AN HNN-EXTENSION WITH CYCLIC ASSOCIATED SUBGROUPS
AND WITH UNSOLVABLE CONJUGACY PROBLEM

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Dedicated to memory of William Werner Boone

ABSTRACT. In this paper, we consider the conjugacy problem for HNN-
extensions of groups with solvable conjugacy problem for which the associated
subgroups are cyclic. An example of such a group with unsolvable conjugacy
problem is constructed. A similar construction is given for free products with
amalgamation.

1. INTRODUCTION

It is known that an HNN-extension of a group with solvable conjugacy prob-
lem may have unsolvable conjugacy problem. Some restrictions placed on the
type of HNN-extensions force the conjugacy problem to be solvable. In this
paper, we investigate HNN-extensions with infinite cyclic associated subgroups.
That is, we consider HNN-extensions of the form

\[ G = \langle H, t; t^{-1} at = b \rangle, \]

where \( a, b \in H \) are of infinite order. We also consider the analogous situation
of free products with cyclic amalgamated subgroups.

The conjugacy problem for HNN-extensions and free products with amalga-
mation of this type has been considered by M. Anshel and P. Stebe [1], L. P.
Comerford and B. Truffault [2], R. D. Hurwitz [3], and S. Lipschutz [4], among
others. All obtained results giving conditions which guarantee that such groups
have solvable conjugacy problem.

In this paper, we approach the problem from the opposite direction. An
example of an HNN-extension \( G = \langle H, t; t^{-1} at = b \rangle \) where \( H \) has solvable
conjucacy problem, \( G \) has solvable word problem, but \( G \) has unsolvable conju-
gacy problem is constructed. A similar example is given involving free products.

In \( \S 2 \), we show the existence of a one-to-one recursive function \( f: N \rightarrow N \)
with nonrecursive range such that \( S_n = \{ f(kn); k = 1, 2, \ldots \} \) is recursive for
each \( n = 2, 3, \ldots \). In §3, we use the function \( f \) to construct group presentations \( H \) and \( G \), each with a recursive set of generators and a recursively enumerable set of defining relators. \( H \) has the following properties.

(i) \( H \) has solvable conjugacy problem;
(ii) \( H \) has an infinite cycle subgroup \( \langle d \rangle \) such that
   
   (a) there is no algorithm to decide if an arbitrary \( W \) in \( H \) is conjugate to an element of \( \langle d \rangle \);
   
   (b) the problem of membership in \( \langle d \rangle \) is decidable;
(iii) If \( W \in H \) then \( W \) is not conjugate in \( H \) to \( W^{-1} \).

Let \( G = \langle H, t; t^{-1} dt = d^{-1} \rangle \). The group \( G \) has solvable word problem by (ii)(b). Straightforward arguments show that for any \( W \in H \), \( W \) is conjugate to \( W^{-1} \) in \( G \) if and only if \( W \) is conjugate in \( H \) to an element of \( \langle d \rangle \). Thus \( G \) has unsolvable conjugacy problem.

We then use the standard HNN embedding into a two-generator group to obtain the result for recursively presented groups. Free products with amalgamation are also considered.

§4 contains the proofs of some assertions used but not proved in §3.

2. The function \( f \)

In this section, we prove the existence of a recursive function \( f \) with certain special properties.

Lemma 1. There is a one-to-one recursive function \( f: N \to N \) with nonrecursive range such that \( S_n = \{ f(kn): k = 1, 2, \ldots \} \) is recursive for each \( n = 2, 3, 4, \ldots \).

Proof. Let \( g \) be any one-to-one recursive function with nonrecursive range. Let \( p_j \) be the \( j \)th prime number. Define \( f \) by

\[
f(1) = 1, f(p_{i_1}^{\alpha_1} p_{i_2}^{\alpha_2} \cdots p_{i_n}^{\alpha_n}) = p_{g(i_1)}^{\alpha_1} p_{g(i_2) + i_2}^{\alpha_2} \cdots p_{g(i_n) + i_n}^{\alpha_n}
\]

where \( i_1 < i_2 < \cdots < i_n \). The function \( f \) is clearly recursive.

We next show that \( f \) is one-to-one. Suppose

\[
f(p_{i_1}^{\alpha_1} p_{i_2}^{\alpha_2} \cdots p_{i_n}^{\alpha_n}) = f(p_{j_1}^{\beta_1} p_{j_2}^{\beta_2} \cdots p_{j_m}^{\beta_m}).
\]

Then

\[
p_{g(i_1)}^{\alpha_1} p_{g(i_2) + i_2}^{\alpha_2} \cdots p_{g(i_n) + i_n}^{\alpha_n} = p_{g(j_1)}^{\beta_1} p_{g(j_2) + j_2}^{\beta_2} \cdots p_{g(j_m) + j_m}^{\beta_m}.
\]

Since \( g(i_1) < g(i_1) + i_2 < g(i_1) + i_3 < \cdots < g(i_1) + i_n \) and

\[
g(j_1) < g(j_1) + j_2 < g(j_1) + j_3 < \cdots < g(j_1) + j_m,
\]

it follows that

\[
g(i_1) = g(j_1), \quad g(i_1) + i_2 = g(j_1) + j_2, \ldots, g(i_1) + i_n = g(j_1) + j_n.
\]

Therefore, \( m = n \) and \( i_k = j_k \) and \( \alpha_k = \beta_k \) for all \( k = 1, \ldots, n \). Hence,

\[
p_{i_1}^{\alpha_1} p_{i_2}^{\alpha_2} \cdots p_{i_n}^{\alpha_n} = p_{j_1}^{\beta_1} p_{j_2}^{\beta_2} \cdots p_{j_m}^{\beta_m}.
\]
Now consider the range of $f$. 

\[ p_i \in \text{range } f \left\{ \begin{array}{ll} \text{iff } p_i = f(p_j) \text{ for some } j, \\ \text{iff } p_i = p_{g(j)} \text{ for some } j, \\ \text{iff } i \in \text{range } g. \end{array} \right. \]

Since the range of $g$ is nonrecursive, the range of $f$ is nonrecursive.

Next consider $S_n$ where $n$ is prime. We claim that 

\[ p_{j_1} p_{j_2}^{\alpha_2} \cdots p_{j_i}^{\alpha_i} \in S_{p_i} \]

if and only if

(i) $j_1 = g(i)$ and $j_i > j_{i-1} > \cdots > j_2 > j_1 + i$; or

(ii) $j_1 = g(i')$ for some $i' < i$, $j_i > \cdots > j_2 > j_1 + i'$, and $j_s - j_1 = i$ for some $s$ between 2 and 1.

It follows immediately from the claim that $S_n$ is recursive if $n$ is prime. We now prove the claim. First assume that $p_{j_1} p_{j_2}^{\alpha_2} \cdots p_{j_i}^{\alpha_i}$ is in $S_{p_i}$. Then

\[ p_{j_1} p_{j_2}^{\alpha_2} \cdots p_{j_i}^{\alpha_i} = f(p_{j_1} p_{j_2}^{\beta_2} \cdots p_{j_i}^{\beta_i}) \]

for some $p_{j_1} p_{j_2}^{\beta_2} \cdots p_{j_i}^{\beta_i}$ where $i_1 < i_2 < \cdots < i_l$ and $i = i_t$ for some $t = 1, \ldots, l$. There are two cases to consider: (1) $t = 1$ and (2) $t \geq 2$.

Case (1). If

\[ p_{j_1} p_{j_2}^{\alpha_2} \cdots p_{j_i}^{\alpha_i} = f(p_{j_1} p_{j_2}^{\beta_2} \cdots p_{j_i}^{\beta_i}) \]

then

\[ p_{j_1} p_{j_2}^{\alpha_2} \cdots p_{j_i}^{\alpha_i} = p_{g(i)} p_{g(i)+i_2}^{\beta_2} \cdots p_{g(i)+i_l}^{\beta_l} \]

Therefore, $j_1 = g(i)$ and $j_2 > j_1 + i$; that is, (i) holds.

Case (2). If

\[ p_{j_1} p_{j_2}^{\alpha_2} \cdots p_{j_i}^{\alpha_i} = f(p_{j_1} p_{j_2}^{\beta_2} \cdots p_{j_i}^{\beta_i}) \]

then

\[ p_{j_1} p_{j_2}^{\alpha_2} \cdots p_{j_i}^{\alpha_i} = p_{g(i)} p_{g(i)+i_2}^{\beta_2} \cdots p_{g(i)+i_l}^{\beta_l} \]

Therefore, $j_1 = g(i)$ for some $i_1 < i$, $j_2 > j_1 + i$, and $i = j_t - j_1$ for some $t$ between 2 and 1. That is, (ii) holds.

Now suppose $n = p_{j_1} p_{j_2}^{\alpha_2} \cdots p_{j_i}^{\alpha_i}$ satisfies (i) or (ii). We will show that $n \in S_{p_i}$.

Suppose $n$ satisfies (i). Let $t_m = j_m - j_1$ for $m = 2, \ldots, l$. Therefore, (i) gives $t_i > t_{i-1} > \cdots > t_2 > i$. We have

\[ f(p_{j_1} p_{j_2}^{\alpha_2} \cdots p_{j_i}^{\alpha_i}) = p_{g(i)} p_{g(i)+t_2}^{\alpha_2} \cdots p_{g(i)+t_l}^{\alpha_l} = p_{j_1} p_{j_2}^{\alpha_2} \cdots p_{j_i}^{\alpha_i}. \]

Hence, $n \in S_{p_i}$.

Suppose $n$ satisfies (ii). Let $t_m = j_m - j_1$ for $m = 2, 3, \ldots, l$. Then $t_i > t_{i-1} > \cdots > t_2 > t_1 > i'$ where $g(i') = j_1$. Therefore,

\[ f(p_{j_1} p_{j_2}^{\alpha_2} \cdots p_{j_i}^{\alpha_i}) = p_{g(i')} p_{g(i') + t_2}^{\alpha_2} \cdots p_{g(i') + t_l}^{\alpha_l} = p_{j_1} p_{j_2}^{\alpha_2} \cdots p_{j_i}^{\alpha_i}. \]
and since \( j_s - j_1 = i \) for some \( s \), we have \( t_s = i \) for some \( s \). Therefore, \( n \in S_{p_i} \). This completes the proof of the claim.

To complete the proof of the lemma, we need to show that \( S_n \) is recursive if \( n > 1 \) and \( n \) is not prime. Let \( p \) be any prime divisor of \( n \). Then, \( x \in S_n \) implies \( x \in S_p \). To determine if \( x \in S_n \), first determine if \( x \in S_p \). If not, then \( x \notin S_n \). If so, we can effectively find the unique number \( a \) such that \( x = f(ap) \). If \( n \) divides \( ap \) then \( x \in S_n \). Otherwise, \( x \notin S_n \). ☐

3. The groups

In this section, we give an example of a recursive group presentation \( N \) with solvable conjugacy problem and with infinite cyclic subgroups \( A_1 \) and \( A_2 \) such that the HNN-extension of \( G \) associating \( A_1 \) and \( A_2 \) has unsolvable conjugacy problem.

If \( W_1 \) and \( W_2 \) are words, we use \( W_1 \equiv W_2 \) to mean that \( W_1 \) and \( W_2 \) are identical words, \( W_1 \cong W_2 \) to mean that \( W_1 \) and \( W_2 \) are equal as elements of the group \( G \), \( W_1 \equiv W_2 \) to mean that \( W_1 \) and \( W_2 \) are freely equal, and \( W_1 \sim W_2 \) to mean that \( W_1 \) and \( W_2 \) are conjugate as elements of \( G \).

Let \( f \) be the function of Lemma 1 and let

\[
H = \langle x_1, z_1, x_2, z_2, \ldots; z_i^{-1}x_j f(i)z_i = z_j^{-1}x_i f(j)z_i; (i = 2, 3, 4, \ldots) \rangle.
\]

We begin with some definitions.

**Definition.** If \( W = x_{i_1}^{\alpha_1} z_{j_1}^{\beta_1} \cdots x_{i_n}^{\alpha_n} z_{j_n}^{\beta_n} \), where \( \alpha_k, \beta_k \) are integers, then \( W \) is said to be in **condensed form** if \( \beta_k = 0 \) implies \( j_k \neq j_{k+1} \) and \( \alpha_k = 0 \) implies \( j_{k-1} \neq j_k \).

**Definition.** If \( W = x_{i_1}^{\alpha_1} z_{j_1}^{\beta_1} \cdots x_{i_n}^{\alpha_n} z_{j_n}^{\beta_n} \), where \( \alpha_k, \beta_k \) are integers, then \( W \) is said to be **reduced** if

(i) \( W \) is freely reduced;

(ii) \( i_s \neq f(j_{s-1}) \) for \( s = 2, 3, \ldots, n \), \( i_s \neq f(j_s) \) for \( s = 1, 2, \ldots, n \) and \( i_1 \neq f(j_n) \); and

(iii) no nontrivial subword of \( W \) or of any cyclic permutation of \( W \) equals 1 in \( H \).

Notice that the free group presented by

\[
B = \langle x_j (j \notin \text{range } f), x_1, z_1, z_2, \ldots; \rangle
\]

is isomorphic to the group presented by \( H \).

We first show that \( H \) has solvable word problem from which it follows that there is an effective procedure to determine if a word is reduced.

**Lemma 2.** \( H \) has solvable word problem.

**Proof.** Assume \( W = x_{i_1}^{\alpha_1} z_{j_1}^{\beta_1} \cdots x_{i_n}^{\alpha_n} z_{j_n}^{\beta_n} \) is freely reduced and in condensed form. If \( i_1, \ldots, i_n \) are not in the range of \( f \), then \( W = 1 \) if and only if \( W = 1 \)
generality, that $\sigma' \neq 1$. By Lemma 1, $S_{\sigma'}$ is recursive. Determine if $i \in S_{\sigma'}$. If not, then (ii) is not satisfied. If so, effectively find $a = m\sigma'$ with $i = f(a)$. The only possible value for $b$ then is $(a\alpha)/\sigma$. If $j = f(a\alpha/\sigma)$, then (ii) is satisfied. Otherwise, (ii) is not satisfied.

**Theorem 1.** There is a group presentation $H$ with a recursive set of generators and a recursively enumerable set of defining relators with solvable conjugacy problem and an HNN-extension $G$ of $H$ associating infinite cyclic subgroups of $H$ that has unsolvable conjugacy problem.

**Proof.** Let $H = \left\{ x_1, z_1, x_2, z_2, \ldots ; z_i^{-1}x_{f(i)}z_i = z_1^{-1}x_1z_1 \ (i = 2, 3, \ldots ) \right\}$ be the group of Lemma 4 and let 

$$G = \langle H, t; t^{-1}z_1^{-1}x_1z_1t = z_1^{-1}x_1^{-1}z_1 \rangle.$$ 

Note that $z_1^{-1}x_1z_1$ has infinite order. We claim that $x_j \sim x_j^{-1}$ if and only if $j \in \text{range of } f$, from which it immediately follows that $G$ has unsolvable conjugacy problem.

We now prove the claim. If $j \in \text{range of } f$, then in $G$

$$x_j \equiv x_{f(n)} = z_nz_1^{-1}x_1^nz_1^{-1}z_n^{-1} \sim z_1^{-1}x_1^n z_1 \sim z_1^{-1}x_1^{-n} z_1$$

$$\sim z_nz_1^{-1}x_1^{-n} z_1^n = x_{f(n)}^{-1} \equiv x_j^{-1}.$$ 

If $x_j \sim x_j^{-1}$, then using the facts that $x_j, x_j^{-1} \in H$ and $G$ is an HNN-extension of $H$, either

(i) $x_j \sim x_j^{-1}$, or

(ii) there is a sequence of words $V_1, V_2, \ldots , V_{2p}$ with $x_j \equiv V_1$ and $x_j^{-1} \equiv V_{2p}$ such that $V_{2j+1} \sim V_{2j+2}$ for $j = 0, \ldots , p - 1$ and, for each $j = 1, \ldots , p - 1$, there is an $e = \pm 1$ with $t^{-e} V_{2j}t^e = V_{2j+1}$.

We will first show that (i) is impossible. If $j \notin \text{range } f$, then $x_j \sim x_j^{-1}$ if and only if $x_j \sim x_j^{-1}$ which is clearly impossible. If $j = f(n)$, then

$$x_j \sim x_j^{-1} \left\{ \begin{array}{ll}
\text{iff } z_nz_1^{-1}x_1^n z_1^n \sim z_1^{-1}x_1^{-n} z_1^n & \text{for } n \neq 0 \\
\text{iff } x_1^n \sim x_1^{-n} & \text{for } n = 0
\end{array} \right.$$ 

which is impossible since $n \neq 0$. Therefore, if $x_j \sim x_j^{-1}$, (ii) must occur and so $x_j$ must be conjugate in $H$ to an element of the form $z_1^{-1}x_1^n z_1$, where $n$ is some integer. If $j \notin \text{range } f$ and $x_j \sim z_1^{-1}x_1^n z_1$, then $x_j \sim z_1^{-1}x_1^n z_1$ which is impossible. Therefore, $x_j \sim x_j^{-1}$ implies that $j \in \text{range } f$. 

We mention without proof that $G$ has solvable word problem.

We now extend the above result to recursive presentations by using the standard HNN embedding into a two-generator group and checking that the appropriate properties are preserved.
Lemma 5. Let \( H = \langle h_1, h_2, \ldots; s_1, s_2, \ldots \rangle \) be a presentation with a recursive set of generators and a recursively enumerable set of defining relators such that

(i) \( H \) has solvable conjugacy problem;
(ii) \( h_i \neq 1 \) for all \( i = 1, 2, 3, \ldots \);
(iii) \( h_i \neq h_j \) for all \( i \neq j, i, j = 1, 2, 3, \ldots \); and
(iv) there is an algorithm which given a word \( W \) of \( H \) decides if \( W = h^i h^j \)
for some \( i, j = 1, 2, \ldots \) and some \( \varepsilon, \delta = \pm 1 \) or 0 and, if so, which ones.

Let \( F = H^*(a, b) \) and
\[
N = \langle F, t; t^{-1} a t = b, t^{-1} b^{-i} a b^i t = h_i a^{-i} b a^i (i = 1, 2, \ldots) \rangle.
\]

Then \( N \) has solvable conjugacy problem.

Proof. It is known that \( N \) has solvable word problem. Let \( G_1 \) be the group generated by \( \{a, b^{-i} a b^i (i = 1, 2, \ldots)\} \) and let \( G_2 \) be the group generated by \( \{b, h_i a^{-i} b a^i (i = 1, 2, \ldots)\} \). Let \( \phi \) be the isomorphism from \( G_1 \) to \( G_2 \) defined by \( \phi(a) = b \) and \( \phi(b^{-i} a b^i) = h_i a^{-i} b a^i \) for \( i = 1, 2, \ldots \). We need the following assertions whose proofs will be delayed until \( \S 4 \).

(1) There is an algorithm to decide if elements of \( G_1 \cup G_2 \) are conjugate in \( N \).
(2) There is an algorithm to decide, given \( V \in F \), if \( V \) is conjugate in \( F \) to an element of \( G_1 \cup G_2 \) and, if so, to find such an element.
(3) There is an effective procedure to determine, for arbitrary \( X, Y \in F \),
if there is a \( Z \in G_2 \) such that \( XZY \in G_1 \) and, if so, to produce the finite number of possible \( Z \).
(4) If \( X, Y \in F - G_2 \), \( D \in G_2 \) is reduced, and \( XDY \in G_2 \), then \( \#_b(D) \leq \#_b(X) + \#_b(Y) + 2 \), where \( \#_b(A) \) is the number of occurrences of the symbol \( b \) in the word \( A \).

Since \( t \)-reduction is effective, we may assume that we are dealing with cyclically \( t \)-reduced words and we may consider the base and nonbase cases separately.

Base Case. We first consider conjugacy in \( N \) of elements of \( F \). Suppose \( U, V \in F \). By Collins’ Lemma, \( U \sim_N V \) if and only if

(a) \( U \sim_F V \); or
(b) there exist words \( W_1, \ldots, W_k \) such that \( U \sim_F W_1 \sim_F W_2 \sim_F W_3 \sim_F \cdots \sim_F W_k \sim_F V \) where \( W_1, \ldots, W_k \in G_1 \cup G_2 \) and \( \sim \) indicates conjugation by \( t \) or \( t^{-1} \).

Case (a) can be decided since \( F \) has solvable conjugacy problem. Case (b) holds if and only if there exist \( W_1, W_k \in G_1 \cup G_2 \) such that \( U \sim_F W_1, V \sim_F W_k \), and \( W_1 \sim_G W_k \). By assertions (1) and (2) this can be decided.
Nonbase Case. We now consider conjugacy in \( N \) of elements of \( N-F \). Assume that
\[
U = x_1 t^{e_1} \cdots x_m t^{e_m} x_{m+1} \quad \text{and} \quad V = y_1 t^{e_1} \cdots y_m t^{e_m} y_{m+1}
\]
both cyclically \( t \)-reduced. We consider three cases:

(i) \( m = 1 \);

(ii) \( m > 1 \) and the \( e_i \) do not cyclically alternate; i.e. \( e_m = e_1 \) or there is an \( i, 1 \leq i \leq m-1 \), such that \( e_i = e_{i+1} \);

(iii) \( m > 1 \) and the \( e_i \) do cyclically alternate.

First consider (i); \( m = 1 \). By Collins’ Lemma, we may assume that \( U \equiv x t \); \( V \equiv y t \); and \( U \sim V \) if and only if there is a \( C \in G_2 \) such that \( C^{-1} x t C = y t \).

But \( C^{-1} x t C = y t \) implies that \( t^{-1} y^{-1} C^{-1} x t C = 1 \). Therefore, \( x \) and \( y \in F \), \( C^{-1} \in G_2 \), and \( y^{-1} C^{-1} x \in G_1 \). By assertion (3), there are only finitely many possible \( C \) and they can be effectively found. Since \( N \) has solvable word problem, \( C^{-1} x t C = y t \) can be effectively tested for each possible \( C \) and so \( U \sim V \) can be decided in case (i).

Now consider (ii); \( m > 1 \) and the \( e_i \) do not cyclically alternate. We may, by Collins’ Lemma and by using \( U^{-1} \) and \( V^{-1} \) instead of \( U \) and \( V \) if necessary, assume that \( e_{m-1} = e_m = -1 \) and \( x_{m+1} = y_{m+1} = 1 \). Then \( U \sim V \) if and only if there is a \( C \in G_1 \) such that

\[
(*) \quad C^{-1} x_1 t^{e_1} \cdots x_{m-1} t^{-1} x_m t^{-1} C t y_m^{-1} t y_{m-1}^{-1} \cdots t^{-e_1} y_1^{-1} = 1.
\]

By considering the first two \( t \)-pinches on the left-hand side of \( (*) \), one obtains that \( C^{-1} U C = V \) implies \( x_m \phi(C) y_m^{-1} \in G_1 \). Therefore, any conjugating element \( C \) has the properties

(a) \( \phi(C) \in G_2 \) and (b) \( x_m \phi(C) y_m^{-1} \in G_1 \).

By assertion (3), there is an effective procedure that will either determine that no such \( C \) exists or produce a finite list containing all \( C \) satisfying (a) and (b). As in case (i), \( U \sim V \) can be decided.

Now consider (iii); the \( e_i \) cyclically alternate. We may, without loss of generality, assume that
\[
U = x_1 t x_2 t^{-1} \cdots x_n t^{-1} x_1 t x_2 t^{-1} t x_n t^{-1};
\]
\[
V = y_1 t y_2 t^{-1} \cdots y_n t^{-1} t y_1 t^{-1} ;
\]
\( n \) is even; and \( U \sim V \) if and only if there is a \( C \in G_1 \) such that \( C^{-1} U C = V \).

Define the homomorphism \( \rho: F \to \langle a, b \rangle \) by \( \rho(h) = 1 \) for all \( h \in H \), \( \rho(a) = a \) and \( \rho(b) = b \). Define the homomorphism \( \tau: F \to F \) by \( \tau(h) = h \).
for all \( h \in H \), \( \tau(a) = b \), and \( \tau(b) = a \). Now,
\[
C^{-1}UC = V \Rightarrow C^{-1}x_1t_{x_2}t^{-1} \cdots x_n t^{-1}Cy_n t^{-1} \cdots y_{n-1}t^{-1}y_{n-1} = 1
\]
\[
\Rightarrow C^{-1}x_1\phi^{-1}(x_2\phi(x_3) \cdots \phi(x_{n-1}\phi^{-1}(x_n\phi(C)y_n^{-1}) \cdots y_3^{-1})y_2^{-1})y_1^{-1} = 1
\]
\[
\Rightarrow \rho(c^{-1}x_1\phi^{-1}(x_2\phi(x_3) \cdots \phi(x_{n-1}\phi^{-1}(x_n\phi(c)y_n^{-1}) \cdots y_3^{-1})y_2^{-1})y_1^{-1})_{(a,b)} = 1
\]
\[
\Rightarrow C^{-1}U_1C \equiv U_2, \text{ where } U_1 = \rho(x_1)(\tau \circ \rho)(x_2)\rho(x_3)(\tau \circ \rho)(x_4) \cdots (\tau \circ \rho)(x_n)
\]
and \( U_2 = \rho(y_1)(\tau \circ \rho)(y_2)\rho(y_3)(\tau \circ \rho)(y_4) \cdots (\tau \circ \rho)(y_n) \)
\[
\Rightarrow C = WmD, \text{ where } D \text{ and } W \text{ are words on } \{a,b\} \text{ such that } D^{-1}U_1D = U_2 \text{ and } U_1 = W^k \text{ for some } k.
\]

Since such \( W \) and \( D \) can be effectively found, it remains only to find a bound for \( m \). For then we will have effectively found the finite number of possible conjugating elements and each can be tested. We know that \( x_n, y_n \notin G_2 \) for otherwise \( U, V \) would not be reduced. If \( C^{-1}UC = V \), then \( \phi(C) \in G_2 \) and \( x_n\phi(C)y_n^{-1} \in G_2 \). By assertion (4), \( \#_b(\phi(C)) \leq \#_b(x_n) + \#_b(y_n) + 2 \). Hence, a bound for \( m \) can be found completing the proof for case (iii). \( \square \)

**Lemma 6.** Let \( H \) and \( N \) be as in Lemma 5 and suppose \( G = \langle H, v; v^{-1}dv = e \rangle \) has solvable conjugacy problem where \( d \) and \( e \) are elements of \( H \) of infinite order. Let \( L = \langle N, v; v^{-1}dv = e \rangle \). Then \( L \) has unsolvable conjugacy problem.

**Proof.** We first show that, for \( y_1, y_2 \in H \), \( y_1 \sim y_2 \) if and only if \( y_1 \sim_N y_2 \). The statement is clear from left to right. If \( y_1 \sim_N y_2 \), then either

1. \( y_1 \sim_F y_2 \) or
2. \( y_1 \sim_{W_1} W_2 \sim_{W_3} \cdots \sim_{W_n} y_2 \).

If (1), then clearly \( y_1 \sim_H y_2 \). If (2), then \( y_1 \sim U \in G_1 \cup G_2 \). This is impossible unless \( y_1 = 1 \) in which case \( y_2 = 1 \) and \( y_1 \sim_H y_2 \).

We will next show that, for \( V_1, V_2 \in G \), \( V_1 \sim V_2 \) if and only if \( V_1 \sim_L V_2 \), thus showing that \( L \) has unsolvable conjugacy problem. The statement from left to right is clear. Suppose that \( V_1 \sim_L V_2 \). We may, without loss of generality, assume that \( V_1 \) and \( V_2 \) are \( v \)-reduced in \( G \).

If \( V_1, V_2 \in H \), then either

1. \( V_1 \sim_{N} V_2 \); or
2. \( V_1 \sim_{v} (d^m \text{ or } e^m) \sim_{v} (e^m \text{ or } d^m) \sim_{N} (d^k \text{ or } e^k) \sim \cdots \sim V_2 \).
Using the fact that, for \( y_1, y_2 \in H \), \( y_1 \sim y_2 \) implies \( y_1 \sim y_2 \), both (i) and (ii) yield \( V_1 \sim V_2 \). Now suppose \( V_1 \sim V_2 \) and \( V_1, V_2 \in G - H \). If \( V_1 \sim V_2 \) then there are cyclic permutations \( V_1' \) and \( V_2' \) of \( V_1 \) and \( V_2 \), respectively, with the same \( v \)-structures such that \( d^{-k} V_1' d^k = V_2' \) or \( e^{-k} V_1' d^k = V_2' \) for some \( k \).

Since all words involved are words of \( G \) and since \( G \) is embedded in \( L \), one of the equalities must hold in \( G \). Hence, \( V_1 \sim V_2 \). Therefore, for \( V_1, V_2 \in G \), \( V_1 \sim V_2 \) if and only if \( V_1 \sim V_2 \) and so \( L \) has unsolvable conjugacy problem. □

**Theorem 2.** There is a finitely generated group \( N \) with solvable conjugacy problem and an HNN-extension \( L = (N, v; v^{-1} d v = e) \) where \( d \) and \( e \) are elements of \( N \) of infinite order such that \( L \) has unsolvable conjugacy problem.

**Proof.** Take \( H \) and \( G \) to be the infinitely generated groups of Theorem 1 and \( N \) to be the group of Lemma 5 constructed from this \( H \). By Lemma 6, \( L \) has unsolvable conjugacy problem. By Lemma 5, to show that \( N \) has solvable conjugacy problem it suffices to verify conditions (ii), (iii), and (iv). (Condition (i) is already known to be true.)

**Condition (ii).** Generators in \( H \) are not equal to 1 in \( H \): If \( i \not\in \text{range } f \), then \( x_i = 1 \) implies \( x_i = 1 \) which is impossible. If \( i = f(j) \), then

\[
x_i = 1 \Rightarrow z_j z_1^{-1} x_j z_1 z_j^{-1} = 1 \Rightarrow x_i^j = 1 \Rightarrow j = 0
\]

which is impossible. Therefore, \( x_i \neq 1 \) for all \( i = 1, 2, \ldots \). Since \( z_i = 1 \) implies \( z_i \neq 1 \) for all \( i = 1, 2, \ldots \), it is clear that \( z_i \neq 1 \) for all \( i = 1, 2, \ldots \).

**Condition (iii).** Distinct generators of \( H \) are not equal in \( H \): If \( i, j \not\in \text{range } f \), then

\[
x_i = x_j \Rightarrow x_i = x_j \Rightarrow i = j.
\]

If \( i \not\in \text{range } f \) and \( j = f(k) \), then

\[
x_i = x_j \Rightarrow x_i = z_k z_1^{-1} x_1 z_1 z_k^{-1}
\]

which is impossible. If \( i, j \in \text{range } f \), it is easy to see that \( x_i = x_j \) implies \( i = j \). It is also easy to see that \( z_i = z_j \) implies \( i = j \) and that \( x_i = z_j \) is impossible.

Therefore, distinct generators of \( H \) are not equal in \( H \).

**Condition (iv).** Given a word \( W \) of \( H \), we must decide if \( W \) is a product of one or two generators or inverses of generators.

Semireduce \( W \) to obtain \( W' \) (see proof of Lemma 2 for definition of semireduction). Then

1. \( W = z_i^{e_1} z_i^{e_2} \Rightarrow W' = z_i^{e_1} z_i^{e_2} \).
2. \( W = x_i^e z_j \Rightarrow W' = x_i^e z_j \) or \( W' = z_k z_1^{-1} x_i^k z_1 z_1^{-1} z_j \) where \( f(k) = i \).
3. \( W = x_i^e x_j \) can be dealt with similarly.
Therefore, $H$ satisfies conditions (ii), (iii), and (iv) of Lemma 5.

**Theorem 3.** There exist group presentations $N_1$ and $N_2$ each with a finite number of generators and a recursively enumerable set of defining relators and each having solvable conjugacy problem such that there is a free product with amalgamation $M$ of $N_1$ and $N_2$ with unsolvable conjugacy problem that amalgamates cyclic subgroups.

**Proof.** Let $N_1 = \langle a, t; r_1, r_2, \ldots \rangle$ and $N_2 = \langle \hat{a}, \hat{t}; \hat{r}_1, \hat{r}_2, \ldots \rangle$ be copies of the group $N$ of Theorem 2. Recall that

$$H = \langle x_1, z_1, x_2, z_2, \ldots ; z_i^{-1} x_i f(i) z_i = z_1^{-1} x_1 z_1 \ (i = 2, 3, \ldots) \rangle.$$

Let $\alpha$ and $\beta$ be the usual embeddings of $H$ into $N_1$ and $H$ into $N_2$, respectively. Let $W_j = \alpha(x_j)$ and $\hat{W}_j = \beta(x_j)$. Let

$$M = \langle N_1 \ast N_2; W_1 = \hat{W}_1 \rangle.$$

We claim that $W_j \sim \hat{W}_j$ if and only if $j \in \text{range } f$. If $j \in \text{range } f$, then $j = f(k)$ for some $k$ and

$$W_j = \alpha(z_k z_1^{-1} x_1^k z_1^{-1}) \sim \alpha(x_1^k) \equiv W_1^k \equiv \hat{W}_1^k = \hat{W}_j,$$

$$\beta(x_1^k) \sim \beta(z_k z_1^{-1} x_1^k z_1^{-1}) = \hat{W}_j.$$

Therefore, if $j \in \text{range } f$ then $W_j \sim \hat{W}_j$.

Now suppose $W_j \sim \hat{W}_j$. Since $M$ is a free product with amalgamation, there is a sequence of words $V_1, V_2, \ldots, V_k$ such that

$$W_j \equiv V_1 \sim V_2 \sim V_3 \sim \cdots \sim V_k \equiv \hat{W}_j,$$

where $V_2, \ldots, V_{k-1}$ are in the amalgamated part and the $\alpha_i$ alternate between $N_1$ and $N_2$. Therefore,

$$W_j \sim W_1^n$$

for some $n \Rightarrow \alpha(x_j) \sim \alpha(x_1^n)$ for some $n \Rightarrow x_j \sim x_1^n$ for some $n \Rightarrow j \in \text{range } f$.

The second implication follows from the fact that, for $y_1, y_2 \in H$, $y_1 \sim y_2$ if and only if $y_1 \sim y_2$ (see proof of Lemma 6). Since $W_j \sim \hat{W}_j$ if and only if $j \in \text{range } f$ and range $f$ is nonrecursive, $M$ has unsolvable conjugacy problem. □

4. **Proof of Assertions**

We now prove the assertions needed in Lemma 5.

(1) There is an algorithm to decide if elements of $G_1 \cup G_2$ are conjugate in $N$. 

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Proof. Suppose \( U, V \in G_1 \cup G_2 \); then \( U \sim V \) if and only if

(i) \( U \sim V \); or

(ii) there are words \( W_1, \ldots, W_m \) such that \( U \sim W_1 \sim W_2 \sim \cdots \sim W_m \sim V \).

Since (i) can be decided we need only consider (ii). By considering cyclic reductions with respect to \( * \) (i.e. with respect to the free product \( F = H * \langle a, b \rangle \)), it can be seen that if \( 1 \neq A \in G_1 \) and \( 1 \neq B \in G_2 \) then \( A \) and \( B \) are not conjugate in \( F \). Therefore, \( W_{2k-1} \) and \( W_{2k} \) must be in the same \( G_i \).

Suppose first that \( U, V \in G_1 \). Then \( W_1, W_2 \in G_2 \) and \( W_3, W_4 \in G_1 \). It can be seen that \( U \sim W_3 \) and \( U \sim W_4 \). Continuing in this fashion, we get

\[
U \sim_{(a,b)} W_{4k-1} \sim_{(a,b)} W_{4k}
\]

for all \( k = 1, \ldots, \lfloor m/4 \rfloor \).

Since \( V \in G_1 \), \( V \) is \( W_{4k-1} \) or \( W_{4k} \) for some \( k \) and so \( U \sim V \). Therefore, (ii) occurs if and only if \( U \sim V \) and so conjugacy in \( N \) of elements of \( G_1 \) is decidable.

Now suppose \( U, V \in G_2 \). Then \( W_1 \in G_1, W_m \in G_1, \) and \( W_1 \sim W_m \).

By the above, \( W_1 \sim_{(a,b)} W_m \). So, for \( U, V \in G_2 \), (ii) occurs if and only if

\[
\phi^{-1}(U) \sim_{(a,b)} \phi^{-1}(V)
\]

The case \( U \in G_1 \) and \( V \in G_2 \) is similar. \( \Box \)

(2) There is an algorithm to decide, given \( V \in F \), if \( V \) is conjugate in \( F \) to an element of \( G_1 \cup G_2 \) and, if so, to find such an element.

Proof. Let \( V \in F \). We may assume that \( V \) is cyclically reduced with respect to \( * \). First (effectively) decide if \( V \in \langle a, b \rangle \). If so, by assumptions (ii) and (iii), \( V \sim U \in G_1 \cup G_2 \) if and only if \( V \sim U \in G_1 \) or \( V \sim b^m \) which can be decided. If \( V \notin \langle a, b \rangle \) then \( V \sim U \in G_1 \cup G_2 \) if and only if \( V \sim U \in G_2 \). Suppose that \( V = A_1H_1 \cdots A_mH_m \) where \( A_i \in \langle a, b \rangle \) and \( H_i \in H \). If \( V \sim U \in G_2 \) then by assumptions (ii) and (iii) each \( A_i \) is a product of at most three words of the forms \( a^{-1}b^{\pm 1}a^l \) or \( b^k \) \( (k \) an integer). It can be effectively determined if each \( A_i \) is such a product and, if so, each \( A_i \) can be effectively written as such a product. The only possible cyclically reduced \( U \) (up to cyclic permutations) can then be determined and tested. \( \Box \)

(3) There is an effective procedure to determine, for arbitrary \( X, Y \in F \), if there is a \( Z \in G_2 \) such that \( XYZ \in G_1 \) and, if so, to produce the finite number of possible \( Z \).

Proof. Suppose \( X = A_1H_1 \cdots A_mH_mA_{m+1} \) and \( Y = B_1K_1 \cdots B_lK_lB_{l+1} \) where \( A_i, B_i \in \langle a, b \rangle \) and \( H_i, K_i \in H \). Since \( XYZ \in G_1 \), all \( H \) factors must cancel and either

(a) \( Z = A_{m+1}^{-1}H_{m}^{-1}A_{m}^{-1} \cdots H_{1}^{-1}Z'^{-1}K_{l}^{-1}B_{l}^{-1} \cdots K_{1}^{-1}B_{1}^{-1} \in G_2 \) and \( A_iZ'B_{l+1} \in G_1 \) or
(b) \( Z = A_m^{-1} H_m^{-1} A_m^{-1} \cdots H_j^{-1} Z' K_j^{-1} B_j^{-1} \cdots K_1^{-1} B_1^{-1} \in G_2 \) where \( 2 \leq j \leq m \), \( 1 \leq q \leq l - 1 \), \( Z' \) is \( h \)-free and \( A_1 H_1 \cdots A_{j-1} H_{j-1} A_j Z' B_{q+1} K_{q+1} \cdots B_l K_{l+1} B_{l+1} \in G_1 \).

If (a) then \( H_i^{-1} = h_{i_1}^{\delta_1} h_{i_2}^{\delta_2} \) and \( K_l^{-1} = h_{l_1}^{\delta_1} h_{l_2}^{\delta_2} \) for some \( \delta_1, t_1 = \pm 1 \) and \( \delta_2, t_2 = 0 \) or \( \pm 1 \). By assumption (iv), there is an effective procedure to determine if \( H_i^{-1} \) and \( K_l^{-1} \) can be written as such and, if so, to so write them. Since \( Z' \) is \( h \)-free and \( Z \in G_2 \), there are four possibilities: \( h_{i_1}^{\delta_1} h_{j_1}^{\delta_2} \in G_2 \), \( Z' h_{i_1}^{\delta_1} \in G_2 \), \( h_{i_2}^{\delta_1} Z' \in G_2 \), \( Z' \in G_2 \). Since \( A_1 Z' B_{l+1} \in G_1 \) implies that the exponent sum of \( b \) in \( A_1 Z' B_{l+1} \) is zero, each of these four possibilities give rise to finitely many possible \( Z' \) which in turn give rise to finitely many possible \( Z \).

If (b) then \( A_f Z' B_{q+1} = 1 \) which implies that

\[
Z = A_m^{-1} H_m^{-1} A_m^{-1} \cdots H_j^{-1} A_j^{-1} B_{q+1} K_{q+1}^{-1} B_q^{-1} \cdots K_1^{-1} B_1^{-1},
\]

again giving rise to only finitely many possible \( Z \). \( \Box \)

(4) If \( X, Y \in F - G_2 \), if \( D \in G_2 \) is reduced and if \( XDY \in G_2 \), then \( \#_b(D) \leq \#_b(X) + \#_b(Y) + 2 \).

**Proof.** Let \( V = v_1 \cdots v_m \) where \( v_i \in \{ b^{\pm 1}, (h\ a^{-i} ba)^{\pm 1} \} \). Suppose \( V \) is freely reduced with respect to the free group \( G_2 \) and \( \#_b(V) \geq 3 \). It is easy to see that if \( W \) is obtained from \( V \) by freely reducing with respect to the free product \( H \ast \langle a, b \rangle \), then the subword of \( V \) containing the three \( b \) can be uniquely recovered from \( W \).

Suppose \( D \in G_2 \), \( XDY \in G_2 \) and \( \#_b(D) \geq \#_b(X) + \#_b(Y) + 3 \). Assume further that \( X, D, \) and \( Y \) are reduced with respect to \( \ast \). It suffices to show that \( X \in G_2 \). Consider the cancellation that occurs in the product \( XDY \). Let

\[
X \equiv X_1 X_2; \quad D \equiv X_2^{-1} D_1 Y_1^{-1}; \quad \text{and} \quad Y \equiv Y_1 Y_2.
\]

We have that \( X_1 D_1 Y_2 \in G_2 \) is reduced with respect to \( \ast \) and \( \#_b(D_1) \geq 3 \). By the comment above, the shortest words \( X_1' \) and \( Y_2' \) such that \( X_1 \equiv X_1' X_1' \), \( Y_2 \equiv Y_2' Y_2' \), and \( X_2''D_1 Y_2' \in G_2 \) can be effectively found. Since \( X_1' \) and \( Y_2' \) are unique, \( X_1' X_2''D_1 Y_2' Y_2'' \in G_2 \), and \( X_2''D_1 Y_2' \in G_2 \), it follows that \( X_1' \) and \( Y_2'' \in G_2 \). Since \( X_2'' \) is the shortest word such that \( X_2''D_1 Y_2' \in G_2 \) and since \( X_2^{-1} D_1 Y_2^{-1} \in G_2 \), it follows that \( X_2^{-1} \) ends in \( X_2'' \). That is, \( X_2^{-1} = X_3 X_1'' \) and \( X_3 \in G_2 \). Therefore, \( X \equiv X_1' X_2'' \) is a word in \( G_2 \) since \( X_1' \) and \( X_3 \in G_2 \). \( \Box \)

5. Conclusion

The existence of a finitely presented group \( H \) and an \( H\text{-NN-extension} \) \( G = \langle H, t; t^{-1} a t = b \rangle \) where \( H \) has solvable conjugacy problem but \( G \) has unsolvable conjugacy problem is open. The analogous question for free products with amalgamation is also open.
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