CONJUGATION AND THE PRIME DECOMPOSITION OF KNOTS IN CLOSED, ORIENTED 3-MANIFOLDS

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ABSTRACT. In this paper we consider the prime decomposition of knots in closed, oriented 3-manifolds. (For classical knots one can easily prove the uniqueness of prime decomposition by using a standard innermost disk argument.) We define a new relation, *conjugation*, between oriented knots in closed, oriented 3-manifolds and prove the following results. (1) The prime decomposition is, roughly speaking, uniquely determined up to conjugation, (2) there is a prime knot $\mathscr{R}$ in $S^1 \times S^2$ such that $\mathscr{R} \# \mathscr{K}_1 = \mathscr{R} \# \mathscr{K}_2$ if $\mathscr{K}_1$ is a conjugation of $\mathscr{K}_2$, and (3) if a knot $\mathscr{K}$ has a prime decomposition which does not contain $\mathscr{R}$, then it is the unique prime decomposition of $\mathscr{K}$.

INTRODUCTION

In [4] it was proved that the prime decomposition of a knot in S^3 is uniquely determined. However, to my knowledge there have been no papers studying the prime decomposition of 3-dimensional *global knots* in the sense of [5], i.e. knots in 3-manifolds. (Cf. [6].) In this paper we will study the prime decomposition of an oriented knot in a closed, oriented 3-manifold. First we obtain:

Existence Theorem. *A nontrivial knot $\mathscr{K}$ is primely decomposable if and only if a meridian of $\mathscr{K}$ is essential in the exterior of $\mathscr{K}$ (Theorem 2.1).*

This follows from a standard innermost disk argument and so-called Haken's finiteness theorem. However, for studying the uniqueness or nonuniqueness we need a new relation, called *conjugation* (defined in §3), between oriented knots in closed, oriented 3-manifolds. This relation is essential for prime decompositions of knots because the decomposition is, roughly speaking, uniquely determined up to conjugation (Theorem 5.4). In particular we have the following results.

Uniqueness Theorem. *Let $\mathscr{R}$ be the prime knot in $S^1 \times S^2$ depicted below. If a knot $\mathscr{K}$ has a prime decomposition which does not contain $\mathscr{R}$, then it is the unique prime decomposition for $\mathscr{K}$ (Corollary 5.6).*
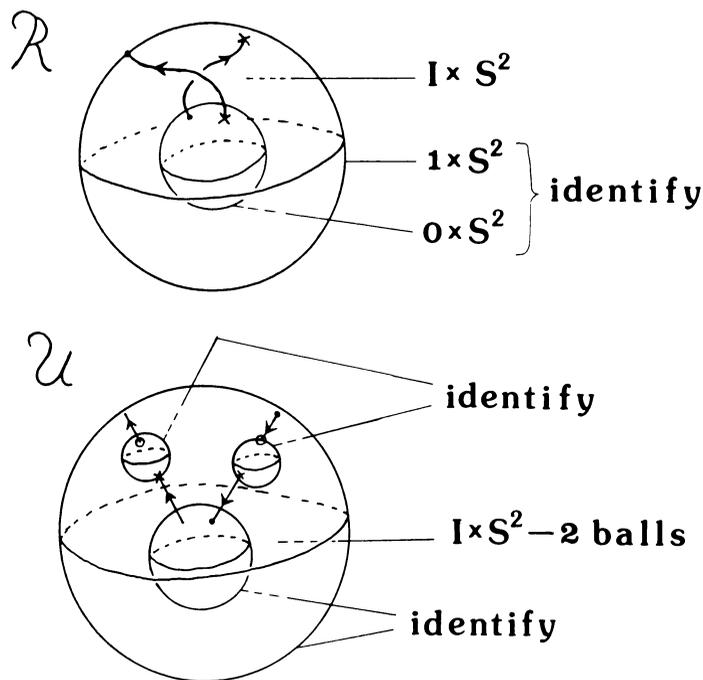
FIGURE 0

Nonuniqueness Theorem. *If a knot $\mathcal{K}_1$ is a conjugation of a knot $\mathcal{K}_2$, then $\mathcal{R}\#\mathcal{K}_1 = \mathcal{R}\#\mathcal{K}_2$ (Theorem 5.7). Let $\mathcal{U}$ be the prime knot in $2S^1 \times S^2$ shown in Figure 0, then $\mathcal{R}\#\mathcal{R}$ is a conjugation of $\mathcal{U}$ (thus $\mathcal{R}\#\mathcal{R}\#\mathcal{R} = \mathcal{R}\#\mathcal{U}$) (Examples 2 and 3 in §5).*

The paper is organized as follows. In §1 we establish terminology. In particular, we classify knots into two classes: *order reversing knots* and *order preserving ones*. In §2 we give a proof of the Existence Theorem (Theorem 2.1). The key idea of this paper is the notion of an *inducing-pair* introduced in §3. Using this, in §3 we define *conjugation*. §4 is devoted to the proof that every order preserving knot has a unique decomposition (Theorem 4.1). The lemma and the remark in §4 will be used in §5. In §5 first we study the order reversing knots. Then we clarify the rules of prime decomposition (Theorems 5.4–5.7).

I would like to thank Masahico Saito for suggesting this problem to me.

1. Terminology

We work throughout in the PL category, and assume that 3-manifolds are compact, connected and oriented.

A *circle* is a simple closed curve, and suitably oriented unless otherwise stated. For a circle α, $-\alpha$ denotes the α with the orientation reversed. Two circles α_1, α_2 in a manifold X are *parallel* if there exists an oriented annulus A in X with $\partial A = \alpha_1 \cup -\alpha_2$.

Let M be a closed 3-manifold, and K a circle in M. We call a pair (M, K) a *knot*. We sometimes call the circle K itself a knot. In this paper a capital letter in script (e.g. $\mathscr{K}$, $\mathscr{K}_i$, $\mathscr{R}$ etc.) stands for a knot, and when dealing with a knot $\mathscr{K}$, the circle of $\mathscr{K}$ is written as K in italic type. The same notation applies to knots $\mathscr{K}_i$, $\mathscr{R}$, etc. Given a knot $\mathscr{K} = (M, K)$, $-\mathscr{K}$ denotes the knot $(M, -K)$. Two knots (M_1, K_1) and (M_2, K_2) are said to have the same *knot type* if there is an orientation preserving homeomorphism $f: (M_1, K_1) \to (M_2, K_2)$. This is denoted by $(M_1, K_1) \cong (M_2, K_2)$.

Let $N(\cdots)$ denote a *regular neighborhood*. The *exterior* of a knot $\mathscr{K} = (M, K)$ is $M - \operatorname{Int} N(K)$, which is denoted by $E(K)$ or $E(\mathscr{K})$. A *meridian* of a knot (M, K) is a circle on $\partial N(K)$ which bounds on oriented disk D in $N(K)$ with $D \cdot K = 1$.

Let N be a 3-manifold with $\partial N = S^2$, and let a be an oriented interval properly embedded in N. We call a pair (N, a) an *arc*. For an arc (N, a), $(A, a)^\wedge$ denotes the knot obtained from (N, a) by capping the boundary pair $(\partial N, \partial a)$ with a trivial ball pair. Conversely, for a knot $\mathscr{K} = (N, a)^\wedge$ $\mathscr{K}^-$ denotes the arc (N, a). Two arcs (N_1, a_1) and (N_2, a_2) are said to have the same *arc type* if $(N_1, a_1)^\wedge$ and $(N_2, a_2)^\wedge$ have the same knot type.

For a 3-manifold N with sphere boundaries, $\widehat{N}$ denotes the closed 3-manifold obtained from N by capping ∂N with 3-balls.

Given two knots $\mathscr{K}_1$ and $\mathscr{K}_2$, we define a *connected sum*

$$\mathscr{K}_1 \# \mathscr{K}_2 = (N_1, a_1) \cup_f (N_2, a_2),$$

where $\mathscr{K}_i^- = (N_i, a_i)$ and $f: (\partial N_2, \partial a_2) \to (\partial N_1, \partial a_1)$ is an orientation reversing homeomorphism. For a knot $(M, K) \cong \mathscr{K}_1 \# \cdots \# \mathscr{K}_{n+1}$ there exists a set of n spheres in M which gives this decomposition. When $n = 1$, this follows from the definition of $\mathscr{K}_1 \# \mathscr{K}_2$. Suppose $n - 1$ spheres $S_1, \ldots, S_{n-1}$ gives a decomposition $(M', K') = \mathscr{K}_1 \# \cdots \# \mathscr{K}_n$. We can isotop a trivial ball pair (B, a) in (M', K') so as to be disjoint from each S_i. Remove (B, a) and attach $\mathscr{K}_{n+1}^-$ we get the knot of the same knot type as (M, K). The set of n spheres $\{S_1, \ldots, S_{n-1}, \partial B\}$ gives the desired decomposition. Note that there is a choice of such a set.

Given a knot $\mathscr{K} = (M, K)$ and a closed 3-manifold N, define a *connected sum* $\mathscr{K} \# N = (M - \operatorname{Int} B_1 \cup_f N - \operatorname{Int} B_2, K)$, where $B_1 \subset M$ and $B_2 \subset N$ are 3-balls missing K, and $f: \partial B_2 \to \partial B_1$ is an orientation reversing homeomorphism.

A *trivial knot* in a 3-manifold is the connected sum of the unknot in S^3 and the 3-manifold.

A knot $\mathscr{K}$ is *irreducible* if $E(\mathscr{K})$ is an irreducible 3-manifold.

A knot $\mathscr{K}$ is *prime* if for each decomposition $\mathscr{K} \cong \mathscr{K}_1 \# \mathscr{K}_2$ either $\mathscr{K}_1$ or $\mathscr{K}_2$ is the unknot in S^3, and $\mathscr{K}$ is not a trivial knot. Note that a prime knot is irreducible. A decomposition $\mathscr{K} \cong \mathscr{K}_1 \# \cdots \# \mathscr{K}_r \# N$, where $\mathscr{K}_1, \ldots, \mathscr{K}_r$ are all prime and N is a 3-manifold, is called a *prime decomposition* of $\mathscr{K}$. A

knot which has a prime decomposition is said to be *primely decomposable*. All nontrivial knots in S^3 are primely decomposable.

Next we classify knots into two classes.

A knot $\mathcal{K}$ is said to be *order reversing* if they are disjoint, properly embedded annuli A_1, A_2 in $E(\mathcal{K})$ satisfying that:

(1) each component of ∂A_i is a meridian;

(2) each component of $\partial E(\mathcal{K}) - \partial A_1$ meets ∂A_2 in a circle. Otherwise, a knot $\mathcal{K}$ is said to be *order preserving*. The above A_1 and A_2 are called *order reversing annuli* in $E(\mathcal{K})$. We say an arc (N, a) is order reversing (resp. order preserving) if a knot $(N, a)\hat{\ }$ is order reversing (resp. order preserving).

Let X_1, X_2 be 3-manifolds with boundaries, and $\alpha_i^1, \ldots, \alpha_i^n$ pairwise disjoint circles on ∂X_i. Choose a regular neighborhood $N_i^j \subset \partial X_i$ of α_i^j with $N_i^j \cap N_i^k = \varnothing$ $(j \neq k)$ and a homeomorphism $f : \bigcup_{j=1}^n N_2^j \to \bigcup_{j=1}^n N_1^j$ such that $f| : (N_2^j, \alpha_2^j) \to (N_1^j, \alpha_1^j)$ is orientation reversing for each j. The oriented homeomorphism class of $X_1 \cup_f X_2$ is independent of the choice of N_i^j and f. Let us denote $X_1 \cup_f X_2$ by $X_1 \cup_{\alpha_1^j = -\alpha_2^j} X_2$.

For a topological space X we denote by $|X|$ the number of connected components of X.

We refer to [1 and 2] for a standard terminology in 3-manifold topology.

2. EXISTENCE

We will give a necessary and sufficient condition for a knot to be primely decomposable.

Theorem 2.1. *A nontrivial knot $\mathcal{K} = (M, K)$ is primely decomposable if and only if a meridian $m \subset \partial E(K)$ is essential in $E(K)$.*

Proof. Suppose that m is null-homotopic. Let $\mathcal{K} = \mathcal{K}_1 \# \cdots \# \mathcal{K}_n \# N$ be a prime decomposition. Now $\pi_1(E(K)) = G$, say, is isomorphic to the amalgamated product

$$G_1 *_{[m_1]=[m_2]} G_2 * \cdots * G_{n-1} *_{[m_{n-1}]=[m_n]} G_n * \pi_1(N),$$

where $G_i = \pi_1(E(K_i))$, and $[m_i]$ denotes some element in G_i corresponding to a meridian of $\mathcal{K}_i$. If each $[m_i]$ generates an infinite cyclic subgroup in G_i, so does $[m]$ in G by [3, Theorem 4.3], a contradiction. Thus we may assume that a meridian, m_1, of $\mathcal{K}_1$ represents a torsion or zero element in G_1. In other words for an annulus neighborhood A $(\subset \partial E(K_1))$ of m_1 $\operatorname{Ker}(\pi_1(A) \to G_1)$ is nontrivial. Then by the loop theorem [2, I.2] m_1 is null-homotopic in $E(K_1)$. By Dehn's lemma m_1 bounds a disk in $E(K_1)$, so that $\mathcal{K}_1$ admits a sphere intersecting K_1 transversely in a single point. Since $\mathcal{K}_1$ is an irreducible knot, $\mathcal{K}_1 \cong (S^1 \times S^2, S^1 \times *)$. Then $\mathcal{K}_1$ can be expressed as the connected sum of $\mathcal{K}_1$ itself and any knot in S^3, contradicting the primeness of $\mathcal{K}_1$.

Suppose that a meridian of $\mathcal{K}$ is essential in $E(K)$. Then if $\{S_1, \ldots, S_n\}$ is a collection of spheres in M which gives a decomposition $\mathcal{K} \cong \mathcal{K}_1 \# \cdots \# \mathcal{K}_{n+1}$, $A_i = S_i - \operatorname{Int} N(K)$ is a properly embedded, incompressible annulus in $E(K)$.

By III.24 in [2], there exists a nonnegative integer N_0 such that either $n < N_0$ or for some $i \neq j$ A_i is parallel to A_j in $E(K)$. If the latter case occurs, then for some i_0 $\mathscr{K}_{i_0}$ is a trivial knot. Thus $\mathscr{K}$ cannot be decomposed into more than N_0 nontrivial knots. Therefore, $\mathscr{K}$ can be decomposed into prime knots and a 3-manifold. This completes the proof of Theorem 2.1.

The next lemma reduces the prime decomposition problem to the case of irreducible knots. For a 3-manifold M, let M^- denote the 3-manifold obtained from M by deleting an open 3-ball.

Lemma 2.2. *Let $\mathscr{K}$ be primely decomposable. If $\mathscr{K} = \mathscr{K}_1 \# \cdots \# \mathscr{K}_n \# N$ and $\mathscr{K} = \mathscr{K}'_1 \# \cdots \# \mathscr{K}'_m \# N'$ are prime decompositions, then $N = N'$, $\#_{i=1}^n \mathscr{K}_i$ is an irreducible knot and $\#_{i=1}^n \mathscr{K}_i \cong \#_{i=1}^m \mathscr{K}'_i$.*

Proof. Set $X = E(\mathscr{K}_1 \# \cdots \# \mathscr{K}_n)$. First we prove that X is irreducible. Let $A_1, \ldots, A_{n-1}$ be properly embedded annuli in X which are on the spheres giving the prime decomposition $\#_{i=1}^n \mathscr{K}_i = \mathscr{K}_1 \# \cdots \# \mathscr{K}_n$. By the proof of Theorem 2.1 each A_i is incompressible. If X is reducible, then there is an incompressible sphere S in X. Since each component of $X - \bigcup A_i$ is irreducible, S may be isotoped so that $S \cap \bigcup_{i=1}^{n-1} A_i = \varnothing$. Hence, some $\mathscr{K}_i$ is not irreducible, contradicting the assumption. Similarly $X' = E(\#_{i=1}^m \mathscr{K}')$ is irreducible. Let $N = \#_{i=1}^s N_i$ and $N' = \#_{i=1}^t N'_i$ be prime decompositions, and let $X \# N = \widetilde{X}$, the exterior of $\mathscr{K}$. We then have two prime decompositions $\widetilde{X} = X \# N_1 \# \cdots \# N_s$ and $\widetilde{X} = X' \# N'_1 \# \cdots \# N'_t$. Note that X^- and X'^- are considered to be submanifolds of $\widetilde{X}$ with $\partial \widetilde{X} \subset \partial X^-$ and $\partial \widetilde{X} \subset \partial X'^-$. The following is proved in [1, 3.21] in the course of proving the unique decomposition theorem for 3-manifolds.

There is an orientation preserving homeomorphism $f: X^- \to X'^-$ with $f|\partial \widetilde{X} = \mathrm{id}$.

Thus $(M, K) \cong (M', K')$ and $N = N'$.

Before going to the uniqueness problem we would like to notice the following.

Remark 2.3. The difficulty in showing the uniqueness of prime decomposition lies in the possible existence of nonseparating annuli properly embedded in $E(K)$ with their boundaries meridional. There are two cases:

(1) There are no such annuli. Then uniqueness can be proved in a standard manner.

(2) There exists such an annulus. Then two prime decompositions are related by conjugation. The uniqueness, however, holds if and only if every such annulus has oppositely oriented boundary components on $\partial N(K)$.

In the next section we define conjugation of knots.

3. INDUCING-PAIRS

Throughout this section we will use the following notation.

Let X denote a 3-manifold with $\partial X = T_1 \cup \cdots \cup T_n$, n tori, and let $\alpha_i \subset T_i$ be an essential circle.
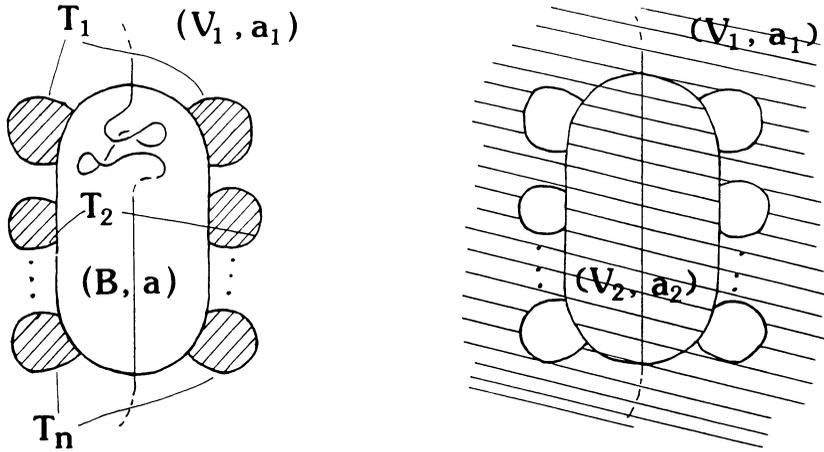
(a) The shaded area is X.   (b) The shaded area is C_n.

FIGURE 1. Cross sections

Let (B,a) be an arc, $\beta_1, \ldots, \beta_n \subset \partial B - \partial a$ pairwise parallel, essential circles with $\mathrm{lk}_{\partial B}(\partial a, \beta_i) = +1$.

Let $(M,a) = X \cup_{\alpha_i = \beta_i} (B,a)$.

D_n denotes a disk-with-n holes with $\partial D_n = s_0 \cup \cdots \cup s_n$, $n+1$ circles. (C_n, J) denotes a pair $(S^1 \times D^2, S^2 \times 0) \cup_f D_n \times S^1$, where $f: s_0 \times S^1 \to S^1 \times \partial D^2$ is a homeomorphism such that $f(p_1 \times S^1) = p_2 \times \partial D^2$ with $p_1 \in s_0$ and $p_2 \in S^1$.

The following lemma and the observations after that are essential for the present paper.

Lemma 3.1. *If $(M,a)\widehat{\ }$ is a prime knot, and $(X, \bigcup_{i=1}^n \alpha_i) \neq (S^1 \times D^2, S^1 \times *)$, $* \in \partial D^2$, then B is a 3-ball and (B,a) is a trivial ball pair.*

Proof. Set $N = \partial X \cap B$ and $N' = \mathrm{Cl}(\partial X - B)$. For $i = 1, 2$ let (V_i, a_i) denote a trivial ball pair. Then

$$(M,a)\widehat{\ } = (V_1, a_1) \cup_{(\partial M, \partial a)} (X \cup (B,a)) \quad \text{(cf. Figure 1(a))}$$

$$= ((V_1, a_1) \cup_{\bigcup N'} X) \cup_{(\partial B, \partial a)} (B,a)$$

$$= ((V_1, a_1) \cup_{\bigcup N'} X)\widehat{\ } \# (B,a)\widehat{\ }$$

$$= ((V_1, a_1) \cup_{\bigcup N'} X \cup_{\bigcup N} (V_2, a_2)) \# (B,a)\widehat{\ }$$

$$= (X \cup_{T_i = s_i \times S^1} (C_n, J)) \# (B,a)\widehat{\ } \quad \text{(cf. Figure 1(b))}$$

Set

$$(X', J) = X \cup_{T_i = s_i \times S^1} (C_n, J).$$

Either (X', J) or $(B, a)\hat{\ }$ must be the unknot in S^3, for $(M, a)\hat{\ }$ is a prime knot. When $n > 1$, X' contains a nonseparating torus T_1, so that (B, a) is trivial ball pair. If $n = 1$ and $X' = S^3$, then J is knotted in S^3 by the assumption that $(X, \alpha_i) \neq (S^1 \times D^2, S^1 \times *)$, so that (B, a) is a trivial ball pair. This completes the proof of Lemma 3.1.

In the rest of this section, (B, a) denotes a trivial ball pair, but the assumption of the above lemma may not be satisfied.

Observaton 1. Now we have the following.

$$(*) \qquad\qquad (M, a)\hat{\ } = (X \cup_{\alpha_i = \beta_i} (B, a))\hat{\ }$$

and

$$(**) \qquad\qquad\qquad = X \cup (C_n, J).$$

The first equality follows from the definition of (M, a). The union in $(**)$ is given by n orientation reversing homeomorphisms $f_i \colon (s_i \times S^1, -l_i) \to (T_i, \alpha_i)$, $1 \le i \le n$, where $l_i = p_i \times S^1$, $p_i \in s_i$, is oriented so that $\mathrm{lk}_{C_n}(l_i, J) = 1$ (C_n is considered to be in S^3). Moreover, even if we interchange β_i and β_j, $i \neq j$, in $(*)$, the resulting knot has the same knot type, for there is an orientation preserving autohomeomorphism f of (C_n, J) such that $f(l_i) = l_j$, $f(l_j) = l_i$, and $f|s_k \times S^1 = \mathrm{id}$ for $k \neq i, j$. Therefore, the knot $(M, a)\hat{\ }$ and the arc (M, a) depend only on X and the circles $\alpha_1, \dots, \alpha_n \subset \partial X$.

We say that a knot $\mathscr{K} = (M, a)\hat{\ }$ or an arc (M, a) is induced from a pair $(X, \bigcup_{i=1}^n \alpha_i)$, and that $(X, \bigcup_{i=1}^n \alpha_i)$ is an *inducing-pair* of $\mathscr{K}$ with *degree* n. When $n = 1$, it is called a *trivial inducing-pair*.

Lemma 3.2. *Suppose that a prime knot $\mathscr{K}$ has a nontrivial inducing-pair with degree n. Then $E(K)$ contains n incompressible tori dividing $E(K)$ into $D_n \times S^1$ and another component.*

Proof. Let $(X, \bigcup_{i=1}^n \alpha_i)$ be the inducing-pair in the assumption. From the equation $(**)$ in Observation 1 we see $E(K) = X \cup D_n \times S^1$, where ∂X is identified with $s_i \times S^1$, $1 \le i \le n$. The torus $s_i \times S^1$ is incompressible in $D_n \times S^1$. Therefore if ∂X were compressible in $E(K)$, there would be a properly embedded disk D in X with ∂D essential in some T_i. Surgery on T_i using D yields a sphere in X. Since $n > 1$, this sphere is nonseparating in $E(K)$, contradicting the primeness of $\mathscr{K}$. Hence, ∂X is the desired tori.

A knot $\mathscr{K}$ is a *conjugation* of $\mathscr{K}'$ through $(X, \bigcup_{i=1}^n \pm\alpha_i)$ if $\mathscr{K}$ is induced from $(X, \bigcup_{i=1}^n \alpha_i)$ and $\mathscr{K}'$ is induced from $(X, \bigcup_{i=1}^n \alpha_i')$ where $\alpha_i = \pm\alpha_i'$ up to order and isotopy in ∂X. Notice that $-\mathscr{K}$ is induced from $(X, \bigcup_{i=1}^n -\alpha_i)$. Similarly we can define conjugation for arcs.

It is immediate from the following observation that the *ambient manifolds of $\mathscr{K}$ and $\mathscr{K}'$ are homeomorphic to each other by an orientation preserving map if $\mathscr{K}$ is a conjugation of $\mathscr{K}'$.*

Observation 2. There is another description of the arc (M,a) induced from $(X, \bigcup_{i=1}^{n} \alpha_i)$. Divide (B,a) into $2n-1$ trivial ball pairs (B_i, a_i), $1 \le i \le 2n-1$, such that for $1 \le i \le n$, B_i is attached to X as a 2-handle along α_i (Figure 2). Then the boundary of $X \cup \bigcup_{i=1}^{n} B_i = X'$, say, consists of n spheres. For $n+1 \le i \le 2n-1$ B_i is attached to X' as a 1-handle joining the distinct components of $\partial X'$. It follows:

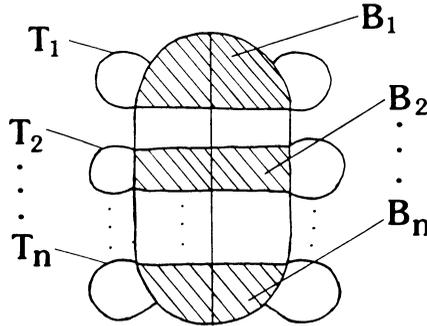$$(M,a) \cong X \cup_{\alpha_1,\ldots,\alpha_n} n(h^2, \text{cocore}) \cup (n-1)(h^1, \text{core}).$$

FIGURE 2. A cross section of (M,a).

Hence, $\widehat{M} \cong \widehat{X'} \#(n-1)S^1 \times S^2$, and this is independent of the orientations of α_i's. Now each cocore is oriented so that $\text{lk}(\alpha_i, \text{cocore}) = +1$. As long as 1-handles are attached so that the obtained arc can be consistently oriented by the cocores, the resulting knot has the same knot type as $(M,a)\hat{\ }$. If the attached 1-handles do not satisfy the consistency of orientations, then the resulting knot is a conjugation of $(M,a)\hat{\ }$. Figure 3 illustrates the case where ∂X consists of three tori.

4. UNIQUENESS

Unique prime decomposition holds for order preserving, primely decomposable knots.

Theorem 4.1. *Let $\mathscr{K}$ be an order preserving, irreducible knot. If $\mathscr{K} = \mathscr{K}_1 \# \cdots \# \mathscr{K}_n$ and $\mathscr{K} = \mathscr{K}_1' \# \cdots \# \mathscr{K}_m'$ are prime decompositions, then $n = m$, $\mathscr{K}_i \cong \mathscr{K}_i'$ up to order, and $\mathscr{K}_i$ and $\mathscr{K}_j'$ are all order preserving knots.*

Proof. It is easily checked that $\mathscr{K}_i$ and $\mathscr{K}_j'$ are order preserving. We will prove the uniqueness by induction on n.

If $n = 1$, then it is trivial. So, suppose $n > 1$. Let $\mathscr{K} = (M, K)$, and let $\{S_1, \ldots, S_{n-1}\}$ be a collection of spheres giving the prime decomposition $\mathscr{K} = \#_{i=1}^{n} \mathscr{K}_i$. Let (M_i, a_i), $1 \le i \le n$, denote the closure of each component of $(M, K) - \bigcup_{i=1}^{n-1} S_i$. We may assume that for each i $\mathscr{K}_i \cong (M_i, a_i)\hat{\ }$, the

Actually α_i is on ∂X.
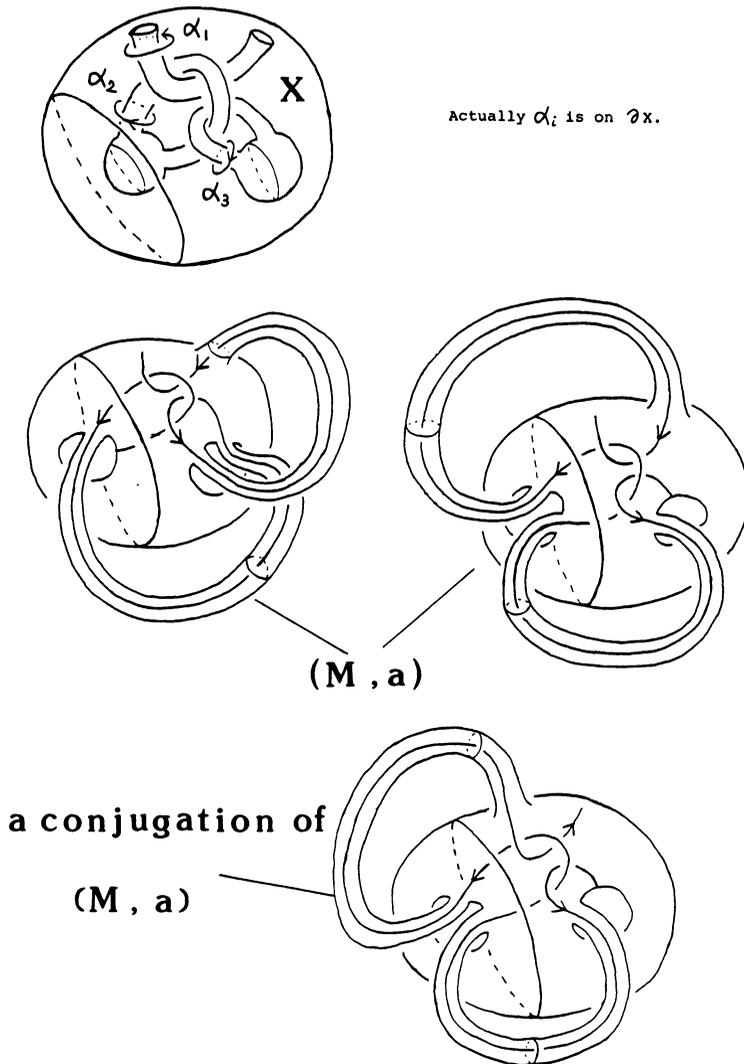
(M , a)

a conjugation of

(M , a)

FIGURE 3

knot obtained from (M_i, a_i) by capping $(\partial M_i, \partial a_i)$ with $|\partial M_i|$ trivial ball pairs. Suppose a collection of spheres $\{S'_j\}_{j=1}^{m-1}$ gives the prime decomposition $\mathscr{K} = \#_{j=1}^m \mathscr{K}'_j$. For this decomposition we define (M'_j, a'_j), $1 \le j \le m$, similarly to (M_i, a_i) above. Moreover, *assume $\partial M_n = S_{n-1}$*. If $M_n \cap \bigcup_{j=1}^{m-1} S'_j = \varnothing$, then (M_n, a_n) is contained in (M'_{j_0}, a'_{j_0}) for some j_0, say $j_0 = m$. From the primeness of $\mathscr{K}'_m$ it is shown that $\mathscr{K}_n \cong \mathscr{K}'_m$ and $\#_{i=1}^{n-1} \mathscr{K}_i \cong \#_{j=1}^{m-1} \mathscr{K}'_j$. By induction the theorem is proved. Hence, *we assume that $M_n \cap \bigcup_{j=1}^{m-1} S'_j = \varnothing$*.

Assertion 1. $\bigcup_{i=1}^{n-1} S_i$ and $\bigcup_{j=1}^{m-1} S'_j$ can be isotoped separately with $|K \cap S_i|$ and $|K \cap S'_j|$ fixed for each i, j so that (1) each S'_j is transverse to S_{n-1},

(2) $M_n \cap \bigcup_{j=1}^{m-1} S'_j$ is a nonempty set of annuli missing K, and (3) each boundary component of these annuli is essential in $S_{n-1} - K$.

Proof of Assertion 1. First isotop $\bigcup_{j=1}^{m-1} S'_j$ with $|K \cap S'_j|$ fixed for each j so as to be disjoint from a_n. Next isotop $\bigcup S_i$ and $\bigcup S'_j$ separately keeping $K \cap \bigcup S_i$ and $K \cap \bigcup S'_j$ fixed so that $\bigcup S'_j$ is transverse to $\bigcup S_i$ and $|(\bigcup S_i) \cap (\bigcup S'_j)|$ is minimal. (If $M_n \cap \bigcup S'_j = \varnothing$ after isotopy, the theorem is proved.)

We claim that $S'_j \cap \bigcup_i S_i$ consists of essential circles in $S'_j - K$ for each j. Part (2) of Assertion 1 follows from our claim. Suppose that for some p $S'_p \cap \bigcup S_i$ contains a circle null-homotopic in $S'_p - K$. Then there is a disk D' such that Int D' is a component of $S'_p - \bigcup S_i$ and $D' \cap K = \varnothing$. The circle $\partial D'$ bounds a disk D on some S_q with $|(D \cup D') \cap K| \leq 1$. Let S denote the 2-sphere $D \cup D'$, and let $s = |S \cap K|$.

If $s = 0$, then S bounds a 3-ball missing K by the irreducibility of $\mathcal{K}$, so that the circle $\partial D'$ is removed by isotoping S_q without producing any new intersection. This contradicts the minimality assumption. If $s = 1$, then S is a 2-sphere meeting K transversely in a single point, contradicting the existence of prime decomposition (Theorem 2.1).

Similarly we see that $S_i \cap \bigcup_j S'_j$ consists of essential circles in $S_i - K$ for each i. So, (3) of Assertion 1 is proved.

Let $\{X_1, \ldots, X_s\}$ be the collection of the closures of components of $M_n - \bigcup_{j=1}^{m-1} S'_j$. The following is an immediate consequence of Assertion 1. (See Figure 4(a).)

Claim. (1) Every annulus component of $\partial M_n \cap X_i$ is essential in $\partial M_n - \partial a_n$, (2) if $X_i \cap a_n = \varnothing$, each component of ∂X_i is a torus, and (3) there is just one X_i such that ∂X_i has a sphere component, which meets K transversely in two points.

In order to apply Lemma 3.1 we need to modify X_i, $1 \leq i \leq s$. Let $\{\partial_1, \ldots, \partial_t\}$ denote the set of the components of ∂X_i. Take an annulus $A_j \subset \partial_j$ for $j = 1, \ldots, t$ such that each of $A_j \cap (\partial X_i \cap \partial M_n)$ and $A_j \cap \mathrm{Cl}(\partial X_i - \partial M_n)$ is an annulus. (There is a choice of A_j.) Set $\widetilde{X}_i = \mathrm{Cl}(X_i - N((\partial X_i - \bigcup_{j=1}^{t} A_j) \cup a_n))$. See Figure 4(b). Let $\{\widetilde{\partial}_1, \ldots, \widetilde{\partial}_t\}$ denote the set of components of $\partial \widetilde{X}_i$, so that $\widetilde{\partial}_j$ is a torus for all j. Note that (1) $(\mathrm{Cl}(M_n - \widetilde{X}_i), a_n)$ is an arc for each i, (2) the closure of $\widetilde{\partial}_j - \partial M_n$ is an essential annulus in both $\partial(M_n - \widetilde{X}_i) - \partial a_n$ and $\widetilde{\partial}_j$.

Assertion 2. There exists just one $i_0 \in \{1, \ldots, s\}$ such that $(\widetilde{X}_{i_0}, \partial M_n \cap \widetilde{X}_{i_0}) \neq (S^1 \times D^2, S^1 \times I)$, where I is a subarc of ∂D^2.

Proof of Assertion 2. Suppose that both ∂X_i and ∂X_j $(i \neq j)$ satisfy the condition above. We apply Lemma 3.1 to $\widetilde{X}_i$ and $(\mathrm{Cl}(M_n - \widetilde{X}_i), a_n)$ to see
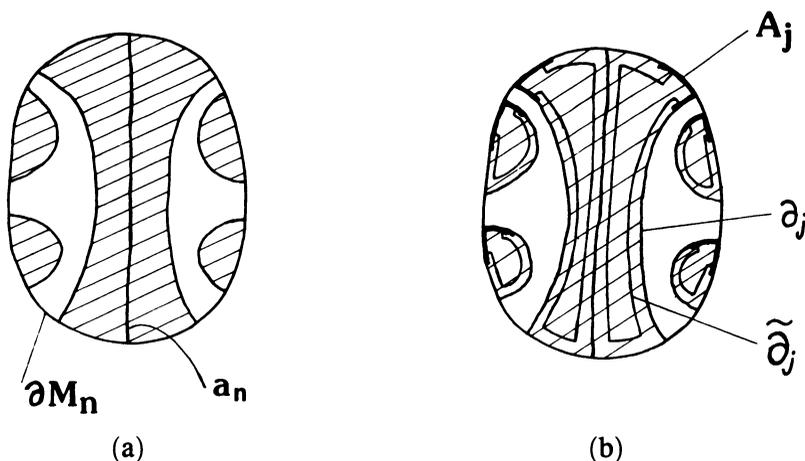
(a)   (b)

FIGURE 4. Cross sections of (M_n, a_n).
The shaded area is some X_i.

the latter is a trivial ball pair. Thus $(\widetilde{X}_j, \partial M_n \cap \widetilde{X}_j) = (S^1 \times D^2, S^1 \times I)$, contradicting the assumption.

If there exists i_0 such that ∂X_{i_0} is disconnected, then X_{i_0} is the desired component. Suppose that ∂X_i is connected for all i. Let Ω denote $\{M_n \cap \bigcup S'_j\}$, a set of annuli. We first claim that each $A \in \Omega$ separates M_n. Take an arbitrary element A_0 of Ω. Let B_1 denote the annulus on ∂M_n bounded by ∂A_0, and let $B_2 = \mathrm{Cl}(\partial M_n - B_1)$. Define $\Omega_j = \{A \in \Omega \mid \partial A \subset B_j\}$, for $j = 1, 2$. Now $E(\mathscr{X}_n) = M_n - \mathrm{Int}\, N(a_n)$ does not contain order reversing annuli, thus $\Omega_1 \cup \Omega_2 = \Omega$ and $\Omega_1 \cap \Omega_2 = \{A_0\}$. Define $\Omega'_j = \{i \mid \partial X_i \subset B_j \cup \text{the union of the elements of } \Omega_j\}$, for $j = 1, 2$. We have $\Omega'_1 \cup \Omega'_2 = \{1, \ldots, s\}$, for ∂X_i is connected for $i = 1, \ldots, s$. Hence, we obtain

$$M_n - A_0 = \bigcup_{i=1}^{s} X_i - A_0 = \left(\bigcup_{i \in \Omega'_1} X_i \cup \bigcup_{i \in \Omega'_2} X_i \right) - A_0.$$

Since $\bigcup_{i \in \Omega'_1} X_i \cap \bigcup_{i \in \Omega'_2} X_i = \bigcup_{\Omega'_1} \partial X_i \cap \bigcup_{\Omega'_2} \partial X_i = A_0$, $M_n - A_0$ is disconnected as we claimed.

Suppose that each X_i whose boundary is a torus satisfies $(\widetilde{X}_i, \partial M_n \cap \widetilde{X}_i) = (S^1 \times D^2, S^1 \times I)$. From the result just proved there exists an outermost annulus in Ω, and each solid torus X_i is attached to ∂M_n with longitudinal annuli (i.e. regular neighborhood of longitudes). Remove all the solid tori X_i's from M_n starting with the outermost ones, then the resulting arc has the same arc type as (M_n, a_n), which is the union of X_i' such that $(X_i, X_i \cap a_n)$ is an arc. By (3) in Claim there is just one i_0 such that $(X_{i_0}, X_{i_0} \cap a_n) \cong (M_n, a_n)$. Since (M_n, a_n) is not a trivial ball pair, $(\widetilde{X}_{i_0}, \partial M_n \cap \widetilde{X}_{i_0}) \neq (S^1 \times D^2, S^2 \times I)$. This completes the proof of Assertion 2.

Assertion 3. There exists $(M_k', a_k')\widehat{\ }$ which is a conjugation of $(M_n, a_n)\widehat{\ }$. For simplicity let us assume $k = m$, then $\#_{i=1}^{n-1}\mathscr{X}_i \cong \#_{j=1}^{m-1}\mathscr{X}_j'$.

Proof of Assertion 3. Let X_{i_0} be the component whose existence is guaranteed by Assertion 2. Let $\partial\widetilde{X}_{i_0} = T_1 \cup \cdots \cup T_r$, r tori. Applying Lemma 3.1 to X_{i_0} and $(\mathrm{Cl}(M_n - \widetilde{X}_{i_0}), a_n) = (B, a_n)$, say, we see that (B, a_n) is a trivial ball pair. Take a core α_i of the annulus $T_i \cap B$ for $1 \le i \le r$. Orient $\bigcup\alpha_i \subset \partial B - \partial a_n$ by a_n so that $\mathrm{lk}_{\partial B}(\partial a_n, \alpha_i) = 1$, then $(\widetilde{X}_{i_0}, \bigcup_{i=1}^r \alpha_i)$ is an inducing-pair of $(M_n, a_n)\widehat{\ }$.
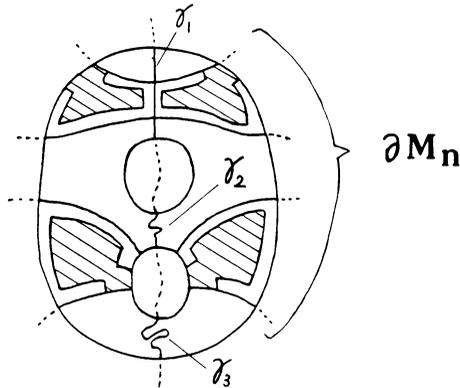

FIGURE 5. A cross section of (M_k', a_k').
The shaded area is X_{i_0}.

On the other hand, $\widetilde{X}_{i_0}$ is contained in just one (M_k', a_k'), where a_k' consists of arcs $\gamma_1, \ldots, \gamma_u$, $u = |\partial M_k'|$, which are oriented by K. Note that for each i $T_i - \partial M_k'$ is an open annulus whose core, β_i, is setwise isotopic to α_i on T_i. Hence, $(\mathrm{Cl}(M_k' - \widetilde{X}_{i_0} - \bigcup_{i=1}^{u-1} N(\gamma_i)), \gamma_u)$ is an arc, (B', γ), say. The proof of Assertion 1 implies that each component of $\partial M_k' \cap \partial M_n$ is essential in $\partial M_k' - \partial a_k'$, so that β_i is essential in $\partial B' - \partial\gamma$ (cf. Figure 5). Therefore, (B', γ) is a trivial ball by Lemma 3.1. Orient $\bigcup_{i=1}^r \beta_i \subset \partial B' - \partial\gamma$ by γ, then $(M_k', a_k')\widehat{\ } = (\widetilde{X}_{i_0} \cup (B', \gamma))\widehat{\ }$ is induced from $(\widetilde{X}_{i_0}, \bigcup\beta_i)$. Since $\beta_i = \pm\alpha_i$ up to isotopy in T_i, the first part of Assertion 3 is proved.

Next we assume, without loss of generality, that S_{m-1}' is one of the boundary components of $\partial M_k'$. Let $Y = M - \mathrm{Int}\,\widetilde{X}_{i_0}$ with $\partial Y = T_1 \cup \cdots \cup T_r$. Paste Y and r solid tori $V_1, \ldots, V_r$ via r homeomorphisms $f_i: T_i \to \partial V_i$, $1 \le i \le r$, such that $f_i(T_i - \partial M_k')$ is a longitudinal open annulus of V_i. Then we obtain $(M'', K'') = \mathscr{X}''$, where $M''\#\widehat{M}_n \cong M$ and $K = K''$. The spheres $\bigcup_{i=1}^{n-2} S_i$ and $\bigcup_{j=1}^{m-2} S_j'$ remain in $\mathscr{X}''$ and give the prime decompositions $\mathscr{X}'' \cong \mathscr{X}_1 \# \cdots \# \mathscr{X}_{n-1}$ and $\mathscr{X}'' \cong \mathscr{X}_1' \# \cdots \# \mathscr{X}_{m-1}'$, respectively. This establishes Assertion 3.

Theorem 4.1 follows from Assertion 3 and the following assertion by induction on n.

Assertion 4. $(M_k', a_k')^\frown \cong (M_n, a_n)^\frown$.

We use the same notation as in the proof of Assertion 3. Suppose that $(M_k', a_k')^\frown \not\cong (M_n, a_n)^\frown$. Then, there exists l such that $\beta_l = -\alpha_l$ up to isotopy in T_l. Take an unoriented circle $C \subset T_l \cap \partial B \cap \partial B'$ which is setwise isotopic to the circles α_l and β_l. Let D be an oriented disk in B such that D meets the arc a_n transversely in a single point with $D \cdot a_n = 1$ and $D \cap \partial B = C$. Similarly define a disk D' and B'. Since $\beta_l = -\alpha_l$, D and D' give C the opposite orientations. We may assume that $S = D \cup D'$ is the oriented immersed sphere in M with $S \cdot K = 2$. Then, Assertion 4 will be proved by the following lemma and the assumption that $\mathcal{H}$ is order preserving.

Lemma 4.2. *Suppose that $\mathcal{H} = (M, K)$ is irreducible and primely decomposable with $\mathcal{H} = \mathcal{H}_1 \# \mathcal{H}_2$, where $\mathcal{H}_2$ is an order preserving knot. If $\mathcal{H}$ admits an oriented immersed sphere S meeting K transversely in two points with $S \cdot K = 2$, then $\mathcal{H}_1$ admits an embedded sphere meeting K_1 in the same way as S meets K, so that $\mathcal{H}_1$ and $\mathcal{H}$ are order reversing.*

To prove this lemma we will use the annulus theorem stated in [2, VIII.13]. Refer to [2] for the definitions of Haken manifold pairs, nondegenerate maps, and essential maps.

Proof of Lemma 4.2. Let S' be the sphere giving the decomposition $(M, K) = \mathcal{H}_1 \# \mathcal{H}_2$. Let (M_1, a_1) and (M_2, a_2) denote the closures of two components of $(M, K) - S'$ with $(M_i, a_i)^\frown = \mathcal{H}_i$. The immersed sphere S can be moved so as to meet a_1 transversely in two points by an ambient isotopy keeping K setwise fixed.

Take some regular neighborhood $N(K)$ so that $S - \text{Int}\,N(K)$ is an immersed annulus, A, in $E(K)$ with ∂A disjoint circles on $\partial E(K)$. Let $f: (S^1 \times I, S^1 \times \partial I) \to (E(K), \partial E(K))$ be a map of pairs such that $f(S^1 \times I) = A$ and $f | S^1 \times \partial I$ is an embedding. If $\partial E(K)$ is compressible, then $\mathcal{H} \cong (V_1 \cup_{\partial V_1 = \partial V_2} V_2, c)$, where $V_1 \cong V_2 \cong S^1 \times D^2$, and c is a core of V_2. Now $\mathcal{H}$ is primely decomposable, so $V_1 \cup V_2 \neq S^1 \times S^2$. Since every 2-sphere in $V_1 \cup V_2$ ($\neq S^1 \times S^2$) is separating, $\mathcal{H}$ does not admit S satisfying the assumption of Lemma 4.2, a contradiction. We may assume that $\partial E(K)$ is incompressible, thus $(E(K), \partial E(K))$ is a Haken manifold pair. The map f is nondegenerate because $f_*: \pi_1(S^1 \times I) \to \pi_1(E(K))$ is injective and the assumption $S \cdot K \neq 0$ assures that f is an essential map of pairs. By the annulus theorem there exists an embedding $h: (S^1 \times I, S^1 \times \partial I) \to (E(K), \partial E(K))$ with $h | S^1 \times \partial I = f | S^1 \times \partial I$. Let us denote the properly embedded annulus $h(S^1 \times I)$ by also A. Note that A gives a sphere whose intersection number with K is two, or equivalently A and one of the components of $\partial N(K) - \partial A$ form a Klein bottle.

Now A can be isotoped with ∂A fixed so that $A \cap M_2$ are properly embedded annuli $A_1, \ldots, A_n$ whose boundary circles are essential in $S' - K$, using the same argument as in the proof of Assertion 1. Let B_i be the annulus on S' bounded by two circles ∂A_i. Since $\mathscr{K}_2$ is order preserving, $A_i \cup B_i$ is a torus for all i. If $B_i \cap B_j \neq \varnothing$, then either $B_i \subset B_j$ or $B_j \subset B_i$.

Modify A as follows. Replace A_i by B_i for all i and push into $\operatorname{Int} M_1$ these annuli B_i's starting with the innermost ones (see Figure 6), then we get a properly embedded annulus in $M_1 - \operatorname{Int} N(K)$. Since $A_i \cup B_i$ is a torus for all i, this annulus gives the sphere S^* in M_1 such that $S^* \cdot K = 2$. The two annuli $\partial N(S^*) - \operatorname{Int} N(K)$ are order reversing ones. This completes the proof of Lemma 4.2.
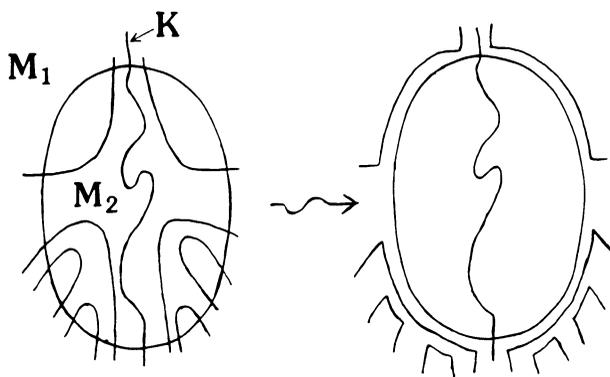

FIGURE 6

In proving Theorem 4.1 we also obtain the following, which will be used in the next section.

Remark 4.3. Suppose that $\mathscr{K} \cong \#_{i=1}^{n} \mathscr{K}_i$ and $\mathscr{K} \cong \#_{i=1}^{m} \mathscr{K}_i'$ are prime decompositions, and some $\mathscr{K}_{i_0}$ is order preserving.

(1) Using the same argument as in the proof of Assertions 1–3, we get $\mathscr{K}_{j_0}'$ a conjugation of $\mathscr{K}_{i_0}$ such that $\mathscr{K}_1 \# \cdots \# \widehat{\mathscr{K}_{i_0}} \# \cdots \# \mathscr{K}_n \cong \mathscr{K}_1' \# \cdots \# \widehat{\mathscr{K}_{j_0}'} \# \cdots \# \mathscr{K}_m'$ where the knots under $\wedge$'s are deleted.

(2) If $\mathscr{K}_{i_0} \not\cong \mathscr{K}_{j_0}'$, then by the proof of Assertion 4 $\mathscr{K}$ admits a sphere S meeting K transversely in two points with $S \cdot K = 2$.

5. NONUNIQUENESS

First we consider typical examples of order reversing knots.

Example 1. A knot $\mathscr{R}$ is constructed from $(I \times S^2, I \times \{p_1, p_2\})$, where $p_1, p_2 \in S^2$, by identifying $\{0\} \times S^2$ and $\{1\} \times S^2$ so as to yield a circle in $S^1 \times S^2$. See Figure 7(a). It is easily checked that $\mathscr{R}$ is order reversing, prime, and $\mathscr{R} \cong -\mathscr{R}$. $E(R)$ is an annulus bundle over S^1 (in fact, a twisted I-bundle over a Klein bottle).
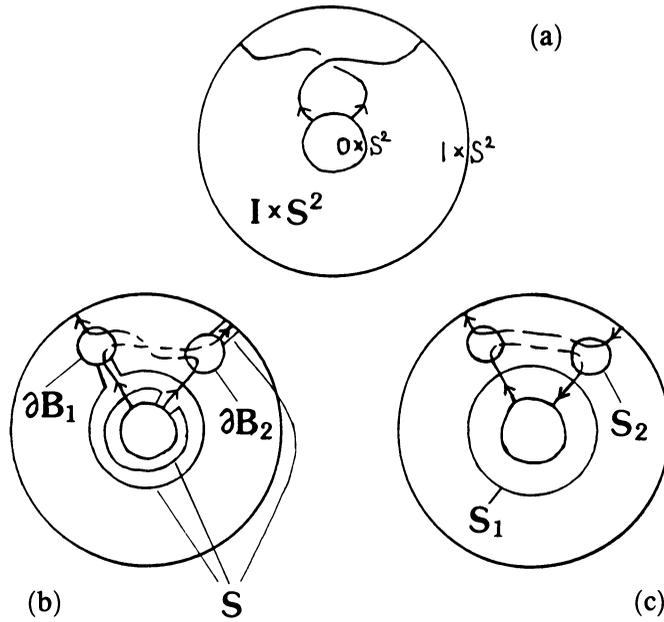
FIGURE 7

Example 2. Let $(X, \alpha_1 \cup \alpha_2) = (S^1 \times S^1 \times I, S^1 \times * \times \{0, 1\})$ where α_1 and α_2 are oriented to be parallel. The knot induced from $(X, \alpha_1 \cup -\alpha_2)$ is constructed from $(S^1 \times S^2, t_1 \cup t_2)$, where $t_i = S^1 \times p_i$, $i = 1, 2$, are oriented to be parallel. That is, first remove two disjoint 3-balls, B_1 and B_2, meeting t_1, t_2 respectively in an unknotted spanning arc. Then identify the resulting two boundary pairs $(\partial B_i, \partial B_i \cap t_i)$ via an orientation reversing homeomorphism to obtain the knot. This is because $(S^1 \times S^2, t_1 \cup t_2) - \text{Int}(B_1 \cup B_2) = X \cup_{\alpha_1, -\alpha_2} 2(h^2, \text{cocore})$. This knot is $\mathscr{R} \# \mathscr{R}$. The sphere S shown in Figure 7(b) gives this decomposition. The fact that each summand is $\mathscr{R}$ is left to the reader as the exercise.

Example 3. With the notation of Example 2, the knot induced from $(X, \alpha_1 \cup \alpha_2)$ is called $\mathscr{U}$. It is easily checked that the knot type of $\mathscr{U}$ is also induced from $(\pm X, \pm(\alpha_1 \cup \alpha_2))$, which are homeomorphic to each other via orientation preserving maps. $\mathscr{U}$ is order reversing, for a pair of S_1 and S_2 shown in Figure 7(c) give the order reversing annuli $S_i - \text{Int} N(U)$ in $E(U)$. Recall Observation 1 of §3, then it follows $E(U) = X \cup D_2 \times S^1 = D_1 \times S^1 \cup D_2 \times S^1$, where D_n denotes a disk-with-n holes as in §3. Inspecting the unions above, we see that $E(U) = (D_1 \cup D_2) \times S^1 \cong \mathring{T}^2 \times S^1$, where $\mathring{T}^2$ is a torus with a hole. Under this homeomorphism from $\mathring{T}^2 \times S^1$ to $E(U)$, a circle $* \times S^1$ in $\partial \mathring{T}^2 \times S^1$ corresponds to a meridian of $\mathscr{U}$.

Lemma 5.1. $\mathscr{U}$ is prime.

Proof. Identify $E(K)$ with $\overset{\circ}{T}^2 \times S^1$ as above. If there is a sphere S giving a nontrivial decomposition of $\mathscr{U}$, then $\overset{\circ}{T}^2 \times S^1$ contains a separating, non-∂-parallel, properly embedded annulus $A = S - \operatorname{Int} N(U)$ such that each component of ∂A is setwise isotopic to $* \times S^1$ in $\partial \overset{\circ}{T}^2 \times S^1$. Using a standard argument, we see that such an annulus does not exist in $\overset{\circ}{T}^2 \times S^1$. Hence $\mathscr{U}$ is a prime knot.

Proposition 5.2. *All the order reversing prime knots are $\mathscr{R}$ and $\mathscr{U}$.*

Proof. Let (M,a) be an arc with $(M,a)\widehat{}$ order reversing and prime. Let A_1 and A_2 be the order reversing annuli of (M,a) as in Figure 8(1). For $i = 1,2$ let T_i denote the closed surface which is the union of A_i and an annulus, A_i', on $\partial M - \partial a$ bounded by ∂A_i, so that T_i is a Klein bottle or a torus.

Case 1. T_1 *or* T_2 *is a Klein bottle.* With no loss of generality we assume that T_1 is a Klein bottle. Let $(B,a) = (M - \operatorname{Int} N(T_1), a)$. Then ∂B is a sphere and $B \cap N(T_1)$ is an annulus. See Figure 8(2). Applying Lemma 3.1 to (B,a) and $N(T_1)$, we obtain (B,a) is a trivial ball pair. Next let S denote a sphere $(\partial M - A_1') \cup A_1$. See Figure 8(3). Then $M - \operatorname{Int} N(S) \cong B$ is a
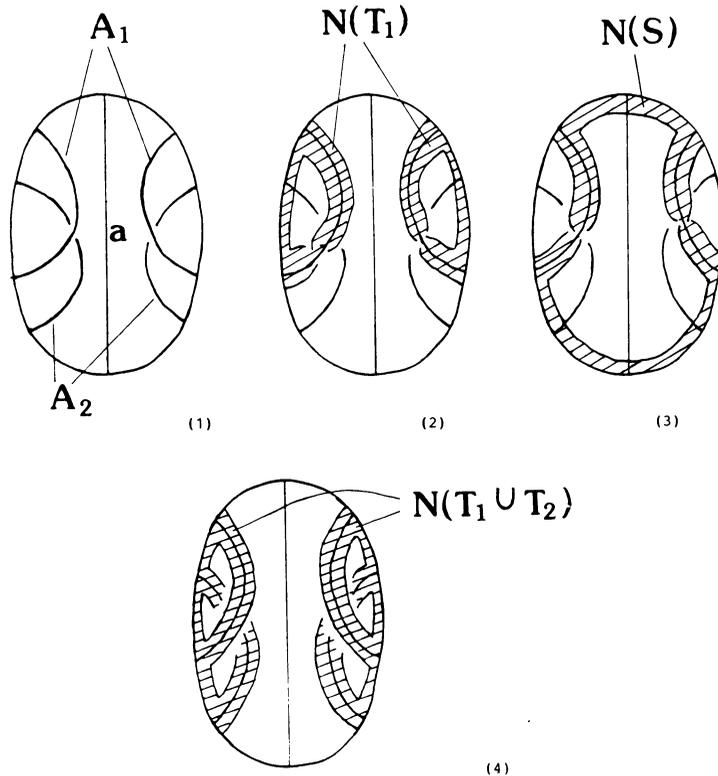

FIGURE 8

3-ball, and $\partial M \cap (M - \text{Int } N(S))$ is an essential annulus in $\partial M - \partial a$. Hence, $(M, a)^\frown - \text{Int } N(S) \cong (I \times S^2, I \times \{p_1, p_2\})$. More easily we have $(N(S), N(S) \cap a) \cong (I \times S^2, I \times \{p_1, p_2\})$. Example 1 shows that $(M, a)^\frown \cong \mathscr{R}$.

Case 2. *Each* T_i *is a torus.* Then each $N(T_i)$ is homeomorphic to $S^1 \times S^1 \times I$. Since $N(T_1) \cap N(T_2) \cong I \times I \times I$, it follows that $N(T_1 \cup T_2) \cong \overset{\circ}{T}{}^2 \times S^1$. (See Figure 8(4).) Applying Lemma 3.1 to $N(T_1 \cup T_2)$ and $(M - \text{Int } N(T_1 \cup T_2), a)$, we see that the latter is a trivial ball pair, so that $N(T_1 \cup T_2) \cong M - \text{Int } N(a)$. Moreover, it follows that $\overset{\circ}{T}{}^2 \times S^1$ is homeomorphic to $E((M, a)^\frown)$ with $* \times S^1 (\subset \partial \overset{\circ}{T}{}^2 \times S^1)$ corresponding to a meridian of $(M, a)^\frown$. The observation in Example 3 shows that $(M, a)^\frown = \mathscr{U}$.

Lemma 5.3. *There exists no order preserving knot which is a conjugation of either* $\mathscr{U}$ *or* $\mathscr{R}$.

Proof. If $\mathscr{R}$ had a nontrivial inducing-pair, $E(R)$ would contain an incompressible, nonseparating torus by Lemma 3.2. On the other hand, an incompressible torus in an annulus bundle over S^1 is ∂-parallel, so separating. Thus $\mathscr{R}$ itself is the only conjugation of $\mathscr{R}$.

We show that $\mathscr{U}$ has just one nontrivial inducing-pair given in Example 3, so that $\mathscr{R} \# \mathscr{R}$ and $\mathscr{U}$ are all the conjugation of $\mathscr{U}$. Let us identify $E(U)$ with $\overset{\circ}{T}{}^2 \times S^1$ in the same way as in Example 3. It is easily checked that each incompressible torus in $\overset{\circ}{T}{}^2 \times S^1$ is isotopic to $l \times S^1$ for some essential circle l in $\overset{\circ}{T}{}^2$. If $E(U)$ contains incompressible tori, arbitrary two of them are parallel. Thus, by Lemma 3.2 a degree of inducing-pair of $\mathscr{U}$ is 1 or 2. Suppose that $\mathscr{U}$ has two inducing-pairs of degree 2. Let $c_i \times S^1 (i = 1, 2)$ or $c_j' \times S^1 (j = 1, 2)$ be a pair of incompressible tori in $E(U)$ corresponding to each inducing-pair, where c_i or c_j' is an unoriented circle in $\overset{\circ}{T}{}^2$. Since there is an orientation preserving homeomorphism $f: \overset{\circ}{T}{}^2 \to \overset{\circ}{T}{}^2$ such that $f|\partial \overset{\circ}{T}{}^2 = \text{id}$ and $f(\{c_i\}) = \{c_j'\}$, these inducing-pairs are the same ones. This completes the proof of Lemma 5.3.

Here we are ready to state and prove our main theorems. A knot $n\mathscr{K}$ will denote $\mathscr{K} \# \cdots \# \mathscr{K}$ (n times).

Theorem 5.4. *Suppose that* $\mathscr{K}$ *is primely decomposable. If*

$$\mathscr{K} \cong \mathscr{K}_1 \# \cdots \# \mathscr{K}_t \# n\mathscr{R} \# m\mathscr{U} \# N$$
$$\cong \mathscr{K}_1' \# \cdots \# \mathscr{K}_{t'}' \# n'\mathscr{R} \# m'\mathscr{U} \# N'$$

are prime decompositions such that neither $\mathscr{K}_i$ *nor* $\mathscr{K}_j' \cong \mathscr{R}$ *or* $\mathscr{U}$, *then*

(1) $t = t'$. $\mathscr{K}_i'$ *is a conjugation of* $\mathscr{K}_i$ *up to order,* $1 \leq i \leq t$.

(2) $n + 2m = n' + 2m'$. Moreover, if $n = 0$, then $n' = 0$, and $\mathcal{K}_i \cong \mathcal{K}_i'$ up to order $1 \leq i \leq t$.

(3) $N \cong N'$.

As immediate consequences we obtain:

Corollary 5.5. *If $\mathcal{K}$ admits prime decompositions with distinct lengths, then every prime decomposition for $\mathcal{K}$ contains $3\mathcal{R}$ or $\mathcal{U}\#\mathcal{R}$.*

Corollary 5.6. *If $\mathcal{K}$ has a prime decomposition which does not contain $\mathcal{R}$, then $\mathcal{K}$ does not admit any other prime decompositions.*

Proof of Theorem 5.4. By Lemma 2.2 $N \cong N'$, thus we assume that $\mathcal{K}$ is irreducible. We will make some observations.

Case 1. $t > 0$. By Remark 4.3 (1) and Lemma 5.3 for any $\mathcal{K}_i$ there exists $\mathcal{K}_j'$ which is a conjugation of $\mathcal{K}_i$, and we have

$$\mathcal{K}_1 \# \cdots \# \widehat{\mathcal{K}_i} \# \cdots \# \mathcal{K}_t \# n\mathcal{R} \# m\mathcal{U} \cong \mathcal{K}_1' \# \cdots \# \widehat{\mathcal{K}_j'} \# \cdots \# \mathcal{K}_{t'}' \# n'\mathcal{R} \# m'\mathcal{U}$$

where the knots under $\wedge$'s are deleted. Suppose $n = 0$. If $\mathcal{K}_i \not\cong \mathcal{K}_j'$, then by Remark 4.3 (2) and Lemma 4.2 there is a sphere S in $2mS^1 \times S^2$ with $S \cdot mU = 2$. This contradicts the fact that mU is homologous to zero, so that $\mathcal{K}_i \cong \mathcal{K}_j$.

Case 2. $t = 0$. Then,

$$\mathcal{K} \cong n\mathcal{R} \# m\mathcal{U} \cong \mathcal{K}_i' \# \cdots \# \mathcal{K}_{t'}' \# n'\mathcal{R} \# m'\mathcal{U}.$$

By Remark 4.3 (1) $\mathcal{K}_1'$ is a conjugation of either $\mathcal{R}$ or $\mathcal{U}$, contradicting Lemma 5.3. Therefore, $t' = 0$ and $n + 2m = n' + 2m'$. If $n = 0$, then $n' = 0$, for the knot mU is homologous to zero.

These observations and Remarks 4.3 (1) establish our theorem by induction on t.

In fact, the conclusion of Theorem 5.4 is best possible.

Theorem 5.7. *If $\mathcal{K}_1$ is a conjugation of $\mathcal{K}_2$, then $\mathcal{R}\#\mathcal{K}_1 \cong \mathcal{R}\#\mathcal{K}_2$. In particular, $\mathcal{R}\#\mathcal{R}\#\mathcal{R} \cong \mathcal{R}\#\mathcal{U}$.*

Proof. Suppose that $\mathcal{K}_1$ is induced from $(X, \bigcup_{i=1}^n \alpha_i)$ and $\mathcal{K}_2$ is induced from $(X, \bigcup_{i=1}^n \alpha_i')$ with α_i and α_i' on the same component of ∂X for all i. Let $s = \#\{i \mid \alpha_i = -\alpha_i'$ up to isotopy in $\partial X\}$. We will prove the theorem by induction on s.

If $s = 0$, it is trivial.

If $s \geq 1$, then we may assume that $\alpha_1 = -\alpha_1'$, and let $\mathcal{K}_3$ be the knot induced from $(X, -\alpha_1 \cup \bigcup_{i=2}^n \alpha_i)$. We claim that $\mathcal{R}\#\mathcal{K}_1 \cong \mathcal{R}\#\mathcal{K}_3$, thus $\mathcal{R}\#\mathcal{K}_1 \cong \mathcal{R}\#\mathcal{K}_2$ by induction.

Let (B, a) be a trivial ball pair. Let $\beta_1, \ldots, \beta_n$ be pairwise parallel, essential circles on $\partial B - \partial a$ such that $\mathrm{lk}(a, \beta_i) = 1$, and β_1 bounds a disk in ∂B missing $\beta_j (j \neq 1)$, and such that there is an annulus ω in ∂B with
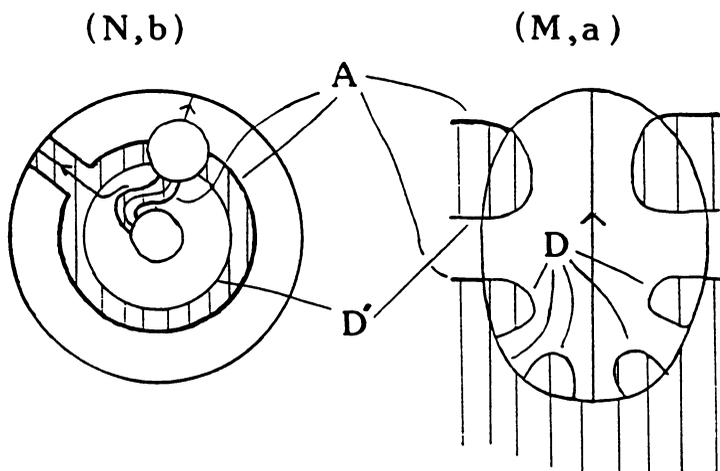
FIGURE 9

$\partial \omega = \beta_1 \cup -\beta_2$ and $\beta_i \cap \omega = \varnothing$ $(i \neq 1, 2)$. Set $(M, a) = X \cup_{\alpha_i = \beta_i} (B, a)$, so $(M, a)^{\wedge} \cong \mathscr{K}_1$. Let $N(\beta_i)$ denote the annulus component of $X \cap B$ containing β_i. Let D denote a disk in ∂B such that ∂D is one of the components of $\partial N(\beta_2)$ with $\beta_2 \subset D$ and $\beta_1 \cap D = \varnothing$.

Next we take a nonseparating disk D' properly embedded in the arc $\mathscr{R}^- = (N, b)$ such that D' meets b transversely in just one point, $\partial D'$ is an essential circle in $\partial N - b$. Take a properly embedded annulus A in $N - b$ such that A and an annulus in ∂N bounded by ∂A form a Klein bottle, and $A \cap D' = \varnothing$ (see Figure 9). Note that (a) $A \cup D'$ separates (N, b) into two trivial ball pairs (B_i, a_i), $i = 1, 2$, (b) $\partial B_i \cap \partial N$ consists of an annulus u_i and a disk v_i, and (c) arbitrary essential circles $c_1 \subset u_i$ and $c_2 \subset v_i - b$ are parallel on ∂B_i if and only if they are not parallel on ∂N.

We obtain $\mathscr{R} \# \mathscr{K}_1$ and the sphere $S = D \cup A \cup N(\beta_1) \cup D'$ in $\widehat{M} \# S^1 \times S^2$ by pasting (M, a) and (N, b) via an orientation reversing homeomorphism $f: (\partial N, \partial b) \to (\partial M, \partial a)$ with $f(\partial D' \cup \partial A) = \partial N(\beta_1) \cup \partial D$. We may assume that $f(u_1) \subset \partial X$. Then S separates $\mathscr{R} \# \mathscr{K}_1$ into two arcs. One is

$$(B_1, a_1) \cup X \cong X \cup_{-\alpha_1, \alpha_2, \ldots, \alpha_n} n(h^2, \text{cocore}) \cup (n - 1)(h^1, \text{core}) \cong \mathscr{K}_3^- ;$$

another is $(B_2, a_2) \cup (B, a) \cong \mathscr{R}^-$, where $B_2 \cap B = u_2 \cup v_2$. Hence, $\mathscr{R} \# \mathscr{K}_1 \cong \mathscr{R} \# \mathscr{K}_3$. This completes the proof.

Remark. We can prove that an arbitrary 3-manifold contains a prime knot $\mathscr{K}$ with $\mathscr{K} \not\cong -\mathscr{K}$. Combining this fact with Theorem 5.7, we obtain: every closed 3-manifold having $S^1 \times S^2$ as a summand contains a knot which can be decomposed into prime knots in more than one way.

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