

RIGIDITY OF PSEUDO-HOLOMORPHIC CURVES OF CONSTANT CURVATURE IN GRASSMANN MANIFOLDS

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ABSTRACT. Rigidity of minimal immersions of constant curvature in harmonic sequences generated by holomorphic curves in Grassmann manifolds is studied in this paper by lifting them to holomorphic curves in certain projective spaces. We prove that for such curves the curvature must be positive, and that all such simply connected curves in CP^n are generated by Veronese curves, thus generalizing Calabi's counterpart for holomorphic curves in CP^n . We also classify all holomorphic curves from the Riemann sphere into $G(2, 4)$ whose curvature is equal to 2 into two families, which illustrates pseudo-holomorphic curves of positive constant curvature in $G(m, N)$ are in general not unitarily equivalent, constructing to the fact that generic isometric complex submanifolds in a Kaehler manifold are congruent.

0. INTRODUCTION

Given a harmonic map f from a Riemann surface M into the complex Grassmann manifold $G(m, N)$, by using the ∂ -transform associated to the map f , S. S. Chern and J. G. Wolfson obtained the following sequence of harmonic maps [6],

$$(0.1) \quad f = f_0 \rightarrow f_1 \rightarrow f_2 \rightarrow \dots$$

where $f_{j+1} \equiv \partial f_j$ for $j = 0, 1, 2, \dots$, and $f_j: M \rightarrow G(m_j, N)$ are harmonic maps; one defines $m_{j+1} = 0$ if f_j is anti-holomorphic. We call f_r a *pseudo-holomorphic curve of position r* generated by f when f is holomorphic, where f is called the directrix of f_r . Such curves with the induced metrics from the associated Grassmannians form a class of minimal immersions. When specialized to $G(1, n+1) = CP^n$, a pseudo-holomorphic curve of position r is just the r th position vector of a Frénet frame of a holomorphic curve projected into CP^n . The importance of such maps comes from the fact that any harmonic map from the Riemann sphere, or any harmonic map of nonzero degree from a torus, into CP^n is pseudo-holomorphic [7]. Furthermore in [6] these curves play a central role, when m_j 's defined in the sequence (0.1) are all equal and f_j 's are orthogonal, called Frenet harmonic sequences, for the study of

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harmonic maps from the Riemann sphere into $G(2, n)$. Henceforth they have attracted much attention.

When a simply connected pseudo-holomorphic curve is of position zero in CP^n , i.e., when it is holomorphic, Calabi showed that the only such curves of constant curvature are obtained, up to unitary equivalence, by the Veronese curves restricted to suitable domains of the Riemann sphere in a linear subspace of CP^n ; furthermore any two isometric holomorphic curves in CP^n must be unitarily congruent [3, 4]. In this paper we generalize Calabi's results in the following way. On the one hand we show that there is no (local) pseudo-holomorphic curve of nonpositive constant curvature in $G(m, N)$ (Theorem 1), and that all simply connected pseudo-holomorphic curves of positive constant curvature in CP^n are unitarily equivalent to the ones generated by the Veronese curves restricted to suitable domains in linear subspaces of CP^n (Theorem 2). Notice that the two theorems are completely local in nature, where no topological assumptions are needed. Theorem 2 implies previous results obtained in [1] that any pseudo-holomorphic curve of constant curvature generated by a rational normal curve in CP^n is unitarily equivalent to one generated by some Veronese curve; and also obtained in Bando and Ohnita, Minimal 2-spheres with constant curvature in $P_n(\mathbb{C})$, (J. Math. Soc. Japan **39** (1987), sent to us by the authors after this present paper was done), that any minimal immersion of the Riemann sphere into CP^n of constant curvature is unitarily equivalent to one generated by some Veronese curve. We would like to point out that the Riemann surfaces being spheres is crucial in Bando and Ohnita's analysis in that they need the facts that the minimal immersions are thus pseudo-holomorphic and that there are no nontrivial harmonic functions on a Riemann sphere.

On the other hand M. Green showed that generic isometric complex submanifolds in a Kaehler manifold are congruent [9]. We shall classify in Theorem 3 the holomorphic curves from the Riemann sphere into $G(2, 4)$ with the induced constant curvature 2 into two classes, up to unitary equivalence, in which none of the curves are congruent. This shows that the holomorphic curves of constant curvature in the Grassmannians are among the "nongeneric" ones, making the classification of such curves pretty interesting.

The unitary equivalence fails to be true in general for any two isometric pseudo-holomorphic curves in CP^n . For example, the complex conjugate of any holomorphic curve is pseudo-holomorphic and is isometric, but not unitarily equivalent, to the holomorphic curve. Nevertheless one can still ask whether they are generated by congruent holomorphic curves when the two pseudo-holomorphic curves are of the same position (In [1] it is proved true in case when the two isometric curves have the same Kaehler angles.) More generally it would be interesting to understand the space of noncongruent pseudo-holomorphic curves in CP^n which are isometric to a given one.¹

¹ The first author has shown that a large class of pseudo-holomorphic curves from a compact Riemann surface into CP^2 are rigid up to unitary equivalence and the complex conjugation.

Our starting point is the lifting holomorphic map defined in (1.17), which transforms the metric of a pseudo-holomorphic curve into that of a holomorphic curve. By studying this lifting map, one can conclude the above theorems via the corresponding counterparts of Calabi on holomorphic curves in CP^n . Another consequence of this construction is that the holomorphic liftings of two isometric pseudo-holomorphic curves must be unitarily congruent, which seems to be a first step towards understanding the rigidity of such curves (Proposition 1).

Basic material is developed in §1, where we collect formulas relating certain invariants of pseudo-holomorphic curves by using the method of moving frames. These invariants have been studied before for various purposes [1, 15]. Construction of the lifting map (1.17) enables us to investigate these invariants, which leads to the proof of Theorem 1 in §2 and Theorem 2 in §3. The proof of Theorem 3 in §4 is based on some previous work done by G. Jensen [12].

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1. PSEUDO-HOLOMORPHIC CURVES AND THE LIFTING MAPS

The complex Grassmann manifold $G(m, N)$ is the set of all m -dimensional linear subspaces of C^N through the origin, which can be realized as the homogeneous space $U(N)/(U(m) \times U(N - m))$, where

$$\begin{aligned} \pi: U(N) &\rightarrow G(m, N), \\ g &\rightarrow [g \cdot o] \end{aligned}$$

is a principal $U(m) \times U(N - m)$ -bundle, o is the $N \times m$ matrix

$$\begin{bmatrix} I_m \\ 0 \end{bmatrix},$$

and I_m is the $m \times m$ identity matrix.

Let W_{AB} be the Maurer-Cartan forms of $U(N)$, $1 \leq A, B \leq N$, and let e be a local section of $\pi: U(N) \rightarrow G(m, N)$. Set

$$(1.1) \quad \Psi_{AB} = e^*(W_{AB}),$$

$$(1.2) \quad ds_{G(m,N)}^2 = \sum_{\substack{1 \leq i \leq m \\ m < \alpha \leq N}} \Psi_{i\alpha} \bar{\Psi}_{i\alpha}.$$

Then $ds_{G(m,N)}^2$ is a $U(N)$ -invariant hermitian metric on $G(m, N)$; when $m = 1$ and $N = n + 1$ this is the Fubini-Study metric on CP^n with constant holomorphic curvature 4.

Suppose now M is a Riemann surface, and $f: M \rightarrow G(m, N)$ is a smooth immersion inducing the metric ds_M^2 . Locally

$$(1.3) \quad ds_M^2 = \theta \bar{\theta},$$

$$(1.4) \quad d\theta_M^2 = i\omega \wedge \theta,$$

around the points where $\det(X_{j,j+1}) \neq 0$ they satisfy [15]

$$(1.11) \quad \Delta \log |\det(X_{j,j+1})| \\ = mK + 2(|X_{j-1,j}|^2 + |X_{j+1,j+2}|^2 - 2|X_{j,j+1}|^2),$$

for $0 \leq j \leq k - 1$. In fact it is shown in [16] that when m_j 's are different one can still associated with $X_{j,j+1}$ a global invariant of analytic type which reduces to $\det(X_{j,j+1})$ when m_j 's are equal. But we will not use this fact later on.

Notice also that when $m = 1$, if we set

$$(1.12) \quad q_j = |X_{j,j+1}| = |\det(X_{j,j+1})|,$$

then (1.10) reduces to

$$(1.13) \quad q_{r-1}^2 + q_r^2 = 1,$$

and (1.11) reduces to

$$(1.14) \quad \Delta \log q_j = K + 2(q_{j-1}^2 + q_{j+1}^2 - 2q_j^2).$$

Now let $P: G(m, N) \rightarrow CP^{(m)}-1$ be the standard Plücker embedding [11]. Then for each i , $P(f_i(p))$ is an $\binom{N}{m_i}$ -dimensional vector defined up to a multiple. Also, for $0 \leq j \leq k$ set

$$F_j = P(f_0) \wedge P(f_1) \wedge \dots \wedge P(f_j).$$

$[F_j]$ is a holomorphic map from M to $CP^{L(j)-1}$, where

$$L(j) = \binom{N}{\sum_{i=0}^j m_i},$$

called the j th associated curve of f ; when $m = 1$, this is just the standard j th associated curve of the holomorphic curve f in CP^{N-1} and the pull-back metric by F_j can be expressed as [1]

$$(1.15) \quad F_j^* ds_{CP^{(N)-1}}^2 = q_j^2 \theta \bar{\theta}.$$

Finally we define $\eta_{j-1,j}: M \rightarrow CP^{N(j)}$, where

$$(1.16) \quad N(j) = L(j - 1)L(j) - 1,$$

for $1 \leq j \leq k$ by letting

$$(1.17) \quad \eta_{j-1,j}(p) = [F_{j-1}(p) \otimes F_j(p)].$$

It is easy to check that this is a well-defined holomorphic curve in $CP^{N(j)}$. We will call it the j th *lifting map* of f , which will play an important role in our analysis in the sequel.

2. NONEXISTENCE OF PSEUDO-HOLOMORPHIC CURVES IN $G(m, N)$
WITH NONPOSITIVE CONSTANT CURVATURE

Without loss of generality one may assume that the metrics induced by pseudo-holomorphic curves are nonsingular on M , since the following argument is local.

A nonconstant holomorphic curve in a complex projective space is a pseudo-holomorphic curve of position zero. For such a curve one has $|X_{0,1}|^2 = 1$ by (1.10). Suppose now that the holomorphic curve has constant curvature. Then by (1.10) all $|X_{j,j+1}|$ are constant, and hence $K = 4/N_0$ for some positive integer.

Now assume that f_r is a pseudo-holomorphic curve of position r with constant Gaussian curvature K generated by a holomorphic curve f in $G(m, N)$.

If $r = 0$, $f_0 = f$ is holomorphic. Via the standard Plücker embedding P , $P \circ f$ is a holomorphic curve in $CP^{(N)-1}$ with the same constant curvature K . Therefore $K = 4/N_0 > 0$ by the above remark.

If $r > 0$, let $\eta_{r-1,r}$ be the r th lifting map defined by (1.17) and let e be the local orthonormal frame defined by (1.6). Then locally

$$(2.1) \quad \eta_{r-1,r} = [(e_{01} \wedge \cdots \wedge e_{(r-1),m_{r-1}}) \otimes (e_{01} \wedge \cdots \wedge e_{rm_r})].$$

By (1.7), (1.8), (1.9), (1.10) and Maurer-Cartan structure equations, one gets

$$(2.2) \quad \begin{aligned} & d\{(e_{01} \wedge \cdots \wedge e_{(r-1),m_{r-1}}) \otimes (e_{01} \wedge \cdots \wedge e_{rm_r})\} \\ &= \left(\sum_{i=0}^{r-1} \text{trace } \varphi_i + \sum_{i=0}^r \text{trace } \varphi_i \right) \{(e_{01} \wedge \cdots \wedge e_{(r-1),m_{r-1}}) \otimes (e_{01} \wedge \cdots \wedge e_{rm_r})\} \\ &+ \sum_{i=1}^{m_r} \sum_{t=1}^{m_{r-1}} q_{i,t}^{r-1,r} \theta \{(e_{01} \wedge \cdots \wedge e_{(r-1),(i-1)} \wedge e_{ri} \\ &\quad \wedge e_{(r-1),(i+1)} \wedge \cdots \wedge e_{(r-1),m_{r-1}}) \otimes (e_{01} \wedge \cdots \wedge e_{rm_r})\} \\ &+ \sum_{i=1}^{m_{r+1}} \sum_{t=1}^{m_r} q_{i,t}^{r,r+1} \theta \{(e_{01} \wedge \cdots \wedge e_{(r-1),m_{r-1}}) \\ &\quad \otimes (e_{01} \wedge \cdots \wedge e_{r,(i-1)} \wedge e_{(r+1),i} \wedge e_{r,(i+1)} \wedge \cdots \wedge e_{rm_r})\} \end{aligned}$$

and hence

$$(2.3) \quad \begin{aligned} \eta_{r-1,r}^* ds_{CP^{N(r)}}^2 &= \left(\sum_{i=1}^{m_r} \sum_{t=1}^{m_{r-1}} |q_{i,t}^{r-1,r}|^2 + \sum_{i=1}^{m_{r+1}} \sum_{t=1}^{m_r} |q_{i,t}^{r,r+1}|^2 \right) \theta \bar{\theta} \\ &= (|X_{r-1,r}|^2 + |X_{r,r+1}|^2) \theta \bar{\theta} = \theta \bar{\theta} = f_r^* ds_{G(m,N)}^2. \end{aligned}$$

Therefore the pull-back metrics induced by f_r and by the lifting map $\eta_{r-1,r}$ are identical, and in particular, their Gaussian curvatures are equal. Since $\eta_{r-1,r}$ is holomorphic, one infers $K = 4/N_0$ for some positive integer N_0 . Thus one can conclude the following.

Theorem 1. *There does not exist any nonconstant (local) pseudo-holomorphic curve of nonpositive constant Gaussian curvature in $G(m, N)$.*

Corollary 1. *All minimal tori of constant curvature in S^n are not pseudo-holomorphic, when they are regarded as curves in CP^n via the standard embedding of S^n into CP^n .*

Minimal tori of constant curvature in S^n have been classified in [2, 13]. These surfaces are of degree zero when imbedded in CP^n , which provide a large class of examples verifying that Eells and Woods' nonzero degree condition mentioned in the introduction cannot be dropped. One also establishes the following proposition.

Proposition 1. *The holomorphic liftings of two isometric pseudo-holomorphic curves in $G(m, N)$ must be unitarily equivalent.*

Proof. Use the fact that two isometric holomorphic curves in a complex projective space are unitarily congruent. \square

3. RIGIDITY OF PSEUDO-HOLOMORPHIC CURVES OF POSITIVE CONSTANT CURVATURE IN CP^n

Restricting to pseudo-holomorphic curves of positive constant curvature in CP^n , where now $N = n + 1$, $m = 1$, and $N(j) = \binom{N}{j} \binom{N}{j+1} - 1$, classical examples are the Veronese curves $V_m: CP^1 \rightarrow CP^m \subset CP^n$ given by

$$V_m[(z, w)] = \left[\left(\binom{m}{0}^{1/2} w^m, \binom{m}{1}^{1/2} zw^{m-1}, \dots, \binom{m}{m}^{1/2} z^m \right) \right]$$

whose curvatures are $4/m$ for $0 \leq m \leq n$. The following theorem says that they generate all pseudo-holomorphic curves of constant curvature in CP^n .

Theorem 2. *Let M be a simply connected Riemann surface not necessarily closed, and let $f: M \rightarrow CP^n$ be a pseudo-holomorphic curve of constant curvature. Then the curve is unitarily equivalent to an open subset of a pseudo-holomorphic curve generated by some Veronese curve in a linear subspace of CP^n after reparameterization.*

Proof. Notation as in §1. Let f_r be a pseudo-holomorphic curve in CP^n of position r with constant curvature K . Without loss of generality, we may assume that f_r is linearly full. As has been shown in §2, $K = 4/N_0$ for some positive integer N_0 , which is also the curvature for the metric $\eta_{r-1,r}^* ds_{CP^{N(r)}}^2$. Since $\eta_{r-1,r}$ is holomorphic, by the theorem of Calabi $\eta_{r-1,r}$ is unitarily equivalent to a portion of V_{N_0} , a Veronese curve in a linear subspace of $CP^{N(r)}$ after reparameterization.

Let $\tilde{F}_r = (F_r^0, F_r^1, \dots, F_r^{\binom{N}{r+1}-1})$ and $\tilde{F}_{r-1} = (F_{r-1}^0, F_{r-1}^1, \dots, F_{r-1}^{\binom{N}{r}-1})$ be local lifts of F_r and F_{r-1} respectively, where F_r^t and F_{r-1}^i are local holomorphic

functions for $0 \leq t \leq \binom{N}{r+1} - 1$, and $0 \leq i \leq \binom{N}{r} - 1$. Then

$$(3.1) \quad \tilde{\eta}_{r-1,r} = \tilde{F}_r \otimes \tilde{F}_{r-1} \\ = (F_r^0 F_{r-1}^0, F_r^0 F_{r-1}^1, \dots, F_r^i F_{r-1}^j, \dots, F_r^{\binom{N}{r+1}-1} F_{r-1}^{\binom{N}{r}-1})$$

is a local lift of $\eta_{r-1,r}$. Since $\eta_{r-1,r}$ and V_{N_0} are unitarily equivalent,

$$(3.2) \quad \tilde{\eta}_{r-1,r} = A \cdot \tilde{V}_{N_0} \cdot h,$$

where h is a nowhere vanishing local holomorphic function, A is a suitable constant unitary matrix, and \tilde{V}_{N_0} is the standard lift of V_{N_0} in $C^{N_0+1} \subset C^{N(r)+1}$.

Comparing components of both sides of (3.2) one sees

$$(3.3) \quad F_r^i F_{r-1}^j(z) = P^{ij}(z)h(z),$$

where P^{ij} are polynomials in z for $0 \leq i \leq \binom{N}{r+1} - 1$ and $0 \leq j \leq \binom{N}{r} - 1$.

Without loss of generality, one may assume that $F_r^0 F_{r-1}^0 \neq 0$. Then

$$(3.4) \quad \frac{F_r^i}{F_r^0} = \frac{F_r^i F_{r-1}^0}{F_r^0 F_{r-1}^0} = \frac{P^{i,0}h}{P^{0,0}h} = \frac{P^{i,0}}{P^{0,0}},$$

and

$$(3.5) \quad \frac{F_{r-1}^j}{F_{r-1}^0} = \frac{F_r^0 F_{r-1}^j}{F_r^0 F_{r-1}^0} = \frac{P^{0,j}h}{P^{0,0}h},$$

so

$$(3.6) \quad F_{r-1} = [(h \cdot P^{0,0}, h \cdot P^{0,1}, \dots, h \cdot P^{0, \binom{N}{r}-1})]$$

and

$$(3.7) \quad F_r = [(P^{0,0}, P^{1,0}, \dots, P^{\binom{N}{r+1}-1,0})].$$

Thus one may assume further that

$$(3.8) \quad \tilde{F}_{r-1} = (hP^{0,0}, hP^{0,1}, \dots, hP^{0, \binom{N}{r}-1})$$

and

$$(3.9) \quad \tilde{F}_r = (P^{0,0}, P^{1,0}, \dots, P^{\binom{N}{r+1}-1,0}).$$

For $v = (v_0, v_1, \dots, v_L)$ in C^{L+1} , define $\|v\|^2 = \sum_{i=0}^L |v_i|^2$. Then by (3.2) one gets

$$(3.10) \quad \|\tilde{\eta}_{r-1,r}\|^2 = \|A \cdot \tilde{V}_{N_0}\|^2 |h|^2 = \|\tilde{V}_{N_0}\|^2 |h|^2 = (1 + |z|^{2N_0}) |h|^2.$$

On the other hand,

$$(3.11) \quad \|\tilde{\eta}_{r-1,r}\|^2 = |h|^2 \left(\sum_{j=0}^{\binom{N}{r}-1} |P^{0,j}|^2 \right) \left(\sum_{j=0}^{\binom{N}{r+1}-1} |P^{j,0}|^2 \right),$$

so

$$(3.12) \quad (1 + |z|^2)^{N_0} = \left(\sum_{j=0}^{\binom{N}{r}-1} |P^{0,j}|^2 \right) \left(\sum_{j=0}^{\binom{N}{r+1}-1} |P^{j,0}|^2 \right).$$

Since functions on both sides of (3.12) are polynomials in variables z and \bar{z} , where $1 + |z|^2$ is an irreducible polynomial, one concludes

$$(3.13) \quad \sum_{j=0}^{\binom{N}{r}-1} |P^{0,j}|^2 = C_1(1 + |z|^2)^{N_1},$$

and

$$(3.14) \quad \sum_{j=0}^{\binom{N}{r+1}-1} |P^{j,0}|^2 = C_2(1 + |z|^2)^{N_2},$$

where C_1 and C_2 are positive constants and N_1, N_2 are nonnegative integers with $N_1 + N_2 = N_0$. Therefore,

$$(3.15) \quad \|\tilde{F}_{r-1}\|^2 = C_1|h|^2(1 + |z|^2)^{N_1},$$

and

$$(3.16) \quad \|\tilde{F}_r\|^2 = C_2(1 + |z|^2)^{N_2}.$$

Now compute the metrics in terms of local coordinates. By (3.8), (3.9), (3.15) and (3.16) one gets

$$(3.17) \quad F_{r-1}^*(ds^2_{CP^{\binom{N}{r}-1}}) = \frac{\partial^2 \log(\|\tilde{F}_{r-1}\|^2)}{\partial z \partial \bar{z}} |dz|^2 = \frac{N_1}{(1 + |z|^2)^2} |dz|^2,$$

$$(3.18) \quad F_r^*(ds^2_{CP^{\binom{N}{r+1}-1}}) = \frac{\partial^2 \log(\|\tilde{F}_r\|^2)}{\partial z \partial \bar{z}} |dz|^2 = \frac{N_2}{(1 + |z|^2)^2} |dz|^2,$$

and

$$(3.19) \quad \begin{aligned} \eta_{r-1,r}^*(ds^2_{CP^{N(r)}}) &= \frac{\partial^2 \log(\|\tilde{\eta}_{r-1,r}\|^2)}{\partial z \partial \bar{z}} |dz|^2 \\ &= \frac{N_0}{(1 + |z|^2)} |dz|^2 = \theta\bar{\theta}. \end{aligned}$$

Therefore

$$(3.20) \quad F_{r-1}^*(ds^2_{CP^{\binom{N}{r}-1}}) = \frac{N_1}{N_0} \theta\bar{\theta}$$

and

$$(3.21) \quad F_r^*(ds^2_{CP^{\binom{N}{r+1}-1}}) = \frac{N_2}{N_0} \theta\bar{\theta}.$$

Comparing (3.20) and (3.21) with (1.15), one has

$$(3.22) \quad q_{r-1} = \sqrt{N_1/N_0}$$

and

$$(3.23) \quad q_r = \sqrt{N_2/N_0}.$$

By (1.14) all q_j are locally constant. Since q_j are continuous, they are globally constant, and $K = 4/(2r(n-r) + n)$. In particular, q_0 is constant. But then

$$(3.24) \quad f^* ds_{CP^n}^2 = q_0^2 f_r^* ds_{CP^n}^2,$$

and with this metric f is a holomorphic curve with constant curvature $K = 4/n$ on M . The Theorem of Calabi now infers that f is unitarily equivalent to a portion of the Veronese curve V_n in CP^n . Pushing f down to f_r , the result follows. \square

As an immediate corollary we recover Calabi’s theorem on the rigidity of minimal 2-spheres of constant curvature in n -spheres.

Corollary 2. *The only minimal 2-spheres of constant curvature in a standard n -sphere are (real) Veronese curves, up to orthogonal equivalence.*

Proof. Embed the n -sphere into CP^n in the standard way, and notice that the minimal 2-spheres are now pseudo-holomorphic in CP^n . By Theorem 2 they are generated by the Veronese curves in CP^n . The rest of the proof proceeds as in [1]. \square

4. HOLOMORPHIC CURVES IN $G(2, 4)$ WITH CONSTANT CURVATURE 2

Holomorphic curves in $G(2, 4)$ have been studied by P. Griffiths [10] and by G. Jensen [12] in details. In this final section, we are going to follow Jensen’s work to classify all holomorphic maps from S^2 to $G(2, 4)$ with constant curvature 2, thus showing that Theorem 2 is no longer true for a general $G(m, N)$. In the following computation we will repeatedly use Maurer-Cartan structure equations without mentioning; details can be found in [12].

Let f be a holomorphic immersion from S^2 into $G(2, 4)$ and let e be a local unitary frame along f such that (1.7) becomes

$$(4.1) \quad \phi = \begin{bmatrix} \varphi_{11} & \varphi_{12} & -t_1 \bar{\theta} & 0 \\ \varphi_{21} & \varphi_{22} & 0 & -t_2 \bar{\theta} \\ t_1 \theta & 0 & \varphi_{33} & \varphi_{34} \\ 0 & t_2 \theta & \varphi_{43} & \varphi_{44} \end{bmatrix}$$

where $t_1 \geq t_2 \geq 0$, and θ is the unitary coframe with respect to the metric $f^* ds_{G(2,4)}^2$.

If $\text{rank } \partial f = 2$, then $t_2 \neq 0$ and f is a pseudo-holomorphic curve of positive zero in $G(2, 4)$. Since Gaussian curvature $K = 2$, $t_1 t_2$ is constant by

[15]. On the other hand $f^* ds_{G(2,4)}^2 = \theta\bar{\theta} = (t_1^2 + t_2^2)\theta\bar{\theta}$, then $t_1^2 + t_2^2 = 1$. Thus $0 < t_2 \leq t_1 < 1$ are constant.

If $t_1 = t_2$, by [12] f is locally, hence by a standard monodromy argument, globally equivalent to the curve $h: S^2 = CP^1 \rightarrow G(2, 4)$ defined by

$$(4.2) \quad h \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} z & I_2 \\ w & I_2 \end{bmatrix}$$

where $z, w \in C$ and I_2 is the 2×2 identity matrix.

Now assume that $t_1 < t_2$, one can take exterior derivatives on both sides of equations $\varphi_{41} = 0$ and $\varphi_{32} = 0$ to get

$$(4.3) \quad (t_2\varphi_{21} - t_1\varphi_{43}) \wedge \theta = 0$$

and

$$(4.4) \quad (t_1\varphi_{12} - t_2\varphi_{34}) \wedge \theta = 0,$$

then

$$(4.5) \quad t_2\varphi_{21} - t_1\varphi_{43} = y\theta$$

and

$$(4.6) \quad t_1\varphi_{12} - t_2\varphi_{34} = z\theta,$$

where y and z are locally defined smooth functions, while $|y|$ and $|z|$ are globally defined on S^2 .

Taking exterior derivatives on both sides of equations $\varphi_{31} = t_1\theta$ and $\varphi_{42} = t_2\theta$, one obtains

$$(4.7) \quad d(t_1\theta) = t_1(\varphi_{11} - \varphi_{33}) \wedge \theta$$

and

$$(4.8) \quad d(t_2\theta) = t_2(\varphi_{22} - \varphi_{44}) \wedge \theta.$$

Using (1.4), (4.7) and (4.8) and observing that $i\omega$, φ_{11} , φ_{22} , φ_{33} and φ_{44} are purely imaginary forms, one has

$$(4.9) \quad i\omega = \varphi_{11} - \varphi_{33} = \varphi_{22} - \varphi_{44}.$$

It then follows from taking exterior derivatives on both sides of (4.5) and (4.6) that

$$(4.10) \quad [dy + y(i\omega - \varphi_{11} + \varphi_{22})] \wedge \theta = 0$$

and

$$(4.11) \quad [dz + z(i\omega + \varphi_{11} - \varphi_{22})] \wedge \theta = 0.$$

By a lemma in [5], $|y|$ and $|z|$ are of analytic type. In particular, either they are identically zero or they only vanish at finitely many points. If yz were not

identically zero, then via a formula in [8], around the points where yz did not vanish, one would get from (4.10), (4.11) and (1.5)

$$(4.12) \quad \Delta \log |yz| = 2K = 4 > 0.$$

Using the maximal principle argument as in [15], one then could conclude that $|yz|$ must be constant, which would lead to a contradiction since then $\Delta \log |yz| = 0$. Hence yz is identically zero.

If $z \equiv 0$, then taking exterior derivatives on both sides of (4.9) and using (4.5), (4.6) and (1.5), one has $|y| \equiv t_1^2 - t_2^2$. By a suitable choice of a local unitary frame, one can take $y = t_1^2 - t_2^2$.

Thus by [12], f is unitarily equivalent to the curve $h: CP^1 \rightarrow G(2, 4)$ defined by

$$(4.13) \quad h \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} z^2 & z\bar{w}(\cos \psi - \sin \psi) \\ -zw(\cos \psi - \sin \psi) & |z|^2 + |w|^2 \sin 2\psi \\ -zw(\cos \psi + \sin \psi) & -|w|^2 \cos 2\psi \\ w^2 & -\bar{z}w(\cos \psi - \sin \psi) \end{bmatrix}$$

where ψ is defined by

$$(4.14) \quad \tan \psi = \frac{t_1 - t_2}{t_1 + t_2}, \quad 1 > t_1 > t_2 > 0.$$

The same argument applies to the case when $y \equiv 0$. In that case, f is unitarily equivalent to the curve $h: CP^1 \rightarrow G(2, 4)$ defined by

$$(4.15) \quad h \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} |z|^2 - |w|^2 \sin 2\Psi & zw(\cos \Psi + \sin \Psi) \\ -z\bar{w}(\cos \Psi + \sin \Psi) & z^2 \\ \bar{z}w(\cos \Psi + \sin \Psi) & w^2 \\ -|w|^2 \cos 2\Psi & zw(\cos \Psi - \sin \Psi) \end{bmatrix}$$

where Ψ is defined in (4.14). Thus, we cover the case when $t_2 \neq 0$, i.e., all pseudo-holomorphic curves of position zero in $G(2, 4)$ with curvature 2.

Now assume that $t_2 = 0$, then $t_1 = 1$ and $\text{rank } \partial f = 1$. Taking exterior derivatives on both sides of equations $\varphi_{41} = 0$, $\varphi_{32} = 0$ and $\varphi_{31} = \theta$, one gets

$$(4.16) \quad i\omega = \varphi_{11} - \varphi_{33},$$

$$(4.17) \quad \varphi_{43} = z\theta$$

and

$$(4.18) \quad \varphi_{12} = y\theta,$$

where again $|z|$ and $|y|$ are well defined and are functions of analytic type on S^2 .

Take exterior derivative of (4.16), then

$$(4.19) \quad |y|^2 + |z|^2 = 1.$$

If $y \neq 0$, by taking exterior derivative of (4.18), the same argument as above shows that

$$(4.20) \quad \Delta \log |y| = 4(1 - |y|^2) \geq 0$$

except for finitely many points on S^2 . Then again $|y|$ is constant, and therefore a suitable choice of the unitary frame e makes $y \equiv 1$. So by [12] f is unitarily equivalent to $h: CP^1 \rightarrow G(2, 4)$ defined by

$$(4.21) \quad h \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} |z|^2 - |w|^2 & \sqrt{2}zw \\ -\sqrt{2}z\bar{w} & z^2 \\ \sqrt{2}\bar{z}w & w^2 \\ 0 & 0 \end{bmatrix}.$$

If however $y = 0$ and $z = 1$, then f is unitarily equivalent to $h: CP^1 \rightarrow G(2, 4)$ defined by

$$(4.22) \quad h \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} z^2 & 0 \\ 0 & 1 \\ \sqrt{2}zw & 0 \\ w^2 & 0 \end{bmatrix}.$$

In (4.13) if $t_1 = t_2$, one gets a curve unitarily congruent to the one defined by (4.2), whereas if $t_2 = 0$ it gives a curve unitarily congruent to the one defined by (4.22). Likewise in (4.15) if $t_2 = 0$, one obtains a curve unitarily congruent to that defined by (4.21). Therefore all these curves can be uniformly written as in (4.13) and (4.15) by letting $0 \leq \psi \leq \pi/4$ in (4.13) and $0 \leq \psi \leq \pi/4$ in (4.15). Since each of them has invariants different from those of the others, they are not unitarily equivalent. In summary, one has

Theorem 3. *There are two families of holomorphic curves with constant curvature 2 from S^2 into $G(2, 4)$, which are not unitarily equivalent, and any holomorphic curve of constant curvature 2 from S^2 into $G(2, 4)$ must be unitarily equivalent to exactly one of them.*

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