TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 314, Number 1, July 1989

ON $H_*(\Omega^{n+2}S^{n+1};\mathbf{F}_2)$

THOMAS J. HUNTER

ABSTRACT. In this paper, we study $H_*\Omega^{n+2}S^{n+1}$. Here ΩX denotes the space of pointed maps $S^1 \to X$, and H_* represents homology modulo 2. We show that the Eilenberg-Moore spectral sequence $\mathrm{Tor}^{**}_{H^*\Omega_0^{n+1}S^{n+1}}(F_2, F_2) \Rightarrow H^*\Omega^{n+2}S^{n+1}$ collapses, and we identify the kernel of the Whitehead product map $\Omega^{n+1}p_*: H_*\Omega^{n+3}S^{2n+1} \to H_*\Omega^{n+1}S^n$. These observations yield two different descriptions of $H_*\Omega^{n+2}S^{n+1}$ up to extension.

1. INTRODUCTION

Let X be a reasonable pointed topological space. That is, let X be a compactly generated topological space with the homotopy type of a locally finite CW-complex, and a nondegenerate base point. Let $\Omega^k X$ denote the space of pointed maps of the k-sphere into X with the compactly generated, compact-open topology. In particular, ΩX is the space of all loops in X. All homology groups will be taken to have coefficients in $\mathbf{F}_2$.

In certain cases, the computation of $H_*\Omega^k X$ in terms of H_*X has been carried out explicitly. In [4], F. Cohen determines $H_*\Omega^k S^l X$ when $k \leq l$. In particular we have $H_*\Omega^k S^l$ is a primitively generated polynomial algebra when $0 < k < l$. (The calculation of $H_*\Omega^k S^l$ for $k < l$ was carried out by Araki and Kudo in [1].) In the case that $0 < k = l$, we have $\Omega^l S^l$ is no longer connected; $\pi_0\Omega^l S^l = \pi_l S^l = Z$. All of the path components of $\Omega^l S^l$ are homotopy equivalent and we shall call the component of the constant map $\Omega_0^l S^l$. It is still the case that $H_*\Omega^l S^l$ is a polynomial algebra, but it is no longer primitively generated. In [4], explicit polynomial generators for $H_*\Omega_0^l S^l$ are constructed along with inductive formulae for calculating the coproduct and the actions of the Steenrod and Dyer-Lashof algebras. In [9], R. Wellington dualizes and analyzes the formulae of [4] in order to produce a good description of $H^*\Omega_0^l S^l$ as an algebra over the Steenrod algebra. See [1, 4 and 9] for details of the descriptions of these Hopf algebras.

Received by the editors March 17, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 55U99.
This paper constitutes a revised portion of the author's Ph.D. disseratation written under Professor Franklin P. Peterson at the Massachusetts Institute of Technology.

If X is not at least a k-fold suspension, $H_*\Omega^k X$ can be hard to calculate. Very little is known about $H_*\Omega^k X$ when X is not even k-connected. In this paper, we study $H_*\Omega^{n+2}S^{n+1}$. (This particular choice of k and l will be justified by a later simplification in the notation.)

The case $n = 0$ and $n = 1$ are both easy. $\Omega^2 S^1$ is contractable. The Hopf map $S^3 \to S^2$ (that is, the generator of $\pi_3 S^2$) induces an isomorphism on π_i for $i > 2$, hence it induces a weak equivalence which is a map of three-fold loop spaces $\Omega^3 S^3 \to \Omega^3 S^2$.

We will consider two approaches to calculating $H_*\Omega^{n+2}S^{n+1}$. The first approach follows from thinking of $\Omega^{n+2}S^{n+1}$ as $\Omega\Omega_0^{n+1}S^{n+1}$. This approach produces the following Eilenberg-Moore spectral sequence which arises from the geometric cobar construction.

$$\mathrm{Tor}^{**}_{H^*\Omega_0^{n+1}S^{n+1}}(F_2, F_2) \Rightarrow H^*\Omega^{n+2}S^{n+1} \tag{1}$$

We shall discuss this spectral sequence in detail in §2. The second approach arises from considering $\Omega^{n+2}S^{n+1}$ as the base space in the following fibration.

$$\Omega^{n+3}S^{2n+1} \xrightarrow{\Omega^{n+1}p} \Omega^{n+1}S^n \xrightarrow{\Omega^{n+1}e} \Omega^{n+2}S^{n+1} \tag{2}$$

This approach produces the following Eilenberg-Moore spectral sequence which arises from the geometric bar construction.

$$\mathrm{Tor}^{H_*\Omega^{n+3}S^{2n+1}}_{**}(H_*\Omega^{n+1}S^n, \mathbf{F}_2) \Rightarrow H_*\Omega^{n+2}S^{n+1} \tag{3}$$

In §3, we will discuss this spectral sequence, and the fibration (2) in detail. In particular, we shall show that the spectral sequence collapses at E^2 and hence is entirely determined by the algebra map $\Omega^{n+1}p_*$.

We will obtain our results by comparing these two geometrically quite different views of the space $\Omega^{n+2}S^{n+1}$. In §2, we will show that the spectral sequence (1) provides an upper bound on the size of $H^*\Omega^{n+2}S^{n+1}$. In §3, we will show that the spectral sequence (3) provides a lower bound on the size of $H_*\Omega^{n+2}S^{n+1}$ which will depend on the size of $H_*\Omega^{n+1}S^n$. In §4 we will show that these bounds coincide.

As consequences we will obtain the following two theorems. Theorem 1.1 describes $H^*\Omega^{n+2}S^{n+1}$ in a way that relates most closely to the cohomology algebra structure.

Theorem 1.1. *For all $n \geq 2$ we have $E_2 = E_\infty$ in the Eilenberg-Moore spectral sequence,* (1).

Theorem 1.2 partially describes the map $\Omega^{n+1}p_*$. If V is a graded vector space, PV will denote the polynomial algebra on V, EV the exterior algebra, and P_iV the polynomial algebra truncated at height 2^i. A sequence, I, of nonnegative integers will be assumed to be admissible (i.e. nondecreasing) unless otherwise stated. αI will denote the first entry in I and ωI will denote the final

entry. We will write Q_I for the product $Q_{\alpha I}\cdots Q_{\omega I}$, where $Q_i\colon H_q \to H_{2q+i}$ is as in [1].

Theorem 1.2. *For $n > 2$, and n odd, $\Omega^{n+1}p_*$ is the zero map. For $n > 2$, and n even, the kernel of $\Omega^{n+1}p_*$ is the subalgebra of*

$$H_*\Omega^{n+3}S^{2n+1} = P(Q_I\iota_{n-2} \mid \alpha I > 0, \omega I < n+3)$$

generated by the $Q_I\iota_{n-2}$ with I having at least one odd entry. When $n = 2$, the kernel of $\Omega^3 p_*$ is equal to the kernal of Sq_{0*} (the dual of the cohomology algebra squaring map.)

A corollary of Theorem 1.2 is the following partial description of the homology algebra $H_*\Omega^{n+2}S^{n+1}$.

Corollary 1.3. *When n is odd and $n > 2$ there is an isomorphism of algebras*

$$H_*\Omega^{n+2}S^{n+1} \cong H_*\Omega^{n+1}S^n \otimes H_*\Omega^{n+2}S^{2n+1}.$$

When n is even and $n \geq 2$, there is an isomorphism of algebras,

$$H_*\Omega^{n+2}S^{n+1} \cong \operatorname{Im}\Omega^{n+1}e_* \otimes P(\overline{Q_I\iota_{n-1}} \mid \mu I \text{ even }, \omega I \leq 2n),$$

where μI is the product of the entries in I, and $\Omega^n p_*\overline{Q_I\iota_{n-1}} = Q_I\iota_{n-1} \in H_*\Omega^{n+2}S^{2n+1}$.

2. An upper bound: The E_2 term for the path-loop spectral sequence

In this section, we will examine the spectral sequence which appeared in the Introduction as (1). This examination will yield an explicit upper bound on the size of the Hilbert series $f(H^*\Omega^{n+2}S^{n+1})$. We will use this upper bound together with the lower bound of §3 to obtain our results in §4.

Wellington [9] gives a description of the cohomology algebra $H^*\Omega_0^{n+1}S^{n+1}$ sufficient for calculating the E_2 term of (1). We will state the theorems and definitions from [9] that we need here. We will freely restrict to the case $p = 2$ and use lower notation for the Dyer-Lashof algebra. Let M_nS^0 be the graded $\mathbf{F}_2$-module generated by the symbols $Q_I\iota_0$, where I ranges over all I with I admissible (i.e. nonnegative and nondecreasing), $\omega I \leq n$, I nonempty, and I not equal to a sequence $(0,\ldots,0)$ of only zeros. The grading in M_nS^0 is defined by treating the symbols $Q_I\iota_0$ as if they were elements of the Dyer-Lashof algebra applied to a zero dimensional class. Let $[n] \in H_0\Omega^{n+1}S^{n+1}$ represent the component of the degree n map, and $*$ be the Pontryagin product. We then have an embedding $M_nS^0 \to H_*\Omega^{n+1}S^{n+1}$ given by mapping $Q_I\iota_0$ to $Q_I[1] * [-2^{l_I}]$. We will identify M_nS^0 with its image under this embedding. Wellington's Lemma 1.6 then becomes the following lemma.

Lemma 2.4 (Cohen). *For $n > 0$, $H_*\Omega_0^{n+1}S^{n+1}$ is the free commutative algebra generated by M_nS^0 modulo the ideal generated by the set $\{Q_0x - x^2\}$.*

Give $M_n^*S^0 = \hom_{\mathbf{F}_2}(M_nS^0, \mathbf{F}_2)$ the structure of a restricted abelian Lie algebra by letting the restriction, ξ, be defined by $\xi x = \mathrm{Sq}_0 x$, with Sq_0 acting via the Nishida formulae. That is, let $\xi_* Q_I \iota_0 = Q_{I/2}\iota_0$. (Here, we let $Q_{I/2}$ denote $Q_{i_1/2}Q_{i_2/2}\cdots$ as long as each entry in I is even, otherwise $Q_{I/2}x$ is defined to be zero.) Let $VM_n^*S^0$ denote the universal enveloping algebra of the restricted Lie algebra $M_n^*S^0$. So $VM_n^*S^0$ is the free commutative algebra generated by $M_n^*S^0$ modulo the ideal generated by the set $\{\mathrm{Sq}_0 x - x^2\}$. The monomial basis given by Lemma 2.4 yields a splitting $j\colon H_*\Omega^{n+1}S^{n+1} \to M_nS^0$. Wellington's Theorem 3.7 then becomes the following theorem.

Theorem 2.5 (Wellington). *For $n > 0$ the inclusion j^* induces a map $\phi\colon VM_n^*S^0 \to H^*\Omega_0^{n+1}S^{n+1}$ which is an isomorphism of algebras.*

This theorem gives the explicit description of $H^*\Omega_0^{n+1}S^{n+1}$ that we will use. We now give suggestive names to each of the elements of $H^*\Omega_0^{n+1}S^{n+1}$, and introduce some more notation that will be useful in the sequel. If I is an admissible sequence with $\omega I \le n$, I nonempty, and I not equal to a sequence $(0, \dots, 0)$ of only zeros, let $u_I = \phi(Q_I\iota_0)^*$ (where the dual is taken with respect to the monomial basis). For an admissible sequence, I, let h_nI be the smallest number so that $2^{h_nI}\omega I > n$. Hence, 2^{h_nI} is exactly the height at which u_I is truncated in $H^*\Omega_0^{n+1}S^{n+1}$. If I is a sequence consisting only of even elements, we shall call I even, otherwise, we shall call I odd. Using this notation, Theorem 2.5 becomes the following equation.

$$H^*\Omega_0^{n+1}S^{n+1} = E(u_I \mid h_nI = 1, I \text{ odd}) \otimes_{i\ge 2} P_i(u_I \mid h_nI = i, I \text{ odd}) \tag{4}$$

In order to calculate $H^*\Omega^{n+2}S^{n+1}$ using the Eilenberg-Moore spectral sequence we need to note that $\pi_1\Omega_0^{n+1}S^{n+1}$ acts nilpotently on $H^*\Omega^{n+2}S^{n+1}$, but this is true if $n \ge 2$ since $\pi_1\Omega_0^{n+1}S^{n+1} = \mathbf{Z}/2\mathbf{Z}$ is a finite 2-group and $H^*\Omega^{n+2}S^{n+1}$ is an $\mathbf{F}_2$ vector space (see [6]).

The following corollary states how these remarks yield an upper bound on $H^*\Omega^{n+2}S^{n+1}$. We introduce some notation here to facilitate comparing the sizes of graded algebras. If A^* is a nonnegatively graded $\mathbf{F}_2$ module, then our notation for the Hilbert series for A^* is given by the following equation.

$$f(A^*) = \sum_{i\ge 0}(\dim_{\mathbf{F}_2} A^i)x^i \in \mathbf{Z}[[x]]$$

Given power series f and g, we will write $f \le g$ if each coefficient in f is less than or equal to the corresponding coefficient in g. If A is bigraded, $f(A)$ will denote the Hilbert series of the singly graded total module of A.

Corollary 2.6. *Let G_n be defined by the following equality*:

$$G_n = f(\mathrm{Tor}^{**}_{H^*\Omega_0^{n+1}S^{n+1}}(\mathbf{F}_2,\mathbf{F}_2)).$$

Then,

$$G_n \geq f(H^*\Omega^{n+2}S^{n+1}),$$

with equality if and only if the spectral sequence,

$$\mathrm{Tor}^{**}_{H^*\Omega_0^{n+1}S^{n+1}}(\mathbf{F}_2,\mathbf{F}_2) \Rightarrow H^*\Omega^{n+2}S^{n+1},$$

collapses.

Note that G_n may be computed explicitly from equation (4).

3. Toward a lower bound: The geometric EHP sequence

In §2, Corollary 2.6 we obtained an upper bound for $f(H^*\Omega^{n+2}S^{n+1})$. In this section we use the geometric EHP sequence and the action of the Dyer-Lashof algebra to make progress toward a good lower bound. In order to complete the computation, it will be necessary to use an induction which will be carried out in §4. In this section the coefficients will still be $\mathbf{F}_2$, but we will work in homology in order to make the formulae for the action of the Dyer-Lashof algebra more familiar.

Our lower bound will arise from a formula for $f(H_*\Omega^{n+2}S^{n+1})$ which depends only on $f(\ker\Omega^{n+1}p_*), f(H_*\Omega^{n+1}S^n)$, and the known quantity, $f(H_*\Omega^{n+3}S^{2n+1})$. We will then give information about $f(\ker\Omega^{n+1}p_*)$ which allows us to write down a lower bound on $f(H_*\Omega^{n+2}S^{n+1})$ which depends only on $f(H_*\Omega^{n+1}S^n)$. In §4, we will use this lower bound, together with the upper bound of §2, to achieve our results.

In subsection 3.1, we produce the promised formula for $f(H_*\Omega^{n+2}S^{n+1})$ which depends on $f(\ker\Omega^{n+1}p_*)$ and $f(H_*\Omega^{n+1}S^n)$. In subsection 3.2, we produce the promised information about $f(\ker\Omega^{n+1}p_*)$ which allows us to write down a lower bound on $f(H_*\Omega^{n+2}S^{n+1})$ which depends on $f(H_*\Omega^{n+1}S^n)$.

3.1 On the homology of the geometric EHP sequence. The geometric EHP sequence is the fiber mapping sequence of the following fibration of spaces localized at 2.

$$S^n \xrightarrow{e} \Omega S^{n+1} \xrightarrow{h} \Omega S^{2n+1} \tag{5}$$

The map h can be constructed using the James model for ΩSX. See, for example, [8]. This construction gives the following formula for the action of h on homology.

$$h_*\iota_n^{2k} = \iota_{2n}^k \tag{6}$$

The map e is the suspension map. Define $p\colon \Omega^2S^{2n+1} \to S^n$ to be the fiber of e. Then the long exact sequence in homotopy arising from the fibration (5) is

the usual EHP sequence, with E, H, and P the maps induced by e, h, and p.

The fibration (5) gives us the following fibration, which occurred in the Introduction as the fibration (2).

$$\Omega^{n+3}S^{2n+1} \xrightarrow{\Omega^{n+1}p} \Omega^{n+1}S^n \xrightarrow{\Omega^{n+1}e} \Omega^{n+2}S^{n+1} \tag{7}$$

We will use the bar construction spectral sequence to analyze the fibration, (7), to obtain a lower bound on the size of $H_*\Omega_0^{n+2}S^{n+1}$. The following lemma shows that this spectral sequence collapses, and that its E^∞ term has a particularly simple form. In this lemma, we use the notation of [7].

Lemma 3.7. *If $n > 2$, the bar construction spectral sequence,*

$$\mathrm{Tor}_{**}^{H_*\Omega^{n+3}S^{2n+1}}(H_*\Omega^{n+1}S^n, \mathbf{F}_2) \Rightarrow H_*\Omega^{n+2}S^{n+1}, \tag{8}$$

collapses at E^2 yielding the following equality of bigraded vector spaces.

$$E^\infty = \mathrm{Tor}_{**}^{H_*\Omega^{n+3}S^{2n+1}\backslash\backslash\Omega^{n+1}p_*}(\mathbf{F}_2, \mathbf{F}_2) \otimes H_*\Omega^{n+1}S^n//\Omega^{n+1}p_* \tag{9}$$

If $n = 2$, the bar construction spectral sequence,

$$\mathrm{Tor}_{**}^{H_*\Omega_0^5S^5}(H_*\Omega_0^3S^2, \mathbf{F}_2) \Rightarrow H_*\Omega_0^4S^3,$$

collapses at E^2 yielding the following equality of bigraded vector spaces.

$$E^\infty = \mathrm{Tor}_{**}^{H_*\Omega_0^5S^5\backslash\backslash\Omega^3p_*}(\mathbf{F}_2, \mathbf{F}_2) \otimes H_*\Omega_0^3S^2//\Omega^3p_*$$

Proof. If $n > 2$, $\Omega^{n+3}S^{2n+1}$ is path connected and

$$H_*\Omega^{n+3}S^{2n+1} = P(Q_I\iota_{n-2} \mid \alpha I > 0, \omega I \le n+2),$$

yielding the following equality.

$$\begin{aligned}\mathrm{Tor}_{**}^{H_*\Omega^{n+3}S^{2n+1}}(H_*\Omega^{n+1}S^n, \mathbf{F}_2) = {} & \mathrm{Tor}_{**}^{H_*\Omega^{n+3}S^{2n+1}\backslash\backslash\Omega^{n+1}p_*}(\mathbf{F}_2, \mathbf{F}_2) \\ & \otimes H_*\Omega^{n+1}S^n//\Omega^{n+1}p_*\end{aligned} \tag{10}$$

$H_*\Omega^{n+3}S^{2n+1}\backslash\backslash\Omega^{n+1}p_*$ is a sub-Hopf algebra of a polynomial Hopf algebra. So it is also a polynomial Hopf algebra. Hence $\mathrm{Tor}_{**}^{H_*\Omega^{n+3}S^{2n+1}\backslash\backslash\Omega^{n+1}p_*}(\mathbf{F}_2, \mathbf{F}_2)$ is an exterior algebra with generators in homological degree 1. Since the spectral sequence (10) is a first quadrant homology spectral sequence, (10) must collapse at E^2. This proves the lemma in the case $n > 2$.

When $n = 2$, we have almost the same situation, but we must take into account the fact that Ω^5S^5 is not path connected, so we apply the bar construction spectral sequence to the fibration (7) restricted to the components of the base points. Theorem 2.4 gives us the following equality.

$$H_*\Omega_0^5S^5 = P(Q_I\iota_0 \mid \iota I > 0, \alpha I > 0, \omega I \le 4)$$

The remarks of the preceding paragraph also apply to this situation, so the lemma is finished. □

Hence we see that the map $\Omega^{n+1}p_*$ completely determines the bar construction spectral sequence for the fibration (7). In fact we see that the size of the E^∞ term is completely determined by the size of

$$\ker \Omega^{n+1}p_* = H_*\Omega_0^{n+3}S^{2n+1}\backslash\backslash\Omega^{n+1}p_*.$$

Lemma 3.8. *For $n > 2$, we have the following equality.*

$$f(H_*\Omega^{n+2}S^{n+1}) = f(\mathrm{Tor}_{**}^{H_*\Omega^{n+3}S^{2n+1}\backslash\backslash\Omega^{n+1}p_*}(\mathbf{F}_2,\mathbf{F}_2)) \cdot \frac{f(H_*\Omega^{n+1}S^n)\cdot f(H_*\Omega^{n+3}S^{2n+1}\backslash\backslash\Omega^{n+1}p_*)}{f(H_*\Omega^{n+3}S^{2n+1})}$$

For $n = 2$, we have the following equality.

$$f(H_*\Omega_0^4S^3) = f(\mathrm{Tor}_{**}^{H_*\Omega_0^5S^5\backslash\backslash\Omega^3p_*}(\mathbf{F}_2,\mathbf{F}_2)) \cdot \frac{f(H_*\Omega_0^3S^2)\cdot f(H_*\Omega_0^5S^5\backslash\backslash\Omega^{n+1}p_*)}{f(H_*\Omega_0^5S^5)}$$

Proof. This follows from the following observations about the abelian category of bicommutative Hopf Algebras. A short exact sequence,

$$\mathbf{F}_2 \to A \to B \to C \to \mathbf{F}_2,$$

yields an equality,

$$f(B) = f(A)f(C).$$

Given any morphism $\phi\colon A \to B$ of bicommutative Hopf algebras, we have the following short exact sequences.

$$\mathbf{F}_2 \to A\backslash\backslash\phi \to A \to \mathrm{Im}\,\phi \to \mathbf{F}_2$$
$$\mathbf{F}_2 \to \mathrm{Im}\,\phi \to B \to B//\phi \to \mathbf{F}_2$$

Hence

$$f(B//\phi) = \frac{f(B)\cdot f(A\backslash\backslash\phi)}{f(A)}. \qquad \square$$

3.2 On the kernel of $\Omega^{n+1}p_*$. In this subsection we produce the promised information about $\ker\Omega^{n+1}p_*$. This will yield a lower bound on $f(\ker\Omega^{n+1}p_*)$ which, in light of Lemma 3.8, will give a lower bound on $f(H_*\Omega_0^{n+2}S^{n+1})$.

First, we note that $\Omega^n p\colon \Omega^{n+2}S^{2n+1} \to \Omega^n S^n$ induces the zero map on homology for all $n \geq 0$ since $\Omega^n e\colon \Omega^n S^n \to \Omega^{n+1}S^{n+1}$ induces a monomorphism in homology for all $n \geq 0$. Although this does not imply that $\Omega^{n+1}p_*$ induces the zero map in homology, we do have the following lemma along the lines of Cohen and Peterson [5].

Lemma 3.9. *For all* $n \geq 0$ $\Omega^{n+1}p_*: H_*\Omega^{n+3}S^{2n+1} \to H_*\Omega_0^{n+1}S^n$ *annihilates all odd dimensional primitives.*

Proof. We have the following diagram in which ϕ is the map induced by the natural projection $\overline{\Omega_0^n S^n} \to \Omega_0^n S^n$ from the universal cover.

$$\begin{array}{ccc}
PH_*\Omega^{n+3}S^{2n+1} & \xrightarrow{\Omega^{n+1}p_*} & PH_*\Omega_0^{n+1}S^n \\
\downarrow & & \downarrow q \\
QH_*\Omega^{n+3}S^{2n+1} & \xrightarrow{\Omega^{n+1}p_*} & QH_*\Omega_0^{n+1}S^n \\
\downarrow & & \downarrow \sigma \\
PH_{*+1}\Omega^{n+2}S^{2n+1} & \xrightarrow{\Omega^n p_*} & PH_{*+1}\overline{\Omega_0^n S^n} \\
\downarrow = & & \downarrow \phi \\
PH_{*+1}\Omega^{n+2}S^{2n+1} & \xrightarrow{0} & PH_{*+1}\Omega_0^n S^n
\end{array}$$

Furthermore, q is a monomorphism on odd dimensional primitives by Milnor-Moore [7], σ is a monomorphism on odd dimensional elements by the $\mathbf{F}_2$ version of a theorem of Clark (see [3]), and $\ker\phi$ is concentrated in odd degrees (see [2]). □

Lemma 3.9 and the Dyer-Lashof action give us the following corollary when n is even and $n > 2$.

Corollary 3.10. *If* n *is even, and* $n > 2$, $\Omega^{n+1}p_*Q_I\iota_{n-2} = 0$ *whenever* I *has any odd entries.*

Proof. $\Omega^{n+1}p_*: H_*\Omega^{n+3}S^{2n+1} \to H_*\Omega_0^{n+1}S^n$ commutes with the Dyer-Lashof operations Q_i for $0 \leq i \leq n$. If n is even and I is admissible and odd with $\omega I \leq n+2$, then I has an odd entry, i, with $i \leq n+1$. So we can write $Q_I = Q_J Q_i Q_K$, with J and K admissible, i odd, $i \leq n+1$, and $\omega J < i \leq \alpha K$. $\Omega^{n+1}p_*Q_iQ_K\iota_{n-2} = 0$ by Lemma 3.9, hence $\Omega^{n+1}p_*Q_I\iota_{n-2} = 0$. □

When $n = 2$, we have to alter the argument for Corollary 3.10 somewhat. The structure of $PH_*\Omega_0^5S^5$ is more complicated than that of $PH_*\Omega^{n+3}S^{2n+1}$ for $n > 2$, but we will be able to get by with a crude description. Representatives for $QH_*\Omega_0^5S^5$ are given by the following set.

$$\{Q_I\iota_0 = Q_I[1]\cdot[-2^{lI}] \mid \alpha I > 0, \omega I < 5\}$$

Since $H_*\Omega_0^5S^5$ is bicommutative Hopf algebra, [7] gives us the following exact sequence.

$$PH_*\Omega_0^5S^5 \xrightarrow{\phi} PH_*\Omega_0^5S^5 \xrightarrow{q} QH_*\Omega_0^5S^5 \xrightarrow{\eta} QH_*\Omega_0^5S^5$$

As noted in §2, the halving map η is given by the following formula.

$$\eta(Q_I[1]\cdot[-2^{lI}]) = \begin{cases} Q_{I/2}[1]\cdot[-2^{lI}] & \text{if } I \text{ is even}, \\ 0 & \text{if } I \text{ is odd}. \end{cases}$$

Let Q_i^a be the a-fold iterate of the operation Q_i. If $Q_I = Q_3^{a_3}Q_4^{a_4}$ with $a_3 > 0$ or $Q_I = Q_1^{a_1}Q_2^{a_2}Q_3^{a_3}Q_4^{a_4}$ with $a_1 > 0$, let p_I be a primitive chosen so that $Q_I[1]\cdot[-2^{lI}] + p_I$ is decomposable. If $Q_I = Q_0^{a_0}Q_1^{a_1}Q_2^{a_2}Q_3^{a_3}Q_4^{a_4}$ with $a_3 > 0$, let p_I be the primitive defined by $p_I = Q_0^{a_0}Q_1^{a_1}Q_2^{a_2}p_{I'}$, where $Q_{I'} = Q_3^{a_3}Q_4^{a_3}$. If I is even, let $q_I = Q_I[1]\cdot[-2^{lI}]$. In this notation, we have

$$\begin{aligned} H_*\Omega_0^5S^5 = {} & P(p_I \mid \alpha I > 0, \omega I < 5, I \text{ odd}) \\ & \otimes P(q_I \mid \alpha I > 0, \omega I < 5, I \text{ even}). \end{aligned} \tag{11}$$

Now consider the following diagram.

$$\begin{tikzcd} PH_*\Omega_0^5S^5 \arrow[r, "\Omega^3 p_*"] \arrow[d, "q"] & PH_*\Omega_0^3S^2 \cong PH_*\Omega_0^3S^3 \arrow[d, "q"] \\ QH_*\Omega_0^5S^5 \arrow[r, "\Omega^3 p_*"] \arrow[d, "\sigma"] & QH_*\Omega_0^3S^2 \cong QH_*\Omega_0^3S^3 \arrow[d, "\sigma\cong"] \\ PH_{*+1}\overline{\Omega^4S^5} \arrow[r, "\Omega^2 p_*"] \arrow[d] & PH_{*+1}\overline{\Omega_0^2S^2} \cong PH_{*+1}\overline{\Omega^2S^3} \arrow[d, "f"] \\ PH_{*+1}\Omega^4S^5 \arrow[r, "0"] & PH_{*+1}\Omega_0^2S^2 \cong PH_{*+1}\Omega^2S^3 \end{tikzcd} \tag{12}$$

In the right-hand column of (12), σ is an isomorphism, f is a monomorphism, and q is a monomorphism on odd dimensional primitives. Hence $\Omega^3 p_*$ is zero on odd dimensional primitives. Hence $\Omega^3 p_*$ is zero on all the p_I as in the proof of Corollary 3.10. We record the preceding remarks in the following corollary.

Corollary 3.11. $\Omega^3 p_* p_I = 0$ *whenever* I *has any odd entries.* □

When n is odd we need to use the Nishida formulae to obtain an appropriate corollary of Lemma 3.9. The following two formulae follow from [4]. If $b \geq 0$ and $a \geq 1$ we have the following equality.

$$\mathrm{Sq}_*^{2^b-1}\, Q_{n+2}^{a+b} = Q_{n+1}^b Q_{n+2}^a \tag{13}$$

If $|u| = n-2$, we have the following equality.

$$\mathrm{Sq}_*^{2^b-1}\, Q_{n+2}^b u = Q_{n+1}^b u \tag{14}$$

We now can prove the following corollary of Lemma 3.9 which shows that when n is odd the spectral sequence (8) takes a particularly simple form.

Corollary 3.12. *When n is odd and $n > 2$, $\Omega^{n+1}p_*$ is the zero map.*

Proof. Lemma 3.9 shows that when n is odd and $n > 2$, $\Omega^{n+1}p_* Q^a_{n+2}\iota_{n-2} = 0$ for all $a \geq 0$. Equations 13 and 14 then give the following equalities.

$$\begin{aligned}\Omega^{n+1}p_* Q^b_{n+1} Q^a_{n+2}\iota_{n-2} &= \Omega^{n+1}p_* \operatorname{Sq}_*^{2^b-1} Q^{a+b}_{n+2}\iota_{n-2} \\ &= \operatorname{Sq}_*^{2^b-1} \Omega^{n+1}p_* Q^{a+b}_{n+2}\iota_{n-2} \\ &= 0\end{aligned}$$

The lemma follows because $\Omega^{n+1}p_*$ commutes with the Dyer-Lashof operation Q_i whenever $0 \leq i \leq n$. □

The preceding corollaries of 3.9 together with the Lemma 3.8 yield an inductive lower bound on the size of $H_*\Omega^{n+2}S^{n+1}$ which we now explain. When n is odd and $n > 2$, the following corollary of Corollary 3.12 shows that we have an explicit formula for $f(H_*\Omega^{n+2}S^{n+1})$ in terms of $f(H_*\Omega^{n+1}S^n)$.

Corollary 3.13. *When n is odd and $n > 2$, we have*

$$f(H_*\Omega^{n+2}S^{n+1}) = f(\operatorname{Tor}_{**}^{H_*\Omega^{n+3}S^{2n+1}}(\mathbf{F}_2, \mathbf{F}_2) \otimes H_*\Omega^{n+1}S^n). \quad \square \tag{15}$$

Let F_n denote the common value of the two expressions in (15) when n is odd. When n is even and $n > 2$, we can write $H_*\Omega^{n+3}S^{2n+1}$ as follows.

$$\begin{aligned}H_*\Omega^{n+3}S^{2n+1} &= P(Q_I\iota_{n-2} \mid I \text{ even}, \alpha I > 0, \omega I < n+3) \\ &\quad \otimes P(Q_I\iota_{n-2} \mid I \text{ odd}, \alpha I > 0, \omega I < n+3)\end{aligned} \tag{16}$$

Corollary 3.10 states that when n is even, the second term of the splitting (16) is annihilated by $\Omega^{n+1}p_*$. The following lemma shows that if $\Omega^{n+1}p_*$ annihilates any more, then we would have a larger E^∞ term in the spectral sequence (8). For n even and $n > 2$, let F_n be defined by the following equality.

$$F_n = \frac{f(\operatorname{Tor}_{**}^{P(Q_I\iota_{n-2} \mid I \text{ odd}, \alpha I > 0, \omega I < n+3)}(\mathbf{F}_2, \mathbf{F}_2)) \cdot f(H_*\Omega^{n+1}S^n)}{f(P(Q_I\iota_{n-2} \mid I \text{ even}, \alpha I > 0, \omega I < n+3))}$$

Lemma 3.14. *If $n > 2$, $F_n \leq f(H_*\Omega_0^{n+2}S^{n+1})$ with equality if and only if $\Omega^{n+1}p_*$ is a monomorphism when restricted to $P(Q_I\iota_{n-2} \mid I$ even, $\alpha I > 0, \omega I < n+3)$.*

Proof. Let

$$E^2 = \operatorname{Tor}_{**}^{H_*\Omega^{n+3}S^{2n+1}\backslash\backslash\Omega^{n+1}p_*}(\mathbf{F}_2, \mathbf{F}_2) \otimes H_*\Omega^{n+1}S^n // \Omega^{n+1}p_*$$

be the E^2 term of the spectral sequence (8). This spectral sequence collapses, so we have $f(E^2) = f(H_*\Omega_0^{n+2}S^{n+1})$. From Corollary 3.10, we have

$$P(Q_I\iota_{n-2} \mid I \text{ odd}, \alpha I > 0, \omega I < n+3) \subseteq H_*\Omega^{n+3}S^{2n+1}\backslash\backslash\Omega^{n+1}p_*, \tag{17}$$

and since $H_*\Omega^{n+3}S^{2n+1}$ is a primitively generated Hopf algebra, we have equality if and only if $\Omega^{n+1}p_*$ is a monomorphism when restricted to

$P(Q_I\iota_{n-2} \mid I$ even, $\alpha I > 0, \omega I < n+3)$. This gives,

$$(18)\qquad \begin{aligned} &f(\mathrm{Tor}_{**}^{P(Q_I\iota_{n-2}\mid I \text{ odd}, \alpha I>0, \omega I<n+3)}(\mathbf{F}_2,\mathbf{F}_2)) \\ &\quad \le f(\mathrm{Tor}_{**}^{H_*\Omega^{n+3}S^{2n+1}\backslash\backslash\Omega^{n+1}p_*}(\mathbf{F}_2,\mathbf{F}_2)), \end{aligned}$$

with equality if and only if $\Omega^{n+1}p_*$ is a monomorphism when restricted to $P(Q_I\iota_{n-2} \mid I$ even, $\alpha I > 0, \omega I < n+3)$. The inequalities (17) and (18) together with Lemma 3.8 prove the lemma. □

As usual, we must alter the argument somewhat when $n = 2$. Here, using the notation of (11), an argument almost identical to that of Lemma 3.14 yields the following lemma. Let F_2 be defined by the following equation, where $|\eta| = 0$.

$$(19)\qquad F_2 = \frac{f(E([p_I] \mid \alpha I > 0, \omega I < 5, I \text{ odd}) \otimes H_*\Omega_0^3S^2 \otimes E(\eta))}{f(P(q_I \mid \alpha I > 0, \omega I < 5, I \text{ even}))}$$

Lemma 3.15. *$F_2 \le f(H_*\Omega^4S^3)$ with equality if and only if $\Omega^{n+1}p_*$ is a monomorphism when restricted to $P(Q_I\iota_{n-2} \mid I$ even, $\alpha I > 0, \omega I < n+3)$.*

Note that, since $\Omega_0^3S^2$ is homotopy equivalent to $\Omega_0^3S^3$, F_2 is an explicit power series that depends only on $H_*\Omega_0^3S^3$.

4. The induction

In this section we show inductively that, for $n \ge 2, F_n = G_n$ where F_n and G_n are as in Corollaries 3.13 and 2.6 and Lemmas 3.14 and 3.15. In view of these lemmas, when these two power series are equal their common value is $f(H_*\Omega^{n+2}S^{n+1})$, and the Eilenberg-Moore spectral sequence which appeared as (1) collapses. This allows us to write down a formula for F_{n+1}, and thus to continue our induction. The proof proceeds in three steps: the even case, the odd case, and the start of the induction. These three steps are achieved by the three lemmas of this section. When $n = 2$, the argument is a straightforward but somewhat complicated counting argument. In the cases $n > 2$ the argument is a straightforward but somewhat complicated counting argument that relies on the formula for $f(H_*\Omega_0^{n+1}S^n)$ which was achieved at the $(n-1)$th stage of the induction. The argument is simplest when n is odd because $\Omega^{n+1}p_*$ is the zero map when n is odd as was shown in Corollary 3.12.

We will start with the case $n > 2$ and n even, because in this case the arguments are the most difficult. Throughout these arguments, it should be kept in mind that in the homology bar construction spectral sequence, total degree is given by the sum of the homological and internal degrees, while in the cohomology Eilenberg-Moore spectral sequence, we must either let homological degree be nonpositive, or we must take total degree to be the difference of the integral degree and the homological degree.

Lemma 4.16. *Suppose that n is even and $E_2 = E_\infty$ in the Eilenberg-Moore spectral sequence below.*

$$\mathrm{Tor}^{**}_{H^*\Omega_0^n S^n}(\mathbf{F}_2, \mathbf{F}_2) \Rightarrow H^*\Omega^{n+1}S^n \tag{20}$$

We then have $F_n = G_n$.

Proof. In order to prove the lemma, we will provide simple systems of generators for the algebras associated to F_n and G_n, then we will produce degree preserving correspondences between these sets. This will show that $F_n = G_n$ because if V is the vector space generated by a simple system of generators for some algebra A, we have $fA = fEV$. Let $\mathscr{S}$ be the set of all admissible sequences I, with I odd, $\alpha I > 0$, and $\omega I \le n+2$. The inductive hypothesis gives the following formula for F_n.

$$F_n = \frac{f(\mathrm{Tor}^{**}_{H^*\Omega_0^n S^n}(\mathbf{F}_2, \mathbf{F}_2)) \cdot f(\mathrm{Tor}^{P(Q_I\iota_{n-2} \mid I\in\mathscr{S})}_{**}(\mathbf{F}_2, \mathbf{F}_2))}{f(P(Q_I\iota_{n-2} \mid I \text{ even}, \alpha I > 0, \omega I \le n+2))} \tag{21}$$

First we simplify our formula for F_n. A simple system of generators for the algebra $\mathrm{Tor}^{**}_{H^*\Omega_0^n S^n}(\mathbf{F}_2, \mathbf{F}_2)$ is given by the set

$$\{[u_I], \gamma_{2^i}[u_I^{2^{h_{n-1}I-1}} \mid u_I^{2^{h_{n-1}I-1}}] \mid \omega I < n, I \text{ odd}\}.$$

A simple system of generators for $\mathrm{Tor}^{P(Q_I\iota_{n-2} \mid I\in\mathscr{S})}_{**}(\mathbf{F}_2, \mathbf{F}_2)$ is given by the set, $\{[Q_I\iota_{n-2}] \mid I \in \mathscr{S}\}$. A simple system of generators for $P(Q_I\iota_{n-2} \mid I$ even, $\alpha I > 0, \omega I \le n+2)$ is given by the following set.

$$\{Q_I\iota_{n-2} \mid I \text{ even}, \alpha I \ge 0, \omega I \le n+2\} \tag{22}$$

We have the following equality for $I = (i_1, i_2, \dots)$, $\alpha I = i_1 > 0$, and $\omega I \le \frac{n}{2} + 1$.

$$|Q_0^a Q_{2I}\iota_{n-2}| = |\gamma_{2^{a+1}}[u_{I'}]|$$

In this equality, $I' = (i_1 - 1, i_2 - 1, \dots, \omega I - 1, \frac{n}{2})$ and $u_{I'}$ is defined even when I' is even by $u_{I'} = (u_{I'/(2^r)})^{2^r}$ with $I'/(2^r)$ odd. This sets up a degree preserving correspondence between the set, (22), and the set,

$$\{\gamma_{2^i}[u_I \mid u_I] \mid \omega I = \tfrac{n}{2}\}, \tag{23}$$

which is equal to the set,

$$\{\gamma_{2^i}[u_I^{2^{h_{n-1}I-1}} \mid u_I^{2^{h_{n-1}I-1}}] \mid i \ge 0, \omega I < n, 2^{h_{n-1}I}\omega I = n, I \text{ odd}\}.$$

This allows us to write F_n as the Hilbert series of the exterior algebra generated by the union of the following graded sets.

$$\{[u_I] \mid \omega I < n, I \text{ odd}\} \tag{24}$$

$$\{\gamma_{2^i}[u_I^{2^{h_{n-1}I-1}} \mid u_I^{2^{h_{n-1}I-1}}] \mid i \ge 0, \omega I < n, 2^{h_{n-1}I}\omega I \ne n, I \text{ odd}\} \tag{25}$$

$$\{[Q_1^{i+1} Q_I\iota_{n-2}] \mid i \ge 0, I \text{ even}\} \tag{26}$$

$$\{[Q_1^i Q_I\iota_{n-2}] \mid i \ge 0, I \text{ odd}, \alpha I > 1\} \tag{27}$$

G_n is the Hilbert series of the exterior algebra generated by the following graded set.

$$\{[u_I], \gamma_{2^i}[u_I^{2^{h_nI-1}}|u_I^{2^{h_nI-1}}]|I \text{ odd}, \omega I \le n, i \ge 0\} \tag{28}$$

We will be done when we have exhibited a degree preserving correspondence between these two sets. In order to present this correspondence, we need to compare h_nI with $h_{n-1}I$. When $\omega I = n$, $h_{n-1}I$ is not defined. When $\omega I < n$, $h_nI = h_{n-1}I$, unless there is some power, 2^j, of two such that $2^j\omega I = n$. Necessarily, if this is the case, we have $j = h_{n-1}I$ and $j+1 = h_nI$. In light of these remarks, rewrite the set (28) as the union of the following sets.

$$\{[u_I] \mid \omega I < n, I \text{ odd}\} \tag{29}$$

$$\{\gamma_{2^i}[u_I^{2^{h_nI-1}}|u_I^{2^{h_nI-1}}]|i \ge 0, \omega I < n, 2^{h_nI}\omega I \ne n, I \text{ odd}\} \tag{30}$$

$$\{\gamma_{2^i}[u^{2^{h_nI-1}}|u_I^{2^{h_nI-1}}]|i \ge 0, \omega I < n, 2^{h_{n-1}I}\omega I = n, I \text{ odd}\} \tag{31}$$

$$\{\gamma_{2^i}[u_I] \mid i \ge 0, \omega I = n\}. \tag{32}$$

Certainly, there is a degree preserving correspondence between the sets (24) and (29). Similarly, there is a correspondence between the sets (25) and (30). Suppose now that $I = (i_1, \dots, i_r, \omega I)$ with $2^{h_{n-1}I}\omega I = 2^{h_nI-1}\omega I = n$. Let $I' = (2^{h_{n-1}I}i_1 + 1, \dots, 2^{h_{n-1}I}i_r + 1)$, and $I'' = (2^{h_{n-1}I}i_1 + 2, \dots, 2^{h_{n-1}I}i_r + 2)$. Then we have the following equalities of degrees.

$$|[u_I^{2^{h_nI-1}}|u_I^{2^{h_nI-1}}]| = |[Q_1Q_{I''}\iota_{n-2}]|$$

These equalities yield a correspondence between the sets (26) and (31). For any I with $\omega I = n$, I odd, and $I = (i_1, \dots, i_r, \omega I)$, let $I' = (i_1 + 2, \dots, i_r + 2)$. Then we have,

$$|\gamma_{2^i}[u_I]| = |[Q_1^iQ_{I'}\iota_{n-2}]|,$$

yielding a correspondence between the sets (27) and (32). These equalities complete the correspondence, so we are finished. □

Lemma 4.17. *Suppose that n is odd and $E_2 = E_\infty$ in the Eilenberg-Moore spectral sequence below.*

$$\mathrm{Tor}^{**}_{H^*\Omega_0^nS^n}(\mathbf{F}_2, \mathbf{F}_2) \Rightarrow H^*\Omega^{n+1}S^n \tag{33}$$

We then have $F_n = G_n$.

Proof. If the spectral sequence of equation (33) collapses, then we have,

$$F_n = f(E([Q_i\iota_{n-2}]|\omega I \le n+2, \alpha I > 0)) \cdot f(\mathrm{Tor}^{**}_{H^*\Omega_0^nS^n}(\mathbf{F}_2\mathbf{F}_2)). \tag{34}$$

If n is odd, $h_nI = h_{n-1}I$ for all I with $\omega I \le n-1$ and $h_nI = 1$ for I with $\omega I = n$. Hence we have

$$H^*\Omega^{n+1}S^{n+1} = H^*\Omega^nS^n \otimes E(u_I \mid \omega_I = n). \tag{35}$$

Comparing equation (34) with equation (35), we see that to finish, all we need to show is

$$f(E([Q_I \iota_{n-2}]|\omega I \leq n+2, \alpha I > 0)) = f(\mathrm{Tor}^{**}_{E(u_I \mid \omega I = n)}(\mathbf{F}_2, \mathbf{F}_2)). \tag{36}$$

This follows since we have the following equality of degrees.

$$|Q_1^{j_1} \cdots Q_{n+2}^{j_{n+2}}[\iota_{n-2}]| = |\gamma_{2^{j_1}}[u_{I''}]|$$

Here we have $Q_{I''} = Q_0^{j_2} Q_1^{j_3} \cdots Q_n^{j_{n+2}+1}$. Hence the proof of Lemma 4.17 is complete. □

Lemma 4.18. *The Eilenberg-Moore spectral sequence,*

$$\mathrm{Tor}^{**}_{H^* \Omega_0^3 S^3}(\mathbf{F}_2, \mathbf{F}_2) \Rightarrow H^* \Omega^4 S^3,$$

collapses.

Note that when $n = 1$ the Hopf map $S^3 \to S^2$ induces a homotopy equivalence $\Omega^3 S^2 \cong \Omega^3 S^3$. This gives us explicit formulae for both F_2 and G_2. The calculations in §2 apply when $n = 2$. So formula (4) implies that the following sets comprise a simple system of generators for the algebra

$$\mathrm{Tor}^{**}_{H^* \Omega_0^3 S^3}(\mathbf{F}_2, \mathbf{F}_2).$$

$$\{\gamma_{2^j}[u_I]|Q_i = Q_0^{a_0} Q_1^{a_1} Q_2^{a_2}; a_1, a_2 > 0; j \geq 0\} \tag{37}$$

$$\{[u_I]|Q_I = Q_0^{a_0} Q_1^{a_1}, a_1 > 0\} \tag{38}$$

$$\{\gamma_{2^j}[u_I^2|u_I^2]|Q_I = Q_0^{a_0} Q_1^{a_1}, a_1 > 0, j \geq 0\} \tag{39}$$

Since $|(w_I)^2| = |q_{2I}|$, and I is an admissible sequence with $\alpha I > 0$ and $\omega I < 3$ exactly when $2I$ is an even admissible sequence with $\alpha(2I) > 0$ and $\omega(2I) < 5$, we may rewrite equation (19) as follows.

$$\begin{aligned} F_2 = f(&E([p_I]|\alpha I > 0, \omega I < 5, I \text{ odd}) \\ &\otimes E(w_I|\alpha I > 0, \omega I < 3) \otimes E(\eta)) \end{aligned}$$

A simple system of generators for the algebra,

$$E([p_I]|\alpha I > 0, \omega I < 5, I \text{ odd}) \otimes E(w_I|\alpha I > 0, \omega I < 3),$$

is given by the union of the following sets.

$$\{[p_I]|Q_I = Q_1^{a_1} Q_2^{a_2} Q_3^{a_3} Q_4^{a_4}; a_3 > 0\} \cup \{\eta\} \tag{40}$$

$$\{w_I|Q_I = Q_1^{a_1} Q_2^{a_2}; a_1 + a_2 > 0\} \tag{41}$$

$$\{[p_I]|Q_I = Q_1^{a_1} Q_2^{a_2} Q_4^{a_4}; a_1 > 0\} \tag{42}$$

We note that there are dimension preserving correspondences between the sets (37) through (39) and the sets (40) through (42), respectively. Hence $F_2 = G_2$ so, by Lemma 3.15 and Corollary 2.6, we are done with the case $n = 2$. □

As consequences we obtain Theorems 1.1 and 1.2. To see how Corollary 1.3 follows, note that the bar construction spectral sequence

$$\mathrm{Tor}_{**}^{H_*\Omega^{n+3}S^{2n+1}}(\mathbf{F}_2,\mathbf{F}_2) \Rightarrow H_*\Omega^{n+2}S^{2n+1}$$

collapses for all $n > 2$, and that we must take $Q_0^{a_0}\cdots Q_{n+1}^{a_{n+1}}\iota_{n-1}$ to represent $[Q_1^{a_0}\cdots Q_{n+2}^{a_{n+1}}\iota_{n-2}]$. The diagram below commutes.

$$\begin{tikzcd}
\Omega^{n+3}S^{2n+1} \arrow[r, "="] \arrow[d, "\Omega^{n+1}p"] & \Omega^{n+3}S^{2n+1} \arrow[d] \\
\Omega_0^{n+1}S^n \arrow[r] \arrow[d, "\Omega^{n+1}e"] & P\Omega^{n+2}S^{2n+1} \arrow[d] \\
\Omega_0^{n+2}S^{n+1} \arrow[r, "\Omega^{n+1}h"] & \Omega^{n+2}S^{2n+1}
\end{tikzcd}$$

This diagram induces a map,

$$\mathrm{Tor}_{**}^{H_*\Omega^{n+3}S^{2n+1}}(H_*\Omega_0^{n+1}S^n,\mathbf{F}_2) \to \mathrm{Tor}_{**}^{H_*\Omega^{n+3}S^{2n+1}}(\mathbf{F}_2,\mathbf{F}_2),$$

so no matter what representative in $H_*\Omega_0^{n+2}S^{n+1}$ we take for the element,

$$[Q_I\iota_{n-2}] \in \mathrm{Tor}_{**}^{H_*\Omega^{n+3}S^{2n+1}}(H_*\Omega_0^{n+1}S^n,\mathbf{F}_2),$$

(with I odd if n is even) it must map to $Q_{I'}\iota_{n-1} \in H_*\Omega^{n+2}S^{2n+1}$, where I and I' are related by the following equalities.

$$Q_I = Q_1^{a_1}\cdots Q_{n+2}^{a_{n+2}}, \quad Q_{I'} = Q_0^{a_1}\cdots Q_{n+1}^{a_{n+2}}$$

Hence we obtain Corollary 1.3.

References

1. T. Kudo and S. Araki, *On $H_*(\Omega^N S^n;Z_2)$*, Proc. Japan Acad. **32** (1956), 333–335.
2. W. Browder, *The cohomology of covering space of H-spaces*, Bull. Amer. Math. Soc. **65** (1959), 140–141.
3. A. Clark, *Homotopy-commutativity and the Moore spectral sequence*, Pacific J. Math. **15** (1965), 65–74.
4. F. R. Cohen, T. J. Lada and J. P. May, *The homology of iterated loop spaces*, Lecture Notes in Math., vol. 533, Springer, 1976.
5. F. R. Cohen and F. P. Peterson, Mimeographed handwritten notes.
6. W. Dwyer, *Strong convergence of the Eilenberg-Moore spectral sequence*, Topology **13** (1974), 255–265.
7. J. W. Milnor and J. C. Moore, *On the structure of Hopf-algebras*, Ann. of Math. **81** (1965), 211–264.

8. H. Toda, *Composition methods in the homotopy groups of spheres*, Princeton Univ. Press, 1962.

9. Robert J. Wellington, *The unstable Adams spectral sequence for free iterated loop spaces*, Mem. Amer. Math. Soc. **36**, no. 258, 1982.

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139

Current address: Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506-0027