

BAND-LIMITED FUNCTIONS: L^p -CONVERGENCE

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ABSTRACT. We consider the set $B_p(\Omega)$ (functions of $L^p(\mathbb{R})$ whose Fourier spectrum lies in $[-\Omega, +\Omega]$). We prove that the prolate spheroidal wave functions constitute a basis of this space if and only if $4/3 < p < 4$. The result is obtained as a consequence of the analogous problem for the spherical Bessel functions. The proof rely on a weighted inequality for the Hilbert transform.

1. INTRODUCTION

In this paper we shall study a set of functions, namely the prolate spheroidal wave functions, which have been considered by several authors because of their relations with several problems in communication theory. We refer to the papers [3, 4 and 10] where Slepian, Landau and Pollak analyse their properties and give a description of their important applications. Those functions, which from now on will be denoted by $b_n(x)$, $n = 0, 1, 2, \dots$, appear naturally related to the uncertainty principle. That is, when one considers in communication theory the impossibility of the simultaneous confinement of a given signal and of its amplitude spectrum. Given a time interval $[-T, +T]$ and a frequency interval $[-\Omega, +\Omega]$ we may ask for signals $f(t)$ of total energy $\|f\|_2 = 1$ and such that

$$\tau^2 = \int_{-T}^T |f(t)|^2 dt, \quad \omega^2 = \int_{-\Omega}^{\Omega} |\hat{f}(\xi)|^2 d\xi,$$

being as close to 1 as possible, where $\hat{f}(\xi) = \int e^{-2\pi i x \xi} f(x) dx$ denotes the Fourier transform.

$\tau = 1$ means that the signal is confined to the period $|t| \leq T$ or that f is time-limited.

$\omega = 1$ means that its power spectrum is confined to the frequency band $|\xi| \leq \Omega$ or that f is bandlimited.

A well-known fact about the Fourier transform is that τ and ω cannot both be equal to 1 at the same time. In [3], Landau and Pollak proved that the couple (τ, ω) corresponding to a signal f , $\|f\|_2 = 1$, describes the planar

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region defined by the inequality

$$\cos^{-1}(\tau) + \cos^{-1}(\omega) \geq \cos^{-1} \sqrt{\lambda_0}, \quad 0 \leq \tau \leq 1, \quad 0 \leq \omega \leq 1,$$

$$\lambda_0 = \sup \left\{ \int_{-T}^T |f(t)|^2 dt / f \in L^2(\mathbf{R}), \|f\|_2 = 1 \text{ supp}(\hat{f}) \subset [-\Omega, +\Omega] \right\}$$

(with the proviso that if τ or $\omega = 1$ ($= 0$), then the other is > 0 (< 1)).

Another interesting result where the functions $b_n(x)$ play an important role is the so-called dimension theorem. Roughly speaking it says that the set of signals band-limited to the interval $[-\Omega, +\Omega]$ and "concentrated" in $[-T, +T]$, has dimension $4\Omega T$. Landau and Pollak [4] and Slepian [8] give precise statements of this theorem.

In the proofs of those results the operator

$$(1) \quad P f(x) = \int_{-T}^T \frac{\sin 2\pi\Omega(x-t)}{\pi(x-t)} f(t) dt$$

appears quite naturally. It is a positive operator and by the standard theory it has a countable family of eigenvalues: $1 > \lambda_0 > \lambda_1 > \dots$ decreasing to 0. The associated family of normalized eigenfunctions are the b_n 's. The functions b_n can be chosen to be real valued and they constitute a complete orthonormal system in the space of band-limited functions:

$$B_2(\Omega) = \{f \in L^2(\mathbf{R}) / \text{supp}(\hat{f}) \subset [-\Omega, +\Omega]\}.$$

Furthermore the family $\{\lambda^{-1/2} b_n(x) \chi_{[-T, +T]}(x)\}$ is also a complete orthonormal system in the space of time-limited functions, i.e. $L^2[-T, +T]$. Further information of these statements are included in the survey article of Slepian, reference [9].

In this work we extend the results mentioned above to the case $L^p(\mathbf{R})$, $p \neq 2$. To do this we consider the spaces

$$B_p(\Omega) = \{f \in L^p(\mathbf{R}) / \text{supp}(\hat{f}) \subset [-\Omega, +\Omega]\}, \quad 1 \leq p \leq \infty.$$

Given $f \in B_p(\Omega)$, one can consider its Fourier coefficient

$$\langle f, b_n \rangle = \int f(x) b_n(x) dx,$$

and its partial sums $\sigma_N f(x) = \sum_{0 \leq n \leq N} \langle f, b_n \rangle b_n(x)$. We can now state our main result:

Theorem 1. $\lim_{N \rightarrow \infty} \|\sigma_N f - f\|_{L^p(\mathbf{R})} = 0$ for every $f \in B_p(\Omega)$ if and only if $4/3 < p < 4$.

This theorem is closely related to the analogous statement for spherical Bessel functions $j_n(x)$: given $f \in B_p(1/2\pi)$ we may consider

$$S_N f(x) = \sum_{0 \leq n \leq N} (2n+1) \pi^{-1} \langle f, j_n \rangle j_n(x),$$

then we have

Theorem 2. $S_N f$ converges to f in $B_p(1/2\pi)$ if and only if $4/3 < p < 4$.

§2 of this paper is devoted to the presentation of some preliminary results and statements. We shall define the functions $b_n(x)$, $j_n(x)$ and describe their required properties as well as those of the spaces $B_p(\Omega)$. §3 contains the proofs of Theorems 1 and 2.

The main point of the proof is to show the uniform boundedness in $B_p(\Omega)$ ($4/3 < p < 4$) of the partial sum operators:

$$\sigma_N f(x) = \int K_N(x, y) f(y) dy \quad \text{where } K_N(x, y) = \sum_{0 \leq n \leq N} b_n(x) b_n(y).$$

We use several clever identities verified by Bessel functions to decompose K_N in several pieces. A typical one is given by $x^{1/2} y^{1/2} (x - y)^{-1} J'_{N+1/2}(x) J_{N+1/2}(y)$ where J_v denotes the Bessel function of order v . Therefore our proof rely on a two-weight inequality for the Hilbert transform. However, to get the needed uniform estimates no one of the known results for the Hilbert transform seems to apply directly, and we are forced to produce a rather careful analysis of the size of both $J_{N+1/2}(x)$ and $J'_{N+1/2}(x)$ in the delicate region, i.e. when $x \sim N$.

2. PRELIMINARY RESULTS AND DEFINITIONS

In the introduction we have considered the family $\{b_n\}$ as eigenfunctions of the operator P defined by (1). If we normalize P by considering the integration in the interval $[-1, +1]$ and taking $c = 2\pi\Omega T$, the eigenfunctions of the corresponding operator P acting on $L^2[-1, +1]$ are exactly those of the following Sturm-Liouville problem:

$$(2) \quad (1 - x^2)y'' - 2xy' + (\chi - c^2x^2)y = 0$$

imposing as boundary conditions that the solutions must be finite or continuous at the extreme points. This equation appears in the study of the three-dimensional wave equation by the method of separation of variables, using a prolate spheroidal coordinate system. This point of view is developed in [2] and [12].

Problem (2) only has solutions for a discrete set of real values $\chi_n(c)$, $n = 0, 1, 2, \dots$. If we denote by $S_n(x)$ the corresponding eigenfunctions, they satisfy the following relations:

$$\begin{cases} 2c\pi^{-1}[R_n(1)]^2 S_n(x) = \int_{-1}^1 \frac{\sin c(x-t)}{\pi(x-t)} S_n(t) dt, \\ 2i^n R_n(1) S_n(x) = \int_{-1}^1 e^{icxt} S_n(t) dt, \end{cases}$$

where $R_n(t)$ is a solution of (2) with $\chi = \chi_n(c)$, normalized in such a way that its behavior at infinity is given by $(1/ct) \cos[ct - (n + 1)\pi/2]$. Obviously the relations given by (3) allow us to extend $S_n(t)$ as an entire function in the complex plane.

Let us define

$$\lambda_n(c) = \lambda_n = 2c\pi^{-1}[R_n(1)]^2, \quad b_n(x, c) = b_n(x) = \sqrt{\lambda_n T^{-1}} u_n^{-1} S_n(xT^{-1}),$$

where $u_n^2 = \int_{-1}^1 [S_n(t)]^2 dt$.

The functions b_n satisfy the following orthogonality relations:

$$\int_{-T}^T b_n(t) b_m(t) dt = \lambda_n \delta_{nm}, \quad \int_{-\infty}^{\infty} b_n(t) b_m(t) dt = \delta_{nm},$$

and they constitute a complete orthogonal system in $L^2[-T, +T]$, which follows easily from the analogous statement for the family of functions $S_n(x)$ on $L^2[-1, +1]$.

We have

$$\hat{b}_n(x) = i^{-n} \sqrt{T\Omega^{-1} \lambda_n^{-1}} b_n(T\Omega^{-1} x) \chi_{[-\Omega, +\Omega]}(x)$$

and the Plancherel's theorem (see [11]) shows that $\{b_n\}$ is a complete orthonormal system in $B_2(\Omega)$.

The spherical Bessel functions $j_n(x) = \sqrt{(\pi/2)x^{-1}} J_{n+1/2}(x)$, where $J_{n+1/2}(x)$ denotes the Bessel function of order $n+1/2$, belong to the space $L^p(\mathbf{R})$, $p > 1$, and their behavior at infinity is given by $x^{-1} \sin[x - (n\pi)/2]$ (see [14]). They satisfy the orthogonality relation:

$$\int_{-\infty}^{\infty} j_n(t) j_m(t) dt = \pi(2n+1)^{-1} \delta_{nm}$$

and their Fourier transforms are given by

$$(4) \quad \hat{j}_n(x) = (-i)^n \pi p_n(2\pi x) \chi_{[+1, -1]}(2\pi x)$$

where p_n is the Legendre polynomial of order n . Formula (4) together with Plancherel's theorem shows that the family of functions $\sqrt{(2n+1)\pi^{-1}} j_n(x)$ also constitutes a complete orthonormal system in $B_2(\Omega)$.

Problem (2) can be considered as a perturbation of the Sturm-Liouville problem for Legendre's equation, and their solutions $S_n(x)$ can be expanded as a series in the normalized Legendre's polynomials $q_n(x) = \sqrt{(2n+1)/2} p_n(x)$. Therefore we can express the functions $b_n(x)$ in terms of the spherical Bessel functions $j_n(x)$. More concretely, if $S_n(x) = \sum_{[-(n-1)/2] \leq k \leq \infty} d(k, n) q_{n+2k}(x)$ then relations (3) and (4) imply that

$$b_n(x) = \sqrt{2\Omega} u_n^{-1} \sum_{[-(n-1)/2] \leq k \leq \infty} (-1)^k d(k, n) \sqrt{2n+4k+1} j_{n+2k}(2\pi\Omega x).$$

In this series expansion the "leading" coefficient is the one corresponding to $k = 0$. In fact, when n is big enough, we have the asymptotic $b_n(x) \sim \sqrt{(2n+1)\pi^{-1}} j_n(x)$ ($\Omega = 1/2\pi$).

If we substitute the expression of $S_n(x)$ as a series of Legendre's polynomials in problem (2), we obtain the relations satisfied by the coefficients $d(k, n)$:

$$\begin{aligned}
 & d(k, n)f(k, n, c) \\
 &= A(k, n)d(k-1, n) + B(k, n)d(k+1, n), \quad k > [-(n-1)/2], \\
 (5) \quad & d([-(n-1)/2], n)(3\chi_n(c) - c^2)c^{-2} \\
 &= 2/\sqrt{5}d([-(n-1)/2] + 1, n) \quad \text{if } n \text{ is even,} \\
 & d([-(n-1)/2], n)(\chi_n(c) - 2 - (3/5)c^2)c^{-2} \\
 &= (2/5)\sqrt{3/7}d([-(n-1)/2] + 1, n) \quad \text{if } n \text{ is odd,}
 \end{aligned}$$

where

$$\begin{aligned}
 A(k, n) &= \sqrt{\frac{2n+4k+3}{2n+4k-1}}\gamma_{n+2k}^1, & B(k, n) &= \sqrt{\frac{2n+4k+1}{2n+4k+5}}\gamma_{n+2k}^{-1}, \\
 \gamma_n^{-1} &= \frac{n(n-1)}{(2n+1)(2n-1)}, & \gamma_n^0 &= \frac{2n(n+1)-1}{(2n+3)(2n-1)}, & \gamma_n^1 &= \frac{(n+1)(n+2)}{(2n+1)(2n+3)}, \\
 f(k, n, c) &= \frac{\chi_n(c) - (n+2k)(n+2k+1) - c^2\gamma_{n+2k}^0}{c^2}.
 \end{aligned}$$

Since $A(k, n)$ and $B(k, n)$ are bounded by $1/2$, uniformly in k and n , in order that (5) be verified one of the following two inequalities must hold:

$$(6) \quad \begin{cases} |d(k, n)||f(k, n, c)| \leq |d(k-1, n)|, \\ |d(k, n)||f(k, n, c)| \leq |d(k+1, n)|, \end{cases} \quad k \geq [-(n-1)/2] + 1.$$

Using the Rayleigh quotient characterization of $\chi_n(c)$ we obtain

$$(7) \quad n(n+1) \leq \chi_n(c) \leq n(n+1) + c^2.$$

This last relation, together with the fact that at least one of the inequalities contained in (6) must hold ((5) in the case $k = [-(n-1)/2]$), yields the existence of an integer $N(c)$ such that if $n \geq N(c)$ we have

$$(8) \quad \left\{ \begin{array}{l} |f(k, n, c)| \geq \begin{cases} \frac{3kn + 4k^2 + 2k}{c^2} \geq 1, & k \geq 1, \\ \frac{-3kn - 4k^2 - 2k}{c^2} \geq 1, & k \in [[-(n-1)/2], -1], \end{cases} \\ |d(k, n)| \leq \begin{cases} \frac{3kn + 4k^2 + 2k}{c^2} |d(k-1, n)|, & k \geq 1, \\ \frac{-3kn - 4k^2 - 2k}{c^2} |d(k+1, n)|, & k \in [[-(n-1)/2], -1]. \end{cases} \end{array} \right.$$

(This is true because the series defining $S_N(x)$ must be convergent at the extreme points.)

Since $u_n^2 = \int_{-1}^1 [S_n(x)]^2 dx = \sum_{[-(n-1)/2] \leq k \leq \infty} [d(k, n)]^2$, (8) produces, for $n \geq N(c)$, the following estimates:

$$(9) \quad \frac{n^2}{n^2 + A(c)} \leq \frac{[d(0, n)]^2}{u_n^2} \leq 1$$

where $A(c)$ is a constant depending only upon the parameter c .

Relations (8) and (9) yield the following: if n is big enough and if $\Omega = 1/2\pi$, we have $b_n(x) \sim \sqrt{2n+1}\pi^{-1}j_n(x)$, as it was stated in the introduction.

In the statements of Theorems 1 and 2 we have introduced the family of spaces $B_p(\Omega)$ which are a natural generalization of $B_2(\Omega)$

$$B_p(\Omega) = \{f \in L^p(\mathbf{R}) / \text{supp}(\hat{f}) \subset [-\Omega, +\Omega]\}.$$

They are Banach's spaces with the induced topology of $L^p(\mathbf{R})$. If $p < q$ then $B_p(\Omega) \subsetneq B_q(\Omega)$ and the inclusion is continuous and dense. We have that $b_n(x) \in B_p(\Omega)$ for every $p > 1$ and if $\Omega = 1/2\pi$ then $j_n(x) \in B_p(\Omega)$, $p > 1$, for every n .

The dual space $(B_p(\Omega))^*$ is equal to $B_q(\Omega)$, $1/p + 1/q = 1$, $1 < p < \infty$, in the standard sense, that is, given $T \in (B_p(\Omega))^*$ there exists a unique $g \in B_q(\Omega)$ such that $T(f) = \langle f, g \rangle$ for every $f \in B_p(\Omega)$. Also we have the norm equivalence: there exists a positive constant $A > 0$ so that $A\|g\|_q \leq \|T\|_{(B_p(\Omega))^*} \leq \|g\|_q$ (the value of the constant A depends on the norm of the Hilbert transform on $L^q(\mathbf{R})$).

In the proof of Theorems 1 and 2 a rather complete analysis of the size of both $J_{n+1/2}(x)$ and $J'_{n+1/2}(x)$ in the "critical region" (i.e. when $x \sim n$) is needed. Both functions have representations as oscillatory integral and the stationary phase method has been used to achieve our estimates.

To simplify notation let us take $\nu = n + 1/2$. The estimates are the following:

$$J_\nu(\nu + x) = \frac{\cos(\theta(x))}{\sqrt{2\pi}(2\nu x + x^2)^{1/4}} + h(x, \nu), \quad x \geq \nu^{1/3},$$

where

$$\theta(x) = \nu \arccos \frac{\nu}{\nu + x} - (2\nu x + x^2)^{1/2} + \frac{\pi}{4},$$

$$|h(\nu, x)| \leq \begin{cases} A \left[\frac{\nu^2}{(2\nu x + x^2)^{7/4}} + \frac{1}{\nu + x} \right], & \nu^{1/3} \leq x \leq \nu, \\ \frac{A}{\nu + x}, & x > \nu, \end{cases}$$

where A is a constant independent of ν .

To get estimates from above we have to divide the intervals $(\nu/2, \nu - \nu^{1/3}]$ and $(\nu + \nu^{1/3}, 2\nu]$ in the following manner:

$$(\nu/2, \nu - \nu^{1/3}] \subset \bigcup_{0 \leq j \leq M-1} (\nu - 2^{j+1}\nu^{1/3}, \nu - 2^j\nu^{1/3}] \text{ with } \nu^{2/3}/2 < 2^M \leq \nu^{2/3},$$

$$(\nu + \nu^{1/3}, 2\nu] \subset \bigcup_{0 \leq j \leq K-1} (\nu + 2^j\nu^{1/3}, \nu + 2^{j+1}\nu^{1/3}] \text{ with } \nu^{2/3} < 2^K \leq 2\nu^{2/3},$$

| x | | $ J_\nu(x) \leq$ | $ J'_\nu(x) \leq$ |
|--|--------------------------------|-----------------------|-----------------------|
| $ x \leq \nu/2$ | | $A\nu^{-1}$ | $A\nu^{-1}$ |
| $\nu - 2^{j+1}\nu^{1/3} < x \leq \nu - 2^j\nu^{1/3}$ | $(j \in \{0, 1, \dots, M-1\})$ | $A2^{-j}\nu^{-1/3}$ | $A2^{-j/2}\nu^{-2/3}$ |
| $\nu - \nu^{1/3} < x \leq \nu + \nu^{1/3}$ | | $A\nu^{-1/3}$ | $A\nu^{-2/3}$ |
| $\nu + 2^j\nu^{1/3} < x \leq \nu + 2^{j+1}\nu^{1/3}$ | $(j \in \{0, 1, \dots, K-1\})$ | $A2^{-j/4}\nu^{-1/3}$ | $A2^{j/4}\nu^{-2/3}$ |
| $ x > 2\nu$ | | $Ax^{-1/2}$ | $Ax^{-1/2}$ |

where A is a constant independent of ν .

The straightforward details are left to the reader.

3. PROOFS

3.1. Proof of Theorem 2.

[A] We shall analyse first the reasons why p has to be contained in the interval $4/3 < p < 4$ in order to have the convergence $\lim_{N \rightarrow \infty} \|S_N f - f\|_p = 0$ for every $f \in B_p(\Omega)$, where $\Omega = 1/2\pi$.

If $\sum_{0 \leq n \leq \infty} (2n+1)\pi^{-1} \langle f, j_n \rangle j_n(x)$ converges for each $f \in B_p(1/2\pi)$, then an application of the Banach-Steinhaus theorem together with the fact that the norm of the operator $T_n(f) = \langle f, j_n \rangle$ on $B_p(1/2\pi)$ is given by $\|j_n\|_q$, $1/p + 1/q = 1$, yields $(2n+1)\|j_n\|_p \|j_n\|_q = O(1)$. Using the estimates for the Bessel's functions discussed in §2, we have

$$(10) \quad \|j_n\|_p \sim \begin{cases} n^{-1+1/p} & \text{if } 1 < p < 4, \\ n^{-3/4} \log n & \text{if } p = 4, \\ n^{-5/6+1/3p} & \text{if } p > 4, \end{cases}$$

(where the symbol $a_n \sim b_n$ has the standard meaning, that is, there exists a universal constant $0 < A < \infty$ such that $A^{-1}a_n \leq b_n \leq Aa_n$).

It is clear that the condition $(2n+1)\|j_n\|_p \|j_n\|_q = O(1)$ is satisfied if and only if $4/3 < p < 4$, which proves our assertion.

[B] To prove the convergence in $B_p(1/2\pi)$, $4/3 < p < 4$, we shall follow the standard strategy, namely: we shall prove that the finite linear combinations of the j_n 's are dense in $B_p(1/2\pi)$ and that the partial sum operators S_N are uniformly bounded there. The first part is the easiest and the heart of the proof lies in the second. Before the discussion of the details we shall present a sketch of the proof.

Sketch of the proof. In order to analyse the kernel corresponding to S_N

$$Q_N(x, y) = \sum_{0 \leq n \leq N} (2n+1)j_n(x)j_n(y)$$

we shall follow an idea of H. Pollard (see [6] and also [1] where it has been used to treat this kind of problems) which allows us to express $Q_N(x, y)$ in terms of the Bessel's functions $J_{N+1/2}(x)$ and $J'_{N+1/2}(x)$. After a certain amount of manipulation, the uniform boundedness of the partial sum operators S_N is reduced to the following inequality:

$$(*) \quad \left(\int_B \left| x^{1/2} J'_v(x) \int_B \frac{y^{1/2} J_v(y) f(y)}{x-y} dy \right|^p dx \right)^{1/p} \leq A \|f\|_p,$$

$4/3 < p < 4$, $\nu = N + 1/2$, $B = \{x/\nu + \nu^{1/3} < x < 2\nu\}$ and A is a universal constant (i.e. independent of ν).

Dividing the interval B in dyadic pieces $B_j = (\nu + 2^j \nu^{1/3}, \nu + 2^{j+1} \nu^{1/3}]$, the "size" of $x^{1/2} J'_v(x)$ in those intervals is balanced by the decrease of $x^{1/2} J'_v(x)$, therefore a "brute force" argument using Minkowski's inequality would yield a logarithmic growth in inequality (*). To eliminate the $\log N$ factor one needs a more suitable analysis, namely the following: the integration with respect to the variable x is decomposed in a sum of integrals, one for each interval B_j . For a fixed j , we divide the y -integration in three parts, corresponding to the sets $\bigcup_{k < j-1} B_k$, $\bigcup_{k > j+1} B_k$ and $B_{j-1} \cup B_j \cup B_{j+1}$. The contribution of the last two terms can be controlled by the standard boundedness properties of the Hilbert transform and the estimates of the functions $J_v(x)$ and $J'_v(x)$. In $\bigcup_{k < j-1} B_k$ we have that

$$\begin{aligned} & |x^{1/2} J'_v(x) H(y^{1/2} J_v(y) f(y) \chi_{\bigcup_{k < j-1} B_k}(y))(x)| \\ & \leq A |x^{1/2} J'_v(x) M(y^{1/2} J_v(y) f(y) \chi_{[\nu + \nu^{1/3}, 2\nu]}(y))(x)| \end{aligned}$$

where H denotes the Hilbert transform, M is the Hardy-Littlewood maximal operator and A is a constant independent of ν . Then we are in conditions to use a two-weight inequality obtained by E. Sawyer [7] to finish the proof.

The details. First, we use the relationship $(2n + 1)x^{-1} j_n(x) = j_{n-1}(x) + j_{n+1}(x)$ (see [14]) to get a simplified expression for the kernels, that is:

$$Q_N(x, y) = \frac{xy}{y-x} \{j_{N+1}(x)j_N(y) - j_N(x)j_{N+1}(y)\} + \frac{\sin(y-x)}{y-x}.$$

Next we use the following equality:

$$Q_N(x, y) + Q_{N+1}(x, y) = 2Q_N(x, y) + (2N + 3)j_{N+1}(x)j_{N+1}(y)$$

to get

$$\begin{aligned} Q_N(x, y) = & -\frac{xy}{2(x-y)} j_{N+1}(y) [j_{N+2}(x) - j_N(x)] \\ & -\frac{xy}{2(x-y)} j_{N+1}(x) [j_N(y) - j_{N+2}(y)] \\ & + \frac{2 \sin(y-x)}{y-x} - \frac{(2N+3)}{2} j_{N+1}(x) j_{N+1}(y). \end{aligned}$$

The operator associated to the third kernel is an ordinary singular integral and it is bounded on $L^p(\mathbf{R})$, $1 < p < \infty$. The fourth is uniformly bounded on $L^p(\mathbf{R})$ if $4/3 < p < 4$ as we have observed in part 2. Since the first two are "duals", taking into account the identity (see [14]):

$$J'_{N+3/2}(x) = \frac{1}{2}(J_{N+1/2}(x) - J_{N+5/2}(x))$$

we may reduce our problem to the proof of the following inequality:

$$\left(\int_{-\infty}^{\infty} \left| x^{1/2} J'_{N+1/2}(x) \int_{-\infty}^{\infty} \frac{y^{1/2} J_{N+1/2}(y) f(y)}{x-y} dy \right|^p dx \right)^{1/p} \leq A \|f\|_p$$

for every $f \in L^p(\mathbf{R})$, $4/3 < p < 4$, where A , independent of N , is a finite constant.

Let

$$\begin{aligned} R_1 &= \{t/|t| \leq (N + 1/2) + (N + 1/2)^{1/3}\}, \\ R_2 &= \{t/(N + 1/2) + (N + 1/2)^{1/3} \leq |t| \leq 2(N + 1/2)\}, \\ R_3 &= \{t/|t| \geq 2(N + 1/2)\}. \end{aligned}$$

Using these sets we obtain the following nine integrals:

$$\left(\int_{R_j} \left| x^{1/2} J'_{N+1/2}(x) \int_{R_k} \frac{y^{1/2} J_{N+1/2}(y) f(y)}{x-y} dy \right|^p dx \right)^{1/p}, \quad j, k = 1, 2, 3.$$

Which, except for the case $j = k = 2$, can be majorized by $A\|f\|_p$ (A independent of N , $4/3 < p < 4$) by the use of the boundedness properties of the Hilbert transform together with the estimates obtained in part 2 for $J_{N+1/2}(x)$ and $J'_{N+1/2}(x)$.

Therefore, if we denote $v_1(x) = |x^{1/2} J'_{N+1/2}(x)|$ and $v_2(x) = |x^{1/2} J_{N+1/2}(x)|$ we have to prove the inequality

$$(11) \quad \|v_1 \chi_{R_2} H(v_2 \chi_{R_2} f)\|_p \leq A \|f\|_p$$

or, what it is the same,

$$\|v_1 \chi_{R_2} H(f)\|_p \leq A \|v_2^{-1} \chi_{R_2} f\|_p$$

with A independent of N , $f \in L^p(\mathbf{R})$ and $4/3 < p < 4$.

This estimate can be considered as a two-weight inequality for the Hilbert transform, namely with weights

$$w(x) = v_1^p(x) \chi_{R_2}(x) \quad \text{and} \quad u(x) = v_2^{-p}(x) \chi_{R_2}(x).$$

As far as we know there is not general result which can be directly applied here. However, if in (11) we substitute H by M , then Sawyer ([7]) obtained a necessary and sufficient condition upon $w(x)$ and $u(x)$ to have the inequality:

$$(12) \quad \|v_1 \chi_{R_2} M(v_2 \chi_{R_2} f)\|_p \leq A \|f\|_p$$

or, what it is the same,

$$\left(\int [Mf(x)]^p w(x) dx \right)^{1/p} \leq A \left(\int |f(x)|^p u(x) dx \right)^{1/p}$$

namely, for every interval I one must have

$$(13) \quad \int_I [M(\chi_I u^{1-q})(x)]^p w(x) dx \leq C \int_I [u(x)]^{1-q} dx < \infty$$

where C is independent of the interval and $1/p + 1/q = 1$. Furthermore C and A are comparable.

To finish the proof we shall assume first that (12) holds for $4/3 < p < 4$ and we prove (11). Then, of course, we will prove (12). Let $\nu = N + 1/2$ and, in the following, we shall assume that A is an absolute constant, not necessarily the same at each occurrence.

Let $B_j = \{x \in \mathbf{R} / \nu + 2^j \nu^{1/3} < |x| \leq \nu + 2^{j+1} \nu^{1/3}\} \cap \mathbf{R}_2$, $j = 0, 1, \dots, K - 1$, where K satisfies $\nu^{2/3} < 2^K \leq 2\nu^{2/3}$. We have

$$\begin{aligned} \|v_1 \chi_{\mathbf{R}_2} H(v_2 \chi_{\mathbf{R}_2} f)\|_p^p &\leq \sum_{0 \leq j \leq K-1} \int v_1^p \chi_{B_j} |H(v_2 f \chi_{\cup_{0 \leq i \leq K-1} B_i})(x)|^p dx, \\ &\leq A \sum_{0 \leq j \leq K-1} \int v_1^p \chi_{B_j} |H(v_2 f \chi_{\cup_{0 \leq i \leq j-2} B_i})(x)|^p dx \\ &\quad + A \sum_{0 \leq j \leq K-1} \int v_1^p \chi_{B_j} |H(v_2 f \chi_{\cup_{j-1 \leq i \leq j+1} B_i})(x)|^p dx \\ &\quad + A \sum_{0 \leq j \leq K-1} \int v_1^p \chi_{B_j} |H(v_2 f \chi_{\cup_{j+2 \leq i \leq K-1} B_i})(x)|^p dx. \end{aligned}$$

To estimate the integrals appearing in the first sum, let us observe that if $x \in B_j$ and $y \in B_i$, $i \leq j - 2$, then $|x - y| > 2^{j-1} \nu^{1/3}$, which shows that, in this case, the Hilbert transform is majorized pointwise by the maximal function $M(v_2 \chi_{\mathbf{R}_2} f)$. Therefore, applying (12) we get

$$\sum_{0 \leq j \leq K-1} \int v_1^p \chi_{B_j} |H(v_2 f \chi_{\cup_{0 \leq i \leq j-2} B_i})(x)|^p dx \leq A \|f\|_p^p, \quad 4/3 < p < 4.$$

In the second sum we apply the estimates obtained in §2 for v_1 and v_2 together with the boundedness of the Hilbert transform, we get

$$\sum_{0 \leq j \leq K-1} \int v_1^p \chi_{B_j} |H(v_2 f \chi_{\cup_{j-1 \leq i \leq j+1} B_i})(x)|^p dx \leq A \|f\|_p^p, \quad 1 < p < \infty.$$

For the third summation we use the inequality $(\sum |a_i|)^p \leq A \sum 2^{ip/2} |a_i|^p$ (where $A = A(p)$ is a finite constant) to get

$$\begin{aligned} &\sum_{0 \leq j \leq K-1} \left| \int v_1^p \chi_{B_j} |H(v_2 f \chi_{\cup_{j+2 \leq i \leq K-1} B_i})(x)|^p dx \right. \\ &\quad \left. \leq A \sum_{0 \leq j \leq K-1} \sum_{0 \leq i \leq K-j+1} 2^{pi/2} \int v_1^p \chi_{B_j} |H(v_2 f \chi_{B_{i+j+2}})(x)|^p dx. \right. \end{aligned}$$

Applying Hölder’s inequality and taking into account the size of v_2 in the interval B_{i+j+2} , we obtain

$$v_1^p \chi_{B_j} |H(v_2 f \chi_{B_{i+j+2}})(x)|^p \leq 2^{i((p/4)-1)} 2^{-j} \nu^{-1/3} \|f \chi_{B_{i+j+2}}\|_p^p$$

and

$$\sum_{0 \leq j \leq K-1} \int v_1^p \chi_{B_j} |H(v_2 f \chi_{\cup_{j+2 \leq i \leq K-1}})(x)|^p dx \leq A \|f\|_p^p, \quad 4/3 < p < 4.$$

Therefore we only need to show (12) to finish the proof. That is, we have to show that Sawyer’s condition (13) is satisfied:

$$\int |M(\chi_1 v_2^q)(x)|^p v_1^p dx \leq A \int_I v_2^q dx$$

$I \subset R_2$ is an interval, $1/p + 1/q = 1$ and $4/3 < p < 4$.

Since v_1 and v_2 are even functions, there is no loss of generality if we assume that $I \subset [\nu + \nu^{1/3}, 2\nu]$. Let $B_j = \{t/\nu + 2_j \nu^{1/3} < t \leq \nu + 2^{j+1} \nu^{1/3}\} \cap R_2$, $j = 0, 1, \dots, K-1$. If $x \in B_j$, using the estimates that we have for v_2 on the sets B_i , we get

$$\begin{aligned} M(\chi_I v_2^q)(x) &\leq A \sum_{0 \leq i \leq K-1} (\nu^{1/6} 2^{-i/4})^q M \chi_{B_i}(x) \\ &\leq A \sum_{0 \leq i \leq j-1} (\nu^{1/6} 2^{-i/4})^q M \chi_{B_i}(x) + A(\nu^{1/6} 2^{-j/4})^q M \chi_{B_j}(x) \\ &\quad + A \sum_{j+1 \leq i \leq K-1} (\nu^{1/6} 2^{-i/4})^q M \chi_{B_i}(x) \\ &\leq \sum_{0 \leq i \leq j-1} (\nu^{1/6} 2^{-i/4})^q 2^{i-j} + A \nu^{q/6} 2^{-qj/4} + A \sum_{j+1 \leq i \leq K-1} (\nu^{1/6} 2^{-i/4})^q \\ &\leq A \nu^{q/6} 2^{-qj/4}. \end{aligned}$$

Now we use the estimates for v_1 and B_j and, without loss of generality, we shall assume that

$$(\nu + 2^{n+1} \nu^{1/3}, \nu + 2^{m-1} \nu^{1/3}) \subset I \subset (\nu + 2^n \nu^{1/3}, \nu + 2^m \nu^{1/3}),$$

$n, m = 0, 1, \dots, K-1$, $m - n > 2$. We have

$$\begin{aligned} \int_I |M(\chi_I v_2^q)(x)|^p v_1^p dx &\leq A \sum_{n \leq j \leq m-1} (2^{j/4} \nu^{-1/6})^p \int_{B_j} |M(\chi_I v_2^q)|^p dx \\ &\leq A \nu^{q/6} \nu^{1/3} \sum_{n \leq j \leq m-1} 2^{j-(qj/4)} = A \nu^{q/6+1/3} 2^{m(1-q/4)}. \end{aligned}$$

Using the asymptotic behavior of $J_\nu(x)$ in the interval $(\nu + 2^{n+1} \nu^{1/3}, \nu + 2^{m-1} \nu^{1/3})$ it is easy to see that $\int_I v_2^q dx \sim \nu^{q/6+1/3} 2^{m(1-q/4)}$, which proves (12) and, therefore, Theorem 2.

3.2. *Proof of Theorem 1.* From the point of view of the proof of Theorem 1 it is clear that the particular value given to Ω is irrelevant. Therefore, to simplify notation, we shall take $2\pi\Omega = 1$.

We shall prove that the family of operators $\sigma_N - S_N$ is uniformly bounded on $L^p(\mathbf{R})$, showing, in particular, that Theorem 2 implies Theorem 1.

The kernels corresponding to $\sigma_N - S_N$ is given by

$$\begin{aligned} T_N(x, y) &= - \sum_{0 \leq n \leq N} d(0, n)d(1, n)u_n^{-2}\sqrt{2n+1}\sqrt{2n+5} \\ &\quad \times (j_n(x)j_{n+2}(y) + j_{n+2}(x)j_n(y)) \\ &\quad - \sum_{0 \leq n \leq N} d(0, n)d(-1, n)u_n^{-2}\sqrt{2n+1}\sqrt{2n-3} \\ &\quad \times (j_n(x)j_{n-2}(y) + j_{n-2}(x)j_n(y)) + R_N(x, y) \\ &= -T_N^1(x, y) - T_N^2(x, y) + R_N(x, y). \end{aligned}$$

To get this expression we have used the series expansion of the functions $b_n(x)$ in terms of the spherical Bessel functions $j_n(x)$.

Since the decay of the coefficients $d(k, n)$ and the behavior of the quotient $[d(k, n)]^2 u_n^{-2}$ is known for n greater than a certain $N(c)$, we may assume, modulus a finite part of the sum which therefore is uniformly bounded, that the summation index begins with $N(c)$.

A direct application of the estimates (8) and (9) shows that $R_N(x, y)$ yields a uniformly bounded family of operators on $L^p(\mathbf{R})$, $1 < p < \infty$.

The study of the operators associated to the kernels $T_N^1(x, y)$ and $T_N^2(x, y)$ is completely similar and, in the following, we shall only consider $T_N^1(x, y)$. Let us assume the following lemma whose proof will be given at the end.

Lemma. *If*

$$P_N(x, y) = \sum_{N(c) \leq n \leq N} \sqrt{2n+1}\sqrt{2n+5}(j_n(x)j_{n+2}(y) + j_{n+2}(x)j_n(y)),$$

then we have

$$\left(\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} P_N(x, y)f(y) dy \right|^p dx \right)^{1/p} \leq AN^{2/3} \|f\|_p$$

for every $f \in L^p(\mathbf{R})$, $1 < p < \infty$, and A is independent of N .

Using (5) with $k = 0$, we can decompose $T_N^1(x, y)$ in two kernels:

$$\begin{aligned} T_N^{1,1}(x, y) &= \sum_{N(c) \leq n \leq N} \frac{A(1, n)}{f(1, n, c)} \frac{[d(0, n)]^2}{u_n^2} \sqrt{2n+1}\sqrt{2n+5} \\ &\quad \times (j_n(x)j_{n+2}(y) + j_{n+2}(x)j_n(y)), \\ T_N^{1,2}(x, y) &= \sum_{N(c) \leq n \leq N} \frac{B(1, n)}{f(1, n, c)} \frac{d(2, n)d(0, n)}{u_n^2} \sqrt{2n+1}\sqrt{2n+5} \\ &\quad \times (j_n(x)j_{n+2}(y) + j_{n+2}(x)j_n(y)). \end{aligned}$$

Since $(\|j_n\|_p \|_{n+2}\|_q + \|j_{n+2}\|_p \|j_n\|_q) = O(n^{-5/6})$ ($1/p + 1/q = 1$) for every $p, 1 < p < \infty$, the estimates (8) give us the uniform boundedness on $L^p(\mathbf{R})$, $1 < p < \infty$, of the operators associated to the kernels $T_N^{1,2}(x, y)$.

Summation by parts allows us to decompose again $T_N^{1,1}(x, y)$ in the following manner:

$$\sum_{N(c) \leq n \leq N-1} \left(\frac{A(1, n)}{f(1, n, c)} \frac{[d(0, n)]^2}{u_n^2} - \frac{A(1, n+1)}{f(1, n+1, c)} \frac{[d(0, n+1)]^2}{u_{n+1}^2} \right) P_n(x, y) + \frac{A(1, N)}{f(1, N, c)} \frac{[d(0, N)]^2}{u_N^2} P_N(x, y).$$

The lemma together with estimates (8) and (9) yields that the second part of the kernel produces a uniformly bounded family of operators on $L^p(\mathbf{R})$, $1 < p < \infty$.

The first part of $T_N^{1,1}(x, y)$ can be further decomposed:

$$\sum_{N(c) \leq n \leq N-1} \frac{A(1, n)}{f(1, n, c)} \left(\frac{[d(0, n)]^2}{u_n^2} - \frac{[d(0, n+1)]^2}{u_{n+1}^2} \right) P_n(x, y) + \sum_{N(c) \leq n \leq N-1} \frac{[d(0, n+1)]^2}{u_{n+1}^2} \left(\frac{A(1, n)}{f(1, n, c)} - \frac{A(1, n+1)}{f(1, n+1, c)} \right) P_n(x, y).$$

Using (9) we obtain $([d(0, n)]^2 u_n^{-2} - [d(0, n+1)]^2 u_{n+1}^{-2}) = O(n^{-2})$; (7), (8) and the definition of $f(1, n, c)$ produces

$$(A(1, n)[f(1, n, c)]^{-1} - A(1, n+1)[f(1, n+1, c)]^{-1}) = O(n^{-2}).$$

These facts together with the lemma prove that the remainder kernels also generate uniformly bounded operators on $L^p(\mathbf{R})$, $1 < p < \infty$.

Proof of the lemma. We use the identity $(2n + 1)x^{-1}j_n(x) = j_{n-1}(x) + j_{n+1}(x)$ to get

$$\sum_{N(c) \leq n \leq N} \frac{(2n + 1)^2 + (2n + 5)^2}{(2n + 3)} (j_n(x)j_{n+2}(y) + j_{n+2}(x)j_n(y)) = (x^{-2} + y^{-2}) \sum_{N(c) \leq n \leq N} (2n + 3)^3 j_n(x)j_n(y) + Q_N^1(x, y)$$

where $Q_N^1(x, y)$ is “essentially” the partial sum kernel associated to $\{\sqrt{2n + 1}j_n(x)\}$.

Since $|x^{1/2}J'_{N+1/2}(x)| \leq A, |x^{1/2}J_{N+1/2}(x)| \leq AN^{1/6}$, $x \in \mathbf{R}$, and A is an absolute constant, the boundedness of the Hilbert transform on $L^p(\mathbf{R})$ together with (10) yield that the operator whose kernel is $Q_N^1(x, y)$ is bounded on $L^p(\mathbf{R})$, $1 < p < \infty$, with a norm majorized by $AN^{1/6}$.

Therefore, to finish the proof of the lemma it is enough to show that the integral operator associated to the first kernel is also bounded on $L^p(\mathbf{R})$, with a norm controlled by $AN^{2/3}$. To see that we apply partial summation to get

$$(x^{-2} + y^{-2}) \sum_{N(c) \leq n \leq N} (2n + 1)^3 j_n(x) j_n(y) = A_N(x, y) + A_N(y, x) + B_N(x, y) + B_N(y, x) + C_N(x, y) + D(x, y),$$

$$A_N(x, y) = \pi(x^{-2} + y^{-2}) x^{1/2} y^{1/2} (x - y)^{-1} \times \sum_{N(c) \leq n \leq N-1} (4n + 6) J'_{n+3/2}(x) J_{n+3/2}(y),$$

$$B_N(x, y) = (\pi/2)(2N + 1)(2N + 3)(x^{-2} + y^{-2}) \times x^{1/2} y^{1/2} (x - y)^{-1} J'_{N+3/2}(x) J_{N+3/2}(y),$$

$$C_N(x, y) = -(1/2)(2N + 1)(2N + 3)(2N + 5)(x^{-2} + y^{-2}) j_{N+1}(x) j_{N+1}(y),$$

where $D(x, y)$ is independent of N and bounded on $L^p(\mathbf{R})$, $1 < p < \infty$.

To control $C_N(x, y)$ we observe that (10) yields

$$(\|j_N x^{-2}\|_p \|j_N\|_q + \|j_N\|_p \|j_N x^{-2}\|_q) = O(N^{-17/6}), \quad 1/p + 1/q = 1, \quad 1 < p < \infty,$$

which implies that the corresponding operator has norm $\ll N^{1/6}$ on $L^p(\mathbf{R})$, $1 < p < \infty$.

Finally, the estimates $|J'_{n+3/2}(x)x^{-3/2}| \leq An^{-3/2}$ and $|J_{n+3/2}(x)x^{-3/2}| \leq An^{-3/2}$, A independent of n , added again to the boundedness of the Hilbert transform produces the bound $AN^{2/3}$ for the norm of the integral operators whose kernels are given by $A_N(x, y)$ or $B_N(x, y)$. Q.E.D.

4. FINAL REMARKS

(1) The analogous problem to the one considered in Theorem 1 but in relation with the time-limited functions, has also been treated. The result is the following:

Theorem 3. *Let $f \in L^p[-T, +T]$, $4/3 < p < 4$, and*

$$\sigma_N f(x) = \sum_{0 \leq n \leq N} \lambda_n^{-1} \langle f, b_n \rangle b_n(x),$$

where $\langle f, b_n \rangle = \int_{-T}^T f(x) b_n(x) dx$. Then we have $\lim_{N \rightarrow \infty} \|\sigma_N f - f\|_{L^p[-T, +T]} = 0$. Furthermore the interval $4/3 < p < 4$ is the best possible.

The techniques of the proof are completely similar to those of Theorem 1, except that, in this case, we use the properties of Legendre's polynomials. It is a well-known fact that these polynomials constitute a basis of $L^p[-1, +1]$ if and only if $4/3 < p < 4$, see references [5 and 6].

(2) It seems natural to extend some of the harmonic analysis of the classical Fourier series to the developments considered in this work, as well as to consider the higher-dimensional case. We have made progress in both areas and have obtained some positive partial results, which we plan to present in a future communication.

REFERENCES

1. A. Askey and S. Wainger, *Mean convergence of expansion in Laguerre and Hermite series*, Amer. J. Math. **87** (1965), 695–708.
2. C. Flamer, *Spherical wave functions*, Stanford Univ. Press, Stanford, Calif., 1957.
3. H. L. Landau and H. O. Pollak, *Prolate spheroidal wave functions, Fourier analysis and uncertainty II*, Bell Systems Tech. J. **40** (1961), 65–84.
4. ———, *Prolate spheroidal wave functions, Fourier analysis and uncertainty III*, Bell Systems Tech. J. **41** (1962), 1295–1335.
5. J. Newman and W. Rudin, *On mean convergence of orthogonal series*. Proc. Amer. Math. Soc. **3** (1952), 219–222.
6. H. Pollard, *The mean convergence of orthogonal series. I*, Trans. Amer. Math. Soc. **62** (1947), 387–403.
7. E. T. Sawyer, *A characterization of a two-weight norm inequality for maximal operators*, Studia Math. **75** (1982), 1–11.
8. D. Slepian, *On bandwidth*, Proc. IEEE **64** (1976), 292–300.
9. ———, *Some comments on Fourier analysis, uncertainty and modeling*, SIAM Rev. **25** (1983), 379–393.
10. D. Slepian, and H. O. Pollak, *Prolate spheroidal wave functions, Fourier analysis and uncertainty I*. Bell Systems Tech. J. **40** (1961) 43–64.
11. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press, Princeton N.J., 1971.
12. J. A. Straton, P. M. Morse, L. J. Chu, J. D. Little and F. J. Corbato, *Spherical wave functions*, Wiley, New York, 1956.
13. G. Szego, *Orthonormal polynomials*, 4th ed., Amer. Math. Soc. Colloq. Publ., Amer. Math. Soc., Providence, R.I., 1975.
14. G. N. Watson, *A treatise on the theory of Bessel functions*, 2nd ed., Cambridge Univ. Press, 1966.

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