THE SPACE OF HARMONIC MAPS OF $S^2$ INTO $S^4$

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Abstract. Every branched superminimal surface of area $4\pi d$ in $S^4$ is shown to arise from a pair of meromorphic functions $(f_1, f_2)$ of bidegree $(d,d)$ such that $f_1$ and $f_2$ have the same ramification divisor. Conditions under which branched superminimal surfaces can be generated from such pairs of functions are derived. For each $d \geq 1$ the space of harmonic maps (i.e. branched superminimal immersions) of $S^2$ into $S^4$ of harmonic degree $d$ is shown to be a connected space of complex dimension $2d + 4$.

Introduction

In a study of minimal surfaces in euclidean spheres, Calabi showed that every minimal immersion of $S^2$ in $S^n$ arises from an isotropic map to projective space $[4], [5]$. This work was used by Bryant who showed that every compact Riemann surface can be superminimally immersed in $S^4$. There exist Calabi-type theorems representing harmonic maps of $S^2$ into other locally symmetric spaces in essentially algebro-geometric terms. These are of interest to people studying $\sigma$-models in physics. In this paper, we study the space of branched superminimal immersions of compact Riemann surfaces into $S^4$.

In §I, we characterize branched superminimal surfaces in $S^4$ by pairs of meromorphic functions with the same ramification divisor. This is done by constructing a contact map between $\mathbb{P}^3$ and $PT(\mathbb{C}P^1 \times \mathbb{C}P^1)$ where $\mathbb{P}^3$ is the blow-up of $\mathbb{C}P^3$ along 2 skew lines. The bidegree of such a pair is related to the degree of the canonical lift of the surface in $\mathbb{C}P^3$. We then show that if in addition the surface is linearly full (i.e. not contained in any strict subspace of $\mathbb{R}^5$) then the pair of meromorphic functions has bidegree $(d,d)$ where $d \geq 3$ and where the 2 functions do not differ by a Möbius transformation.

In §II, we analyze the space of harmonic maps of $S^2$ into $S^4$. By examining the projective geometry of certain Grassmann varieties, we show that the space $H_d$ of harmonic maps of $S^2$ into $S^4$ of degree $d$ is a connected space of complex dimension $2d + 4$. We also construct examples of unbranched superminimal surfaces of genus 0 in $S^4$ of area $4\pi d$ for $d \geq 3$.
In §111, we consider branched superminimal surfaces of genus $g$. We discuss conditions under which a pair of meromorphic functions on a Riemann surface $\Sigma$ can give rise to a branched superminimal immersion of $\Sigma$ into $S^4$.

This paper is based on the author's Ph.D thesis [13]. The author would like to thank Blaine Lawson for all the help and advice he has given me.

**Preliminaries**

Let $\Sigma$ be a compact Riemann surface and $\psi : \Sigma \rightarrow S^4$ an immersion into the unit 4-sphere. Let $B$ denote the second fundamental form of $\psi$. Then $\psi$ is a minimal immersion if the mean curvature $H := \text{trace } B$ vanishes identically. More generally, $\psi$ is a branched minimal immersion if it is minimal away from the set of isolated singular points. These are precisely the nonconstant conformal harmonic maps. Observe that any harmonic map $\psi : S^2 \rightarrow S^4$ is automatically conformal. Thus, branched minimal immersions of $S^2$ in $S^4$ are just the nonconstant harmonic maps from $S^2$ to $S^4$ (Eells-Lemaire [7]).

Let $\psi : \Sigma \rightarrow S^4$ be a (branched) minimal immersion of a compact Riemann surface in $S^4$. Let $x$ and $y$ denote the local isothermal coordinates on $\Sigma$. Consider the holomorphic quartic form $\Phi \in H^0(\Sigma; (\Omega^1)^4)$ defined by $\Phi := \varphi \cdot \varphi \cdot d z^4$ where

$$\varphi = \frac{1}{2} \left\{ B \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right) - i B \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \right\},$$

and where "•" is the complex bilinear extension of the dot product to $\mathbb{C}^5$. We say that $\psi$ is a branched superminimal immersion if $\Phi$ vanishes identically. This means that $\psi$ has a holomorphic horizontal lift, $\bar{\psi}$, to $\mathbb{C}P^3$ (Bryant [3], Chern-Wolfson [6], Lawson [10]). Observe that since $S^2$ has no nontrivial holomorphic quartic differentials, every branched minimal immersion (i.e. harmonic map) of $S^2$ into $S^4$ is automatically branched superminimal.

Consider the Calabi-Penrose fibration $\pi : \mathbb{C}P^3 \rightarrow S^4 = \mathbb{H}P^1$. This fibration can be obtained via a quotient of 2 Hopf maps. Choose homogeneous coordinates $(z_0, z_1, z_2, z_3)$ for $\mathbb{C}P^3$. Consider $\mathbb{C}^4 \cong \mathbb{H}^2$ as a quaternion vector space with left scalar multiplication, where the identification is given by $(z_0, z_1, z_2, z_3) \mapsto (z_0 + z_1 j, z_2 + z_3 j)$. The Kähler form of the Fubini-Study metric is given by $\omega = \partial \bar{\partial} \log \|z\|^2$. The Calabi-Penrose fibration is then given by the quotient

$$\begin{array}{ccc}
\mathbb{C}^4 - \{0\} & \rightarrow & \mathbb{H}^2 - \{0\} \\
\text{Hopf}_C \downarrow & & \downarrow \text{Hopf}_H \\
\mathbb{C}P^3 & \xrightarrow{\pi} & \mathbb{H}P^1
\end{array}$$
with fiber $\mathbb{CP}^1$. The horizontal 2-plane field $\mathcal{H}$ for $\pi$ is given by a 1-form whose lifting to $\mathbb{C}^4 - \{0\}$ is

$$\Omega := \frac{1}{\|z\|^2}(z_0 dz_1 - z_1 dz_0 + z_2 dz_3 - z_3 dz_2).$$

Superminimal surfaces in $S^4$ are just the projections to $S^4$ of nonsingular holomorphic curves in $\mathbb{CP}^3$ which are integral curves of $\mathcal{H}$. Unfortunately, it is difficult to find integral curves of $\mathcal{H}$ directly. Our search for superminimal surfaces would be vastly simplified if we can find a contact manifold $(M, \mathcal{F})$ birationally equivalent to $\mathbb{CP}^3$, where it is easy to find integral curves of the contact plane field $\mathcal{F}$. Robert Bryant has found a birational correspondence between $\mathbb{CP}^3$ and the projectivized tangent bundle of $\mathbb{CP}^2$ carrying $\mathcal{H}$ to the contact plane field of $PT(\mathbb{CP}^2)$. Using that, he was able to prove the following result:

**Theorem** (Bryant [3]). *Every compact Riemann surface admits a superminimal immersion into $S^4$.*

In this paper, I will be using another contact manifold—$PT(P^1 \times P^1)$. From now on, I will let $P^n$ denote $\mathbb{CP}^n$.

I. **Some projective geometry**

1. **Holomorphic contact structures.** Let $V$ be a complex $(2n + 1)$-manifold. A *holomorphic contact structure* on $V$ is a nondegenerate holomorphic distribution $\mathcal{F}$ of hyperplanes on $V$ (i.e. the orthogonal spaces of some twisted holomorphic 1-form). (cf. Arnold [1], LeBrun [12]).

Let $M$ be a complex $n$-manifold. Then the projectivized cotangent bundle of $M$ has a canonical holomorphic contact structure. Now let $\pi : PT^*M \to M$ denote the projection map onto the base space. A point $\varphi \in PT^*M$ defines a hyperplane $P_\varphi$ in $T_{\pi(\varphi)}M$. The contact hyperplane at $\varphi$ is given by $(\pi^{-1}_*)\varphi(P_\varphi)$. Thus the canonical contact 2-plane field $\mathcal{H}$ at a point $y \in PT(P^1 \times P^1) \cong PT^*(P^1 \times P^1)$ is given by $(\pi^{-1}_*)y(L_y)$ where $L_y$ denotes the tangent line at $\pi(y)$ corresponding to $y$.

The Calabi-Penrose fibration $p : P^3 \to S^4$ has a contact 2-plane field $\mathcal{H}$ orthogonal to the fibers of $p$ with respect to the Fubini-Study metric. The 2-plane field $\mathcal{H}$ for $p$ is given by a 1-form whose lifting to $\mathbb{C}^4 - \{0\}$ is $\Omega = \|z\|^{-2}(z_0 dz_1 - z_1 dz_0 + z_2 dz_3 - z_3 dz_2)$. Let $\omega := dz_0 \wedge dz_1 + dz_2 \wedge dz_3$ denote the standard holomorphic symplectic form on $\mathbb{C}^4$. Let

$$\xi := z_0 \frac{\partial}{\partial z_0} + z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + z_3 \frac{\partial}{\partial z_3}.$$

Then $\Omega = \|z\|^{-2} \xi \wedge \omega$. 

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2. Projection to $\mathbb{P}^1 \times \mathbb{P}^1$. Consider the two distinguished skew lines in $\mathbb{P}^3$ defined by $L_1 := p^{-1}(N) = \{ [0,0,z_2,z_3] | [z_2,z_3] \in \mathbb{P}^1 \}$ and $L_2 := p^{-1}(S) = \{ [z_0,z_1,0,0] | [z_0,z_1] \in \mathbb{P}^1 \}$, where $N$ and $S$ denote the north and south poles of $\mathbb{S}^3$ respectively.

Lemma 1.1. There is a well-defined projection map $\text{pr}: \mathbb{P}^3 - (L_1 \cup L_2) \to \mathbb{P}^1 \times \mathbb{P}^1$ with $\mathbb{P}^1$ as fiber.

Proof. It suffices to show that there is a unique line $L$ through each point $x \in \mathbb{P}^3 - (L_1 \cup L_2)$ which intersects $L_1$ and $L_2$. The intersection of $L$ with $L_1$ and $L_2$ (identifying $L_1 \times L_2$ with $\mathbb{P}^1 \times \mathbb{P}^1$) gives us the desired projection map. For each $x \in \mathbb{P}^3 - (L_1 \cup L_2)$ consider the planes $P_1$ and $P_2$ in $\mathbb{P}^3$ defined by $P_1 = \text{span}(x,L_1)$ and $P_2 = \text{span}(x,L_2)$. Since $L_1$ and $L_2$ are skew, $P_1$ and $P_2$ intersect in a line $L$ which contains the point $x$ and which intersects both $L_1$ and $L_2$. 

Proposition 1.2. The fibers of $\text{pr}: \mathbb{P}^3 - (L_1 \cup L_2) \to \mathbb{P}^1 \times \mathbb{P}^1$ are horizontal with respect to $p$ (i.e. the fibers of $\text{pr}$ are integral curves of $\mathbb{H}$).

Proof. Let $(x,y) \in L_1 \times L_2$. Let $L$ denote the line through $x$ and $y$, i.e. $L = \text{pr}^{-1}(x,y)$. Denote the inverse images of $L$, $L_1$, $L_2$, $x$ and $y$ to $\mathbb{C}^4 - \{0\}$ by $P$, $P_1$, $P_2$, $l_x$ and $l_y$ respectively.

Note. $P_1$ and $P_2$ are orthogonal with respect to $\omega$. Let $A \in P_1$ and $B \in P_2$. Then $A = (0,0,a,b)$ and $B = (c,d,0,0)$ for some $a,b,c,d \in \mathbb{C}$. It is clear from the definition of $\omega$ that $\omega(A,B) = 0$. Since $\omega$ is skew, we also have $\omega(A,A) = \omega(B,B) = 0$.

Now pick nonzero vectors $X \in l_x \subset P_1$ and $Y \in l_y \subset P_2$. Observe that $P$ is spanned by $X$ and $Y$. Now let $V_1 = \alpha X + \beta Y$ and $V_2 = \gamma X + \delta Y$ be 2 vectors in $P$. Then by the note, $\omega(V_1,V_2) = 0$. Thus $\omega$ vanishes on $P$. Let $p: \mathbb{C}^4 - \{0\} \to \mathbb{P}^3$. Since $\xi$ is tangent to the fibers of $p$ and $\Omega|_L = \|z\|^{-2}(\xi \perp \omega)|_p$, we see that $\Omega$ vanishes on $L$. Thus $L$ is horizontal with respect to $p$. 

3. The contact map. Let $X$ denote the blow up of $\mathbb{P}^3$ along $L_1$ and $L_2$, i.e. $X:=\{(z_0,z_1,z_2,z_3) | \{[y_0,y_1],[y_2,y_3] | z_0y_1 = z_1y_0, z_2y_3 = z_3y_2 \} \}$. Note that $X$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1 \times \mathbb{P}^1$: $\tilde{\pi}: X \to \mathbb{P}^1 \times \mathbb{P}^1$ where

$\tilde{\pi}([z_0,z_1,z_2,z_3],[y_0,y_1],[y_2,y_3]) = ([y_0,y_1],[y_2,y_3]).$

For ease of notation, let $Y$ denote $\text{PT}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \text{PT}(\mathbb{P}^1 \times \mathbb{P}^1)$. Let $\psi: X \to Y$ be defined by

$\psi([z_0,z_1,z_2,z_3],[v_0,v_1],[v_2,v_3])$
$= ([y_0,y_1],[y_2,y_3],[z_0dy_1-z_1dy_0,z_2dy_3-z_3dy_2]).$
We have the following diagram:

\[
\begin{array}{ccc}
P^3 & \xleftarrow{\beta} & X & \xrightarrow{\psi} & Y \\
p \downarrow & & \gamma \downarrow & & \pi \\
S^4 & \xrightarrow{\gamma} & P^1 \times P^1 & \xrightarrow{\pi} & P^1 \times P^1
\end{array}
\]

Observe that $\gamma$ extends to all of $X$, and for $x \in X$, $\bar{\gamma}_x(\mathcal{H}_x)$ is a tangent line in $T_{\gamma(x)}(P^1 \times P^1)$, i.e. $\bar{\gamma}_x(\mathcal{H}_x) \in \pi T_{\gamma(x)}(P^1 \times P^1)$. Furthermore, $\gamma = \pi \circ \psi$ where $\pi$ is the projection to $P^1 \times P^1$. Now let $l := \bar{\gamma}_x(\mathcal{H}_x)$. Then $\gamma^{-1}(l)$ is the contact plane at $l \in Y$. Now $l = \bar{\gamma}_x(\mathcal{H}_x) = (\pi \circ \psi)(\mathcal{H}_x) = \gamma \circ \psi(\mathcal{H}_x)$. Thus, $\gamma^{-1}(l) = \psi_* (\mathcal{H}_x)$. We thus have

**Lemma 1.3.** $\psi$ is a contact map, i.e. $\psi_*$ sends the horizontal plane field $\mathcal{H}$ in $X$ to the contact plane field $\gamma$ in $Y$.

The blow ups, $\sigma_1$ and $\sigma_2$, of the two distinguished skew lines $L_1, L_2 \subset P^3$ are given by

\[
\sigma_1 := \left\{ ([0,0,z_2,z_3], [y_0,y_1], [z_2,z_3]) \mid [y_0,y_1] \in P^1 \text{ and } [z_2,z_3] \in P^1 \right\}
\]

and

\[
\sigma_2 := \left\{ ([z_0,z_1,0,0], [z_0,z_1], [y_2,y_3], [z_2,z_3]) \mid [z_0,z_1] \in P^1 \text{ and } [y_2,y_3] \in P^1 \right\}.
\]

We observe that

\[
\psi(\sigma_1) = \left\{ ([y_0,y_1], [z_2,z_3], [1,0]) \mid [y_0,y_1] \in P^1 \text{ and } [z_2,z_3] \in P^1 \right\}
\]

and

\[
\psi(\sigma_2) = \left\{ ([z_0,z_1], [y_2,y_3], [0,1]) \mid [z_0,z_1] \in P^1 \text{ and } [y_2,y_3] \in P^1 \right\}.
\]

**Proposition 1.4.** $\psi$ is a branched 2-fold covering map. It is branched precisely along $\sigma_1$ and $\sigma_2$.

This proposition will be proved in the next subsection.

4. The involutions on $X$ and $S^4$. We first define an involution $\alpha: X \to X$ by

\[
\alpha([z_0,z_1,z_2,z_3], [y_0,y_1], [y_2,y_3]) = ([z_0,z_1, - z_2, - z_3], [y_0,y_1], [y_2,y_3]).
\]

(Actually, $\alpha$ is an involution on $P^3$ which is extended to $X$ in a trivial manner.)

**Note.**

1. $\alpha|_{\sigma_1} = \text{Id}$, $\alpha|_{\sigma_2} = \text{Id}$ and $\alpha^* \Omega = \Omega$.
2. By Note 1, $\alpha_*$ maps the horizontal plane $\mathcal{H}_x$ at $x \in X$ to the horizontal plane $\mathcal{H}_{\alpha(x)}$ at $\alpha(x)$.
3. Let $u \in L_1$ and $v \in L_2$. Denote by $l_{uv}$ the line in $P^3$ uniquely defined by $u$ and $v$. Since $\alpha(u) = u$ and $\alpha(v) = v$, we have $\alpha(l_{uv}) = l_{uv}$.
Consequently, $\hat{\pi} \circ \alpha = \hat{\pi}$. (This actually follows immediately from the definition of $\alpha$ and $\hat{\pi}$.)

1. Since $\hat{\pi}_*(\mathcal{H}_x) = \pi_\circ \psi_*(\mathcal{H}_x) = \psi(x)$, we have
   
   $\psi(\alpha(x)) = \hat{\pi}_*(\mathcal{H}_{\alpha(x)}) = \hat{\pi}_*(\mathcal{H}_x)$
   
   by Note 2

   $= (\hat{\pi} \circ \alpha)_*(\mathcal{H}_x)$

   $= \hat{\pi}_*(\mathcal{H}_x)$

   by Note 3

   $= \psi(x)$.

Thus $\psi \circ \alpha = \psi$, i.e. $\psi$ is $\alpha$-invariant.

Notes 1–4 imply that $\psi$ is at least 2-to-1 except along $\sigma_1$ and $\sigma_2$. From the definition of $\psi$, it is clear that $\psi$ is 1-to-1 on $\sigma_1$ and $\sigma_2$. Let us now examine the map $\psi$ explicitly in local coordinates. Assume that $x \notin \sigma_1 \cup \sigma_2$. We can then set $z_i = y_i$, for $i = 0, 1, 2, 3$. Without loss of generality, we can suppose that $z_0 = y_0 = 1$ and $z_2 \neq 0$. Set $s = y_1$ and $t = y_3/y_2$. Then $ds = dy_1$ and $dt = z^{-2}(z_2 dy_3 - z_3 dy_2)$. Thus, $z_2^2 dt = z_2 dy_3 - z_3 dy_2$.

Hence, $\psi([1, z_1, z_2, z_3], s, t) = (s, t, [ds, z_2^2 dt])$. We also have

$\psi([1, z_1, -z_2, -z_3], s, t) = (s, t, [ds, z_2^2 dt])$.

From the above local coordinate expression for $\psi$, it is clear that $\psi$ is 2-to-1 away from $\sigma_1$ and $\sigma_2$. Now, $\psi$ is a holomorphic map with finite fibers between compact complex 3-folds. Thus, it is a branched covering map of degree 2. This proves Proposition 1.4.

Let us now examine the inverse image of $\psi$ locally. Choose a point $y \in Y - (S_1 \cup S_2)$ where $S_1$ and $S_2$ are the images under $\psi$ of $\sigma_1$ and $\sigma_2$ respectively. Locally, $y$ has coordinates $(s, t, a)$. Recall that $\psi([1, z_1, z_2, z_3], s, t) = (s, t, [ds, z_2^2 dt])$ where $s = z_1$ and $t = z_3/z_2$. Then

$\psi^{-1}(y) = \psi^{-1}(s, t, a) = ([1, s, \sqrt{a}, \sqrt{at}], s, t)$.

The involution $\alpha$ on $X$ corresponds to a permutation of the roots. Thus,

**Proposition 1.5.** $\psi: X \to Y$ is equivalent to the projection map $p: X \to X/\mathbb{Z}_2$ where the $\mathbb{Z}_2$-action on $X$ is given by the involution $\alpha$.

The involution on $\mathbb{P}^3$ descends to an involution on $S^4$. Identifying $S^4$ with $\mathbb{HP}^1$, the stereographic projections to $\mathbb{R}^4 = \mathbb{H}^1$ from the south and north poles are respectively given by $\varphi_1([q_1, q_2]) = q_1^{-1}q_2$ and $\varphi_2([q_1, q_2]) = q_2^{-1}q_1$, with transition functions $q \mapsto q^{-1}\|q\|^{-2}q$. Now $p([z_0, z_1, z_2, z_3]) = [z_0 + z_1 j, z_2 + z_3 j] \in \mathbb{HP}^1$, where $[z_0, z_1, z_2, z_3] \in \mathbb{CP}^3$. Thus,

$p(\alpha[z_0, z_1, z_2, z_3]) = p([z_0, z_1, -z_2, -z_3]) = [z_0 + z_1 j, -(z_2 + z_3 j)]$.

The involution $\alpha$ thus descends to an involution on $S^4 = \mathbb{HP}^1$ as follows: $\alpha([q_1, q_2]) = [q_1, -q_2]$ for all $[q_1, q_2] \in \mathbb{HP}^1$. (We will let $\alpha$ denote the involution on both $X$ as well as $S^4$.)
Now, $\varphi_1 \circ \alpha([q_1, q_2]) = \varphi_1([q_1, -q_2]) = -q_1^{-1}q_2$ and $\varphi_2 \circ \alpha([q_1, q_2]) = \varphi_2([q_1, -q_2]) = -q_2^{-1}q_1$. Hence the action of $\alpha$ on a point $x \in S^4$ is just the antipodal map on the $S^3 \subset S^4$ obtained by the intersection of the horizontal 4-plane through $x$ with $S^4$. (This $S^3$ is the “latitudinal $S^3$”.) Thus, the geodesic 3-sphere in $S^4$ passing through the north and south poles is invariant under $\alpha$.

5. Some degree computations. We now compute the degree of the total preimage in $P^3$ of a holomorphic curve in $Y$. Recall the diagram:

$$
P^3 \xleftarrow{\beta} X \xrightarrow{\psi} Y$$

$$S^4 \xleftarrow{\pi} P^1 \times P^1 \xrightarrow{\pi} P^1 \times P^1$$

Let $l_1$ and $l_2$ (resp. $l_1'$ and $l_2'$) denote the preimages in $X$ (resp. $Y$) of the first and second factors of $P^1 \times P^1$ respectively under the map $\tilde{\pi}: X \to P^1 \times P^1$ (resp. $\pi: Y \to P^1 \times P^1$). Let $S_1$ and $S_2$ denote the 2 distinguished sections of $Y$ corresponding to lines tangent to the second and first factors of $P^1 \times P^1$ respectively. Recall that $\psi_*(\sigma_1) = S_1$ and $\psi_*(\sigma_2) = S_2$. Note that $\psi_*(l_i) = 2l_i'$, $i = 1, 2$. Let $H$ be a hyperplane in $P^3$. Then $\beta^*H = \sigma_1 + l_1 = \sigma_2 + l_2$. Thus $\sigma_1 - \sigma_2 = l_2 - l_1$. Also, $S_1 - S_2 = \psi_*(\sigma_1 - \sigma_2) = \psi_*(l_2 - l_1) = 2(l_2' - l_1')$. Hence, the Picard group of $X$ and $Y$ are given by

$$\text{Pic}(X) = \mathbb{Z}\{l_1, l_2, \sigma_1, \sigma_2\} / \langle \sigma_1 - \sigma_2 = l_2 - l_1 \rangle$$

and

$$\text{Pic}(Y) = \mathbb{Z}\{l_1', l_2', S_1, S_2\} / \langle S_1 - S_2 = 2(l_2' - l_1') \rangle.$$
where ‘deg’ refers to the intersection number of \( \tilde{F}(\Sigma) \) with the relevant generators. Let \( \tilde{C} := \psi^{-1}(C') \subset X \) and \( \gamma := \beta_*(\tilde{C}) \subset P^3 \). Then for a generic hyperplane \( H \) in \( P^3 \), we have

\[
\text{deg} \gamma = H \cdot \beta_*(\tilde{C}) = \beta^*H \cdot \tilde{C} = (\sigma_1 + I_1) \cdot (\psi^{-1}C')
\]

\[
= \psi_*(\sigma_1 + I_1) \cdot C' = (S_1 + 2I_1') \cdot F_*(\Sigma)
\]

\[
= \text{deg} F^*(S_1 + 2I_1') = 2g - 2 + 2n + 2m.
\]

Suppose \( \text{deg} f_1 = \text{deg} f_2 = d \) and \( \text{Ram}(f_1) = \text{Ram}(f_2) \). Then the curve \( C = F(\Sigma) \) has singular points with the property that \( \text{deg} F^*(S_1) = \text{deg} F^*(S_2) = 0 \). Consequently, \( \text{deg} \gamma = 2d \).

6. Conjugate branched superminimal surfaces. Let us suppose that \( f: \Sigma \to S^4 \) is a branched superminimal immersion of a compact Riemann surface in \( S^4 \). Generically, \( f(\Sigma) \) misses a pair of antipodal points in \( S^4 \) (say the north and south poles). Also, generically, \( \alpha(f(\Sigma)) \neq f(\Sigma) \), i.e. \( f(\Sigma) \) is not \( \alpha \)-invariant. Let \( \tilde{f}: \Sigma \to P^3 \) be the holomorphic horizontal lift of \( f \) to \( P^3 \).

**Proposition 1.6.** A generic branched superminimal surface \( f(\Sigma) \) in \( S^4 \) has the property that its lift \( \tilde{f}(\Sigma) \) in \( P^3 \) is not \( \alpha \)-invariant.

**Proof.** The proposition follows immediately from the definition of the involution \( \alpha \) and the fact that \( \alpha \)-invariance in \( P^3 \) descends to \( \alpha \)-invariance in \( S^4 \). \( \square \)

**Note.** The converse is not necessarily true. For example, the totally geodesic \( S^2 \) of area \( 4\pi \) contained in the equator of \( S^4 \) is obviously \( \alpha \)-invariant. However, its lift in \( P^3 \) is a curve \( \gamma \) of degree 1 (and hence \( \gamma \cong P^1 \)) which avoids \( L_1 \) and \( L_2 \), and thus is not \( \alpha \)-invariant. Observe that \( \alpha(\gamma) \) projects down to the same geodesic \( S^2 \) (but with the opposite orientation).

**Corollary 1.7.** Given a generic branched superminimal surface \( f(\Sigma) \) in \( S^4 \), we obtain a conjugate branched superminimal surface, \( \alpha \circ f(\Sigma) \), in \( S^4 \).

**Proof.** Since \( f(\Sigma) \) is generic, it avoids the poles and hence its lift \( \tilde{f}(\Sigma) \) avoids \( L_1 \) and \( L_2 \). Thus, \( \tilde{f}(\Sigma) \) is diffeomorphic to its image \( \tilde{f}'(\Sigma) \) in \( X \) under the blow up of \( P^3 \) along \( L_1 \) and \( L_2 \). Now by notes 1-4 in §1.4, we have \( \tilde{\pi} \circ \tilde{f}'(\Sigma) = \tilde{\pi} \circ (\alpha \circ \tilde{f}'(\Sigma)) \) and that \( \alpha \circ \tilde{f}(\Sigma) \) is holomorphic and horizontal in \( P^3 \) and thus projects to a branched superminimal surface in \( S^4 \), i.e. we obtain conjugate branched superminimal surfaces for free! \( \square \)

7. Bidegrees and ramification divisors. Let \( f(\Sigma) \) be a generic branched superminimal surface in \( S^4 \). Its lift \( \tilde{f}(\Sigma) \) is a holomorphic horizontal curve \( \gamma \) in \( P^3 \). The homology degree of \( \gamma \subset P^3 \) is the fundamental class \( [\gamma] \in H_2(P^3; Z) \cong Z \). This degree is also the intersection number of \( \gamma \) with a generic \( P^2 \) in \( P^3 \) (i.e. homology degree = algebraic degree). Let \( \tilde{\pi} = (\tilde{\pi}_1, \tilde{\pi}_2) \) denote the projection map of \( P^3 - (L_1 \cup L_2) \) to \( P^1 \times P^1 \). Define \( f_1, f_2: \Sigma \to P^1 \) by \( f_1 := \tilde{\pi}_1 \circ \tilde{f} \) and \( f_2 := \tilde{\pi}_2 \circ \tilde{f} \).
Proposition 1.8. Suppose that \( \deg(y) = d \). Then the holomorphic curve \( C = \tilde{\pi} \circ \tilde{f}(\Sigma) \) in \( \mathbb{P}^1 \times \mathbb{P}^1 \) has bidegree \((d,d)\), i.e. \( \deg f_1 = \deg f_2 = d \). Furthermore, \( \text{Ram}(f_1) = \text{Ram}(f_2) \).

Proof. Let \( x_1 \in L_1 \). The fiber \( \tilde{\pi}_1^{-1}(x_1) \subset \mathbb{P}^3 \) is the plane \( P_1 = \text{span}(x_1, L_2) \). Since \( \deg y = d \), \( P_1 \) has \( d \) intersection points with \( y \). Similarly, for \( x_2 \in L_2 \), the plane \( P_2 = \tilde{\pi}_2^{-1}(x_2) \) has \( d \) intersection points with \( y \). Thus \( C = \tilde{\pi}(\gamma) \) has bidegree \((d,d)\).

Let \( z_0 \) be a ramification point of \( f_1 \). Let \( p \in \gamma \) denote the point \( \tilde{f}(z_0) \). Then the point \( x = \tilde{\pi}_1(p) \) is a branch point of \( f_1 \). Let \( y = \tilde{\pi}_2(p) \) and let \( L_{xy} \) denote the line in \( \mathbb{P}^3 \) through \( x \) and \( y \). Finally, let \( H_x \) denote the plane \( \{ v \in \mathbb{T}_p \mathbb{P}^3 \mid \tilde{\pi}_1^*(v) = 0 \} \). Now \( x \) is a branch point of \( f_1 \) and \( y \) is an integral curve of \( \mathscr{H}_p \), so the tangent line to the curve \( y \) at \( p \) must be \( L_{xy} \)—the intersection of \( \mathscr{H}_p \) and \( H_x \). We thus have \( \tilde{\pi}_1^*(L_{xy}) = \tilde{\pi}_2^*(L_{xy}) = 0 \). Hence, \( y \) is a branch point of \( f_2 \) and so \( z_0 \) is in the ramification locus of both \( f_1 \) and \( f_2 \). By genericity, \( \text{Ram}(f_1) = \text{Ram}(f_2) \).

Lemma 1.9. A holomorphic map \( F = (f_1, f_2): \Sigma \to \mathbb{P}^1 \times \mathbb{P}^1 \) has a canonical Gauss lift \( \tilde{F} \) to \( Y = \mathcal{P}T(\mathbb{P}^1 \times \mathbb{P}^1) \).

Proof. First suppose \( (df_1(z), df_2(z)) \neq (0, 0) \). Then the lift is given by \( \tilde{F}(z) = (f_1(z), f_2(z), [f_1'(z), f_2'(z)]) \). We are thus left with a finite set of singular points. Without loss of generality, suppose \( 0 \) is a singular point. Then \( f_1'(z) = z^q g_1(z) \) and \( f_2'(z) = z^p g_2(z) \) for some \( p \), \( q \) and where \( g_1(0) \neq 0 \) and \( g_2(0) \neq 0 \). We may assume that \( 1 \leq p \leq q \). So

\[
\tilde{F}(z) = \left( f_1(z), f_2(z), [g_1(z), z^{q-p} g_2(z)] \right)
\]

for \( z \) in a neighborhood of \( 0 \).

Proposition 1.10. Suppose \( f: \Sigma \to S^4 \) is a generic superminimal immersion. Let \( \tilde{f}: \Sigma \to \mathbb{P}^3 \) be the holomorphic horizontal lift of \( f \), and let \( f_1 := \tilde{\pi}_1 \circ \tilde{f} \) and \( f_2 := \tilde{\pi}_2 \circ \tilde{f} \). Suppose that \( \deg f_1 = \deg f_2 = d \geq 2 \). Then \( f_2 \neq A \circ f_1 \) for any \( A \in PSL(2, \mathbb{C}) \).

Proof. Suppose \( f_2 = A \circ f_1 \) for some \( A \in PSL(2, \mathbb{C}) \). Then \( F = (f_1, f_2) = (f_1, A \circ f_1): \Sigma \to \mathbb{P}^1 \times \mathbb{P}^1 \) factors through \( \mathbb{P}^1 \) as follows:

\[
\Sigma \xrightarrow{f_1} \mathbb{P}^1 \xrightarrow{G = (\text{Id}, A)} \mathbb{P}^1 \times \mathbb{P}^1.
\]

Since \( G \) has bidegree \((1,1)\), it is nonsingular and its canonical lift \( \tilde{G} \) to \( Y \) avoids the two sections \( S_1 \) and \( S_2 \). The map \( f_1 \) is necessarily branched since \( \deg f_1 \geq 2 \). Hence, the canonical lift \( \tilde{F} \) of \( F \) is a branched covering map of \( \Sigma \) into \( \tilde{G}(\mathbb{P}^1) \cong \mathbb{P}^1 \), i.e. \( \tilde{F}(\Sigma) \) is branched. Consequently, its lift to \( \mathbb{P}^3 \), \( \tilde{F}(\Sigma) \), is branched and hence projects to a branched superminimal surface in \( S^4 \). This contradicts the assumption that \( f(\Sigma) \subset S^4 \) is unbranched.
Note that for \( d = 1 \), \( \Sigma \) must have genus zero and so \( f(\Sigma) \) is totally geodesic in \( S^4 \).

We thus have

**Theorem A.** Every superminimal immersion \( f: \Sigma \hookrightarrow S^4 \) arises from a pair of meromorphic functions \( f_1, f_2 \) on \( \Sigma \) such that

1. \( \deg f_1 = \deg f_2 = d \) for some integer \( d \geq 1 \).
2. \( \text{Ram}(f_1) = \text{Ram}(f_2) \)
3. For \( d \geq 2 \), \( f_1 \neq A \circ f_2 \) for any \( A \in \text{PSL}(2, \mathbb{C}) \).

We would like to generate superminimal surfaces in \( S^4 \) by considering pairs of meromorphic functions on \( \Sigma \) which satisfy the three conditions in Theorem A. Suppose \( F = (f_1, f_2) \) is such a pair. Let \( \tilde{\gamma} = \tilde{f}(\Sigma) \subset \bar{Y} \). Our degree computations in §1.5 show that the total preimage curve \( \gamma = \beta \circ \psi^{-1}(\tilde{\gamma}) \) in \( \mathbb{P}^3 \) has degree \( 2d \). Suppose \( \gamma \) consists of \( 2 \) connected (or irreducible) components \( \gamma_1 \) and \( \gamma_2 \). Then \( \alpha(\gamma_1) = \gamma_2 \) and consequently \( \deg \gamma_1 = \deg \gamma_2 = d \). Under suitable conditions (to be discussed later), \( \gamma_1 \) and \( \gamma_2 \) will project to a conjugate pair of superminimal surfaces in \( S^4 \).

II. Genus zero

1. **Meromorphic functions, Grassmannians and resultants.** Let \( f: \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) be a holomorphic map of degree \( d \) (i.e. \( f \) is a meromorphic function of degree \( d \) ). Then \( f \) can be expressed as a rational function of the form \( P(z)/Q(z) \) where \( P(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0 \) and \( Q(z) = b_d z^d + b_{d-1} z^{d-1} + \cdots + b_1 z + b_0 \), \( a_i, b_i \in \mathbb{C} \). Note that the map \( f \) is of degree \( d \) if \( \min\{\deg P(z), \deg Q(z)\} = d \) and if the resultant of the 2 polynomials does not vanish. Let \( P = (a_d, a_{d-1}, \ldots, a_1, a_0) \) and \( Q = (b_d, b_{d-1}, \ldots, b_1, b_0) \) denote the coefficient vectors of \( P(z) \) and \( Q(z) \) respectively. Then the resultant \( \mathcal{R}(P, Q) \) of \( P(z) \) and \( Q(z) \) is the determinant of the matrix

\[
M = \begin{pmatrix}
A_1 & A_2 \\
B_1 & B_2
\end{pmatrix}
\]

where

\[
A_1 = \begin{pmatrix}
a_d & a_{d-1} & \cdots & a_1 \\
0 & a_d & \cdots & a_2 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_d
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
a_0 & 0 & \cdots & 0 \\
0 & a_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
a_{d-1} & a_{d-2} & \cdots & a_0
\end{pmatrix},
\]

\[
B_1 = \begin{pmatrix}
b_d & b_{d-1} & \cdots & b_1 \\
0 & b_d & \cdots & b_2 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & b_d
\end{pmatrix}, \quad B_2 = \begin{pmatrix}
b_0 & 0 & \cdots & 0 \\
b_1 & b_0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
b_{d-1} & b_{d-2} & \cdots & b_0
\end{pmatrix}.
\]
The resultant is a homogeneous polynomial of bidegree \((d, d)\) in the \(a_i\) and the \(b_j\). Furthermore, \(\mathcal{R}(P, Q)\) is irreducible over any arbitrary field (cf. [18]). We thus require that \((P, Q) \in \mathbb{C}^{d+1} \times \mathbb{C}^{d+1} - \mathcal{R}\), where \(\mathcal{R}\) is the irreducible resultant divisor. Observe that \((\lambda P, \lambda Q)\) describes the same function as \((P, Q)\) for any \(\lambda \in \mathbb{C}^*\). Thus the space of meromorphic functions of degree \(d\) is

\[
M_d := P(\mathbb{C}^{d+1} \times \mathbb{C}^{d+1} - \mathcal{R}) \subset \mathbb{P}^{2d+1}.
\]

We next define an action of \(GL(2, \mathbb{C})\) on \(\mathbb{C}^{d+1} \times \mathbb{C}^{d+1}\) as follows:

\[
g \cdot (P, Q) := (\alpha P + \beta Q, \gamma P + \delta Q) \quad \text{for} \quad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2, \mathbb{C}).
\]

Let \(N_d := \mathbb{C}^{d+1} \times \mathbb{C}^{d+1} - \Delta\) where \(\Delta = \{(P, Q) \mid P \wedge Q = 0\}\). Observe that for \((P, Q) \in N_d\), \(g \cdot (P, Q) = (\alpha P + \beta Q, \gamma P + \delta Q) = (P_1, Q_1)\), and \(P_1 \wedge Q_1 = (\alpha P + \beta Q) \wedge (\gamma P + \delta Q) = (\alpha \delta - \beta \gamma) P \wedge Q \neq 0\). Thus, \(GL(2, \mathbb{C})\) acts on \(N_d\). In fact, we have a free action on \(N_d:\) \(g \cdot (P, Q) = (\alpha P + \beta Q, \gamma P + \delta Q) = (P, Q)\) implies that \(g = I\) since \(P \wedge Q \neq 0\). Note that we can identify \(N_d\) with the Stiefel manifold of 2-frames in \(\mathbb{C}^{d+1}\). For \((P, Q) \in N_d\), let \([P \wedge Q]\) denote the 2-plane in \(\mathbb{C}^{d+1}\) spanned by \(P\) and \(Q\). Let \(P_1, Q_1 \in [P \wedge Q]\). Then \(P_1 = \alpha P + \beta Q\) and \(Q_1 = \gamma P + \delta Q\) for some \(\alpha, \beta, \gamma, \delta \in \mathbb{C}\). If \(P_1 \wedge Q_1 \neq 0\), then \(0 \neq P_1 \wedge Q_1 = (\alpha \delta - \beta \gamma) P \wedge Q\), i.e. \((\alpha \delta - \beta \gamma) \neq 0\). Thus, \(GL(2, \mathbb{C})\) acts transitively on pairs of noncollinear vectors in \([P \wedge Q]\). It follows that \(N_d/GL(2, \mathbb{C}) = G(2, d+1)\) and \(\pi: N_d \to G(2, d+1)\) is a principal \(GL(2, \mathbb{C})\)-bundle (where \(\pi(P, Q) = [P \wedge Q]\)).

**Lemma 2.1.** \(\mathcal{R}(g \cdot (P, Q)) = (\det g)^d \mathcal{R}(P, Q)\).

**Proof.** Let \((\tilde{P}, \tilde{Q})\) denote \(g \cdot (P, Q)\). The resultant of \((\tilde{P}, \tilde{Q})\) is given by the determinant of the matrix

\[
\tilde{M} = \begin{pmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{B}_1 & \tilde{B}_2 \end{pmatrix}.
\]

Since \((\tilde{P}, \tilde{Q}) = (\alpha P + \beta Q, \gamma P + \delta Q)\), we observe that \(\tilde{A}_1 = \alpha A_1 + \beta B_1\), \(\tilde{A}_2 = \alpha A_2 + \beta B_2\), \(\tilde{B}_1 = \gamma A_1 + \delta B_1\), \(\tilde{B}_2 = \gamma A_2 + \delta B_2\), i.e.

\[
\begin{pmatrix} \tilde{A}_1 \\ \tilde{B}_1 \end{pmatrix} = \begin{pmatrix} \alpha I & \beta I \\ \gamma I & \delta I \end{pmatrix} \cdot \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}, \quad \begin{pmatrix} \tilde{A}_2 \\ \tilde{B}_2 \end{pmatrix} = \begin{pmatrix} \alpha I & \beta I \\ \gamma I & \delta I \end{pmatrix} \cdot \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}
\]

where \(I \in GL(d, \mathbb{C})\) is the identity matrix. It is straightforward to verify that

\[
\det\begin{pmatrix} \alpha I & \beta I \\ \gamma I & \delta I \end{pmatrix} = (\alpha \delta - \beta \gamma)^d = (\det g)^d.
\]

Thus, \(\det \tilde{M} = (\det g)^d \det M\), i.e. \(\mathcal{R}(g \cdot (P, Q)) = (\det g)^d \cdot \mathcal{R}(P, Q)\). \(\Box\)

It follows that \(\mathcal{R} \subset \mathbb{C}^{d+1} \times \mathbb{C}^{d+1}\) is fixed under the action of \(GL(2, \mathbb{C})\). Let \(\text{Reg}(\mathcal{R})\) denote the regular part of \(\mathcal{R}\). Since \(\mathcal{R}\) is irreducible, \(\text{Reg}(\mathcal{R})\)
is connected. Note that \( \Delta = \{(P,Q) \mid P \wedge Q = 0\} \subset \mathcal{R} \) and that \( \Delta \) has codimension \( d \) in \( \mathbb{C}^{d+1} \times \mathbb{C}^{d+1} \). So \( \Delta \) cannot disconnect \( \text{Reg}(\mathcal{R}) \) (which has dimension \( 2d + 1 \)). Consequently, \( (\text{Reg}(\mathcal{R})) \cap N_d \) is connected, i.e. \( \mathcal{R} \cap N_d \) is irreducible. For ease of notation, we shall let \( \mathcal{R} \) to denote \( \mathcal{R} \cap N_d \) also. By Lemma 2.1, \( \dim(\mathcal{R}/GL(2, \mathbb{C})) = \dim(\pi(\mathcal{R})) = 2d - 3 \). Furthermore, since \( \text{Reg}(\mathcal{R}) \) is connected and \( \pi: N_d \to G(2, d+1) \) is a principal \( GL(2, \mathbb{C}) \)-bundle, \( \pi(\text{Reg}(\mathcal{R})) = \text{Reg}(\pi(\mathcal{R})) \) is connected. Thus, \( \pi(\mathcal{R}) \) is an irreducible divisor in \( G(2, d + 1) \).

Observe that the space of meromorphic functions of degree \( d \) is \( M_d = P(N_d - \mathcal{R}) \). We thus have a free action of \( PSL(2, \mathbb{C}) \) on \( M_d \). Furthermore, \( M_d/PSL(2, \mathbb{C}) \subset G(2, d + 1) \), the Grassmannian of 2-planes in \( \mathbb{C}^{d+1} \).

2. The ramification divisor. Let \( f: \mathbb{P}^1 \to \mathbb{P}^1 \) be a holomorphic map of degree \( d \). Recall that \( z_0 \in \mathbb{P}^1 \) is a ramification point of \( f \) if \( f_\ast(v) = 0 \) for all \( v \in T_{z_0} \mathbb{P}^1 \). Expressing \( f \) as a rational function \( P(z)/Q(z) \), we have \( f'(z) = (Q(z)P'(z) - P(z)Q'(z))/(Q(z))^2 \). Then the ramification points of \( f \) are given by the zero locus of \( Q(z)P'(z) - P(z)Q'(z) \), a polynomial of degree \( 2d - 2 \). Observe that if \( \deg(Q(z)P'(z) - P(z)Q'(z)) = k < 2d - 2 \), then \( \infty \) is a ramification point of order \( 2d - 2 - k \).

Define a map \( \Psi^d: M_d = P(N_d - \mathcal{R}) \to \mathbb{P}^{2d-2} \) by

\[
[(P, Q)] \mapsto \text{coeff}(Q(z)P'(z) - P(z)Q'(z)),
\]

where \( \text{coeff}(R(z)) \) denotes the coefficient vector of the polynomial \( R(z) \). The ramification map \( \Psi^d \) is well defined since

\[
(\lambda P, \lambda Q) \mapsto [\lambda^2 \cdot \text{coeff}(Q(z)P'(z) - P(z)Q'(z))]
= [\text{coeff}(Q(z)P'(z) - P(z)Q'(z))],
\]

and if \( Q(z)P'(z) - P(z)Q'(z) \equiv 0 \), we have

\[
\frac{P'(z)}{P(z)} = \frac{Q'(z)}{Q(z)}, \quad \text{i.e.} \quad \log P(z) = \log Q(z) + C = \log(\tilde{C}Q(z)).
\]

Thus \( P(z) = \tilde{C}Q(z) \) and so \( [(P, Q)] \notin M_d \).

Lemma 2.2. \( PSL(2, \mathbb{C}) \) preserves the fibers of \( \Psi^d \)

Proof. Let \( g \in PSL(2, \mathbb{C}) \). Let \( (\alpha, \beta, \gamma, \delta) \) be a representative of \( g \). Then

\[
\Psi^d(g \cdot [(P, Q)]) = \Psi^d([\alpha P(z) + \beta Q(z), \gamma P(z) + \delta Q(z)])
= \text{coeff}((\gamma P(z) + \delta Q(z))(\alpha P'(z) + \beta Q'(z)) - (\alpha P(z) + \beta Q(z))(\gamma P'(z) + \delta Q'(z)))
= \text{coeff}((\alpha\delta - \beta\gamma)(Q(z)P'(z) - P(z)Q'(z)))
= \text{coeff}(Q(z)P'(z) - P(z)Q'(z))
= \Psi^d([(P, Q)]). \quad \square
\]
Corollary 2.3. \( \text{PSL}(2, \mathbb{C}) \) acts freely on the fibers of \( \Psi^d \).

Proof. \( \text{PSL}(2, \mathbb{C}) \) acts freely on \( M_d = \mathbb{P}(\mathbb{C}^{d+1} \times \mathbb{C}^{d+1} - R) \), and by Lemma 2.2, it preserves fibers. \( \square \)

We thus have an induced map \( \Psi_d : G(2, d+1) \to \mathbb{P}^{2d-2} \) where

\[ [P \land Q] \mapsto [\text{coeff}(Q(z)P'(z) - P(z)Q'(z))]. \]

This map is well defined.

Note that for \( d = 2 \), \( G(2, 3) \cong G(1, 3) = \mathbb{P}^2 \).

Proposition 2.4. \( \Psi_2 : G(2, 3) \cong \mathbb{P}^2 \to \mathbb{P}^2 \) is a biholomorphism.

Proof. Let \( [P \land Q] \in G(2, 3) \). Then \( [P \land Q] \) can be represented by one of the following matrices:

\[
\begin{pmatrix}
1 & 0 & a \\
0 & 1 & b
\end{pmatrix}, \quad \begin{pmatrix}
1 & a & 0 \\
0 & 0 & 1
\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

where \( P \) and \( Q \) correspond to the rows of the matrices. For the first matrix, \( P(z) = z^2 + a, \) and \( Q(z) = z + b. \) Then

\[
\Psi_2([P \land Q]) = [\text{coeff}(Q(z)P'(z) - P(z)Q'(z))]
\]

\[ = [\text{coeff}((z + b)(2z) - (z^2 + a))] = [1, 2b, -a] \]

i.e.

\[
\begin{pmatrix}
1 & 0 & a \\
0 & 1 & b
\end{pmatrix} \mapsto [1, 2b, -a].
\]

Similarly, we have

\[
\begin{pmatrix}
1 & a & 0 \\
0 & 0 & 1
\end{pmatrix} \mapsto [0, 2, a] \quad \text{and} \quad \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \mapsto [0, 0, 1].
\]

Note that in the second case, \( \infty \) is a ramification point and that the third case is a degenerate case since \( (P, Q) \in R \). From the explicit computations, it is clear that \( \Psi_2 \) is one-to-one, nonsingular and is hence a biholomorphism. \( \square \)

A consequence of the proposition is that \( \Psi^2 : M_2 \to \mathbb{P}^2 \) has connected fibers.

Corollary 2.5. Let \( f \) be a meromorphic function of degree 2. Let \( g \) be any other meromorphic function of degree 2 with the property that \( \text{Ram}(f) = \text{Ram}(g) \). Then \( g = A \circ f \) for some \( A \in \text{PSL}(2, \mathbb{C}) \).

Corollary 2.6. There is no superminimal surface in \( S^4 \) whose lifting to \( \mathbb{P}^3 \) is a curve of degree 2.

Proof. The genus 0 case follows immediately from Proposition 1.10 and Corollary 2.5. The following argument proves the general case. Let \( \gamma \) be a holomorphic horizontal curve in \( \mathbb{P}^3 \) of degree 2. Suppose \( \gamma \) is not a projective line. Pick any 3 noncollinear points \( A, B, C \) on \( \gamma \). Let \( L_{AB} \) and \( L_{AC} \) denote the
lines through $A&B$ and $A&C$ respectively. Let $P$ denote the plane spanned by these two lines. Since $\deg(\gamma) = 2$ and $P$ contains the points $A, B$ and $C$, necessarily, $\gamma \subset P$, i.e. $\gamma$ is planar. Since there are no horizontal planes in $P^3$ (otherwise, that horizontal $P^2$ would be diffeomorphic to $S^4!$), $P$ (and hence $\gamma$) is in fact a projective line. Since $\deg(\gamma) = 2$, $\gamma$ is necessarily branched. (Nevertheless, $\gamma$ projects to a totally geodesic surface in $S^4$.) □

3. The orbits in the fibers of $\Psi^d$. Let $N = \frac{1}{2}(d + 2)(d - 1) = \left(\frac{d + 2}{2}\right) + 1 = \dim(P^\left(\bigwedge^2 C^{d+1}\right))$. Let $P = (a_d, \ldots, a_0)$ and $Q = (b_d, \ldots, b_0)$ be two vectors in $C^{d+1}$ which span a plane, $(\frac{P}{Q})$, in $C^{d+1}$. Then the Plücker embedding $G(2, d + 1) \hookrightarrow P^N = P^\left(\bigwedge^2 C^{d+1}\right)$ is given by $(\frac{P}{Q}) \mapsto [P \wedge Q]$. Choose Plücker coordinates $x_{ij}$ on $P^N$ where $i > j$, $i = 1, \ldots, d$, $j = 0, \ldots, d - 1$. Let $P(z) = a_dz^d + \cdots + a_1z + a_0$ and $Q(z) = b_dz^d + \cdots + b_0$. Then

$$Q(z)P'(z) - P(z)Q'(z) = \alpha_2a_2z^{2d-2} + \cdots + \alpha_nz^n + \cdots + \alpha_1z + \alpha_0$$

where

$$\alpha_n = \sum_{i+j=n+1}^{i>j} (i-j)x_{ij}, \quad n = 0, \ldots, 2d - 2.$$

Consider the linear map $L: C^{N+1} \rightarrow C^{2d-1}$ given by

$$(x_{ij}) \mapsto (\alpha_2, \ldots, \alpha_n, \ldots, \alpha_0).$$

Observe that since $\alpha_n$ contains only the $x_{ij}$'s which satisfy the condition $i + j = n + 1$, $L$ has maximal rank. Let $K$ denote the kernel of $L$. Then $\dim K = \frac{1}{2}(d^2 + d) - 2d + 1 = \frac{1}{2}(d - 2)(d - 1)$. Let $\kappa := PK$, a projective $\frac{1}{2}d(d - 3)$-plane in $P^N$. Note that the image of $G(2, d + 1)$ in $P^N$, $G^{2d-2}$, does not intersect $\kappa$ by construction. Thus the map $\Psi^d$ can be given in Plücker coordinates by

$$\Psi^d([P \wedge Q]) = [\alpha_2, \ldots, \alpha_n, \ldots, \alpha_0].$$

So $\Psi^d$ can be thought of as the restriction to $G^{2d-2}$ of a “map” from $P^N$ to $P^{2d-2}$. We can extend $\Psi^d$ to a map from $P^N - \kappa$ to $P^{2d-2}$. Let $\hat{P}^N$ denote the blow-up of $P^N$ along $\kappa$. Let $q \in P^{2d-2}$. Let $\hat{\Psi}_d$ denote the map induced on $\hat{P}^N$. Then $\Lambda_q = (\hat{\Psi}^{-1}_d)(q)$ is a projective $\frac{1}{2}(d - 2)(d - 1)$-plane in $P^N$, i.e. a plane of dimension complementary to that of $G^{2d-2}$. Therefore the number of points of intersection of $\Lambda_q$ with $G^{2d-2}$ is the degree of $G^{2d-2}$ in $P^N$, which is $(2d - 2)!/(d - 1)!d!$. As a consequence, there are generically $(2d - 2)!/(d - 1)!d!$ distinct $PSL(2, C)$-orbits of holomorphic maps of degree $d$ from $P^1$ to $P^1$ which have the same ramification divisor. We thus have

**Theorem B.** Let $f$ be a generic meromorphic function of degree $d \geq 2$. Then, under the action of $PSL(2, C)$, there are $(2d - 2)!/(d - 1)!d!$ distinct orbits of meromorphic functions of degree $d$ with ramification divisor $\text{Ram}(f)$.
Note that when $d = 2$ we have only 1 orbit. This is consistent with our previous result (Corollary 2.5).

4. The space $\mathcal{H}_d$. Let $F = (f_1, f_2): \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ be a holomorphic map of bidegree $(d, d)$ such that $\text{Ram}(f_1) = \text{Ram}(f_2)$. By our previous results, the curve $\tilde{F}(\mathbb{P}^1) \subset Y = \mathbb{P}^T(\mathbb{P}^1 \times \mathbb{P}^1)$ avoids the 2 distinguished sections, $S_1$ and $S_2$ of $Y$. Since $\psi: \mathbb{P}^3 - (\sigma_1 \cup \sigma_2) \to Y - (S_1 \cup S_2)$ is a covering map of degree 2 and since $\pi_1(\mathbb{P}^1) = 0$, the map $\tilde{F}: \mathbb{P}^1 \to \mathbb{P}^3 - (\sigma_1 \cup \sigma_2)$.

Let $\gamma_1 := \beta \circ \tilde{F}(\mathbb{P}^1)$ and $\gamma_2 := \beta \circ \alpha \circ \tilde{F}(\mathbb{P}^1) = \alpha(\gamma_1)$. Then $\gamma_1$ and $\gamma_2$ project to a conjugate pair of branched superminimal surfaces, $\Sigma_1$ and $\Sigma_2$, in $S^4$. If $\tilde{F}$ is an immersion, then the pair of surfaces are unbranched. We also showed that for $d \geq 2$, a necessary condition for $\Sigma_1$ and $\Sigma_2$ to be unbranched is that $f_1$ and $f_2$ belong to different orbits of $\text{PSL}(2, \mathbb{C})$. Our search for unbranched superminimal surfaces is thus aided by the following immediate consequence of Theorem B:

**Theorem C.** For each $d \geq 3$, there is a branched superminimal surface of genus 0 in $S^4$ which arises from a pair of meromorphic functions $(f_1, f_2)$, each of degree $d$ such that $\text{Ram}(f_1) = \text{Ram}(f_2)$ and that $f_1$ and $f_2$ belong to distinct $\text{PSL}(2, \mathbb{C})$-orbits.

**Proof.** By Theorem B, there are $(2d - 2)!/(d - 1)!d!$ distinct orbits for each generic ramification divisor. □

Recall that a branched superminimal immersion of $S^2$ into $S^4$ is just a harmonic map. Also, a (branched) superminimal surface of degree $d$ in $S^4$ is a surface of area $4\pi d$ whose lifting to $\mathbb{P}^3$ is a holomorphic, horizontal curve of degree $d$. We say that a harmonic map $f: S^2 \to S^4$ has harmonic degree $d$ if $f(S^2)$ has area $4\pi d$. Let $\mathcal{H}_d$ denote the space of harmonic maps of $S^2$ into $S^4$ of harmonic degree $d$.

**Theorem D.** For each $d \geq 1$, $\mathcal{H}_d$ is parametrized by a space of complex dimension $2d + 4$.

**Proof.** A meromorphic function of degree $d$ is determined by $2d + 1$ complex parameters. The theorem follows immediately from the fact that the fibers of $\Psi^d$ are 3-dimensional. □

Note. Theorem D is in agreement with the results of Verdier [17]. Verdier in fact shows that $\mathcal{H}_d$ is naturally equipped with the structure of a complex algebraic variety of pure dimension $2d + 4$, and for $d \geq 3$, $\mathcal{H}_d$ possesses three irreducible components. We will show that $\mathcal{H}_d$ is connected.

5. Connectivity of $\mathcal{H}_d$. Recall that a meromorphic function of degree $d$ can be considered as an element of $M_d = \mathbb{P}(N_d) - \mathcal{R}$ where $N_d = \mathbb{C}^{d+1} \times \mathbb{C}^{d+1} - \{(P, Q) | P \wedge Q = 0\}$ and where $\mathcal{R}$ is the resultant divisor. We have a ramification map $\Psi^d: M_d \to \mathbb{P}^{2d-2}$. The action of $\text{PSL}(2, \mathbb{C})$ on $M_d$ induces a
map $\Psi_d : G(2, d + 1) - \pi(\mathcal{R}) \to \mathbb{P}^{2d-2}$, where $\pi(\mathcal{R}) = \mathcal{R}/PSL(2, \mathbb{C})$ is an irreducible variety of codimension 1. For ease of notation, we will let $\mathcal{R}$ and $\mathcal{R}'$ denote $\pi(\mathcal{R})$ and $\Psi_d(\pi(\mathcal{R}))$ respectively for the rest of this section. Now, $\Psi_d : G(2, d + 1) \to \mathbb{P}^{2d-2}$ is a branched covering map. Let $\mathfrak{X}$ and $\mathfrak{X}'$ denote the ramification locus of $\Psi_d$ and the branch locus of $\Psi_d$ respectively. Then

$$\Psi_d : G(2, d + 1) - \mathfrak{X} - \mathfrak{X}' \to \mathbb{P}^{2d-2} - \mathfrak{X} - \mathfrak{X}'$$

is a covering map. Now consider the diagonal map

$$\delta : \mathbb{P}^{2d-2} \to \mathbb{P}^{2d-2} \times \mathbb{P}^{2d-2}.$$

Let $\mathcal{M}_d := G(2, d + 1) - \mathcal{R}$. From the diagram

$$\begin{array}{ccc}
\delta^*(\mathcal{M}_d \times \mathcal{M}_d) & \to & \mathcal{M}_d \times \mathcal{M}_d \\
\downarrow & & \downarrow \\
\mathbb{P}^{2d-2} & \to & \mathbb{P}^{2d-2} \times \mathbb{P}^{2d-2}
\end{array}$$

we see that modulo the action of $PSL(2, \mathbb{C})$, an element of $\delta^*(\mathcal{M}_d \times \mathcal{M}_d)$ is a pair of meromorphic functions of degree $d$ with the same ramification divisor. We will show that the space $\delta^*(\mathcal{M}_d \times \mathcal{M}_d)$ is connected and as a consequence $\delta_d$, the space of pairs of meromorphic functions of degree $d$ with the same ramification divisor, is connected.

**Lemma 2.7.** $\mathcal{R}$ is not a component of $\mathfrak{X}$. Thus, $\dim(\mathcal{R} \cap \mathfrak{X}) \leq 2d - 4$.

**Proof.** In §II.1, we showed that $\mathcal{R}$ is irreducible. Thus, it suffices to show that there exists an $x \in \mathcal{R}$ such that $x \notin \mathfrak{X}$. Now in ambient coordinates,

$$\Psi^d(P, Q) = \Psi^d(a_d, \ldots, a_0, b_d, \ldots, b_0) = (c_{2d-2}, \ldots, c_0)$$

where

$$c_m = \sum_{j=0}^{m+1} (2j - m - 1)a_j b_{m-j+1}$$

$$= \sum_{k=0}^{m+1} (m - 2k + 1)a_{m-k+1} b_k, \quad m = 0, \ldots, 2d - 2.$$ 

Thus,

$$\frac{\partial c_m}{\partial a_j} = \begin{cases} (2j - m - 1)b_{m-j+1}, & \text{for } j = 0, \ldots, m + 1; m - j + 1 \leq d, \\ 0, & \text{for } j > m + 1, \end{cases}$$

and

$$\frac{\partial c_m}{\partial b_k} = \begin{cases} (m - 2k + 1)a_{m-k+1}, & \text{for } k = 0, \ldots, m + 1; m - k + 1 \leq d, \\ 0, & \text{for } k > m + 1. \end{cases}$$
Let $P(z) = z^d + z^2$, $Q(z) = z$. Certainly $[P \wedge Q] \in \mathcal{H} \subset G(2, d + 1)$. Then
\[
\frac{\partial c_m}{\partial a_j} |_{(P, Q)} \neq 0, \quad \text{if} \ j = m = 0, 2, 3, \ldots, d.
\]

Also,
\[
\frac{\partial c_m}{\partial b_k} |_{(P, Q)} \neq 0, \quad \text{if} \ m = d + k - 1, \text{or} \ m = k + 1
\]
i.e. this derivative does not vanish for $k = 0, m = 1$; $k = 0, m = d - 1$; $k = 1, m = d$; $\ldots$; $k = d - 1, m = 2d - 2$. Consequently, $d\Psi^d |_{(P, Q)}$ has maximal rank. Thus, $[P \wedge Q] \notin \mathcal{R}$. □

Recall that an element of $\delta^* (\mathcal{H} \times \mathcal{H})$ is (up to a Möbius transformation) a pair of meromorphic functions of degree $d$ with the same ramification divisor. Thus, if $q \in \mathcal{H}$, the diagonal pair $(q, q)$ is obviously in $\delta^* (\mathcal{H} \times \mathcal{H})$. Since $\mathcal{H}$ is connected, it is clear that a diagonal point $(q, q) \in \delta^* (\mathcal{H} \times \mathcal{H})$ is path connected to any other diagonal point $(q', q') \in \delta^* (\mathcal{H} \times \mathcal{H})$. Thus, to show that $\delta^* (\mathcal{H} \times \mathcal{H})$ is path connected, it suffices to show that any point $(x, y) \in \delta^* (\mathcal{H} \times \mathcal{H})$ is path connected to the point $(y, y)$.

Now let $(x, y) \in \delta^* (\mathcal{H} \times \mathcal{H})$. Let $\Psi_d(x) = \Psi_d(y) = * \in \mathbb{P}^{2d-2} - \mathcal{R}'$. Without loss of generality, $* \in \mathbb{P}^{2d-2} - \mathcal{B} - \mathcal{R}'$, and so, $x, y \notin \mathcal{R}$. (If $* \in \mathcal{B}$, we can find a path $C$ in $\mathbb{P}^{2d-2} - \mathcal{R}'$ so that $C(0) = *$ and $C(1) = *' \notin \mathcal{B}$.) Since $G(2, d+2) - \mathcal{R} - \mathcal{R}$ is connected, there is a path $\tilde{y} \subset G(2, d+1) - \mathcal{R} - \mathcal{R}$ so that $\tilde{y}(0) = x$, $\tilde{y}(1) = y$. Then $\gamma := \Psi_d(\tilde{y})$ is a based loop in $\mathbb{P}^{2d-2} - \mathcal{B} - \mathcal{R}'$, i.e. $[\gamma] \in \pi_1(\mathbb{P}^{2d-2} - \mathcal{B} - \mathcal{R}', *)$. Thus $\gamma : S^1 \rightarrow \mathbb{P}^{2d-2} - \mathcal{B} - \mathcal{R'} \subset \mathbb{P}^{2d-2}$. Since $\mathbb{P}^{2d-2}$ is simply connected, we can extend $\gamma$ to a map $\gamma' : D^2 \rightarrow \mathbb{P}^{2d-2}$. By Thom transversality and Lemma 2.7, we can make $\gamma'$ transversal to $\text{Reg}(\mathcal{B})$, $\text{Reg}(\mathcal{R}')$ and $\Psi_d(\mathcal{B} \cap \mathcal{R}) = \mathcal{B} \cap \mathcal{R}'$, i.e.
\[
\gamma'(D^2) \cap \{ \text{Sing}(\mathcal{B}) \cup \text{Sing}(\mathcal{R'}) \cup \{ \mathcal{B} \cap \mathcal{R}' \} \} = \emptyset.
\]

Then $\gamma'(D^2)$ intersects $\text{Reg}(\mathcal{B})$ and $\text{Reg}(\mathcal{R}')$ in a finite number of points, say $\gamma'(D^2) \cap \text{Reg}(\mathcal{B}) = \{ z_1, \ldots, z_n \}$ and $\gamma'(D^2) \cap \text{Reg}(\mathcal{R}') = \{ \zeta_1, \ldots, \zeta_m \}$ where $z_i \neq \zeta_j$ for any $i, j$. Let $\sigma_i$ and $\tau_j$ be tiny based loops around $z_i$ and $\zeta_j$ respectively. Then $\gamma$ is homotopic to a composition of the $\sigma_i$'s and the $\tau_j$'s. Observe that the $\tau_j$'s act trivially on $F = \Psi_d^{-1}(*)$. Let $x_1 := x$ and $x_{n+1} := y$. Since $[\gamma](x) = y$, we have $[\sigma_j](x_1) = x_2$, $[\sigma_j](x_2) = x_3$, \ldots, $[\sigma_j](x_n) = x_{n+1} = y$ for some $x_2, \ldots, x_n \in F$. Let $\tilde{\sigma}_i$ be the lifting of $\sigma_i$ so that $\tilde{\sigma}_i(0) = x_i$ and $\tilde{\sigma}_i(1) = x_{i+1}$. As $\sigma_i$ traces along the boundary of a tiny disc $D_i$ around the branch point $z_i$, $\tilde{\sigma}_i$ traces a path around some ramification point $y_i \in \Psi_d^{-1}(z_i)$. Let $\tilde{D}_i$ denote the contractible disc in $G(2, d+1) - \mathcal{R}$ around $y_i$ which projects to $D_i$. Suppose $\sigma_i(t)$ traces $\partial D_i$ for $t \in [t_{a_i}, t_{b_i}]$. Let $u_i = \tilde{\sigma}_i(t_{a_i})$ and $v_i = \tilde{\sigma}_i(t_{b_i})$. Let $\tilde{\alpha}_i$ be a path from $u_i$ to $v_i$ and let $\tilde{\beta}_i$ be
a path from \( y_i \) to \( v_i \). Say \( \tilde{\alpha}_i(t_{a_i}) = u_i \), \( \tilde{\beta}_i(t_{b_i}) = v_i \) and \( \tilde{\alpha}_i(t_{e_i}) = \tilde{\beta}_i(t_{e_i}) = y_i \) for some \( t_{a_i}, t_{b_i}, \) \( t_{e_i} \in (t_{a_i}, t_{b_i}) \). Consider the modified path \( \tilde{\sigma}_i' \) defined as follows:

\[
\tilde{\sigma}_i'(t) = \begin{cases} 
\tilde{\sigma}_i(t), & \text{for } t \in [0, t_{a_i}], \\
\tilde{\alpha}_i(t), & \text{for } t \in [t_{a_i}, t_{e_i}], \\
\tilde{\beta}_i(t), & \text{for } t \in [t_{e_i}, t_{b_i}], \\
\tilde{\sigma}_i(t), & \text{for } t \in [t_{b_i}, 1].
\end{cases}
\]

Let \( \sigma_i' := \Psi_d(\tilde{\sigma}_i') \). Observe that \( \sigma_i' \) is a homotopically trivial loop in \( \mathbb{P}^{2d-2} - \mathcal{R}' \). Let \( \tilde{\sigma}_i'' \) denote the lifting of \( \sigma_i' \) so that \( \tilde{\sigma}_i''(0) = \tilde{\sigma}_i''(1) = y \). Let \( \gamma_i \) denote the path \( \left( \tilde{\sigma}_i', \tilde{\sigma}_i'' \right) \) in \( \delta^*(\mathcal{M}_d \times \mathcal{M}_d) \) from \( (x_i, y) \) to \( (x_{i+1}, y) \). We have thus constructed a path \( \gamma_n \circ \gamma_{n-1} \circ \cdots \circ \gamma_1 \) in \( \delta^*(\mathcal{M}_d \times \mathcal{M}_d) \) from \( (x, y) \) to \( (y, y) \).

Thus,

**Theorem E.** For each \( d \geq 1 \), \( \mathcal{H}_d \) is connected.

6. **Examples.** Consider the map \( F_d = (f_1, f_2): \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \) \( (d > 2) \) where

\[
f_1(z) = \frac{P_1(z)}{Q_1(z)} = \frac{z^d + dz + 1}{z^{d-1} + z + (d - 2)} \quad \text{and} \quad f_2(z) = \frac{P_2(z)}{Q_2(z)} = \frac{z^d - dz + 1}{z^{d-1} + z - (d - 2)}.
\]

We will show that for \( d > 2 \), \( F_d \) gives rise to a conjugate pair of unbranched superminimal surfaces in \( S^4 \).

Observe that \( f_1 \) and \( f_2 \) belong to different \( \text{PSL}(2, \mathbb{C}) \)-orbits.

**Lemma 2.8.** For \( d > 2 \), \( F_d \) has bidegree \( (d, d) \). Furthermore, \( \text{Ram}(f_1) = \text{Ram}(f_2) \).

**Proof.** We must first show that \( P_i(z) \) and \( Q_i(z) \) have no common zeroes \( (i = 1, 2) \).

Suppose \( \zeta \) is a common zero of \( P_1(z) \) and \( Q_1(z) \). Certainly \( \zeta \) must be a zero of \( P(z) = zQ_1(z) - P_1(z) = z^2 - 2z - 1 \). But \( P(z) \) has roots \( 1 \pm \sqrt{2} \) which are certainly not roots of \( P_1(z) \) or \( Q_1(z) \). Thus, \( \deg(f_1) = d \). A similar argument shows that \( \deg(f_2) = d \). Now

\[
f_1'(z) = \frac{R(z)}{Q_1^2(z)} = \frac{z^{2d-2} + (d - 1)z^{d} - (d - 1)z^{d-2} + d(d - 2) - 1}{z^{d-1} + z + (d - 2)]^2}
\]

and

\[
f_2'(z) = \frac{R(z)}{Q_2^2(z)} = \frac{z^{2d-2} + (d - 1)z^{d} - (d - 1)z^{d-2} + d(d - 2) - 1}{z^{d-1} + z - (d - 2)]^2}.
\]

Thus, \( \text{Ram}(f_1) = \text{Ram}(f_2) \). \( \Box \)
Proposition 2.9. The map $F_d$ is generically one-to-one onto its image. Hence, it is not a branched covering map.

Proof.

Note that 0 is not a ramification point of either $f_1$ or $f_2$. We shall compute

$$F_d^{-1} \left( \frac{1}{d - 2}, -\frac{1}{d - 2} \right).$$

This amounts to solving the simultaneous equations

$$zd + dz + 1 = \frac{1}{d - 2} \quad \text{and} \quad zd - dz + 1 = \frac{1}{d - 2}.$$

We obtain

$$(d - 2)(zd + dz + 1) - (zd - dz + 1) = 0 \quad \text{and} \quad (d - 2)(zd - dz + 1) - (zd - dz + 1) = 0.$$

Thus, we have to solve the simultaneous equations

$$g_1(z) = (d - 2)zd - zd^2 + (d(d - 2) - 1)z = 0 \quad \text{and} \quad g_2(z) = (d - 2)zd - zd^2 - (d(d - 2) - 1)z = 0.$$

Observe that if $\zeta$ is a common zero of $g_1$ and $g_2$, then certainly it is a zero of $(g_1 + g_2)(z) = 2(d - 2)zd^2$ ($d > 2$). But $g_1 + g_2$ has 0 as its only zero. Thus

$$F_d^{-1} \left( \frac{1}{d - 2}, -\frac{1}{d - 2} \right) = \{0\},$$

i.e. $F_d$ is generically one to one onto its image. □

Proposition 2.10. The map $F_d^\ast : \mathbb{P}^1 \to \mathcal{P}(\mathbb{P}^1 \times \mathbb{P}^1)$ is nonsingular.

Proof. It suffices to show that $F_d^\ast$ does not vanish at the ramification points. We will consider three cases.

Case 1. Assume that the zeroes of $Q_1(z)$ and $Q_2(z)$ are not ramification points. Then $F_d^\ast$ can be described locally by

$$F_d^\ast(z) = (f_1(z), f_2(z), G(z))$$

where

$$G(z) = \frac{f_1'(z)}{f_2'(z)} = \left( \frac{zd - dz + 1}{zd - dz + 1} \right)^2.$$

It suffices to show that $G'$ does not vanish at the ramification points. Now

$$G'(z) = 2 \left( \frac{zd - dz + 1}{zd - dz + 1} \right)^3 \cdot 2(d - 2)h(z)$$
where \( h(z) = (d - 1)z^{d-2} + 1 \). Observe that \( h(z) \) vanishes when \( z^{d-2} = -1/(d-1) \). Let \( \zeta \) be a (\( d - 2 \))th root of \( -1/(d-1) \). Then

\[
R(\zeta) = \zeta^{2d-2} + (d - 1)\zeta^d - (d - 1)\zeta^{d-2} + d(d-2) - 1
\]

\[
= \zeta^2\left(\zeta^{2(d-2)} + (d-1)\zeta^{d-2} - (d-1)\zeta^{d-2} + d(d-2) - 1\right)
\]

\[
= \zeta^2\left(\left(\frac{1}{d-1}\right)^2 - 1\right) + d(d-2) \neq 0.
\]

Thus, the zeroes of \( G' \) do not coincide with the ramification points, i.e. \( \hat{F}_d \) is nonsingular.

**Case 2.** Suppose \( \zeta \) is a common zero of \( R(z) \) and \( Q_1(z) \). Let \( \hat{f}_1(z) = Q_1(z)/P_1(z) \). Then locally,

\[
\hat{F}_d(z) = (\hat{f}_1(z), \hat{f}_2(z), G(z)) \quad \text{where} \quad G(z) = \frac{\hat{f}_1'(z)}{\hat{f}_2'(z)} = -\left(\frac{Q_2(z)}{P_1(z)}\right)^2.
\]

Then \( G'(z) = -2[Q_2(z)/P_1^3(z)] \cdot \Delta(z) \) where

\[
\Delta(z) = P_1(z)Q_2'(z) - Q_2(z)P_1'(z)
\]

\[
= -z^{2d-2} + (1-d)z^d + d(2d-4)z^{d-1} + (d-1)z^{d-2} + d + d(d-2) + 1.
\]

Let \( S(z) = R(z) + \Delta(z) = d(2d-4)z^{d-1} + 2d(d-2) \). First observe that \( Q_1(z) \) and \( Q_2(z) \) have no common zeroes since \( Q_1(z) + Q_2(z) = 2(d-2) \neq 0 \) for \( d > 2 \). Thus \( G'(\zeta) \) is zero if and only if \( \Delta(\zeta) \) is zero. Suppose that \( \zeta \) is a common zero of \( \Delta \) and \( R \). Then \( \zeta \) must be a zero of \( S \). But \( S(z) \) vanishes when \( z^{d-1} = -2d(d-2)/d(2d-4) = -1 \). Then \( \zeta \) must be a (\( d - 1 \))th root of \( -1 \). But \( Q_1(\zeta) = -1 + \zeta + (d-2) = \zeta + d - 3 \neq 0 \) for \( d > 2 \), contradicting our assumption that \( \zeta \) was a zero of \( Q_1(z) \). Thus, \( G'(\zeta) \neq 0 \).

**Case 3.** Suppose \( \zeta \) is a common zero of \( R(z) \) and \( Q_2(z) \). Let \( \hat{f}_2(z) = Q_2(z)/P_2(z) \). Then locally,

\[
\hat{F}_d(z) = (f_1(z), \hat{f}_2(z), G(z)) \quad \text{where} \quad G(z) = \frac{\hat{f}_1'(z)}{\hat{f}_2'(z)} = -\left(\frac{P_2(z)}{Q_1(z)}\right)^2.
\]

Then \( G'(z) = -2[P_2(z)/Q_1^3(z)] \cdot \Delta(z) \) where

\[
\Delta(z) = Q_1(z)P_2'(z) - P_2(z)Q_1'(z)
\]

\[
= z^{2d-2} + (d-1)z^d + d(2d-4)z^{d-1} - (d-1)z^{d-2} - d(d-2) - 1.
\]

Let \( S(z) = R(z) - \Delta(z) = -d(2d-4)z^{d-1} + 2d(d-2) \). If \( \zeta \) is a common zero of \( \Delta \) and \( R \), certainly it is a zero of \( S \). But \( S(z) \) vanishes when \( z^{d-1} = 2d(d-2)/d(2d-4) = 1 \), i.e. \( \zeta \) is a (\( d - 1 \))th root of \( 1 \). But \( Q_2(\zeta) = \zeta - (d+3) \neq 0 \) for \( d > 2 \), a contradiction. Thus, \( G'(\zeta) \neq 0 \). \( \square \)
Thus the total preimage \( \beta \circ \psi^{-1}(\tilde{F}_d(P^1)) \) is a conjugate pair of nonsingular holomorphic, horizontal curves in \( P^3 \) which project to a conjugate pair of superminimal surfaces, each of area \( 4\pi d \), in \( S^4 \) \((d \geq 3)\).

III. HIGHER GENUS

We now consider branched superminimal immersions of a compact Riemann surface \( \Sigma \) of genus \( g > 0 \) into \( S^4 \).

Let \( f: \Sigma \hookrightarrow S^4 \) be a branched superminimal immersion such that \( f(\Sigma) \) has area \( 4\pi d \). Recall that generically, \( f(\Sigma) \) misses a pair of antipodal points on \( S^4 \), say the north and south poles. We have shown that \( f \) arises from a pair of meromorphic functions \((f_1, f_2)\) of bidegree \((d, d)\) such that \( \text{Ram}(f_1) = \text{Ram}(f_2) \). Moreover, \( f \) is linearly full (i.e. \( f(\Sigma) \) is not contained in any strict linear subspace of \( \mathbb{R}^5 \)) provided \( d \geq 3 \) and \( f_2 \neq A \circ f_1 \) for any \( A \in \text{PSL}(2, \mathbb{C}) \).

For each \( d \geq 3 \), we wish to construct linearly full branched superminimal immersions from such pairs of functions. Let \( F = (f_1, f_2) \) be such a pair of functions. Let \( \hat{\mathcal{C}} \) denote the curve \( \hat{\mathcal{F}}(\Sigma) \). We require that \( \psi^{-1}(\hat{\mathcal{C}}) \) consist of two connected components, \( \gamma_1 \) and \( \gamma_2 \), such that \( \alpha(\gamma_1) = \gamma_2 \) and \( \psi(\gamma_1) = \psi(\gamma_2) = \hat{\mathcal{C}} \). If this is the case, then the curves \( \gamma_1 \) and \( \gamma_2 \) project to a conjugate pair of (branched) superminimal surfaces in \( S^4 \).

Let \( X := P^3 - (\sigma_1 \cup \sigma_2) \cong P^3 - (L_1 \cup L_2) \) and \( Y := PT(P^1 \times P^1) - (S_1 \cup S_2) \). Note that \( \pi_1 X = 0 \) and \( \psi: X \rightarrow Y \) is a covering map of degree 2. The maps that we are considering, \( F = (f_1, f_2): \Sigma \rightarrow P^1 \times P^1 \), are such that \( \tilde{F}(\Sigma) \subset Y \).

Observe that \( \tilde{F} \) lifts to a map \( \tilde{F}: \Sigma \rightarrow X \) if and only if \( \tilde{F}_* (\pi_1 \Sigma) = 0 \). If \( \tilde{F}_* (\pi_1 \Sigma) = 0 \), then we have 2 maps, \( \tilde{F} \) and \( \alpha \circ \tilde{F} \), from \( \Sigma \) to \( X \). Thus

**Theorem F.** Suppose \( F = (f_1, f_2): \Sigma \rightarrow P^1 \times P^1 \) is a holomorphic map of bidegree \((d, d)\) of a compact Riemann surface of genus \( g \) to \( P^1 \times P^1 \) such that \( \text{Ram}(f_1) = \text{Ram}(f_2) \) and \( f_2 \neq A \circ f_1 \) for any \( A \in \text{PSL}(2, \mathbb{C}) \). Let \( \tilde{F}: \Sigma \rightarrow PT(P^1 \times P^1) - (S_1 \cup S_2) \) be the canonical Gauss lift of \( F \). Then \( F \) gives rise to a conjugate pair of linearly full branched superminimal surfaces of genus \( g \) in \( S^4 \) provided \( \tilde{F}_* (\pi_1 \Sigma) = 0 \).

**Note.** The condition \( \tilde{F}_* (\pi_1 \Sigma) = 0 \) is automatically satisfied if \( \Sigma \) has genus 0. However, if \( \tilde{F}_* (\pi_1 \Sigma) \neq 0 \), then we do not have a lift of \( \Sigma \) to \( X \). Nevertheless, there is a two-fold cover \( \hat{\Sigma} \) of \( \Sigma \) which lifts to \( X \) (where genus \( (\hat{\Sigma}) = 2g - 1 \)). We then obtain a branched superminimal surface in \( S^4 \) of genus \( 2g - 1 \).

An easy way to satisfy the lifting criterion is by factoring through \( P^1 \):

\[
F = (F_1, F_2): \varphi P^1 \xrightarrow{(f_1, f_2)} P^1 \times P^1
\]

where \( \varphi \) is a holomorphic map of degree \( d_1 \) and \((f_1, f_2)\) is a holomorphic map of bidegree \((d_2, d_2)\) which gives rise to a linearly full branched superminimal immersion of \( P^1 \) into \( S^4 \). Note that \( F \) has bidegree \((d_1 d_2, d_1 d_2)\). Certainly,
\( \text{Ram}(F_1) = \text{Ram}(F_2) \) and \( F_2 \neq A \circ F_1 \) for any \( A \in PSL(2, \mathbb{C}) \) (since \( (f_1, f_2) \) is linearly full). Let \( \tilde{F} : \Sigma \to Y \) be the canonical Gauss lift of \( F \). Then \( \tilde{F}_* (\pi, \Sigma) = 0 \) and by Theorem F, \( \tilde{F} \) lifts to a holomorphic horizontal map, \( \tilde{F} \), to \( \mathbb{P}^3 \). Note however that \( \tilde{F}(\Sigma) \) is necessarily branched. Nevertheless, it projects to a branched superminimal surface in \( S^4 \) of area \( 4\pi d_1 d_2 \). We thus have lots of branched superminimal immersions of \( \Sigma \) into \( S^4 \).

The construction in the previous paragraph gives us superminimal surfaces of genus \( g > 0 \) in \( S^4 \) which were necessarily branched. It would be interesting if the map \( F \) can be deformed (in the space of branched superminimal immersions of \( \Sigma \) into \( S^4 \) of degree \( d_1 d_2 \)) to a map \( F' \) so that \( F' \) gives rise to an unbranched superminimal surface in \( S^4 \).

It has come to the author’s attention that Verdier has obtained a result similar to Theorem E (which was his conjecture in [17]).

**References**

17. J. L. Verdier, *Applications harmoniques de \( S^2 \) dans \( S^4 \)* (preprint).

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