NONUNIQUENESS FOR SOLUTIONS OF THE KORTEWEG-DeVRIES EQUATION

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Abstract. Variants of the inverse scattering method give examples of nonuniqueness for the Cauchy problem for KdV. One example gives a nontrivial $C^\infty$ solution $u$ in a domain $\{(x,t): 0 < t < H(x)\}$ for a positive nondecreasing function $H$, such that $u$ vanishes to all orders as $t \downarrow 0$. This solution decays rapidly as $x \to +\infty$, but cannot be well behaved as $x$ moves left. A different example of nonuniqueness is given in the quadrant $x \geq 0, t \geq 0$, with nonzero initial data.

1. Introduction and summary

The initial value problem

\begin{align}
  u_t - 6u u_x + u_{xxx} &= 0, \\
  u(x,0) &= U_0(x)
\end{align}

for the Korteweg-deVries equation specifies data on a characteristic line. Thus it is natural to look for nonuniqueness, possibly associated with poor behavior as $|x| \to +\infty$. Here we present two nonuniqueness results.

First, we construct a $C^\infty$ function $u_1$ such that

(i) $u_1$ solves KdV in a neighborhood of the form $\mathcal{Z}_1 = \{(x,t): 0 < t < h(x)\}$ for positive nondecreasing $h$ with $\lim_{x \to +\infty} h(x) = +\infty$.

(ii) $u_1(x,0) = 0$ for all $x$.

(iii) $u_1$ cannot vanish identically in $\mathcal{Z}_1$.

The domain $\mathcal{Z}_1$ can be made arbitrarily big in the sense that, given any point $(x_0,t_0)$ with $t_0 > 0$, we can find such $u_1$ and $\mathcal{Z}_1$ with $(x_0,t_0) \in \mathcal{Z}_1$. This construction can be adapted to produce two different solutions to KdV arising from any reasonably nice choice of initial data.

While the first method treats initial data given on the full line at $t = 0$, it gives a solution which is not global in time. The second method treats initial data on a half-line $x \geq x_0$, but produces solutions for all $t > 0$. More precisely

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we construct distinct solutions to KdV in \( Q = \{(x, t): x_0 < x < \infty, 0 < t < \infty\} \) which agree at \( t = 0 \) on \( x_0 < x < \infty \).

Both constructions use variants of the inverse scattering method introduced by Gardner, Greene, Kruskal, and Miura [6]. This method exploits the relationship between KdV and the linear Airy equation
\[
\Omega_t + \Omega_{xxx} = 0.
\]

Tanaka's argument in [13] proves the following result.

**Theorem T.** Let \( \mathcal{D} \) be a domain of the form
\[
\mathcal{D} = \{(x, t): 0 < t < h(x), x \in \mathbb{R}\},
\]
where \( h \) is nondecreasing and valued in \([0, \infty]\). Suppose that \( \Omega \) is a \( C^\infty \) solution of (1.3) in \( \mathcal{D} \) and that all derivatives of \( \Omega \) decay rapidly as \( x \to +\infty \). Suppose that for each \((x, t)\) in \( \mathcal{D} \), \( B(x, \cdot, t) \) solves the Marchenko equation
\[
B(x, y, t) + \Omega(x + y, t) + \int_0^\infty \Omega(x + y + z, t)B(x, z, t) \, dz = 0.
\]
Then the function \( u \) defined by \( u(x, t) = -\partial_x B(x, 0, t) \) solves KdV in \( \mathcal{D} \).

Tanaka proved Theorem T for \( \mathcal{D} = \mathbb{R} \times \mathbb{R}^+ \), i.e. for \( h(x) \equiv +\infty \), but it is straightforward to check that his proof carries over to these more general domains \( \mathcal{D} \).

The crucial efforts in this paper are to obtain the appropriate solutions \( \Omega \) of (1.3) and to show that the corresponding Marchenko equations (1.4) have solutions. These efforts are easy to outline.

We begin with a special case of Hörmander's Theorem 5.2.5 in [8] to obtain a real-valued function \( \Omega^*(x, t) \) with the following properties:

\[
\begin{align*}
(1.5.a) \quad & \Omega^* \in C^\infty(\mathbb{R} \times \mathbb{R}^+) , \\
(1.5.b) \quad & \Omega^*_t + \Omega^*_{xxx} = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^+ , \\
(1.5.c) \quad & \Omega^*(x, 0) \equiv 0 \quad \text{for } t \leq 0 , \\
(1.5.d) \quad & (0, 0) \in \text{supp}(\Omega^*) ; \text{ indeed, there is a } t_1 > 0 \text{ such that } \\
& \Omega^*(0, t) \neq 0 \text{ whenever } 0 < t < t_1 .
\end{align*}
\]

One can verify that \( \Omega^* \) and all its derivatives decay faster than exponentially as \( x \to +\infty \) for fixed positive \( t \), and that \( \Omega^* \) and all its derivatives decay to the zero function as \( t \downarrow 0 \) both in \( L^\infty \) and \( L^1 \) on each half-line \([x_0, \infty)\).

**Results 1.** Nonuniqueness for (1.1), (1.2) with \( u_0 \equiv 0 \) on \( \mathbb{R} \). Pick \((x_0, t_0)\) with \( t_0 > 0 \). For a positive parameter \( \varepsilon \) to be chosen later, let \( \Omega(x, t; \varepsilon) = \varepsilon \Omega^*(x, t) \) and consider the Marchenko equation
\[
B(x, y, t) + \Omega(x + y, t; \varepsilon) + \int_0^\infty \Omega(x + y + z, t; \varepsilon)B(x, z, t) \, dz = 0 .
\]
The kernel $\Omega(x,t;\varepsilon)$ is not the usual one suggested by the inverse scattering method for (1.1), (1.2) with $u_0 \equiv 0$: the classical inverse scattering construction would use $\Omega \equiv 0$ since the reflection coefficient for the potential $u_0 \equiv 0$ is identically 0. To solve (1.5) one estimates $\Omega^\#$ carefully and then chooses $\varepsilon$ artfully to make sure that the integral operator defined by the kernel $\Omega(x,t;\varepsilon)$ with $(x,t) = (x_0,t_0)$ has norm less than 1, and then finds $h(x)$ so all integral operators with $0 < t < h(x)$ also have operator norm less than 1. Let $B_1(x,y,t)$ denote the solution to (1.6); set $u_1(x,t) = -\partial_x B_1(x,0,t)$. By Theorem T $u_1$ solves KdV in $\mathcal{H}_1$. Analysis of $\Omega$ shows that $u_1$ and all its derivatives approach 0 as $t$ decreases to 0. By using the relation

$$B_{xx} - B_{xy} = u(x,t)B,$$

called the "wave equation" by Deift and Trubowitz [3], one shows that if $u \equiv 0$ in $\mathcal{H}_1$, then $\Omega(x,t;\varepsilon) \equiv 0$ in $\mathcal{H}_1$, which contradicts (1.5.d). Details of this argument are given in §2.

In §3 we discuss the further application of this construction to nonuniqueness for (1.1), (1.2) with more general initial data $u_0$.

**Result 2.** Nonuniqueness for all $t > 0$ in $x \geq x_0$. Since KdV is invariant under translation in $x$, it suffices to work in

$$Q_0 = \{(x,t): 0 < t < \infty, 0 < x < \infty\}.$$

Careful analysis of $\Omega^\#$ gives us positive constants $K$ and $\tau$ such that

$$\Omega_2(x,t) := \Omega^\#(x,t) + 2K e^{\tau t - x^2/2} > 0$$

in $Q_0$. This $\Omega_2$ solves the Airy equation, is smooth, and decays fast as $x \to +\infty$. Consider the Marchenko equation

$$B(x,y,t) + \Omega_2(x+y,t) + \int_0^\infty \Omega_2(x+y+z,t)B(x,z,t)dz = 0. \tag{1.6.2}$$

If $(x,t) \in Q_0$, the integral operator is symmetric and positive in $L^2(\mathbb{R}^+)$; thus (1.5.2) can be solved. Let $B_2$ denote the solution, and set $u_2(x,y) = -\partial_x B_2(x,0,t)$. Theorem T tells us that $u_2$ solves KdV in $Q_0$. Next set

$$\Omega_3(x,t) := 2K e^{\tau t - x^2/2}$$

and consider the Marchenko equation (1.6.3), which is (1.6) with $\Omega_3$ in place of $\Omega$. As before we can solve (1.6.3) to get $B_3$, set $u_3 = -\partial_x B_3(x,0,t)$, and use Theorem T to see that $u_3$ solves KdV. Since $\Omega^\#(x,0) \equiv 0$, we see that $B_2(x,0,0) = B_3(x,0,0)$ in $x > 0$, whence $u_2(x,0) = u_3(x,0)$ in $x > 0$. In §4 we present details of this argument and also show that $u_2$ cannot be identically equal to $u_3$ in $Q_0$.

It is not clear whether our solution $u_1$ can be extended to a solution of KdV in any strip $\{(x,t): 0 < t < T\}$. If any such extension were possible, it could
not evolve in any of the uniqueness classes described by Lax [11], Temam [14],
Menikoff [12], Kruzhkov and Faminskii [10], and others. Clearly the \( L^2 \) norm
could not be a conserved quantity for such an extended \( u_x \). Either \( u_x \) cannot
exist all the way to the left, or it blows up as \( x \to -\infty \). The behavior of \( \Omega^# \) as
\( x \to -\infty \) is discussed in §3, as are some speculations about \( u_x \) as \( x \to -\infty \).

Most notation will be standard. The following definitions have been used in
our previous papers and should be made explicit here:

\[
\begin{align*}
L_N^1(\mathbb{R}) &= \{ f : \int_{-\infty}^{\infty} |f(x)| (1 + |x|)^N \, dx < \infty \} \quad \text{for} \ N \geq 1. \\
L_N^p(\mathbb{R}^+) &= \{ f : \int_0^{\infty} |f(x)| (1 + |x|)^N \, dx < \infty \} \quad \text{for} \ N \geq 1. \\
L^p(+\infty) &= \{ f : f \in L^p([a, +\infty)) \text{ for all finite } a \}.
\end{align*}
\]

\( \| \cdot \|_{L^p, L^q(\mathbb{R}^+)} \) denotes the operator norm in \( L^p(L^q(\mathbb{R}^+), L^p(\mathbb{R}^+)) \).

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2. The First Nonuniqueness Result

Theorem 2.1. Choose any point \((x_0, t_0)\) with \( t_0 > 0 \). There is a function \( u_x \)
and a domain \( \mathcal{U}_1 \) such that

(a) \( \mathcal{U}_1 \) has the form \( \{(x, t) : 0 < t < h(x), x \in \mathbb{R}\} \) for a positive nondecreasing
    function \( h \) with \( \lim_{x \to +\infty} h(x) = +\infty \).
(b) \((x_0, t_0) \in \mathcal{U}_1 \).
(c) \( u_x \) is a \( C^\infty \) solution of KdV in \( \mathcal{U}_1 \).
(d) \( u_x \) and all its derivatives approach 0 as \( t \downarrow 0 \).
(e) \( u_x \) is not identically zero in \( \mathcal{U}_1 \).

Proof. The rest of this section is devoted to this proof. It begins with Hörman-
der's nonuniqueness theorem for the Airy equation.

A. The Airy Equation

Following Hörmander's Theorem 5.2.5 of [8] define a function \( \Omega^# \) by

\[
\Omega^#(x, t) = \int_{it = -\infty}^{it = +\infty} e^{ixr(s)} e^{-i(s/3)^{2/3}} \, ds,
\]

where \( \tau > 0 \) and \( r(s) = (-s)^{1/3} \). To be precise we take

\[
r(s) = |s|^{1/3} e^{i(\arg s)/3} \quad \text{with} \quad 0 < s < \pi
\]

and

\[
(s/i)^{2/3} = |s|^{2/3} e^{i(2\arg(s/i))/3} \quad \text{with} \quad -\pi/2 < \arg(s/i) < \pi/2,
\]

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since \( s = \sigma + i\tau \) with \( \sigma \in \mathbb{R} \) and \( \tau > 0 \). With these choices, Hörmander's theorem tells us that

\begin{align*}
(2.2a) \quad & \Omega^* \text{ is independent of } \tau, \\
(2.2b) \quad & \Omega^* \text{ is } C^\infty \text{ in } \mathbb{R} \times \mathbb{R}, \\
(2.2c) \quad & \Omega^* \text{ solves } \Omega_t + \Omega_{xxx} = 0 \text{ in } \mathbb{R} \times \mathbb{R}, \\
(2.2d) \quad & \Omega^*(x, t) = 0 \quad \text{whenever } t \leq 0, \\
(2.2e) \quad & (0, 0) \in \text{supp}(\Omega^*); \text{ indeed there is a positive } t_1 \text{ such that } \\
& \quad \Omega^*(0, t) \neq 0 \text{ whenever } 0 < t < t_1.
\end{align*}

One can verify that

\begin{align*}
(2.2f) \quad & \Omega^* \text{ is real-valued}
\end{align*}

since the integrand at \( s = -\sigma + i\tau \) is the conjugate of the integrand at \( s = \sigma + i\tau \).

Further note that \( \Omega^*(z, t) \) is analytic in \( z \) near the real-\( z \)-axis. Thus it follows from (2.2e) that

\begin{align*}
(2.2g) \quad & \text{there is a positive } t_1 \text{ such that if } 0 < t < t_1, \Omega^*(x, t) \text{ cannot} \\
& \quad \text{vanish identically on any half-line } [x_0, +\infty).
\end{align*}

We now study the decay of \( \Omega^* \) as \( x \to +\infty \), and its convergence to 0 as \( t \downarrow 0 \).

**Lemma 2.2.** Keep \( \tau > \tau_0 = (2/\sqrt{3})^3 \) as (2.1). Then for all \( x \geq 0 \) and \( t \geq 0 \)

\[ |\Omega^*(x, t)| \leq K_1(\tau)e^{t-x} , \]

where \( K_1(\tau) = \int_{it}^{it+\infty} e^{-|s|^{2/3}/2} \, ds \). Note that \( K_1(\tau) \to 0 \) as \( \tau \to +\infty \).

**Proof.** From (2.1)

\[ |\Omega^*(x, t)| \leq e^{t} \int_{it-\infty}^{it+\infty} e^{-s} e^{-\text{Re}((s/i)^{2/3})} \, ds . \]

Note that

\[ |s|^{1/3} \sqrt{3} \leq \text{Im}(r(s)) = |s|^{1/3} \sin \left( \frac{\pi + \text{Arg}(s)}{3} \right) \leq |s|^{1/3} \]

since \( 0 < \text{Arg} < \pi \), and that

\[ \frac{1}{2} |s|^{2/3} \leq \text{Re}((s/i)^{2/3}) = |s|^{2/3} \cos \left( \frac{2 \text{Arg}(s/i)}{3} \right) \leq |s|^{2/3} \]

since \( -\pi/2 < \text{Arg}(s/i) < \pi/2 \). Keep \( x \geq 0 \). Now

\[ |\Omega^*(x, t)| \leq e^{t} \int_{it-\infty}^{it+\infty} e^{-x|s|^{1/3}\sqrt{3}/2} e^{-|s|^{2/3}/2} \, ds . \]
Since $|s| \geq \tau \geq (2/\sqrt{3})^3$, we get

$$|\Omega^#(x, t)| \leq e^{\tau t - x} \int_{t - \infty}^{t + \infty} e^{-|s|^{2/3}/2} \, ds. \quad \square$$

**Lemma 2.3.** Suppose $x_1 < 0$. Keep $\tau > \tau_1 = 8(|x_1| + 1)^3$ as in (2.1). If $x_1 < x \leq 0$ and $t > 0$, then

$$|\Omega^#(x, t)| \leq e^{\tau t} K_2(\tau),$$

where $K_2(\tau) = \int_{t - \infty}^{t + \infty} e^{-|s|^{1/3}} \, ds$. Note that $K_2(\tau) \to 0$ as $\tau \to +\infty$.

**Proof.** From (2.1)

$$|\Omega^#(x, t)| \leq e^{\tau t} \int_{t - \infty}^{t + \infty} e^{-x \cdot \text{Im}(r(s))} e^{-\text{Re}((s/i)^{2/3})} \, ds.$$ 

As before $(\sqrt{3}/2)|s|^{1/3} \leq \text{Im}(r(s)) \leq |s|^{1/3}$ and $(1/2)|s|^{2/3} \leq \text{Re}((s/i)^{2/3}) \leq |s|^{2/3}$. Here $x \leq 0$, so

$$|\Omega^#(x, t)| \leq e^{\tau t} \int_{t - \infty}^{t + \infty} e^{[x]^{1/3}(|x| - |s|^{1/3}/2)} \, ds.$$ 

Since $\tau > \tau_1 = 8(|x_1| + 1)^3$ and $|x| < |x_1|$, we get $|x| - |s|^{1/3}/2 < |x_1| - \tau^{1/3}/2 < -1$. Thus

$$|\Omega^#(x, t)| \leq e^{\tau t} \int_{t - \infty}^{t + \infty} e^{-|s|^{1/3}} \, ds. \quad \square$$

**Corollary 2.4.** Fix $x_0$ in $\mathbb{R}$. Choose $\tau_0$ and $\tau_1$ as above. For $x \geq \max\{\tau_0, \tau_1\}$

$$|\Omega^#(x, t)| \leq e^{\tau t}[K_1(\tau) + K_2(\tau)] \leq e^{-x} \quad \text{whenever } x \geq x_0.$$ 

**Proof.** This follows from the previous two lemmas. \(\square\)

**Corollary 2.5.** Let $\nu$ be a positive integer, $x_0 \in \mathbb{R}$, and $t \geq 0$. Keep $\tau \geq \tau_0$. There is a constant $M_\nu(x_0, \tau)$ such that

$$|\partial^\nu_x \Omega^#(x, t)| \leq M_\nu(x_0, \tau) e^{\tau t} e^{-x} \quad \text{whenever } x \geq x_0 \text{ and } 0 \leq t \leq t_0.$$ 

**Proof.** By (2.1)

$$\partial^\nu_x \Omega^#(x, t) = \int_{t - \infty}^{t + \infty} (ir(s))^\nu e^{i(xr(s) - ts)} e^{-(s/i)^{2/3}} \, ds.$$ 

So for $x > 0$

$$|\partial^\nu_x \Omega^#(x, t)| \leq \int_{t - \infty}^{t + \infty} |s|^{\nu/3} e^{-x} e^{\tau t} e^{-|s|^{2/3}/2} \, ds \
\leq e^{\tau t} e^{-x} \int_{t - \infty}^{t + \infty} |s|^{\nu/3} e^{-|s|^{2/3}/2} \, ds.$$ 

The integral defines $M_\nu(x_0, \tau)$ for all nonnegative $x_0$. For $x < 0$, one appeals to the case $x = 0$ and to the continuity of $\partial^\nu_x \Omega^#$. \(\square\)

By Hörmander's Theorem 5.2.5 [8] we know that $\partial^\nu_x \Omega(x, 0) = 0$ for all $x$ and all orders $\nu$. We need to see how $\partial^\nu_x \Omega^#(x, t)$ approaches $0$ as $t \downarrow 0$. 

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Lemma 2.6. Pick $\nu \geq 0$ and $x_0 \in \mathbb{R}$. Then as $t \downarrow 0$

\begin{align*}
(2.4a) \quad &\partial^\nu_x \Omega^#(x,t) - \partial^\nu_x \Omega^#(x,0) = 0 \quad \text{in } L^\infty([x_0,\infty)), \\
(2.4b) \quad &\partial^\nu_x \Omega^#(x,t) - \partial^\nu_x \Omega^#(x,0) = 0 \quad \text{in } L^1([x_0,\infty)), \\
(2.4c) \quad &\partial^\nu_x \Omega^#(x,t) - \partial^\nu_x \Omega^#(x,0) = 0 \quad \text{in } L^2([x_0,\infty)).
\end{align*}

Proof. Since $\Omega^#$ is $C^\infty$ in $\mathbb{R} \times \mathbb{R}$ it suffices to treat $x_0 = 0$. (If $x_0 < 0$, then $\partial^\nu_x \Omega^#(x,t) - \partial^\nu_x \Omega^#(x,0) = 0$ uniformly in $[x_0,0]$.) Keep $x \geq 0$ and $\tau \geq \tau_0$. Now

\[ \partial^\nu_x \Omega^#(x,t) - \partial^\nu_x \Omega^#(x,0) = \int_{t=\tau_0}^{t+\infty} (ir(s))^\nu e^{ixr(s)} \left( e^{-its} - 1 \right) e^{-s/3^3} ds. \]

So

\[ |\partial^\nu_x \Omega^#(x,t) - \partial^\nu_x \Omega^#(x,0)| \leq \int_{t=\tau_0}^{t+\infty} |s|^{\nu/3} e^{-xIm(r(s))} |e^{-its} - 1| e^{-|s|^{2/3}/2} ds \leq e^{-x} \int_{t=\tau_0}^{t+\infty} |s|^{\nu/3} |e^{its} - 1| e^{-|s|^{2/3}/2} ds. \]

Recall that $\tau \geq \tau_0$ and $x \geq 0$ imply that $-x \text{Im}(r(s)) \leq -x$ for all $s$ with $\text{Im} s = \tau$. Applying the Lebesgue dominated convergence theorem we see that

\[ |\partial^\nu_x \Omega^#(x,t) - \partial^\nu_x \Omega^#(x,0)| \leq e^{-x} Q_\nu(t,\tau), \]

where $Q_\nu(t,\tau) \to 0$ as $t \to 0$. Results (2.4a) and (2.4b) now follow immediately, and (2.4c) follows from (2.4a) and (2.4b). \(\Box\)

B. Analysis of the Marchenko equation

For a positive parameter $\epsilon$ to be chosen later, let

\[ \Omega(x,t) = \Omega(x,t;\epsilon) = \epsilon \Omega^#(x,t). \]

Consider the Marchenko equation

\begin{equation}
(2.5) \quad B(x,y,t) + \Omega(x+y,t;\epsilon) + \int_0^\infty \Omega(x+y+z,t;\epsilon)B(x,z,t)dz = 0
\end{equation}

for parameter points $(x,t)$ in $\mathbb{R} \times \mathbb{R}^+$ are variable $y$ in $\mathbb{R}^+$. In the classical inverse scattering construction of Faddeev [5] one shows that (2.5) has a solution $B(x,\cdot,t)$ for each parameter pair $(x,t)$ because of the form of the kernel as a sum of two terms: one, the inverse Fourier transform of a function (the reflection coefficient) which is almost everywhere smaller than 1 in absolute value; the other, a positive linear combination of decaying exponentials. In this paper the kernel $\Omega$ does not have this form, and we need a different proof of solvability for (2.5).
For $x \in \mathbb{R}$ and $t > 0$ let $\Omega^t_x$ denote the operator in $L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$ given by

$$\Omega^t_x[g](y) = \int_0^\infty \Omega(x + y + z, t) g(z) dz.$$ 

Let $\omega^t_x$ be the function defined by

$$\omega^t_x(y) = \begin{cases} \Omega(x + y, t) & \text{if } y \geq 0, \\ 0 & \text{if } y \leq 0. \end{cases}$$

By (2.3) we see that $\omega^t_x \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. For $g \in L^2(\mathbb{R}^+)$ let

$$\check{g}(z) = \begin{cases} 0 & \text{if } z \geq 0, \\ g(-z) & \text{if } z < 0. \end{cases}$$

It follows that

$$(2.6) \quad \Omega^t_x[g] = \omega^t_x \ast \check{g},$$

where $\ast$ denotes convolution. One notes that (2.6) implies

$$\|\Omega^t_x\|_{op} \leq \|\omega^t_x\|_{L^1(\mathbb{R})} = \int_{s=-x}^\infty |\Omega(s, t)| ds,$$

where $\| \cdot \|_{op}$ denotes the operator norm on $L^2(\mathbb{R}^+)$. Equation (2.5) is now equivalent to

$$(I + \Omega^t_x) [B(x, \cdot, t)] = -\omega^t_x.$$ 

Since $\Omega^t_x$ is selfadjoint and compact, the existence of $(I + \Omega^t_x)^{-1}$ would follow from a proof that $I + \Omega^t_x$ is one-to-one in $L^2(\mathbb{R}^+)$. Such a proof for arbitrary $(x, t)$ would have to rely on very specific properties of the kernel in $\Omega^t_x$. It is not sufficient to use only the fact that $\Omega(x, t)$ decays at least exponentially as $x \to +\infty$: the equation

$$g(y) - \int_0^\infty 2ae^{-a(y+z)} g(z) dz = 0$$

has nontrivial solution $g(y) = e^{-\alpha y}$ when $\alpha > 0$.

**Proposition 2.7.** Given any $(x_0, t_0)$ with $t_0 > 0$, there is a domain $\mathcal{U}_1$ of the form required by Theorem 2.1(a) containing $(x_0, t_0)$ such that the $L^2(\mathbb{R}^+)$-operator norm of $\Omega^t_{x_0}$ is less than 1 on $\mathcal{U}_1$.

**Proof in the case $x_0 \geq 0$.** Recall that

$$\|\Omega^t_{x_0}\|_{op} \leq \epsilon \int_0^\infty |\Omega^x(s, t)| ds.$$ 

So for any $\tau \geq \tau_0$,

$$\|\Omega^t_{x_0}\|_{op} \leq \epsilon \int_{x_0}^\infty e^{\xi_0 \tau} K_1(\tau) e^{-s} ds = e^{\xi_0 \tau} K_1(\tau) e^{-x_0}.$$
Choose $\tau_2$ so that $\tau_2 \geq \tau_0$, and $K_1(\tau) \leq 1/2$ whenever $\tau \geq \tau_2$. Next choose $\varepsilon$ so that $0 < \varepsilon < 1$ and $e^{\varepsilon t_2-x} K_1(\tau_2) < 1$. For any $x \geq 0$ and $t > 0$ we now get

$$\|\Omega_x^t\|_{\text{op}} \leq e^{\varepsilon t_2-x} K_1(\tau_2) < e^{\varepsilon t_2-x}/2.$$  

Note that

$$e^{\varepsilon t_2-x}/2 < 1 \iff t < \frac{\ln 2 + \ln(1/\varepsilon) + x}{\tau_2}.$$  

Let

$$h(x) := \frac{\ln 2 + \ln(1/\varepsilon) + x}{\tau_2} \quad \text{for all } x \geq 0.$$  

We now extend the definition of $h$ to $\mathbb{R}^-$. Choose a sequence $(\tau_0)_0^\infty$ such that

$$\tau_0 \geq \max\{\tau_0, \tau_1, \tau_2\}, \quad \tau_n \geq \tau_{n-1} \quad \text{for all } n,$$  

and

$$K_2(\tau) \leq 1/4(n+1) \quad \text{whenever } \tau \geq \tau_n.$$  

Consider a negative $x$. There is a nonnegative integer $n$ such that $-(n+1) < x < -n$. For this $x$ and any $t > 0$

$$\|\Omega_x^t\|_{\text{op}} \leq e^{\varepsilon t(3/4)} \cdot |x| K_2(\tau) + K_1(\tau)$$

for any $\tau \geq \tau_n$. Thus

$$\|\Omega_x^t\|_{\text{op}} \leq e^{\varepsilon t_2(3/4)}.$$  

Note that

$$e^{\varepsilon t_2(3/4)} < 1 \iff t < \frac{\ln(3/4) + \ln(1/\varepsilon)}{\tau_n}.$$  

We can now define

$$h(x) := \frac{\ln(3/4) + \ln(1/\varepsilon)}{\tau_n} \quad \text{if } -(n+1) \leq x < -n.$$  

Thus we have defined $h$ on all $\mathbb{R}$.

It is easy to verify that $h(x) > 0$ for all $x$, that $h$ is nondecreasing, and that $h(x) \to +\infty$ at least linearly as $x \to +\infty$. Let

$$\mathcal{Z}_1 := \{(x,t): 0 < t < h(x), \ x \in \mathbb{R}\}.$$  

We have built in the properties that $(x_0, t_0) \in \mathcal{Z}_1$ and $\|\Omega_x^t\|_{\text{op}} < 1$ whenever $(x,t) \in \mathcal{Z}_1$.

Proof in the case that $x_0 < 0$. Choose a positive integer $N$ such that $-(N+1) \leq x_0 < -N$. We know that

$$\|\Omega_{x_0}^{t_0}\|_{\text{op}} \leq e^{\varepsilon t_0} \{x_0|K_2(\tau) + K_1(\tau)\}$$

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for all \( \tau \geq \max\{\tau_0, \tau_1\} \). Choose \( \tau_* \) so \( \tau_* \geq \max\{\tau_0, \tau_1\} \) and \((N + 1)K_2(\tau) + K_1(\tau) \leq 3/4\) for all \( \tau \geq \tau_* \). Choose \( \varepsilon \) so \( 0 < \varepsilon < 1 \) and \( e^{e^{\varepsilon t}} \{(N + 1)K_2(\tau_*) + K_1(\tau_*)\} < 1 \). Suppose \( x \geq 0 \) and \( t > 0 \). Then

\[
\|\Omega_{\tau}^{\tau}\|_{\text{op}} \leq e^{e^{\varepsilon t}K_1(\tau)}e^{x} \leq 3e^{e^{\varepsilon t}x}/4
\]

whenever \( \tau > \tau_* \). Note that

\[
e^{e^{\varepsilon t}x}(3/4) < 1 \quad \text{iff} \quad t < \frac{\ln(4/3) + \ln(1/\varepsilon) + x}{\tau_*}.
\]

Thus we define

\[
h(x) := \frac{\ln(4/3) + \ln(1/\varepsilon) + x}{\tau_*}
\]

for \( x \geq 0 \).

Choose a sequence \( (\tau_n)_{n=0}^{\infty} \) so that \( \tau_n \geq \tau_* \), \( \tau_n \leq \tau_{n+1} \) for all \( n \geq 0 \), and \((n + 1)K_2(\tau) + K_1(\tau) \leq 3/4\) whenever \( \tau \geq \tau_n \). Recall that

\[
\|\Omega_{\tau}^{\tau}\|_{\text{op}} \leq e^{e^{\varepsilon t}(|x|K_2(\tau) + K_1(\tau))}
\]

whenever \( \tau \geq \tau_* \). Suppose \( x < 0 \) and \( t > 0 \). Pick the positive integer \( n \) such that \(-(n + 1) \leq x < -n \). Then

\[
\|\Omega_{x}^{\tau}\|_{\text{op}} \leq e^{e^{\varepsilon t}(3/4)}.
\]

Note that

\[
e^{e^{\varepsilon t}(3/4)} < 1 \quad \text{iff} \quad t < \frac{\ln(4/3) + \ln(1/\varepsilon)}{\tau_n}.
\]

Finally we define \( h \) on the rest of \( \mathbb{R} \) by

\[
h(x) := \frac{\ln(4/3) + \ln(1/\varepsilon)}{\tau_n}
\]

for \(-(n + 1) \leq x < -n \).

It is straightforward to check that this \( h \) has the desired properties. \( \square \)

For \((x, t)\) in \( \mathcal{Z}_1 \) we can now solve (2.5) by setting

\[
B_1(x, t) := -(I + \Omega_{\tau})^{-1}[\omega_x] \quad \text{in } L^2(\mathbb{R}^+).
\]

Because of the regularity of \( \Omega \) and its decay as \( x \to +\infty \), one finds from (2.5) that

\[
B_1(x, t) \in L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)
\]

and \( B_1(x, y, t) \) is \( C^\infty \) on \( \mathcal{Y}_1 := \{(x, y, t): y > 0, (x, t) \in \mathcal{Z}_1\} \). For \((x, t)\) in \( \mathcal{Z}_1 \) set

\[
(2.8) \quad u_1(x, t) := -B_1(x, 0, t).
\]

This \( u_1 \) is clearly \( C^\infty \) in \( \mathcal{Z}_1 \). By Theorem T we know that such a \( u_1 \) solves KdV in \( \mathcal{Z}_1 \). It remains to show that all derivatives of \( u_1 \) converge to 0 as \( t \downarrow 0 \), and that \( u_1 \) is not identically zero in \( \mathcal{Z}_1 \).
Lemma 2.8. (a) The operator $\Omega^I_x$ is bounded in $\mathcal{L}(L^\infty(\mathbb{R}^+), L^\infty(\mathbb{R}^+))$ for any $(x, t)$.

(b) For each $x_0$ there is a $t_0 > 0$ such that $(I + \Omega^I_x)$ is invertible on $L^\infty(\mathbb{R}^+)$ whenever $x \geq x_0$ for all $0 < t < t_0$. Furthermore, $t_0$ can be chosen so that

$$\|(I + \Omega^I_x)^{-1}\|_{\text{op}, L^\infty(\mathbb{R}^+)} < 10$$

for such $(x, t)$.

Proof. (a) It suffices to note that

$$\sup_{y \geq 0} \left| \int_{z=0}^{\infty} \Omega(x + y + z, t) g(z) \, dz \right| \leq \sup_{z \geq 0} |g(z)| \int_{s=x}^{\infty} |\Omega(s, t)| \, ds.$$ 

(b) By (a) we have

$$\|\Omega^I_x\|_{\text{op}, L^\infty(\mathbb{R}^+)} \leq \int_{s=x}^{\infty} |\Omega(s, t)| \, ds.$$ 

By (2.4a), we may pick $t_0$ so that $\int_{s=x_0}^{\infty} |\Omega(s, t)| \, ds < 9/10$ when $0 < t < t_0$. Thus $(I + \Omega^I_x)^{-1}$ has a convergent Neumann series in $L^\infty(\mathbb{R}^+)$ and

$$\|(I + \Omega^I_x)^{-1}\|_{\text{op}, L^\infty(\mathbb{R}^+)} \leq 10$$

for $x \geq x_0$ and $0 \leq t \leq t_0$. □

Lemma 2.9. Let $\nu \in \{0, 1, 2, \ldots\}$ and $x_0 \in \mathbb{R}$. Then

$$\lim_{t \to 0} \partial_x^\nu u_1(x, t) = 0 \equiv \partial_x^\nu u_1(x, 0)$$

uniformly on $[x_0, +\infty)$.

Proof. Fix $x_0$. Because of (2.8) it suffices to show that

$$\lim_{t \to 0} \partial_x^n B_1(x, 0, t) = \partial_x^n B_1(x, 0, 0) \equiv 0$$

for all $n \geq 0$, uniformly in $[x_0, +\infty)$. Note that $\Omega(s, 0) \equiv 0$, so (2.5) yields $B_1(x, 0, 0) \equiv 0$ and $\partial_x^n B_1(x, 0, 0) \equiv 0$. Recall that

$$B_1(x, \cdot, t) = -(I + \Omega^I_x)^{-1}[\omega_x^I]$$

for $(x, t)$ in $\mathcal{U}_1$. For $x \geq x_0$ and for sufficiently small $t$, we get

$$\|B_1(x, \cdot, t)\|_{L^\infty(\mathbb{R}^+)} \leq 10\|\omega_x^I\|_{L^\infty(\mathbb{R}^+)} \leq 10 \sup_{s \geq x_0} |\Omega(s, t)|.$$

By (2.4a) $B_1(x, \cdot, t) \to 0$ in $L^\infty(\mathbb{R}^+)$ uniformly in $x \geq x_0$. This proves (2.9) with $n = 0$.

Suppose (2.9) holds for all $n \leq N$, uniformly in $x \geq x_0$; we must prove (2.9) with $n = N + 1$. By (2.5)

$$\partial_x^{N+1} B_1(x, y, t) + \partial_y^{N+1} \Omega(x + y, t) + \sum_{n=0}^{N+1} \binom{N+1}{n} \Psi^n(x, y, t),$$
where $\partial_1$ denotes the derivative with respect to the first argument, and

$$\Psi_n := \int_0^\infty \partial_1^{N+1-n} \Omega(x + y, t) \partial_x^N B_1(x + z, t) \, dz.$$ 

Thus

$$\partial_1^{N+1} B_1(x, \cdot, t) = -(i + n'x)^{-1} \left( \partial_1^{N+1} \Omega(x + \cdot, t) + \sum_{n=0}^N \binom{N+1}{n} \Psi_n(x, \cdot, t) \right),$$

whence

$$||\partial_1^{N+1} B_1(x, \cdot, t)||_{L^\infty(\mathbb{R}^+)} \leq 10 ||\partial_1^{N+1} \Omega(x + \cdot, t)||_{L^\infty(\mathbb{R}^+)} + 10 \sum_{n=0}^N \binom{N+1}{n} ||\Psi_n(x, \cdot, t)||_{L^\infty(\mathbb{R}^+)}$$

for all $x > x_0$ and for all sufficiently small $t$. By Lemma 2.6 the first norm on the right goes to zero as $t \downarrow 0$. Now

$$||\Psi_n(x, \cdot, t)||_{L^\infty(\mathbb{R}^+)} \leq ||\partial_x^n B_1(x, \cdot, t)||_{L^\infty(\mathbb{R}^+)} \int_0^\infty \partial_1^{N+1} \Omega(x + y + z, t) \, dz$$

$$\leq ||\partial_x^n B_1(x, \cdot, t)||_{L^\infty(\mathbb{R}^+)} ||\partial_1^{N+1-n} \Omega(\cdot, t)||_{L^1((x_0, \infty))}.$$ 

Since $n < N$, these terms go to zero by the induction hypothesis and Lemma 2.6. Thus (2.9) holds uniformly on $x > x_0$ when $n = N + 1$. \hfill $\square$

**Lemma 2.10.** The function $u_1$ defined by (2.8) is not identically zero in $\mathcal{U}_1$.

**Proof.** It follows from (2.5) that $B_1$ satisfies

$$B_{xx}(x, y, t) - B_{xy}(x, y, t) = u_1(x, t) B(x, y, t)$$

for $y \geq 0$ and $(x, t)$ in $\mathcal{U}_1$. Suppose now that $u_1(x, t) \equiv 0$ in $\mathcal{U}_1$. Then $B_1$ solves a linear partial differential equation which has general solution

$$B_1(x, y, t) = \varphi_1(x + y, t) + \varphi_2(y, t).$$

But the assumption $u_1 \equiv 0$ together with (2.8) forces

$$0 = \partial_x B_1(x, 0, t) \varphi_1'(x, t) \quad \text{in } \mathcal{U}_1,$$

where $(\cdot)'$ denotes the derivative with respect to the first argument. Thus $\varphi_1(x, t) = \varphi_3(t)$ and

$$B_1(x, y, t) = \varphi_3(t) + \varphi_2(y, t) =: \psi(y, t).$$

Assume for the moment that

$$\lim_{x \to +\infty} B_1(x, y, t) = 0 \quad \text{for all } y, \text{ and all small enough } t.$$ 

Then it would follow that $\psi(y, t) \equiv 0$ and that $B_1(x, y, t) \equiv 0$ in $\mathcal{U}_1$. But then the Marchenko equation (2.5) yields $\Omega(x + y, t) \equiv 0$, which contradicts (2.2g).
It remains to verify (2.10). The definition of $B_1$ and Lemma 2.8 tells us that

$$||B_1(x, \cdot, t)||_{L^\infty(R^+)} \leq 10||\Omega(\cdot, t)||_{L^\infty(R^+)}$$

for all $x \geq 0$ and all small enough $t$. Lemma 2.2 now yields

$$||B_1(x, \cdot, t)||_{L^\infty(R^+)} \leq e^{10K_1(t)e^{rt-x}}.$$

Fixing $y$ and $t$, we get

$$B_1(x, y, t) \leq (\text{constant})e^{-x}$$

and (2.10) surely follows. □

This concludes the proof of Theorem 2.1.

3. Possible behavior of $u_1$ as $x$ moves left

Since $u_1$ is a smooth function of rapid decrease as $x \to +\infty$, the misbehavior as $|x| \to \infty$ usually associated with nonuniqueness in a characteristic initial value problem must occur as $x$ moves left. The region $\mathcal{U}_1$ where $u_1$ is defined does not necessarily include any strip

$$\mathcal{B}_T = \{(x, t): x \in \mathbb{R}, 0 < t < T\}$$

with $T > 0$. We do not know whether $u_1$ can be extended to a solution of (1.1), (1.2) on any strip $\mathcal{B}_T$. Thus the expected misbehavior may be that $u_1$ cannot be extended to any such strip. For the rest of this section we assume that $u_1$ can be so extended, say to $u^\#$, and discuss what follows about $u^\#$ from the various uniqueness results known for KdV.

The first point to notice is that the $L^2$ norm of $u^\#$ is not a conserved quantity: it is zero at $t = 0$, but not zero later. Indeed we will see that $u^\#$ does not evolve in $L^2(\mathbb{R})$.

Lax [11], in 1968, argued by classical methods that (1.1), (1.2) has at most one solution evolving in the class $\mathcal{E}$ of functions $v(x, t)$ such that

(3.1a) $\partial_x^m \partial_t^n v(x, t)$ is continuous for $m + 3n \leq 3$,

(3.1b) $v(x, t)$ and $v_{xx}(x, t)$ approach 0 as $x \to \pm \infty$

[fast enough that $v$ and $v_x$ and $v_{xx}$ are in $L^2(\mathbb{R})$].

Let $z(x, t)$ denote the constant zero solution of KdV. Clearly $z \in \mathcal{E}$, and $z(x, 0) = 0 = u^\#(x, 0)$. Therefore $u^\#$ does not evolve in $\mathcal{E}$. Since $u^\#$ and its derivatives decay exponentially as $x \to +\infty$, one must conclude that $u^\#$ is badly behaved as $x \to -\infty$ in the sense that (3.1a) or (3.1b) must fail as $x \to -\infty$.

Temam [14], in 1969, considered the problem (1.1), (1.2) with periodic initial data. $u_0(x + 1) = u_0(x)$. He showed uniqueness for solutions evolving in the Sobolev space $H^2((0, 1))$. His proof remains valid for the Sobolev space $H^2(\mathbb{R})$. It follows that $u^\#$ does not evolve in $H^2(\mathbb{R})$. 
In 1972, Menikoff [12] showed uniqueness for classical solutions of KdV in a class allowing linear growth at infinity, namely the class of functions $v(x,t)$ such that $\partial^n_x v(x,t) = O(|x|^{-n})$ as $x \to \pm \infty$ for $n \in \{0, 1, 2, \ldots, 7\}$. Since $u^*$ decays rapidly to the right, $u^*$ may grow faster than linearly as $x \to -\infty$, or $\partial^*_x u^*$ may be unbounded, or some higher derivative may fail to decay fast enough. Later we suggest that $u^*$ probably does blow up as $x$ moves left.

S. Kruzhkov and A. Faminskii have a very general uniqueness theorem for generalized solutions of KdV [10]. Let $\mathcal{H}_T$ denote the class of functions $v(x,t)$ such that

$$\text{ess sup}_{0 < t < T} \left[ \int_{-\infty}^{\infty} |v(x,t)|^2 \, dx + \int_{0}^{\infty} x^{3/2} |v(x,t)| \, dx \right] < \infty.$$ 

Kruzhkov and Faminskii show that if $u_1$ and $u_2$ are generalized solutions of KdV which belong to $\mathcal{H}_T$, then

$$\int_{-\infty}^{\infty} \min\{e^x, 1\}|u_1(x,t) - u_2(x,t)|^2 \, dx \leq \gamma_T \int_{-\infty}^{\infty} |u_1(x,0) - u_2(x,0)|^2 \, dx$$

for almost all $t$ in $0 < t < T$, where $\gamma_T$ is some constant.

Again let $z(x,t)$ denote the zero solution to KdV. Clearly $z \in \mathcal{H}_T$ and $z(x,0) \equiv 0 \equiv u^*(x,0)$. So $u^*$ cannot be in $\mathcal{H}_T$ for any $T$. Since $u^*$ decays rapidly as $x \to +\infty$, one concludes that $u_1(\cdot, t)$ is not in $L^2(\mathbb{R})$.

It would be interesting to know if the analysis in [10] allows one to conclude not only that

$$\int_{-\infty}^{0} |u_1(x,t)|^2 \, dx = \infty$$

but also that

$$\int_{-\infty}^{0} e^x |u_1(x,t)|^2 \, dx = \infty.$$ 

Finally one might try to get direct control over $u^*$ as $x \to -\infty$ by studying the nontrivial solution $\Omega^*$ of the Airy equation. Recall that $\Omega^*$ had the properties (1.4). Gel'fand and Shilov [7] have shown that the problem

$$(3.2) \quad \Omega_t + \Omega_{xxx} = 0, \quad \Omega(x,0) = 0$$

has a unique solution in any class $U(C,b)$ of measurable functions satisfying

$$|f(x,t)| \leq C \exp(b|x|^{3/2}) \quad \text{for all } (x,t).$$

This is a special case of their Theorem 1 on page 42 of [7]. The function $\Omega^*$ defined by (2.1) must therefore not belong to any $U(C,b)$. Indeed since all $\partial^*_x \Omega^*$ also satisfy (3.2) none of these derivatives can belong to any $U(C,b)$. But all $\partial^*_x \Omega^*$ decay fast as $x \to +\infty$. So for all $\nu$

$$\limsup_{x \to -\infty} |\partial^*_x \Omega^*(x,t) \exp(-b|x|^{3/2})| = +\infty.$$ 

By stationary phase arguments one sees that \( \Omega^u \) and \( \partial_x \Omega^u \) are subject to growing oscillations as \( x \to -\infty \).

Unfortunately it is not easy to transfer this information about \( \Omega^u \) to the analysis of the Marchenko equation

\[
B(x, y, t) + \Omega(x + y, t) + \int_0^\infty \Omega(x + y + z, t) B(x, z, t) \, dz = 0,
\]

where the kernel is given by \( \Omega = e^{i\xi x} \) as in \( \S 2 \). Let us suppose now not only that \( \sigma \) extends to \( \sigma^u \) in the strip \( \mathbb{B}_T \), but that (3.3) can be solved in \( \mathbb{B}_T \) and \( \sigma^u(x, t) = -\partial_x B(x, 0, t) \) there. Thus

\[
\begin{align*}
-u^u(x, t) + \varepsilon \partial_x \Omega^u(x, t) + \int_0^\infty \varepsilon \partial_x \Omega^u(x + z, t) B(x, z, t) \, dz \\
+ \int_0^\infty \varepsilon \Omega^u(x + z, t) \partial_x B(x, z, t) \, dz = 0,
\end{align*}
\]

whence

\[
|u^u(x, t) - \partial_x \Omega^u(x, t)| \leq \int_0^\infty |\partial_x \Omega^u(x, t) B(x, z, t)| \, dz \\
+ \int_0^\infty |\Omega^u(x + z, t) \partial_x B(x, z, t)| \, dz \\
\leq \sup_{s \geq x} |\partial_s \Omega^u(s, t)| \cdot ||B(x, \cdot, t)||_{L^1(\mathbb{R}^+)} \\
+ \sup_{s \geq x} |\Omega^u(s, t)| \cdot ||\partial_x B(x, \cdot, t)||_{L^1(\mathbb{R}^+)}.\]

We know that \( \sup_{s \geq x} |\partial_s \Omega^u(s, t)| \) and \( \sup_{s \geq x} |\Omega^u(s, t)| \) grow rapidly as \( x \to -\infty \), but so do the available bounds on

\[
||B(x, \cdot, t)||_{L^1(\mathbb{R}^+)} \quad \text{and} \quad ||\partial_x B(x, \cdot, t)||_{L^1(\mathbb{R}^+)}.\]

As a result we can suspect that \( u^u(x, t) \) is subject to growing oscillations as \( x \to -\infty \), but not prove it using this approach.

4. Nonuniqueness for nontrivial initial profile

Here we consider the KdV problem

\[
\begin{align*}
(4.1) \quad & \quad u_t - 6uu_x + u_{xxx} = 0, \\
(4.2) \quad & \quad u(x, 0) = U_1(x)
\end{align*}
\]

under the assumptions

\[
\begin{align*}
(4.3a) \quad & \quad U_1 \text{ has sufficient decay to permit the construction of a solution of (4.1), (4.2) by the classical inverse scattering method,} \\
(4.3b) \quad & \quad U_1 \text{ is small enough, in a sense to be made precise later.}
\end{align*}
\]

For example, for (4.3a) it suffices that \( u_1 \) belong to \( L^1(\mathbb{R}) \cap L^N(\mathbb{R}^+) \) with \( N \geq 11/4 \). More generally, to ensure (4.3a) one could take the initial condition
The solution of (4.1), (4.2) guaranteed by (4.3a) is given by

\[ u_1(x, t) = -\partial_x B(x, 0, t), \]

where \( B \) is determined by the Marchenko equation

\[
B(x, y, t) + Q(x + y, t) + \int_{0}^{\infty} \Omega(x + y + z, t)B(x, z, t)\,dz = 0
\]

in which the kernel \( \Omega \) is a particular function \( \Omega_1 \) constructed from the scattering data of the Schrödinger equation (4.4). One can verify that \( \Omega_1 \) solves \( \Omega_t + \Omega_{xxx} = 0 \). The precise assumption on the size of \( U_1 \) is

\[ U_1 \text{ is small enough that } ||\Omega_1(\cdot, 0)||_{L^1(\mathbb{R})} = 1 - \varepsilon \]

for an \( \varepsilon \) with \( 0 < \varepsilon < 1 \).

Let \( \Omega_2 = \Omega_1 + \varepsilon \Omega^{\#} \), where \( \Omega^{\#} \) is the special nontrivial solution of the Airy equation considered in \( \S2 \) and \( \varepsilon \) is given by (4.3b). We wish to solve

\[
B(x, y, t) + \Omega_2(x + y, t) + \int_{0}^{\infty} \Omega_2(x + y + z, t)B(x, z, t)\,dz = 0.
\]

For this we need to invert \( (I + \Omega_{1x}^t + \varepsilon \Omega_{1x}^{\#}) \). For all \( x \),

\[ ||\Omega_{1x}^0||_{op} \leq \int_{x}^{\infty} ||\Omega_1(s, 0)||\,ds \leq 1 - \varepsilon \]

and

\[ ||(I + \Omega_{1x}^0)^{-1}||_{op} \leq 1/\varepsilon := \mathcal{A}. \]

Now because of the behavior of \( \Omega_1 \) as \( x \to +\infty \) and as \( t \downarrow 0 \), we find that for each \( n \in \{0, 1, 2, \ldots\} \) there is a \( T_n \) such that

\[ ||(I + \Omega_{1x}^t)^{-1}||_{op} \leq 2\mathcal{A} \]

whenever \( x \geq -n \) and \( 0 \leq t \leq T_n \). We may assume that the \( T_n \) are decreasing. Let

\[ \mathcal{R}_1 := \bigcup_{0}^{\infty} \{(x, t) : x > -n \text{ and } 0 \leq t \leq T_n \}. \]

Following earlier arguments we can find that \( \mathcal{R}_2 := \{(x, t) : 0 < t < H(x) \} \) for some positive nondecreasing \( H \) small enough that \( ||\varepsilon \Omega_{1x}^{\#}||_{op} < 1/4\mathcal{A} \) in \( \mathcal{R}_2 \). Thus, in \( \mathcal{R} := \mathcal{R}_1 \cap \mathcal{R}_2 \),

\[ ||-\varepsilon \Omega_{1x}^{\#}(I + \Omega_{1x}^t)^{-1}||_{op} \leq 1/2, \]

\[
(I + \Omega_{2x}^t)^{-1} = (I + \Omega_{1x}^t + \varepsilon \Omega_{1x}^{\#})^{-1} = (I + \Omega_{1x}^t)^{-1} - \sum_{\nu=0}^{\infty} [-\varepsilon \Omega_{1x}^{\#}(I + \Omega_{1x}^t)^{-1}]^\nu,
\]

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Now we can set
\[ u_2(x, t) = -\partial_x B_2(x, 0, t) \] in \( \mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2 \), where \( B_2 \) solves (4.5.2). Since \( \Omega_2 \) still solves \( \Omega_t + \Omega_{xxx} \), \( u_2 \) still solves KdV. Because \( \Omega_2(x, 0) = \Omega_1(x, 0) + 0 \), \( u_2 \) and \( u_1 \) both satisfy (4.2). However, since \( \Omega_2 \neq \Omega_1 \) in \( \mathcal{R} \), we expect that \( u_2 \neq u_1 \) in \( \mathcal{R} \).

**Theorem 4.1.** Under assumption (4.3) there exist two distinct classical solutions \( u_1(x, t) \) and \( u_2(x, t) \) of (4.1), (4.2) in the region \( \mathcal{R} \).

**Proof.** We prove somewhat more by showing that the functions \( u_1 \) and \( u_2 \) constructed above cannot be identical on any region of the form
\[ Q(X, T) = \{(x, t): X < x < \infty, 0 < t < T\} \]
with \( (X, T) \) in \( \mathcal{R} \), and \( T \) less than the \( t_1 \) of (2.2d) or (2.2e).

Suppose that \( u_1 \equiv u_2 \) in \( Q(X, T) \). Write \( u \equiv u_1 \equiv u_2 \) there. Then from the Marchenko equations
\[ B_2(x, y, t) + \int_0^\infty \Omega_2(x + y + z, t)B_2(x, z, t)dz = 0 \quad (4.6) \]
and
\[ B_1(x, y, t) + \int_0^\infty \Omega_1(x + y + z, t)B_1(x, z, t)dz = 0 \quad (4.7) \]
it follows that both \( B_1 \) and \( B_2 \) satisfy
\[ B_{xx}(x, y, t) - B_{xy}(x, y, t) = u(x, t)B(x, y, t) \quad (4.8) \]
in \( Q^-(X, T) = \{(x, y, t): x \geq X, y \geq 0, 0 < t < T\} \) and that
\[ -\partial_x B_1(x, 0, t) = u(x, t) = -\partial_x B_2(x, 0, t) \quad \text{in } Q(X, T). \quad (4.9) \]
Further by (4.2), (4.3), and the decay of \( \Omega_1 \) and \( \Omega_2 \) as \( x \to +\infty \), one finds that \( \partial_x B_j(x, y, t) \to 0 \) as \( x + y \to +\infty \) for any fixed \( t \) and for \( \nu = 1, 2 \). Now freeze \( t \) with \( 0 < t < T \) and pick \( x_0, y_0 \) with \( x_0 > X, y_0 > 0 \). Integrate (4.8) over
\[ R(x_0, y_0) = \{(x, y): 0 < y < y_0, x > x_0 + y_0 - y\}. \]
The result is
\[ B(x_0, y_0, t) - B(x_0 + y_0, 0, 0, t) = \int_{R(x_0, y_0)} u(x, t)B(x, y, t)dx\,dy. \]
By (4.9) it follows that both \( B_1 \) and \( B_2 \) satisfy
\[ B(x_0, y_0, t) = \int_{x_0+y_0}^\infty u(x, t)dx + \int_0^{y_0} \int_{x=x_0+y_0-y}^{\infty} u(x, t)B(x, y, t)dx\,dy. \]
But this integral equation is of Volterra type and has a unique solution because of the decay of \( u(x, t) \) as \( x \to +\infty \). Thus we see that, since \( u_1 \equiv u_2 \) in \( Q(X, T) \), \( B_1 \equiv B_2 \) in \( Q^{-}(X, T) \). Write \( B = B_1 = B_2 \) there. Now it follows from (4.6) and (4.7) that

\[
\Omega^{\#}(x + y, t) + \int_{\Omega}^{\infty} \Omega^{\#}(x + y + z, t) B(x, z, t) dz = 0
\]

in \( Q^{-} \) since \( \Omega_2 - \Omega_1 = \varepsilon \Omega^{\#} \).

Consider the operator \( B_x' \) in \( \mathcal{L}(L^2(\mathbb{R}^+), L^2(\mathbb{R}^+)) \) defined by

\[
B_x'[g](y) = \int_{\Omega}^{\infty} B_2(x, z, t)g(y + z) dz.
\]

So

\[
\int_{\Omega}^{\infty} \Omega^{\#}(x + [\cdot] + z, t) B_2(x, z, t) dz = B_x'[\Omega^{\#}(x + [\cdot])] \quad \text{in} \mathbb{R}^+.
\]

By a method analogous to the proof of (2.11) we get, for each \( X_0 \) in \( \mathbb{R} \) and each positive \( T_0 \), a constant \( K(T_0, X_0) \) such that

\[
\|B_x'[\cdot]\|_{op} \leq K(T_0, X_0)e^{-x}
\]

for \( 0 < t < T_0 \) and \( x > X_0 \). We can now pick an \( X_1 \) such that

\[
\|B_x'[\cdot]\|_{op} \leq 0.5 \quad \text{and} \quad \|(I + B_x')^{-1}\|_{op} \leq 2
\]

uniformly in \( 0 < t < T, x \geq X_1 \). But since (4.10) says that \( (I + B_x')\Omega^{\#} = 0 \), it follows that \( \Omega^{\#}(x, t) = 0 \) for \( x \geq X_1 \) and \( 0 < t < T \). This contradicts the property (2.2e) of \( \Omega^{\#} \). Therefore \( u_2(x, t) \) cannot agree identically with \( u_1(x, t) \) on any such region \( Q(X, T) \). \( \square \)

5. The second nonuniqueness result: global in time, but on a half-line in space

**Theorem 5.1.** There are two \( C^\infty \) solutions \( u_1 \) and \( u_2 \) to KdV in \( Q_0 = \{(x, t): x > 0, t > 0\} \) such that

\[
u_1(x, 0) = u_2(x, 0) \quad \text{for} \ x \geq 0,
\]

\[
u_1 \neq u_2 \quad \text{in} Q_0.
\]

**Remark.** Following the line of argument used in §4 one can show that there are many pairs of solutions \( v_1 \) and \( v_2 \) with the properties stated in Theorem 5.1. One takes a regular solution \( u_3(x, t) \) of KdV evolving in Schwartz space. Using the forward problem we get a bounded solution \( \Omega_3(x, t) \) satisfying the Airy equation. By adding \( \Omega_3 \) to both \( \Omega_1 \) and \( \Omega_2 \) as constructed in the proof of Theorem 5.1, and by choosing the appropriate \( K \) in the bounds on \( \Omega_1 \) and \( \Omega_2 \), we can follow the same arguments to obtain new solutions \( v_1 \) and \( v_2 \) satisfying KdV in \( t \geq 0, x \geq 0 \) such that \( v_1(x, 0) = v_2(x, 0) \neq u_1(x, 0) = u_2(x, 0) \).
Proof. The rest of this section is devoted to this proof. We begin by stating a lemma whose proof is deferred until after it is applied.

**Lemma 5.2.** There is a constant $K = K(\tau)$ such that

$$|\Omega^#(x, t)| \leq Ke^{\tau - x \tau^{1/3}}$$

for all $x \geq 0$ and all $t \geq 0$. Indeed one may take

$$K(\tau) := 2 \int_{\sigma=0}^{\infty} e^{-\sigma^2 + (\tau^2 + \sigma^2)^{1/3}} d\sigma.$$

In this section, as in the previous one, we consider two kernels for the Marchenko equation. This time let

$$\Omega_1(x, t) = \Omega^#(x, y) + 2Ke^{\tau - x \tau^{1/3}}, \quad \Omega_2(x, t) = 2Ke^{\tau - x \tau^{1/3}}.$$

Both $\Omega_1$ and $\Omega_2$ are $C^\infty$ solutions of the Airy equation and have rapid decay as $x \to +\infty$. Since $\Omega^#(x, 0) \equiv 0$, $\Omega_1(x, 0) \equiv \Omega_2(x, 0)$. Further it is clear that both $\Omega_1$ and $\Omega_2$ are positive in the quadrant $Q_0$. Thus the operators $(I + \Omega_1^t)$ and $(I + \Omega_2^t)$ are positive symmetric operators on $L^2(\mathbb{R}^+)$ which can be inverted. For $i = 1, 2$ set $u_i(x, t) = -\partial_x B_i(x, 0, t)$, where

$$B_i(x, y, t) + \int_0^\infty \Omega_i(x + y, z, t)B_i(x, z, t) dz = 0.$$

By Theorem T, $u_1$ and $u_2$ both solve KdV in $Q_0$. Since $\Omega_1 = \Omega_2$ at $t = 0$, $B_1(x, 0, t) = B_2(x, 0, t)$ and $u_1 = u_2$ at $t = 0$. The proof in §4 that $u_1 \neq u_2$ in $Q_0$ carries over.

Thus to complete the proof of Theorem 5.1, it remains only to verify Lemma 5.2.

**Proof of Lemma 5.2.** For any $\tau \geq t_0$

$$\Omega^#(x, t) = \int_{\tau - \infty}^{\tau + \infty} e^{i(xr(s) - ts)} e^{-(s/\iota)^{2/3}} ds$$

$$= 2e^{\tau \int_{\sigma=0}^{\infty} e^{-A(\sigma)} \cos(\theta(\sigma)) d\sigma},$$

where

$$A(\sigma) = x(\sigma^2 + \tau^2)^{1/6} \sin \left(\frac{\text{Arg}(\sigma + i\tau) + \pi}{3}\right)$$

$$+ (\sigma^2 + \tau^2)^{1/3} \cos \left(\frac{2\text{Arg}(\sigma - i\tau)}{3}\right)$$

$$= xB(\sigma) + C(\sigma).$$

The term $\theta(\sigma)$ is real; its form is unimportant. Note that $B(\sigma) \geq 0$. Thus

$$|\Omega^#(x, t)| \leq 2e^{\tau \int_{\sigma=0}^{\infty} e^{-xB(\sigma)} e^{-C(\sigma)} d\sigma}.$$
With $\tau > 0$ and $\sigma \geq 0$ we get
$$0 \leq \frac{2}{3} \arg(\tau - i\sigma) < \frac{2\pi}{3} = \pi.$$  
Thus
$$\frac{1}{2}(\sigma^2 + \tau^2)^{1/3} < (\sigma^2 + \tau^2)^{1/3} \cos\left(\frac{2\arg(\tau - i\sigma)}{3}\right) < (\sigma^2 + \tau^2)^{1/3},$$
$$e^{-C(\sigma)} \leq e^{-(\sigma^2 + \tau^2)^{1/3}/2},$$
and $e^{-C(\sigma)}$ is integrable over $0 < \sigma < \infty$.

Suppose we could show that $B(\sigma) \geq B(0) = \tau^{1/3}$ for all $\sigma \geq 0$. Then
$$|\Omega^g(x, t)| \leq \left\{2 \int_0^\infty e^{-C(\sigma)} d\sigma \right\} e^{\tau - \tau^{1/3}}$$
which would complete the proof of the lemma.

Lemma 5.3. $dB(\sigma)/d\sigma > 0$ for $\sigma > 0$; $dB(\sigma)/d\sigma = 0$ for $\sigma = 0$. Thus $B(0) < B(\sigma)$ whenever $0 < \sigma$.

Proof. Computation yields
$$\frac{dB(\sigma)}{d\sigma} = \frac{1}{6} \frac{2\sigma}{(\sigma^2 + \tau^2)^{5/6}} \sin\left(\frac{\arg(\sigma + i\tau) + \pi}{3}\right)$$
$$+ (\sigma^2 + \tau^2)^{1/6} \cos\left(\frac{\arg(\sigma + i\tau) + \pi}{3}\right) \frac{1}{3} \frac{d\arg(\sigma + i\tau)}{d\sigma}.$$

Further computation yields
$$\frac{d}{d\sigma}\{\arg(\sigma + i\tau)\} = \frac{d}{d\sigma}\left\{\arctan\left(\frac{\tau}{\sigma}\right)\right\} = \frac{-\tau}{\sigma^2 + \tau^2}.$$

Thus
$$\frac{dB(\sigma)}{d\sigma} = \frac{1}{3} \frac{1}{(\sigma^2 + \tau^2)^{5/6}} \left(\sigma \sin\left(\frac{\arg(\sigma + i\tau) + \pi}{3}\right) - \tau \cos\left(\frac{\arg(\sigma + i\tau) + \pi}{3}\right)\right).$$

Case 1. $\sigma \geq \tau$ (recall $\tau \geq \tau_0 > 0$). In this case
$$0 < \arg(\sigma + i\tau) \leq \pi/4,$$
whence
$$\frac{\pi}{4} < \frac{\pi}{3} < \frac{\arg(\sigma + i\tau) + \pi}{3} \leq \frac{\pi}{12} + \frac{\pi}{3} = \frac{5\pi}{12} < \frac{\pi}{2},$$
$$\cos\left(\frac{\arg(\sigma + i\tau) + \pi}{3}\right) < \sin\left(\frac{\arg(\sigma + i\tau) + \pi}{3}\right)$$
and
$$\frac{dB(\sigma)}{d\sigma} > \frac{1}{3} \frac{1}{(\sigma^2 + \tau^2)^{5/6}} \left(\sigma - \tau\right) \sin\left(\frac{\arg(\sigma + i\tau) + \pi}{3}\right) \geq 0.$$  

Case 2. $0 < \sigma < \tau$. Introduce the notation
$$D(\sigma) = \sigma \sin\left(\frac{\arg(\sigma + i\tau) + \pi}{3}\right) - \tau \cos\left(\frac{\arg(\sigma + i\tau) + \pi}{3}\right).$$
Note that $D(0) = 0$. Thus

$$\frac{dB(0)}{d\sigma} = \frac{1}{3} \frac{1}{(\sigma^2 + \tau^2)^{5/6}} \int_0^\sigma \frac{dD(\omega)}{d\omega} \, d\omega.$$ 

It will suffice to show that $dD(\sigma)/d\sigma \geq 0$. Now

$$\frac{dD(\sigma)}{d\sigma} = \sin \left[ \frac{\text{Arg}(\sigma + i\tau) + \pi}{3} \right]$$

$$\quad + \frac{\sigma \cos \left[ \frac{\text{Arg}(\sigma + i\tau) + \pi}{3} \right]}{3} \frac{(-\tau)}{3(\sigma^2 + \tau^2)}$$

$$\quad + \frac{\tau \sin \left[ \frac{\text{Arg}(\sigma + i\tau) + \pi}{3} \right]}{3} \frac{(-\tau)}{3(\sigma^2 + \tau^2)}.$$ 

Since $0 \leq \sigma \leq \tau$ we get

$$\frac{5\pi}{12} \leq \frac{\text{Arg}(\sigma + i\tau) + \pi}{3} \leq \frac{\pi}{2}$$

and

$$0 \leq \cos \frac{\text{Arg}(\sigma + i\tau) + \pi}{3} \leq \cos \frac{5\pi}{12} < \sin \frac{5\pi}{12} \leq \sin \frac{\text{Arg}(\sigma + i\tau) + \pi}{3}.$$ 

Now

$$\frac{dD(\sigma)}{d\sigma} \geq \sin \left[ \frac{\text{Arg}(\sigma + i\tau) + \pi}{3} \right] \left( 1 - \frac{\sigma \tau + \tau^2}{3(\sigma^2 + \tau^2)} \right).$$

Recalling that $0 \leq \sigma < \tau$ in this case, we get

$$1 - \frac{\sigma \tau^2 + \tau^2}{3(\sigma^2 + \tau^2)} \geq 1 - \frac{2\tau^2}{3(\sigma^2 + \tau^2)} \geq \frac{1}{3},$$

and finally

$$\frac{dD(\sigma)}{d\sigma} \geq \frac{1}{3} \sin \left[ \frac{\text{Arg}(\sigma + i\tau) + \pi}{3} \right] \geq 0. \quad \square$$

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