EQUIVALENT CONDITIONS TO THE SPECTRAL DECOMPOSITION PROPERTY FOR CLOSED OPERATORS

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Abstract. The spectral decomposition property has been instrumental in developing a local spectral theory for closed operators acting on a complex Banach space. This paper gives some necessary and sufficient conditions for a closed operator to possess the spectral decomposition property.

In the monograph [3] and in a sequel of papers by the authors, a local spectral theory has been built for closed operators on the sole assumption of the spectral decomposition property. As an abstraction of Dunford’s concept of “spectral reduction” [2, p. 1927] and that of Bishop’s “duality theory of type 3” [1], an operator $T$ endowed with the spectral decomposition property produces a spectral decomposition of the underlying space, pertinent to any finite open cover of the spectrum $\sigma(T)$. In this paper we obtain some conditions equivalent to the spectral decomposition property. Some of them generalize results from [4].

1. Preliminaries

Given a Banach space $X$ over the complex field $\mathbb{C}$, we denote by $C(X)$ the class of closed operators with domain $D_f$ and range in $X$, and we write $C_d(X)$ for the subclass of the densely defined operators in $C(X)$. For a subset $Y$ of $X$, $Y^\perp$ denotes the annihilator of $Y$ in $X^*$ and for $Z \subset X^*$, we use the symbol $1^\perp Z$ for the preannihilator of $Z$ in $X$. For the rest, the terminology and notation conform to that employed in [3].

We shall adopt and adjust some concepts and ideas from Bishop’s “duality theory of type 4” [1, §4]. A couple $U_1$ and $U_2$ of a bounded and an unbounded Cauchy domain, related by $U_2 = (U_1)^c$, are referred to as complementary simple sets. $W_1$ and $W_2$ are the sets of analytic functions from $U_1$ to $X$ and from $U_2$ to $X^*$, respectively, which vanish at $\infty$. The seminorms

$$\|f\|_{K_1} = \max\{\|f(\lambda)\| : f \in W_1, \lambda \in K_1, K_1 (\subset U_1) \text{ is compact}\}$$

and

$$\|g\|_{K_2} = \max\{\|g(\lambda)\| : g \in W_2, \lambda \in K_2, K_2 (\subset U_2) \text{ is compact}\}$$

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induce a locally convex topology on $W_1$ and $W_2$, respectively. For $i = 1, 2$ let $V_i$ be the subset of $W_i$ on which every function can be extended to be continuous on $\overline{U}_i$. For $f \in V_1$, $g \in V_2$, the norms
\[ \|f\|_{V_1} = \sup\{\|f(\lambda)\| : \lambda \in U_1\}, \quad \|g\|_{V_2} = \sup\{\|g(\lambda)\| : \lambda \in U_2\} \]
make $(V_1, \cdot, \|\cdot\|_{V_1})$ and $(V_2, \cdot, \|\cdot\|_{V_2})$ Banach spaces. For $x \in X$, $\mu \in U_1$ and $\lambda \in U_2$, define
\[ (1.1) \quad \alpha(x, \mu, \lambda) = (\mu - \lambda)^{-1} x. \]
For fixed $x \in X$ and $\lambda \in U_2$, $\alpha(x, \cdot, \lambda)$ is called an elementary element of $V_1$. Denote by $V$ the subspace of $V_1$ which is spanned by the elementary elements of $V_1$. For $x^* \in X^*$, $\mu \in U_1$ and $\lambda \in U_2$, define
\[ (1.2) \quad \alpha(x^*, \mu, \lambda) = (\mu - \lambda)^{-1} x^*. \]
For fixed $x^* \in X^*$ and $\mu \in U_1$, call $\alpha(x^*, \mu, \cdot)$ and elementary element of $V_2$. For $f \in V_1$ and $g \in V_2$, with continuous extensions to $\Gamma = \partial U_1 = \partial U_2$, endow $\Gamma$ with the clockwise orientation and ascertain that the bilinear functional
\[ (1.3) \quad \Phi(f) = (f, g) = \frac{1}{2\pi i} \int_{\Gamma} \langle f(\lambda), g(\lambda) \rangle d\lambda \]
is jointly continuous.
For $g \in V_2$, (1.3) defines a bounded linear functional $\Phi$ on $\hat{V}$, i.e. $\Phi \in V^*$. For $f = \alpha(x, \cdot, \lambda)$, one obtains
\[ (1.4) \quad \Phi(f) = \frac{1}{2\pi i} \int_{\Gamma} (\mu - \lambda)^{-1} (x, g(\mu)) d\mu = (x, g(\lambda)). \]
The last equality holds because $g(\infty) = 0$.

The proof of the following lemma is similar to that of [3, Lemma 9.1].

**1.1. Lemma.** Let $U_1, U_2$ be complementary simple sets. With $V$, $V_i$, and $W_i$ ($i = 1, 2$), as defined above, there exists a linear manifold $Y$ in $W_2$ and a norm on $Y$ such that

(i) $Y$ is a Banach space isometrically isomorphic to $V^*$;
(ii) $V_2 \subset Y$;
(iii) the mappings $V_2 \to Y$ and $Y \to W_2$ are continuous;
(iv) the inner product between $V$ and $\hat{V}$, defined by (1.3), can be extended to an inner product between $V$ and $Y$ in conjunction with the isometric isomorphism between $Y$ and $V^*$, as asserted by (i).

**2. Some dual properties**

For an operator $T \in C_d(X)$, define an operator $H$ on $V$ by
\[ D_H = \{ f \in V : Tf(\mu) \in V \}, \quad (Hf)(\mu) = (\mu - T)f(\mu). \]
2.1. Lemma. The operator $H$ is closed and densely defined on $V$.

Proof. For $f = \alpha(x, \cdot, \lambda)$ with $x \in D_T$, one has $f \in D_H$ and

$$ (Hf)(\mu) = (\mu - \lambda)^{-1}(\mu - T)x = x + (\mu - \lambda)^{-1}(\lambda - T)x. $$

The linear span of all elementary elements being dense in $V$, the operator $H$ is densely defined.

Let $\{f_n\}$ be a sequence in $D_H$ such that $f_n \to f$ and $Hf_n \to g$, for some functions $f$ and $g$. $T$ being closed, it follows from

$$ (Hf_n)(\mu) = (\mu - T)f_n(\mu), $$

that $f \in D_H$ and

$$ (Hf)(\mu) = (\mu - T)f(\mu) = g(\mu), \quad \mu \in \overline{U}_1. $$

Thus $H$ is closed. $\square$

The next lemma defines the dual $H^*$ of $H$. Henceforth, $g$ will denote a typical element of $V^* = Y$.

2.2. Lemma. The dual operator of $H$ is defined by

$$ (H^*g)(\lambda) = -\lim_{\lambda \to \infty} \lambda g(\lambda) + (\lambda - T^*)g(\lambda), \quad g \in D_{H^*}. $$

Proof. For $f(\mu) = \alpha(x, \mu, \lambda)$ with $x \in D_T$ and $\lambda \in U_2$ fixed, and $g \in D_{H^*}$, (2.1), (1.3) and (1.4) imply

$$ (f, H^*g) = (Hf, g) = (x, g) + (\alpha((\lambda - T)x, \cdot, \lambda), g) = (x, g) + ((\lambda - T)x, g(\lambda)), $$

where $x$, as a function of $\mu \in \overline{U}_1$, is an element of $V_1$. It follows from

$$ x = \frac{1}{2\pi i} \int_{\Gamma} (\mu - \lambda)^{-1} x \, d\lambda = \frac{1}{2\pi i} \int_{\Gamma} \alpha(x, \mu, \lambda) \, d\lambda, $$

that $x \in V$, where $\Gamma$ is a closed $C^2$-Jordan curve with the clockwise orientation that contains $\Gamma$ in its interior. Furthermore, with the help of (1.4), one obtains

$$ (x, g) = \frac{1}{2\pi i} \int_{\Gamma} (\alpha(x, \cdot, \lambda), g) \, d\lambda = \frac{1}{2\pi i} \int_{\Gamma} (x, g(\lambda)) \, d\lambda $$

$$ = \lim_{z \to \infty} z \left( -\frac{1}{2\pi i} \int_{\Gamma} (\lambda - z)^{-1} (x, g(\lambda)) \, d\lambda \right) = -\lim_{z \to \infty} z (x, g(z)) $$

$$ = -\lim_{\lambda \to \infty} \lambda (x, g(\lambda)) = \left( x, -\lim_{\lambda \to \infty} \lambda g(\lambda) \right). $$

It follows from (2.3) and (2.4) that

$$ (f, H^*g) = \left( x, -\lim_{\lambda \to \infty} \lambda g(\lambda) \right) + (\lambda - T)x, g(\lambda)) = \left( x, -\lim_{\lambda \to \infty} \lambda g(\lambda) + (\lambda - T)^*g(\lambda) \right). $$
In fact, \( f = \alpha(x, \cdot, \lambda) \) and since \( \langle f, H^* g \rangle \) and \( \langle x, -\lim_{\lambda \to \infty} \lambda g(\lambda) \rangle \) are bounded linear functionals of \( x \), so is \( \langle (\lambda - T)x, g(\lambda) \rangle \). Thus \( g(\lambda) \in D_{T^*} \), for every \( \lambda \in U_2 \) and hence the last equality of (2.5) holds. Now (2.5) combined with (1.3) and (1.4), gives \( \langle f, H^* g \rangle = \langle x, (H^* g)(\lambda) \rangle \) and hence \( H^* \) is expressed by (2.2). \( \Box \)

Define the map \( \tau: V^* \to X^* \) by \( \tau g = \lim_{\lambda \to \infty} \lambda g(\lambda) \). Then \( H^* \), given by (2.2), is expressed by

\[
(H^* g)(\lambda) = -\tau g + (\lambda - T^*) g(\lambda).
\]

2.3. Lemma. Let \( x^* \in X^* \). Then \( x^* \in D_{T^*} \) iff there exists \( g \in D_{H^*} \) such that \( \tau g = x^* \).

Proof. First, assume that there is \( g \in D_{H^*} \) such that \( \tau g = x^* \). Since \( H^* g \in V^* \), the following limit exists

\[
\lim_{\lambda \to \infty} T^* \lambda g(\lambda) = \lim_{\lambda \to \infty} \lambda T^* g(\lambda).
\]

Furthermore, \( \lim_{\lambda \to \infty} \lambda g(\lambda) \) also exists and since \( T \) is closed, we have

\[
x^* = \tau g = \lim_{\lambda \to \infty} \lambda g(\lambda) \in D_{T^*}.
\]

Conversely, for every \( x^* \in D_{T^*} \), the corresponding elementary element \( \alpha(x^*, \mu, \cdot) \) with \( \mu \in U_1 \) fixed, is in \( D_{H^*} \). It follows from (1.2), that

\[
\tau(-\alpha) = -\lim_{\lambda \to \infty} \lambda \alpha(x^*, \mu, \lambda) = x^*
\]

and the proof reaches its conclusion by setting \( g = -\alpha \). \( \Box \)

3. Norms on the dual spaces

We introduce the norm

\[
\|f_1, f_2\|_\eta = (\eta \|f_1\|^2 + \|f_2\|^2)^{1/2}, \quad \eta > 0,
\]

in \( V \oplus V \). This induces the norm

\[
\|(g_1, g_2)\|_\eta = (\eta^{-1} \|g_1\|^2 + \|g_2\|^2)^{1/2}
\]

in \( V^* \oplus V^* \). Let \( G(H) \) and \( G(H^*) \) be the graphs of \( H \) and \( H^* \), respectively. \( G(H) \), as a subspace of \( V \oplus V \), is endowed with the norm

\[
\|(f, Hf)\|_\eta = (\eta \|f\|^2 + \|Hf\|^2)^{1/2}.
\]

It follows from

\[
(G(H))^\perp = \nu G(H^*), \quad \text{where} \quad \nu(g_1, g_2) = (-g_2, g_1),
\]

that \( \nu G(H^*) \) is the dual of \( (V \oplus V)/G(H) \). The latter is equipped with the norm

\[
\|(f_1, f_2)\|_\eta = \inf\{\|f_1 - f\|^2 + \|f_2 - Hf\|^2)^{1/2} : f \in D_H\},
\]
where \((f_1, f_2)\) denotes a typical element of \((V \oplus V)/G(H)\). To the norm \((3.2)\), there corresponds the following norm in \(\nu G(H^*)\):

\[
\|(-H^*g, g)\|_\eta = (\eta^{-1} \|H^*g\|^2 + \|g\|^2)^{1/2}.
\]

3.1. **Lemma.** The norm

\[
\|x^*\|_{T^*} = (\|x^*\|^2 + \|Tx^*\|^2)^{1/2}
\]

in \(D_{T^*}\) is equivalent to the norm

\[
\|x^*\|_\eta = \inf\{(\eta^{-1} \|H^*g\|^2 + \|g\|^2)^{1/2} : \tau g = x^*\}.
\]

Furthermore, \(D_{T^*}\) equipped with the norm \((3.3)\) or \((3.4)\) is the dual of a Banach space.

**Proof.** First, we prove that \(D_{T^*}\) endowed with the norm \((3.4)\) is a Banach space. Let \(\{x^*_n\}\) be a Cauchy sequence with respect to the norm \((3.4)\). Without loss of generality, we may suppose that

\[
\sum_{n=0}^{\infty} \|x^*_{n+1} - x^*_n\|_\eta < \infty, \quad x_0 = 0.
\]

For each \(x_n\), we may choose \(g_n \in D_{H^*}\) such that

\[
(\eta^{-1} \|H^* (g_{n+1} - g_n)\|^2 + \|g_{n+1} - g_n\|^2)^{1/2} \leq 2 \|x^*_{n+1} - x^*_n\|_\eta
\]

and \(\tau g_n = x^*_n\). Relations \((3.5)\) and \((3.6)\) imply that both \(\{g_n\}\) and \(\{H^* g_n\}\) converge. Put \(g = \lim_{n \to \infty} g_n\). \(H^*\) being closed, one has \(g \in D_{H^*}\) and \(H^* g = \lim_{n \to \infty} H^* g_n\). Then Lemma 2.3 implies that \(x^* = \tau g \in D_{T^*}\). Since

\[
\|x^*_n - x^*\|_\eta \leq (\eta^{-1} \|H^*(g_n - g)\|^2 + \|g_n - g\|^2)^{1/2} \to 0 \quad n \to \infty,
\]

it follows that \(D_{T^*}\), endowed with the norm \((3.4)\), is a Banach space.

To show that the norms \((3.3)\) and \((3.4)\) are equivalent, let \(x^* \in D_{T^*}\) and \(g = \alpha(x^*, \mu, \cdot)\) with \(\mu \in U_1\) fixed. Since \(\tau g = x^*\), one has

\[
\|x^*\|_\eta \leq (\eta^{-1} \|H^* g\|^2 + \|g\|^2)^{1/2}
\]

\[
\leq \frac{1}{\delta} (\eta^{-1} (|\mu| \cdot \|x^*\| + \|T^* x^*\|)^2 + \|x^*\|^2)^{1/2},
\]

where \(\delta = \text{dist}(\mu, U_2)\). In view of \((3.7)\), there exists \(K_\eta > 0\) such that

\[
\|x^*\|_\eta \leq K_\eta (\|x^*\|^2 + \|T^* x^*\|^2)^{1/2} = K_\eta \|x^*\|_{T^*}.
\]

\(D_{T^*}\) being complete with respect to both \(\|\cdot\|_\eta\) and \(\|\cdot\|_{T^*}\). \((3.8)\) implies that the two norms are equivalent. \(D_{T^*}\) equipped with the norm \((3.3)\) is isometrically isomorphic to \(\nu G(T^*) = G(T)^+\). Since \(\nu G(T^*)\) is the dual of \(X \oplus X/G(T)\), so is \(D_{T^*}\). □

\(D_{T^*}\) equipped with either of the two norms \((3.3), (3.4)\), will be denoted by \(D\). To obtain a further property of \(\tau\), we need the following.
3.2. **Lemma.** Let $Y$, $Z$ be Banach spaces and let $S$ be a bounded surjective map of $Y$ onto $Z$. In $Z$ we define the norm

$$
(3.9) \quad \|z\|_S = \inf\{\|y\|: y \in Y, Sy = z\}, \quad z \in Z.
$$

Then, the corresponding norm in the dual space $Z^*$ is

$$
(3.10) \quad \|z^*\|_{S^*} = \|S^* z^*\|, \quad z^* \in Z^*.
$$

**Proof.** Let $N(S)$ be the null space of $S$. Then $N(S)^\perp$ is the dual of $Y/N(S)$. Let $y_0 \in Y$, $z = Sy_0$ and let $\tilde{y}_0$ be the equivalence class of $y_0$ in $Y/N(S)$. In terms of the norm (3.9), one has

$$
\|\tilde{y}_0\| = \inf\{\|y_0 - w\|: w \in N(S)\} = \inf\{\|y\|: Sy = z\} = \|z\|_S.
$$

The dual norm of $\|\tilde{y}_0\|$ in $N(S)^\perp$ is the usual norm in $Y^*$, restricted to $N(S)^\perp$. Note that $S^*$ is a surjective map from $Z^*$ onto $N(S)^\perp$. Therefore, the corresponding norm of $\|\cdot\|_S$ in $Z^*$ is the one expressed by (3.10). \qed

We define an operator $K$ from $\nu G(H^*)$ into $D_T$, by $K(-H^* g, g) = \tau g$. Let $D^*$ be the dual of $D$. Then $K^*$, the dual of $K$, is an operator from $D^*$ into the dual of $\nu G(H^*)$, i.e., from $D^*$ into $(V^{**} \oplus V^{**})/(\nu G(H^*))^{\perp}$. For every $x \in X$, define a continuous linear functional $\psi$ on $D$, by

$$
(3.11) \quad \psi(x^*) = \langle x, x^* \rangle, \quad x^* \in D.
$$

3.3. **Lemma.** The linear functional $\psi$ (3.11) is a zero functional only if $x = 0$.

**Proof.** Assume that $\psi = 0$. Then $\langle x, x^* \rangle = 0$ for every $x^* \in D$. Thus, we have $\langle 0, -T^* x^* \rangle + \langle x, x^* \rangle = 0$, $x^* \in D$, equivalently, $(0, x) \in G(T^*)$, i.e., $(0, x) \in G(T)$. Consequently, $x = 0$. \qed

In view of Lemma 3.3, we may consider $X$ as a subset of $D^*$. In the following, we shall have a closer look at $K^* x$ for $x \in X$.

For $x^* \in D$ and fixed $\mu \in U_1$, put $g = \alpha(x^*, \mu, \cdot)$. Then, one obtains

$$
\langle K^* x(-H^* g, g) \rangle = \langle x, K(-H^* g, g) \rangle = \langle x, \tau g \rangle = \langle x, x^* \rangle = \frac{1}{2\pi i} \int_\Gamma (\mu - \lambda)^{-1} \langle x, x^* \rangle d\lambda = \langle x, g \rangle,
$$

where $\Gamma$ has a clockwise orientation.

We know that $(0, x)$ is an element of $X \oplus X$. We may also consider $(0, x)$ as an element of $V \oplus V$ and hence $(0, x)$ can be assumed to be an element of $V^{**} \oplus V^{**}$.

Thus, we have

$$
\langle x, g \rangle = \langle (0, x), (-H^* g, g) \rangle = \langle (0, x)^\sim, (-H^* g, g) \rangle,
$$

where $(0, x)^\sim$ is the equivalence class of $(0, x)$ in $(V^{**} \oplus V^{**})/(\nu G(H^*))^{\perp}$. Consequently, $K^* x = (0, x)^\sim$.

Denote by $(0, x)^\sim$ the equivalence class of $(0, x)$ in $(V \oplus V)/G(H)$.
3.4. Lemma. Let $J$ be the natural embedding of $(V \oplus V)/G(H)$ into 
$$(V^{**} \oplus V^{**})/(\nu G(H^*))^\perp.$$ 
Then $J(0, x) = (0, x)^\sim$.

Proof. For any $(-H^* g, g) \in \nu G(H^*)$, one has 
$$\langle (0, x), (-H^* g, g) \rangle = \langle (0, x), (H^* g, g) \rangle = \langle -H^* g, g \rangle, (0, x) \rangle$$ 
$$= \langle (H^* g, g), (0, x)^\sim \rangle.$$ 
Note that while in $\langle (0, x), (-H^* g, g) \rangle$, $(0, x) \in V \oplus V$; in $\langle -H^* g, g \rangle$, $(0, x)$ is considered an element of $V^{**} \oplus V^{**}$. It follows from the 
above equalities that $J(0, x) = (0, x)^\sim$. □

In particular, Lemma 3.4 implies 
$$|| (0, x) \rangle = || (0, x) \rangle ||.$$ 
On the other hand, $(0, x)^\sim = 0$ implies $(0, x) \in G(H)$ and the latter implies 
$x = 0$. Accordingly, we may define the following norm on $X$:
$$|| x ||_{\eta} = || (0, x) \rangle || = \inf \{ (\eta || f ||^2 + || x - Hf ||^2)^{1/2} : f \in D_H \}.$$ 
In view of Lemma 3.4, we may consider $K^# x = (0, x)^\sim$ as a point of 
$(V \oplus V)/G(H)$.

3.5. Lemma. The norm $|| \cdot ||_{\eta}$, defined by (3.13), is the restriction of the norm on $D^*$. 

Proof. The space $(V^{**} \oplus V^{**})/(\nu G(H^*))^\perp$ is the conjugate of $\nu G(H^*)$ and $K$ 
is an operator from $\nu G(H^*)$ into $D$. It follows from Lemma 3.2, that the dual 
norm on $D^*$ is given by 
$$|| x^\# || = || K^# x^\# ||_{**}$$ 
where $x^\# \in D^*$ and $|| \cdot ||_{**}$ is the norm on $V^{**} \oplus V^{**}/[\nu G(H^*)]^\perp$. If $x^\# = x \in X$, then the norm (3.14) becomes $|| x ||_{\eta} = || K^# x || = ||(0, x)^\sim ||$ and it 
follows from (3.12) that the restriction of the norm (3.14) to $X$ is that given 
by (3.13). □

4. A DUALITY PROPERTY OF SOME SPECTRAL-TYPE MANIFOLDS

Define the following linear manifolds in $X$:

$N = \{ x \in X : \text{for every } \varepsilon > 0, \text{ there exists } f \in D_H \text{ with } || x - Hf || < \varepsilon \}$,
$M = \{ x^\# \in D : \text{there exists } g \in D_H, \text{ such that } H^* g = 0, \tau g = x^\# \}.$

4.1. Lemma. The manifolds $N$ and $M$ have the following characterizations:

(4.1) $N = \{ x \in X : || x ||_{\eta} \to 0 \text{ as } \eta \to 0 \},$
(4.2) $M = \{ x^\# \in D : || x^\# ||_{\eta} \leq R \text{ for } n > 0 \text{ and } R \text{ depends on } x^\# \}.$
Proof. First, we establish (4.1). Let \( x \in N \). Since, for every \( \varepsilon > 0 \), there is \( f \in D_H \) such that \( \|x - Hf\| < \varepsilon \), it follows from (3.13) that \( \lim_{\eta \to 0} \|x\|_\eta \leq \varepsilon \). \( \varepsilon \) being arbitrary, it follows that \( \lim_{\eta \to 0} \|x\|_\eta = 0 \).

Conversely, suppose that \( \|x\|_\eta \to 0 \) as \( \eta \to 0 \). Then, for every \( \varepsilon > 0 \), there exists \( \eta > 0 \) such that \( \|x\|_\eta < \varepsilon \) and hence \( \|x - Hf\| < \varepsilon \) for some \( f \in D_H \).

Next, we prove (4.2). It is a straightforward consequence of (3.4) that

\[
M \subset \{x^* \in D : \|x^*\|_\eta \text{ is bounded for } \eta > 0\}.
\]

Conversely, suppose that \( x^* \in D \) and \( \|x^*\|_\eta \) is bounded for \( \eta > 0 \), i.e. there exists \( R > 0 \) such that

\[
\inf\{(n\|H^* g\|^2 + \|g\|^2)^{1/2}, \tau g = x^*\} < R, \quad n = 1, 2, \ldots.
\]

Then, for every \( n \), there exists \( g_n \in D_H \) satisfying conditions

\[
(n\|H^* g\|^2 + \|g\|^2) \leq R^2 \quad \text{and} \quad \tau g_n = x^*.
\]

In view of (4.3), the sequences \( \{g_n\} \) and \( \{H^* g_n\} \) are bounded and hence the sequence \( \{(H^* g_n, g_n)\} \) is bounded. Consequently, \( \{(H^* g_n, g_n)\} \) has a cluster point \( (h, g) \) in \( V^* \oplus V^* \), with respect to the weak* topology of \( V^* \oplus V^* \). Since \( vG(H^*) \) is closed with respect to the same topology, one has \( \{h, g\} \in vG(H^*) \), i.e. \( h = -H^* g \). It follows from \( \|H^* g_n\| \leq R^2/n \) that \( \|H^* g\| = 0 \).

On the other hand, for every \( x \in X \), \( K^* x = (0, x) \in (V \oplus V)/G(H) \).
Therefore,

\[
\langle x, x^* \rangle = \langle x, \tau g_n \rangle = \langle x, K(-H^* g_n, g_n) \rangle = \langle K^* x, (-H^* g_n, g_n) \rangle.
\]

Since \( (-H^* g, g) \) is also a cluster point of \( \{(-H^* g_n, g_n)\} \) in the weak* topology of \( vG(H^*) \), the latter being the dual space of \( (V \oplus V)/G(H) \), we have

\[
\langle x, x^* \rangle = \langle K^* x, (-H^* g, g) \rangle = \langle x, K(-H^* g, g) \rangle = \langle x, \tau g \rangle.
\]

Thus \( \tau g = x^* \) and hence \( x^* \in M \). Expression (4.2) is obtained. \( \Box \)

4.2. Theorem. \( N \) and \( M \), as defined above, are related by

\[
N^\perp = \overline{M^w},
\]

where \( ^w \) denotes the weak* closure in \( X^* \).

Proof. Let \( x \in N \) and \( x^* \in M \). It follows from Lemmas 3.5 and 4.1, that

\[
|\langle x, x^* \rangle| \leq \|x\|_\eta \cdot \|x^*\|_\eta \to 0 \quad \text{as } \eta \to 0.
\]

Therefore, \( N^\perp \supseteq \overline{M^w} \).

Next, we prove the opposite inclusion. For \( x \notin N \) (\( x \in X \)), Lemma 4.1 implies that there exists \( \eta_n \downarrow 0 \) such that

\[
\|x\|_{\eta_n} > C > 0
\]

for some constant \( C \). In view of (4.4), we can find \( x^*_n \in D \) such that \( \|x^*_n\|_{\eta_n} \leq 1 \) and \( |\langle x, x^*_n \rangle| > C \). The sequence \( \{\eta_n\} \) being nonincreasing, so is the norm
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(3.4), i.e. \( \| x_n^* \|_{H_n} \leq \| x_n^* \|_{\eta_n} \). Consequently, \( \{ x_n^* \} \) is bounded in the norm \( \| \cdot \|_{\eta_i} \)-topology. For every \( n \), there exists \( g_n \in D_{H^*} \) such that

\[
\eta_n^{-1} \| H^* g_n \|^2 + \| g_n \|^2 \leq 2 \| x_n^* \|^2_{\eta_n}.
\]

Thus \( \{(x_n^*, g_n, H^* g_n)\} \) is bounded in \( D \oplus G(H^*) \). By Lemma 3.1 and the previous paragraph, \( D \oplus G(H^*) \) is the dual of a Banach space and \( \{(x_n^*, g_n, H^* g_n)\} \) has a cluster point \( (x^*, g, H^* g) \) in the weak* topology of \( D \oplus G(H^*) \). Since (3.11) defines a continuous linear functional on \( D \) for every \( x \in X \), it follows that \( x^* \) is also a cluster point of \( \{x_n^*\} \) in the weak* topology of \( X^* \). Now it follows from the inequalities

\[
(x, x_n^*) = (x, \tau g_n) = (x, K(-H^* g_n, g_n)) = (K^* x, (-H^* g_n, g_n))
\]

that, for \( x \in X \), one has

\[
(x, x^*) = (K^* x, (-H^* g, g)) = (x, K(-H^* g, g)) = (x, \tau g).
\]

Thus \( x^* = \tau g \). By the definition of \( M \), \( x^* \in M \). Hence \( N \subseteq M \), or equivalently, \( N \subseteq \overline{M} \). □

5. The main theorem

We recall the definition of the central topic of this paper.

5.1. Definition. An operator \( T \in C(X) \) is said to have the spectral decomposition property (SDP) if, for every finite open cover \( \{G_i\}_{i=0}^n \) of \( C \) (or \( \sigma(T) \), where \( G_0 \) is a neighborhood of infinity (i.e. its complement \( G_0^c \) is compact in \( C \)), there exists a system \( \{Y_i\}_{i=0}^n \) of invariant subspaces under \( T \) satisfying the following conditions:

\[
\begin{align*}
(\text{I}) & \quad X_i \subset D_T \text{ if } G_i \ (1 \leq i \leq n) \text{ is relatively compact;} \\
(\text{II}) & \quad \sigma(T|X_i) \subset G_i \ (0 \leq i \leq n); \\
(\text{III}) & \quad X = \sum_{i=0}^n X_i.
\end{align*}
\]

The theory based on this property is greatly simplified by the fact [3, Corollary 6.3] that \( T \) has the SDP iff it has the two-summand spectral decomposition property that corresponds to \( n = 1 \). The theory also involves the concept of the spectral manifold \( X(T, H) = \{x \in X : \sigma(x, T) \subset H\} \), where \( H \subset C \) and \( \sigma(x, T) \) is the local spectrum at \( x \in X \), and the concept of the \( T \)-bounded spectral maximal space \( \Xi(T, F) \) for \( F \subset C \) compact. The \( T \)-bounded spectral maximal space \( \Xi(T, F) \) is associated to \( X(T, F) \) [3, Theorem 4.34] by

\[
X(T, F) = \Xi(T, F) \oplus X(T, \emptyset) \quad \text{and} \quad \sigma(T|\Xi(T, F)) = \sigma(T|X(T, F)).
\]

The given operator \( T \) may enjoy two specific properties:

\( T \) is said to have property \( (\beta) \) [1, Definition 8 and 3, Definition 5.5] if, for any sequence \( \{f_n : G \to D_T\} \) of analytic functions, the condition \( (\lambda - T)f_n(\lambda) \to 0 \) (as \( n \to \infty \)) in the strong topology of \( X \) and uniformly on every compact
subset of $G$ implies that $f_n(\lambda) \to 0$ in the strong topology of $X$ and uniformly on every compact subset of $G$.

$T$ is said to have property $(\kappa)$ [3, Definition 5.4] if $T$ has the single valued extension property and $X(T,F)$ is closed for every closed $F$.

Property $(\beta)$ implies property $(\kappa)$, as follows from [3, Proposition 5.6].

5.2. Lemma. Suppose that $S \in C(X^*)$. Then $S$ is the dual of a closed and densely defined operator $T \in C_d(X)$ iff $G(S)$ is closed in the weak* topology of $X^* \oplus X^*$ and $D_S$ is total.

Proof. Only if: Suppose that $S$ is the dual of $T \in C_d(X)$, i.e. $S = T^*$. The equality

$$\nu G(S) = \nu G(T^*) = (G(T))^\perp$$

implies $G(S)$ is closed in the weak* topology of $X^* \oplus X^*$. To prove that $D_S$ is total, let $x \in X$ and $\langle x, x^* \rangle = 0$ for all $x^* \in D_S$. Then

$$\langle x, x^* \rangle = 0 = \langle 0, Sx^* \rangle$$

is equivalent to

$$0 \oplus x \in \perp (\nu G(S)) = G(T)$$

and hence $x = T(0) = 0$, so $D_S$ is total.

If: Assume that $G(S)$ is closed in the weak* topology of $X^* \oplus X^*$ and $D_S$ is total. Letting $W = \perp (\nu G(S))$, one has $W = \perp \nu G(S)$. Let $0 \oplus y \in W$. For every $x^* \in D_S$, one has $0 \oplus y \perp (-Sx^*) \oplus x^*$, or equivalently,

$$(5.1) \quad 0 = \langle 0, Sx^* \rangle = \langle y, x^* \rangle \quad \text{for all } x^* \in D_S.$$

$D_S$ being total, (5.1) implies that $y = 0$ and hence $W$ is the graph of an operator $T$. $W$ being closed, $T$ is a closed operator.

To show that $T$ is densely defined, let $x^* \in X^*$ satisfy condition

$$\langle x, x^* \rangle = 0 \quad \text{for all } x \in D_T.$$

Then

$$x \oplus Tx \perp x^* \oplus 0 \quad \text{for all } x \in D_T$$

and hence $x^* \oplus 0 \in (G(T))^\perp = W^\perp = \nu G(S)$. Therefore $x^* = -S(0) = 0$ and hence $T$ is densely defined. □

5.3. Lemma. Suppose that $T \in C(X)$ and $Y$ is invariant under $T$. Then $T/Y$ is closed iff $G(T/Y)$ is topologically isomorphic to $G(T)/G(T|Y)$.

Proof. Only if: Assume that $T/Y$ is closed. For $x \in D_T$, the following mapping $x \oplus Tx + G(T|Y) \to (x \oplus Y) + (Tx + y)$ is bijective from $G(T)/G(T|Y)$ onto $G(T/Y)$. It follows from the inequalities

$$\|x \oplus Tx + G(T|Y)\| = \inf\{\|x \oplus Tx + y \oplus Ty\|: y \in D_T|y\}$$

$$\geq \inf\{\|(x + y_1) \oplus (Tx + y_2)\|: y_1, y_2, \in Y\}$$

$$= \|(x + Y) \oplus (Tx + Y)\|$$
and from the open mapping theorem that $G(T/Y)$ and $G(T)/G(T|Y)$ are topologically isomorphic.

If: Assume that $G(T/Y)$ and $G(T)/G(T|Y)$ are topologically isomorphic. Then $G(T/Y)$ is a Banach space and hence it is closed in $X/Y \oplus X/Y$. Thus $T/Y$ is closed. □

5.4. Lemma. Given $T \in C_d(X)$, let $Z \subset D_T$ be an invariant subspace under $T$. Then

(i) $Z^\perp$ is invariant under $T^*$;

(ii) $T^*/Z^\perp$ is the dual of $T|Z$ iff $T^*/Z^\perp$ is closed.

Proof. (i) is evident.

(ii): If $T^*/Z^\perp$ is the dual of $T|Z$ then clearly $T^*/Z^\perp$ is closed. Conversely, assume that $T^*/Z^\perp$ is closed. Then, it follows from Lemma 5.3 that $G(T^*/Z^\perp)$ is topologically isomorphic to $G(T^*)/G(T^*|Z^\perp)$. The following equalities

$$vG(T^*) = (G(T))^\perp; \quad G(T^*|Z^\perp) = G(T^*) \cap (Z^\perp \oplus Z^\perp)$$

imply that both $G(T^*)$ and $G(T^*|Z^\perp)$ are closed in the weak* topology of $X^* \oplus X^*$. Then, it follows easily that $G(T^*/Z^\perp)$ is closed in the weak* topology of $X^*/Z^\perp \oplus X^*/Z^\perp$.

It follows from Lemma 5.2 that $D_T$ is total and hence $D_{T^*/Z^\perp}$ is total. Quoting again Lemma 5.2, it follows that $T^*/Z^\perp$ is the dual of a densely defined closed operator $U \in C_d(Z)$.

The assumption $Z \subset D_T$ implies that $T|Z$ is bounded. Let $(x^*)^\sim$ be the equivalence class of $x^* \in X^*$ in $X^*/Z^\perp$. Then, for every $x^* \in D_T$, and $x \in D_U$, one has

$$\langle Tx, x^* \rangle = \langle x, T^*x^* \rangle = \langle x, (T^*/Z^\perp)(x^*)^\sim \rangle = \langle Ux, (x^*)^\sim \rangle = \langle Ux, x^* \rangle.$$

Since $D_{T^*}$ is total, (5.2) implies that $Tx = Ux$, for each $x \in D_U$. Since $T|Z$ is bounded and $U$ is a densely defined closed operator, it follows that $U = T|Z$ and hence $T^*/Z^\perp$ is the dual of $T|Z$. □

Now we are in a position to prove our main theorem.

5.5. Theorem. Given $T \in C_d(X)$, the following assertion are equivalent:

(i) $T$ has the SDP;

(ii) for every pair of open disks $G$, $H$ with $\overline{G} \subset H$, there exist invariant subspaces $X_G$ and $X_H$ such that

$$X = X_G + X_H; \quad X_H \subset D_T;$$

$$\sigma(T|X_H) \subset H \quad \text{and} \quad \sigma(T|X_G) \subset G^c;$$

(iii) for every pair of open disks $G$, $H$ with $\overline{G} \subset H$, there exist invariant subspaces $Y$, $Z$ such that

(a) $\sigma(T|Y) \subset G^c$; $T/Y$ is bounded and $\sigma(T/Y) \subset H$;
(b) $Z \subset D^T$, $\sigma(T|Z) \subset H$, $T/Z$ is closed and $\sigma(T|Z) \subset G^c$;

(c) $T^*|Z^\perp$ is closed;

(iv) both $T$ and $T^*$ have property $(\beta)$;

(v) $T$ has property $(\beta)$ and $T^*$ has property $(\kappa)$.

Proof. The proof will be carried out through the following scheme of implications:

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iv)  

(i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii) is evident.

(i) $\Rightarrow$ (iii): Given $T$ with the SDP, let $G$, $H$ be open disks with $\overline{G} \subset H$ and let $L$ be an open set satisfying inclusions $\overline{G} \subset L \subset L \subset H$. For $Y = X(T,G^c)$ and $Z = \Xi(T,L)$, we have $Y = X + Z$. Then, in view of [3, Proposition 3.4 and Corollary 3.3], conditions (a) and (b) of (iii) are satisfied. Furthermore, it follows from (i) and [3, Theorem 9.8 (II,ii)], that

$$Z^\perp = X^*(T^*,L^c).$$

Consequently, (iii, (c) follows from [3, Proposition 3.4].

(ii) $\Rightarrow$ (iv): Let $G$ and $H$ be a pair of open disks with $\overline{G} \subset H$. There exists invariant subspaces $X_G$ and $X_H$ satisfying conditions (5.3) and (5.4). It follows from [3, Proposition 3.4] that $T/X_G$ is bounded and

$$\sigma(T/X_G) \subset \sigma(T|X_H) \cup \sigma(T|X_G \cap X_H) \subset H.$$

Then [3, Theorem 5.8] implies that $T$ has property $(\beta)$.

To show that $T^*$ has property $(\beta)$, let $\{f^*_n\}$ be a sequence of $D_T^*$-valued analytic functions defined on an open set $G \subset \mathbb{C}$ such that

$$(\lambda - T^*)f^*_n(\lambda) \to 0 \quad (n \to \infty)$$

uniformly on every compact subset of $G$ in the strong topology of $X^*$. Without loss of generality, we may suppose that $G = \{\lambda: |\lambda| < r\}$ for some $r > 0$ and that $K \subset G$ is compact. Let $G_0$ and $H_0$ be open disks satisfying inclusions

$$K \subset G_0 \subset \overline{G}_0 \subset H_0 \subset \overline{H}_0 \subset G.$$

Since $T$ has property $(\beta)$, the subspaces $X(T,G_0^c)$, $\Xi(T,H_0)$ are defined. In view of conditions (5.3) and (5.4) applied to the open disks $G_0$, $H_0$, one obtains

$$X = X(T,G_0^c) + \Xi(T,H_0).$$

Since $K \subset \rho(T|X(T,G_0^c))$, for $\lambda \in K$ and $x \in X(T,G_0^c)$, one has

$$|\langle x, f^*_n(\lambda) \rangle| = |\langle R(\lambda;T)|X(T,G_0^c)\rangle x, (\lambda - T^*)f^*_n(\lambda) \rangle| \leq M_0 \|\lambda - T^*\| f^*_n(\lambda) \cdot \|x\|,$$

where $M_0 > 0$ is a constant independent of $\lambda \in K$. Then for every $\varepsilon > 0$, there exists $N_0 > 0$ such that

$$|\langle x, f^*_n(\lambda) \rangle| \leq \varepsilon \|x\|, \quad \text{for all } \lambda \in K \text{ as } n > N_0.$$
Let $C_0 = \{\lambda : |\lambda| = r_0\} \subset G$ with $\overline{H}_0$ in the interior of the disk bounded by $C_0$, for some $r_0 > 0$. Then $C_0 \subset \rho(T|\Xi(T, \overline{H}_0))$ and hence for $\lambda \in C_0$ and $x \in \Xi(T, \overline{H}_0)$ one has

$$|\langle x, \hat{f}_n^\ast(\lambda) \rangle | = |\langle R(\lambda; \Xi(T, \overline{H}_0))x, (\lambda - T^\ast)\hat{f}_n^\ast(\lambda) \rangle | \leq M_1\| (\lambda - T^\ast)\hat{f}_n^\ast(\lambda) \| ,$$

where $M_1 > 0$ is a constant independent of $\lambda \in C_0$. Then there is $N_1$ such that

$$|\langle x, \hat{f}_n^\ast(\lambda) \rangle | \leq \frac{\text{dist}(K, C_0)}{r_0}\|x\| \quad \text{for all } \lambda \in C_0 \text{ as } n > N_1.$$

It follows from the Cauchy integral formula that

$$(5.7) \quad |\langle x, \hat{f}_n^\ast(\lambda) \rangle | \leq \frac{1}{2\pi} \int_{|\xi| = r_0} \frac{|\langle x, \hat{f}_n^\ast(\lambda) \rangle |}{|\xi - \lambda|} \, |d\xi| \leq \varepsilon \|x\| ,$$

for all $\lambda \in K$ as $n > N_1$.

The decomposition (5.5) and the inequalities (5.6), (5.7) imply that there is a constant $M > 0$ such that

$$|\langle x, \hat{f}_n^\ast(\lambda) \rangle | \leq \varepsilon M\|x\| \quad \text{for all } x \in X, \lambda \in K \text{ as } n > \max\{N_0, N_1\}.$$

Thus it follows that $\{\hat{f}_n^\ast(\lambda)\}$ converges to zero uniformly on $K$ in the strong topology of $X^\ast$ and hence $T^\ast$ has property $(\beta)$.

(iii) $\Rightarrow$ (iv): Condition (iii,a) and [3, Theorem 5.8] imply that $T$ has property $(\beta)$. By Lemma 5.4, $Z^\ast = Z^\perp$ is invariant under $T^\ast$ and then

$$\sigma(T^\ast|Z^\perp) = \sigma(T/Z) \subset G^c.$$

Again, by Lemma 5.4, $T^\ast/Z^\perp$ is bounded and hence so is $T|Z$. We have

$$\sigma(T^\ast/Z^\perp) = \sigma(T|Z) \subset H.$$

Thus [3, Theorem 5.8] applies again and states that $T^\ast$ has property $(\beta)$.

(iv) $\Rightarrow$ (v) is evident.

(v) $\Rightarrow$ (i): Let $\{G_0, G_1\}$ be an open cover of $C$, where $G_0$ is a neighborhood of infinity and $G_1$ is relatively compact. Let $U_1, U_2$ be a couple of Cauchy domains with $U_1$ bounded, $U_2$ unbounded such that $U_2 = (\overline{U_1})^c$. Furthermore, we request that $U_2$ verify inclusions

$$G_1^c \subset U_2 \subset \overline{U_2} \subset G_0.$$

Next, we define the linear manifolds $N$ and $M$ as in §4. We claim that the following inclusions hold:

$$(5.8) \quad \begin{align*}
(a) \quad N &\subset \overline{X}(T, G_0), \\
(b) \quad \overline{M}^\omega &\subset \Xi^\ast(T^\ast, G_1).
\end{align*}$$

To prove (5.8a), let $x \in N$. For $n = 1, 2, 3, \ldots$ choose $f_n \in D_H$ such that $\|x - H f_n\| < 1/n$. Since $T$ has property $(\beta)$, $\{f_n\}$ converges uniformly on compact sets in $U_1$. Put $f(\lambda) = \lim_{n \to \infty} f_n(\lambda)$, for $\lambda \in U_1$. Then $f(\lambda) \in D_T$ and $(\lambda - T)f(\lambda) = x$, $\lambda \in U_1$. Consequently,

$$\sigma(x, T) \subset \overline{U_1}^c = \overline{U_2} \subset G_0$$

and (5.8a) follows.
To prove (5.8b), let $x^* \in M$. There exists $g \in D_{H^*}$ such that $H^* g = 0$ and $\tau g = x^*$, or equivalently,
\[(\lambda - T^*) g(\lambda) = \tau g = x^*, \quad \lambda \in U_2.\]
Thus it follows that
\[\sigma(x^*, T^*) \subset U_2 \subset \overline{G_1}\]
and hence $x^* \in X^*\left(T^*, \overline{G_1}\right)$. Since $g(\lambda) \in V^*$ implies $\lim_{\lambda \to -\infty} \|g(\lambda)\| = 0$, it follows from [3, Lemma 5.11] that $x^* \in \Xi^*\left(T^*, \overline{G_1}\right)$. Therefore, $M \subset \Xi^*\left(T^*, \overline{G_1}\right)$. Now [3, Theorem 9.4] implies that $\Xi^*\left(T^*, \overline{G_1}\right)$ is weak* closed and hence $M' \subset \Xi^*\left(T^*, \overline{G_1}\right)$. Now (5.8) and Theorem 4.2 imply
\[(5.9) \quad (X(T, G_0))' \subset N_1 = M' \subset \Xi^*\left(T^*, \overline{G_1}\right).\]

With $G_0$ fixed, we may choose a sequence of open sets $\{G_n\}$ such that $\bigcap_{n=1}^\infty G_n = G_0 = F_0$ and $\{G_0, G_n\}$ covers $C$ for every $n$. Then (5.9) implies that
\[\left(\{X(T, G_0)\}\right)' \subset \Xi^*\left(T^*, \overline{G_n}\right) \quad \text{for every } n.\]
Consequently,
\[(5.10) \quad \left(\left\{X(T, G_0)\right\}\right)' \subset \bigcap_{n=1}^\infty \Xi^*\left(T^*, \overline{G_n}\right) = \Xi^*\left(T^*, F_0\right).\]
Combining (5.10) with the evident inclusion $(X(T, G_0))' \supset \Xi^*\left(T^*, F_0\right)$, one finds
\[(5.11) \quad (X(T, G_0))' = \Xi^*\left(T^*, F_0\right).\]
Since $\Xi^*\left(T^*, F_0\right)$ is invariant under $T^*$, (5.11) implies that $X(T, G_0)$ is invariant under $T$. In fact, for every $x \in X(T, \overline{G_0}) \cap D_T$ and $x^* \in \Xi^*\left(T^*, F_0\right)$, one has $\langle Tx, x^* \rangle = \langle x, T^* x^* \rangle = 0$ so that $X(T, G_0)$ is invariant under $T$. Furthermore, we shall show that
\[(5.12) \quad \sigma(T, X(T, G_0)) \subset \overline{G_0}.\]
Let $x \in X(T, G_0)$ and choose a sequence $\{x_n\} \subset X(T, G_0)$ such that $x_n \to x$. Let $x_n(\cdot)$ denote the local resolvent of $T$ at $x_n$. By property $\beta$, the convergence
\[\left(\lambda - T\right)x_n(\lambda) = x_n \to x, \quad \lambda \in \left(\overline{G_0}\right)^c\]
implies $x_n(\lambda) \to f(\lambda)$ and $(\lambda - T)f(\lambda) = x$.
Therefore $\sigma(x, T) \subset \overline{G_0}$. On the other hand, for every $\lambda \in \left(\overline{G_0}\right)^c$, we have
\[\sigma(x_n(\lambda), T) = \sigma(x_n, T) \subset G_0,\]
so $x_n(\lambda) \in X(T, G_0)$ and hence $x(\lambda) \in X(T, G_0)$ for $\lambda \in \left(\overline{G_0}\right)^c$. Then, by a known property [5, see also 3, Proposition 2.7], inclusion (5.12) follows.
Now we are in a position to show that $T$ has the SDP. Let $\{G_0, G_1\}$ be an open cover of $C$ with $G_0$ a neighborhood of infinity and $G_1$ relatively compact.
Let $H_0$ be another open neighborhood of infinity such that $\overline{G_1} \cap \overline{H_0} = \emptyset$ and $H_0 \subset G_0$. Then $\widetilde{G}_0 = G_1 \cup H_0$ is a neighborhood of infinity and in virtue of (5.11) one writes

$$X(T,G_0) = \mathbb{E}^*(T^*,F_0),$$

where $\tilde{F}_0 = (G_0)^c$ and both $\mathbb{E}^*(T^*,F_0)$, $\mathbb{E}^*(T^*,\tilde{F}_0)$ are closed in the weak* topology of $X^*$. Similarly, $\mathbb{E}^*(T^*,F_0 \cup \tilde{F}_0)$ is closed in the weak* topology. Since $F_0 \cap \tilde{F}_0 = \emptyset$ ($F_0 = G_0^c$), we have

$$\mathbb{E}^*(T^*,F_0 \cup \tilde{F}_0) = \mathbb{E}^*(T^*,F_0) \oplus \mathbb{E}^*(T^*,\tilde{F}_0).$$

Set $Z^* = \mathbb{E}^*(T^*,F_0 \cup \tilde{F}_0)$.

Let $x \in X$, $x^* \in Z^*$ and denote by $x^*_0$ the projection of $x^*$ onto $\mathbb{E}^*(T^*,F_0)$, in conjunction with (5.13). The linear functional $x_0$ on $Z^*$, defined by

$$(5.14) \quad \langle x_0, x^* \rangle = \langle x, x^*_0 \rangle$$

is continuous in the weak* topology. Use the Hahn-Banach theorem on locally convex spaces to extend $x_0$ to a linear functional on $X^*$, that is continuous in the weak* topology. Therefore $x_0 \in X$. Since the projection $x^*_0$ of $x^* \in \mathbb{E}^*(T^*,\tilde{F}_0)$ onto $\mathbb{E}^*(T^*,F_0)$ is zero, it follows from (5.14) that $\langle x_0, x^* \rangle = 0$ for $x^* \in \mathbb{E}^*(T^*,\tilde{F}_0)$. Thus, $x_0 \in \mathbb{E}^*(T^*,\tilde{F}_0) = (X(T,G_0))$. Put $x_1 = x - x_0$ and for $x^* \in \mathbb{E}^*(T^*,F_0)$, use (5.14) to obtain $\langle x_1, x^* \rangle = 0$. Then $x_1 \in \mathbb{E}^*(T^*,F_0) = \overline{X(T,G_0)}$. Since $x \in X$ is arbitrary, the representation $x = x_0 + x_1$ with $x_0 \in X(T,G_0)$, $x_1 \in X(T,G_0)$ implies

$$X = \overline{X(T,G_0)} + X(T,G_0).$$

As regarding $X(T,G_0)$, it follows from (5.12) that

$$\sigma(T|X(T,G_0)) \subset \overline{G_0} = G_1 \cup H_0.$$ 

Having $G_1 \cap H_0 = \emptyset$ and $G_1$ relatively compact, the functional calculus for closed operators produces the following decomposition

$$X(T,G_0) = Y_1 \oplus Y_2,$$

$$\sigma(T|Y_1) \subset \overline{G_1}, \quad \sigma(T|Y_2) \subset H_0.$$ 

Since $H_0 \subset G_0$, $Y_2 \subset \overline{X(T,G_0)}$, (5.15) and (5.16) imply

$$X = Y_1 + \overline{X(T,G_0)}.$$ 

In view of (5.16b), (5.12), (5.17a) and (5.18), $T$ has the SDP.

**Remark.** A more restrictive version of property ($\beta$) is used in [6, Lemma 4.6]. Given $T \in C(X)$, a function $f: G \to D_T$ defined on an open subset $G$ of the compactified complex plane $C_\infty$, is said to be $T$-analytic if both $f$ and $Tf$ are
analytic on $G$. $T$ has property (β), in this stronger version, if for any sequence of $T$-analytic functions $\{f_n : G \to D_T\}$, the condition $(\lambda - T)f_n(\lambda) \to 0$ (as $n \to \infty$) in the strong topology of $X$ and uniformly on every compact subset of $G$ implies that $f_n(\lambda) \to 0$ in the strong topology of $X$ and uniformly on every compact subset of $G$.

It follows from the definition of the operator $H$ and Lemma 2.2 in §2 that both $Tf(\mu)$ and $T^*g(\lambda)$ are analytic. Consequently, Theorem 5.5 holds if we use the above-mentioned stronger version of property (β) in (iv) and (v).

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