiplicities $2p + 1$ and $2n - 2p - 1$, respectively.

Let $N_p(r) = \pi(M_p(r))$ and U denote the distinguished vector on $N_p(r)$ viewed as a real hypersurface of $\mathbf{C}H^n$. Let H' and H denote the second fundamental forms of $M_p(r)$ and $N_p(r)$, respectively. We can write

$$U_{\pi(z)} = -J(\pi_*(\xi(z))) = \pi_*(-i\xi(z)), \quad \text{i.e.,} \quad (U^\sim)_z = -i\xi(z).$$

An explicit calculation of $H'(U^\sim)_z$ using (4.3) followed by an application of (3.4) shows that $U_{\pi(z)}$ is principal in $T_{\pi(z)}(N_p(r))$ with curvature $2\coth(2r)$. Subsequent calculations yield the other principal curvatures $\tanh(r)$ and $\coth(r)$ of multiplicities $2p$ and $2n - 2p - 2$, respectively, each having a ϕ-invariant eigendistribution. From the work of §§2 and 3 we see that $N_p(r)$ is congruent to a tube of radius r about a totally geodesic complex space form in $\mathbf{C}H^n$ isometric to a $\mathbf{C}H^p$. In particular, we shall need that $N_0(r)$ is congruent to a geodesic hypersphere of radius r.

Example 5. Consider the function $G: \mathbf{C}_1^{n+1} \to \mathbf{R}$ given by $G(z) = |z_0 - z_1|^2$, where $z = (z_0, \ldots, z_n)$, and the level hypersurface of $\mathbf{C}_1^{n+1}$, $\widetilde{M} = G^{-1}(1)$. The gradient of G in $\mathbf{C}_1^{n+1}$ can be written as

$$\tilde{\nabla} G(z) = 2(z_1 - z_0, z_1 - z_0, 0, \ldots, 0)$$

for all $z \in \widetilde{M}$.

The level hypersurface $N = \widetilde{M} \cap H_1^{2n+1}$ is the model hypersurface N of example 4.2 of [13]. The gradient of G in H_1^{2n+1} is given by

$$\nabla G(z) = 2(z_1, 2z_1 - z_0, z_2, \ldots, z_n)$$

for all $z \in N$. Hence,

$$\xi(z) = (z_1, 2z_1 - z_0, z_2, \ldots, z_n)$$

is a unit normal to N in H_1^{2n+1}.

Set $M_n^* = \pi(N)$, as in example 4.2 of [13]. Explicit calculations using (4.3) and (3.4) show that U is principal on M_n^* with principal curvature 2 and that 1 is a principal curvature of multiplicity $2n-2$; that is, the second fundamental form of M_n^* acts as the identity transformation on $\ker(f)$. Thus, according to Theorem 2, M_n^* is our candidate for a horosphere. In order to see this (and thereby establish the existence of a horosphere in CH^n analytically) we will show that M_n^* is a limiting hypersurface of a specific family of geodesic hyperspheres.

Let $P = (1, 0, \ldots, 0) \in H_1^{2n+1}$ and consider the geodesic emanating from $\pi(P)$ in CH^n given by $\gamma(r) = \pi(\cosh(r), \sinh(r), 0, \ldots, 0)$. (See [10, Vol. II, p. 285].) As in example 3, each geodesic hypersphere of radius r centered at $\gamma(r)$ contains the point $\pi(P)$. We will see that as $r \to \infty$, these hyperspheres converge to a limiting hypersurface, namely M_n^*.

Earlier in this section, we discovered that the hypersurface of H_1^{2n+1} defined by

$$M_0(r) = \left\{ z \in H_1^{2n+1} \mid \tanh^2(r)|z_0|^2 = \sum_{i=1}^{n} |z_i|^2 \right\}$$

is actually the lift (up to a congruence, of course) of a geodesic hypersphere of radius r. Notice that $\gamma(r) \in M_0(r)$ and that $\pi(P)$ is equidistant from every point on $M_0(r)$. So $\pi(P)$ plays the role of center of $\pi(M_0(r))$ in CH^n.

In particular, we see that the family of hypersurfaces $\{\pi(M_0(r)) \mid r > 0\}$ is not our candidate for the convergent family. However, all is not lost, for we should be able to find a rigid movement of CH^n, induced by an $A(r) \in U(1, n)$, that for each $r > 0$ will translate $\pi(M_0(r))$ to a geodesic hypersphere of radius r and center $\gamma(r)$ in such a way that the family $\{\pi(A(r)[(M_0(r))]) \mid r > 0\}$ will converge to a limiting hypersurface. This limiting hypersurface will be $M_n^* = \pi^{-1}(N)$, which must then be a horosphere through $\pi(P)$.

For each $r > 0$, let $A(r) \in U(1, n)$ be defined by

$$A(r) = \mathrm{diag}\left\{ \begin{bmatrix} \cosh(r) & -\sinh(r) \\ \sinh(r) & -\cosh(r) \end{bmatrix}, I_{n-1} \right\}.$$

$A(r)$ is a rigid motion that maps $M_0(r)$ onto the lift of the geodesic hypersphere of radius r centered at $\gamma(r)$ and therefore induces a rigid motion of CH^n that moves the geodesic hypersphere $\pi(M_0(r))$ that has radius r and center $\pi(P)$
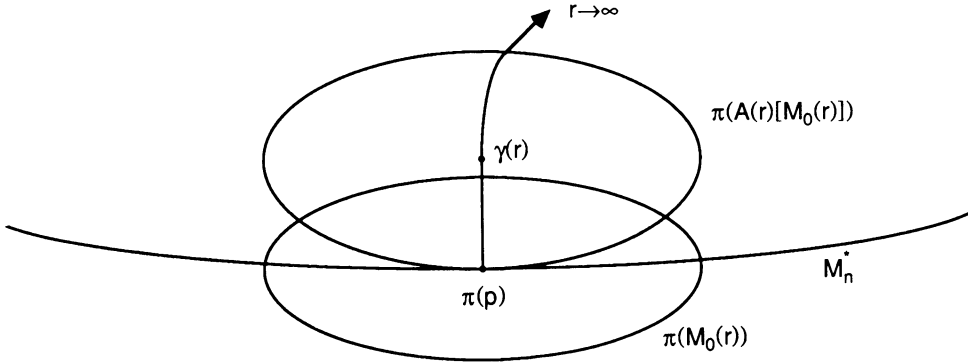
FIGURE 4

onto the geodesic hypersphere that has radius r and center $\gamma(r)$ and contains $\pi(P)$.

Proposition. *N is the limiting hypersurface of the family* $\{A(r)[M_0(r)] \mid r > 0\}$ *of hypersurfaces in* H_1^{2n+1}, *that is,*

$$\lim_{r \to \infty} \{A(r)[M_0(r)]\} = N$$

and is therefore an S^1-*fiber bundle over a horosphere.*

Proof. For any $z = (z_0, z_1, \ldots, z_n) \in M_0(r)$ we have

$$|z_0|^2 = \cosh^2(r) \quad \text{and} \quad \sum_{j=1}^{n} |z_j|^2 = \sinh^2(r).$$

Let $w = (w_0, w_1, \ldots, w_n) \in A(r)[M_0(r)]$. Then

$$w = (\cosh(r)z_0 - \sinh(r)z_1, \ \sinh(r)z_0 - \cosh(r)z_1, z_2, \ldots, z_n)$$

for some $z \in M_0(r)$. In particular, we have

$$|w_0 - w_1| = (\cosh(r) - \sinh(r))|z_0 + z_1| = e^{-r}|z_0 + z_1|.$$

Thus,

$$|w_0 - w_1| \le e^{-r}(|z_0| + |z_1|) \le e^{-r}(\cosh(r) + \sinh(r)) = 1,$$

which shows that the limiting hypersurface $\lim_{r \to \infty}\{A(r)[M_0(r)]\}$ must satisfy $|z_0 - z_1| \le 1$ for any $z = (z_0, \ldots, z_n) \in \lim_{r \to \infty}\{A(r)[M_0(r)]\}$.

To see the reverse inequality, let $R > 0$ be given. For each $r > R$, consider the disc

$$S(r, R) = \{w \in A(r)[M_0(r)] \mid d(P, w) < R\}$$

on the translated lift of the geodesic hypersphere $\pi(M_0(r))$. Let $w \in S(r, R)$. Since $A(r)$ is an isometry, there is a $z \in M_0(r)$ such that $w = A(r)z$ and

$$d((\cosh(r), \sinh(r), 0, \ldots, 0), z) < R.$$

Hence, $|z_0 - \cosh(r)| < R$, $|z_1 - \sinh(r)| < R$, and $|z_k| < R$ for $k \geq 2$. Thus, for r sufficiently large,

$$|w_0 - w_1| = e^{-r}|z_0 + z_1| \geq e^{-r}(\cosh(r) + \sinh(r) - R) = 1 - e^{-r}R.$$

Now we see that for a point z in the lift of the horosphere within R units of P, we must have $|z_0 - z_1| \geq 1$. But R was an arbitrary choice so that

$$N = \bigcup_{R>0} \left\{ \lim_{r \to \infty} S(r, R) \right\} = \lim_{r \to \infty} \{A(r)[M_0(r)]\}.$$

This establishes the existence of a horosphere analytically in CH^n.

As one would expect, the representation of a horosphere as a submersed level hypersurface depends both on the choice of $P \in H_1^{2n+1}$ and on the geodesic emanating from P; equivalently: on the choice of normal to the lift of a horosphere at P. It is interesting to note that we obtain different bounds for $|z_0 - z_1|^2$ for a limiting hypersurface of a convergent family of S^1-fiber bundles over geodesic hyperspheres if a different geodesic emanating from P is selected.

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