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OPTIMAL L^p AND HÖLDER ESTIMATES FOR THE KOHN SOLUTION OF THE $\bar{\partial}$-EQUATION ON STRONGLY PSEUDOCONVEX DOMAINS

DER-CHEN E. CHANG

ABSTRACT. Let Ω be an open, relatively compact subset in $\mathbf{C}^{n+1}$, and assume the boundary of Ω, $\partial\Omega$, is smooth and strongly pseudoconvex. Let $\mathrm{Op}(K)$ be an integral operator with mixed type homogeneities defined on $\overline{\Omega}$: i.e., K has the form as follows:

$$\sum_{k,l\geq 0} E_k H_l,$$

where E_k is a homogeneous kernel of degree $-k$ in the Euclidean sense and H_l is homogeneous of degree $-l$ in the Heisenberg sense. In this paper, we study the optimal L^p and Hölder estimates for the kernel K. We also use Lieb-Range's method to construct the integral kernel for the Kohn solution $\overline{\partial^*}\mathbf{N}$ of the Cauchy-Riemann equation on the Siegel upper-half space and then apply our results to $\overline{\partial^*}\mathbf{N}$. On the other hand, we prove Lieb-Range's kernel gains 1 in "good" directions (hence gains 1/2 in all directions) via Phong-Stein's theory. We also discuss the transferred kernel from the Siegel upper-half space to Ω.

1. INTRODUCTION

To study the solution of the inhomogeneous Cauchy-Riemann equation $\overline{\partial \mu} = f$ on a given domain Ω in $\mathbf{C}^{n+1}$ with good bounds has been one of the main themes in the theory of complex analysis for many years. Suppose Ω is a bounded, smooth domain in $\mathbf{C}^1$ and $f \in C^1(\overline{\Omega})$ (hence in L^2), then we have the following integral representation for f: for $z \in \Omega$ and $\zeta \in \partial\Omega$,

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\dot{\gamma}(\zeta)\,d\sigma(\zeta) + \frac{1}{2\pi i}\int_{\Omega} \frac{\overline{\partial f/\partial\overline{\zeta}}}{\zeta - z}\,d\zeta\wedge d\overline{\zeta}$$

where $dz = \dot{\gamma}(\zeta)d\sigma(\zeta)$, $\zeta = \gamma(s)$ is the arc length parametrization of $\partial\Omega$, and $d\sigma$ is linear measure (arc length) on $\partial\Omega$. The first is called Cauchy projection $\mathbf{H}(f)$ of f. It is well known that

$$\mu(z) = \frac{1}{2\pi i}\int_{\Omega}\frac{f(\zeta)}{\zeta - z}\,d\zeta\wedge\overline{d\zeta}$$

is one solution to $\overline{\partial\mu} = f$, and any other solution is obtained by adding holomorphic functions to the original solution. We define a "good" solution

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μ (sometimes called the Kohn solution of $\overline{\partial \mu} = f$) by requiring that $\mu \perp$ {holomorphic functions} with respect to the inner product in $L^2(\Omega)$. Does the solution μ as we defined above by the integral formula give us the Kohn solution? In general, it is not. On the other hand, we can subtract the Szegö projection $S(\mu)$ from μ to minimize the L^2-norm of μ. (The Szegö projection $S: L^2(\Omega) \to H^2(\Omega)$ is the orthogonal projection defined by the Szegö kernel $S(z, \zeta)$.) As we discussed above, the Cauchy kernel is a projection to $H^2(\Omega)$ with an explicit expression. The problem reduces to whether this kernel

$$H(z, \zeta) = \frac{1}{2\pi i} \frac{1}{\zeta - z} \dot{\gamma}(\zeta)$$

is equal to the Szegö kernel $S(z, \zeta)$. Both $H(z, \zeta)$ and $S(z, \zeta)$ have the reproducing property; however the reproducing property of $S(z, \zeta)$ comes from Hilbert space considerations, not from Stokes' theorem. By the results obtained by Kerzman-Stein [11, 12], $H(z, \zeta) = S(z, \zeta)$ *is true only if* Ω *is a disc.* However, from their results, we know that S can be written in terms of the Cauchy kernel H and its conjugate: Suppose A is an operator defined by the kernel: $A(z, \zeta) = \overline{H(z, \zeta)} - H(z, \zeta)$, then

$$S = \sum_{j=0}^{\infty} \mathbf{H}\mathbf{A}^j, \qquad S(\mu) = \lim_{N \to \infty} \sum_{j=0}^{N} \mathbf{H}\mathbf{A}^j(\mu),$$

converges in L^2 and gives the Szegö projection of μ onto $H^2(\Omega)$. Then $\mu - S(\mu)$ gives the Kohn solution.

There is no perfect analogue of the Cauchy kernel in cases when $n \geq 1$: What comes closest to it are certain so-called *Cauchy-Fantappie* (*C-F*) *kernels.* The crucial property of the Cauchy kernel $H(z, \zeta)$ that makes it reproduce holomorphic functions is that, for fixed z,

$$d\left(\frac{1}{2\pi i} \frac{1}{\zeta - z} \dot{\gamma}(\zeta) d\sigma(\zeta)\right) = 0,$$

hence the C-F generalization to $\mathbf{C}^{n+1}$ has two parts:

(i) The construction of a differential form ψ with "correct" number of $d\zeta$ and $\overline{d\zeta}$'s that satisfy $d\psi = 0$; this leads to the direction of the work by Leray [9] and Norguet [22], etc.

(ii) The above forms involves certain functions $g_1(z, \zeta), g_2(z, \zeta), \ldots, g_{n+1}(z, \zeta)$ which can be chosen with great freedom so long as they are not expected to be holomorphic in z. For example, $g_i(z, \zeta) = \overline{z_i - \zeta_i}$ leads to the Bochner-Martinelli kernel. One can also construct $g_i(z, \zeta)$ which are holomorphic in z. This was achieved by Henkin-Ramirez [9], Kerzman-Stein [12] under the assumption that the domain is strongly pseudoconvex.

From now on, we always consider the problem on a bounded, strongly pseudoconvex domain with smooth boundary. Once again, we want to know how to find an integral representation for the Kohn solution. We look at the

Henkin's construction of the integral representation:

$$\begin{aligned} f(z) = c_1 &\int_{(\zeta,t)\in\partial\Omega\times[0,1]} \overline{\partial f}(\zeta)\wedge K_1(z,\zeta,t) \\ &- c_2 \int_{(\zeta,t)\in\partial\Omega\times\{0\}} f(\zeta)K_2(z,\zeta,t) \\ &+ c_3 \int_{(\zeta,t)\in\Omega\times\{1\}} \overline{\partial f}(\zeta)\wedge K_3(z,\zeta,t) \end{aligned}$$

where $K_i(z,\zeta,t)$, $i = 1, 2, 3$, are a combination of a Bochner-Martinelli kernel and a Henkin type kernel. The kernel K_2 is a reproducing kernel. Sometimes we call the operator $\mathbf{H}: L^2(\Omega) \to H^2(\Omega)$ defined by K_2 the Henkin projection. In general, $\mathbf{H}$ is not an orthogonal projection. So we want to look at this problem by other methods.

The purpose of this paper is to discuss the integral representation for the Kohn solution of the Cauchy-Riemann equation, using more direct methods than those described above. We look at the problem in the following way (as [1, 5, 8, 18, 19]): When $n \geq 1$, the equation $\overline{\partial}\mu = f$ is over-determined. It can be solved only when f satisfies the consistency condition $\overline{\partial} f = 0$. This is equivalent to the $\overline{\partial}$-Neumann problem:

$$\Box u = (\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial})u = f$$

with two boundary conditions: (1) $u \in \operatorname{domain}(\overline{\partial}^*)$; and (2) $\overline{\partial}u \in \operatorname{domain}(\overline{\partial}^*)$. Since $\overline{\partial} f = 0$, the $\overline{\partial}$-Neumann problem is equivalent to solving:

$$\overline{\partial}\,\overline{\partial}^* u = f, \qquad u \in \operatorname{domain}(\overline{\partial}^*). \tag{1.1}$$

Now we just need to put $\mu = \overline{\partial}^* u$ to get the Kohn solution. We will not explain the development of this problem, the existence and regularity properties of this problem can be found in Kohn [5], Greiner-Stein [8], Beals-Greiner-Stanton [2], Phong [24], Chang [3], etc. What we are interested in here is the integral representation of the parametrix $\mathbf{N}$ (the Neumann operator) of the $\overline{\partial}$-Neumann problem and the Kohn solution $\overline{\partial}^*\mathbf{N}$. The integral representation of the Neumann operator was first done by Phong [24], using the techniques in PDEs. He found an asymptotic expansion of $\mathbf{N}$ on the Siegel upper-half space under the assumption that the underlying metric is a Levi metric. If we consider f as a $(0,1)$-form, then the Neumann operator $\mathbf{N}$ will depend on the metric but the Kohn solution $\overline{\partial}^*\mathbf{N}$ (which is a function) will not. Once we prove the regularity properties for $\overline{\partial}^*\mathbf{N}$ for one metric then it is true for all Hermitian metrics. We refer readers to the paper by Beals-Fefferman-Grossman [1] about Phong's theorem. Phong's theorem shows that the Neumann operator $\mathbf{N}$ generates singular integrals with mixed type homogeneities. In fact, Krantz [14, 15, 16] first discussed the regularity properties of those operators when he considered the optimal estimates for the Henkin solution of $\overline{\partial}$-equations. As we mentioned above, the kernel $K_i(z,\zeta,t)$ which appears in the Henkin solution of the $\overline{\partial}$-equation has mixed type homogeneities already (the Euclidean homogeneity E_k comes from the Bochner-Martinelli kernel and the Heisenberg homogeneity H_l comes from the Henkin type kernel). Later on, when E. M. Stein and his collaborators introduced the method of osculating the boundary $\partial\Omega$ by the Heisenberg group, such phenomena were even more apparent, see Folland-Stein [6], Greiner-Stein [8], Rothschild-Stein [28], Phong-Stein [25], Chang [3]. Unlike the parametrix of boundary complex Laplacian $\Box_b$, Phong's result cannot be transferred to Ω directly by *standard Heisenberg coordinates.* After a couple of years effort, Phong and Stein [26, 27] achieved this goal by using *special Heisenberg coordinates.* In their paper [26, 27], they also conquer the difficulties of the L^p estimates for the "type 0" operators (the kernel $E_k H_l$ has critical exponents with $k = 2n$ and $l = 4$) by looking at the singular Radon transforms, oscillatory integrals and Hilbert integrals. In those papers, they also open a new direction by discussing singular integrals whose singularity is not a point!

On the other hand, Lieb and Range [18, 19, 20] studied this problem by different methods via looking at the integral representations. They found an integral representation for a function u which defined on $\overline{\Omega}$. The kernels of the Lieb-Range's representation are defined explicitly in terms of the geodesic distance function for the given Hermitian metric (the Euclidean homogeneity part) and the Levi polynomial of a strictly plurisubharmonic defining function for Ω (the Heisenberg homogeneity part). It is easy to get an integral representation for the Kohn solution from Lieb-Range's theorem by putting $\overline{\partial}^*\mathbf{N}f$ (f is a given $(0,1)$-form) in the position of u.

These two methods are based on totally different philosophy, but they are related: Lieb-Range's method gives us more geometric feeling of this problem and allows us to generalize it to more general domains, we will discuss it in a forthcoming paper; but Phong-Stein's technique involves deep analysis for the operators with mixed type homogeneities which give us optimal estimates. In this paper, the author will combine these two methods to look at this problem. In §2, we use Lieb-Range's method to construct the integral kernel of $\overline{\partial}^*\mathbf{N}$ on the Siegel upper-half space: Let f be a $(0,1)$-form with the property $\overline{\partial} f = 0$, then the $\ker(\overline{\partial}^*\mathbf{N})$ of the Kohn solution for $\overline{\partial} u = f$ is a vector $(K_1, K_2, \dots, K_{n+1})$. The principal part of each K_i has a form as follows:

$$\sum_{k,l\geq 0} E_k H_l + \tilde{\psi}^{(2n+1)/2},$$

where $E_k = c_1 P/\psi^a$ is homogeneous of degree $-k$ in the Euclidean sense, and $H_l = c_2 Q/\varphi^b$ is homogeneous of degree $-l$ in the Heisenberg sense. (We will define $\psi, \varphi, \tilde{\psi}$ in §2.) The crucial terms for each K_i are $E_{2n-1}H_4$ and $E_{2n-3}H_6$, but when we consider the differentiation on those kernels, there will be a critical case, i.e., $E_{2n}H_4$. This is the reason why we need to compute the kernel explicitly. When we try to apply Phong-Stein's theory to those kernels, we need to know whether those kernels $E_{2n}H_4$ satisfy the mean-value zero property, i.e.,

$$\int_{a<|z'|<b} E_k H_l(z',t;\rho)\,dV(z') = 0, \qquad \forall 0 < a \le b < \infty, \text{ and } \forall(t;\rho) \in \mathbf{R}^1 \times \mathbf{R}^+.$$

In §3 and §4, we will prove the optimal L^p and Hölder estimates for operators with mixed type homogeneities. These results are new and allow us to apply them to $\overline{\partial}^*\mathbf{N}$. We also apply Phong-Stein's theorem to prove $\overline{\partial}^*\mathbf{N}$ gains 1 in "good" directions in §3 which improve Lieb-Range's result to standard Sobolev spaces. (In [20], Lieb and Range considered the regularity properties for the operator $\overline{\partial}^*\mathbf{N}$ on weighted Sobolev spaces $L^p(\mathrm{dist}^\alpha(z,\partial\Omega)dV)$, for some $\alpha >$ 0.) We state our main theorems as follows:

Main Theorem (Theorem (3.12)). *Let U be a boundary neighborhood of* $0 \in \partial\mathbf{D} \subset \mathbf{C}^{n+1}$, $f \in C^\infty_{(0,1),0}(U) \cap \mathrm{dom}(\overline{\partial}^*)$ *with* $\overline{\partial f} = 0$. *Then the Kohn solution* $\overline{\partial}^*\mathbf{N}f$ *of the Cauchy-Riemann equation* $\overline{\partial u} = f$ *satisfies: If* $f \in L^p_k(U)$ *then* $\overline{\partial}^*\mathbf{N}f \in L^p_{k+1/2}(U) \cap S^p_{k+1}(U)$, $k = 0, 1, 2, \ldots$.

Main Theorem 2 (Theorem (3.17)).

(i) $\overline{\partial}^*\mathbf{N}\colon L^p(U) \to L^q(U)$, $1/q = 1/p - 1/(2n+4)$, *if* $1 < p < 2n+4$;

(ii) $|\iint_U \exp(\alpha|(\overline{\partial}^*\mathbf{N})(f)/\|f\|_{L^{2n+4}}|^{(2n+4)/(2n+3)})\,d(w')\,ds\,d\mu| \le C < \infty$, α *is a sufficiently small constant; i.e.,*

$$\mathrm{Op}(\overline{\partial}^*\mathbf{N})\colon L^{2n+4}(U) \to L(\exp(\alpha|\cdot|^{(2n+4)/(2n+3)}))(U).$$

(iii) $\overline{\partial}^*\mathbf{N}\colon L^1(U) \to L^{(2n+4)/(2n+3)-\varepsilon}(U)$, $\forall\varepsilon > 0$;

(iv) $\overline{\partial}^*\mathbf{N}\colon L^{2n+4}(\log^+ L)^{2n+3+\varepsilon}(U) \to L^\infty(U)$, $\forall\varepsilon > 0$.

Main Theorem 3 (Theorem (4.10)).

(i) $\overline{\partial}^*\mathbf{N}\colon L^p(U) \to \Lambda_{1/2-(2n+4)/2p}(U)$, $\overline{\partial}^*\mathbf{N}\colon L^p(U) \to \Gamma_{1-(2n+4)/p}(U)$, *if* $2n+4 < p < \infty$;

(ii) $\overline{\partial}^*\mathbf{N}\colon L^\infty(U) \to \Lambda_{1/2} \cap \Gamma_1(U)$;

(iii) $\overline{\partial}^*\mathbf{N}\colon \Lambda_\alpha(U) \to \Lambda_{\alpha+1/2}(U)$, $\forall\alpha > 0$.

In §5, we show how standard Heisenberg coordinates can be used to transfer the kernel $\overline{\partial}^*\mathbf{N}$ from the model $\mathbf{D}$ to Ω and prove that the results of Main Theorem 1 to 3 are true for the "transferred kernel" $(\overline{\partial}^*\mathbf{N})_D$, where D is a boundary neighborhood of Ω.

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2. The kernel of $\overline{\partial}^*\mathbf{N}$ on the Siegel upper-half space

In this section we apply the Lieb-Range's theorem (see [18, 19]) to find the integral representation for the Kohn solution of the Cauchy-Riemann equation on the Siegel upper-half space. Using the same notations as in [18, 19], we define operators T_q and E_q as follows:

$$T_0^a g = \langle g, \vartheta_z \mathscr{T}_0 -{}^* \overline{K}_0 \rangle,$$
$$T_0^i g = \langle g, \overline{\partial^*}\Gamma_0 \rangle, \qquad g \in C^1_{(0,q),0}(\overline{U});$$
$$E_0 g = \langle g, {}^*\partial_z \overline{K}_0 \rangle, \qquad g \in C^1_0(\overline{U});$$
$$T_q^a g = \langle g, \vartheta_z \mathscr{T}_q - \partial_w \mathscr{T}_{q-1} \rangle;$$
$$T_q^i g = \langle g, \overline{\partial}\Gamma_q \rangle, \qquad g \in C^\infty_{(0,q+1)}(\overline{U}),\ q > 1;$$
$$T_q = T_q^a + T_q^i;$$
$$E_q g = \langle g, \vartheta_w \partial_w \mathscr{T}_0 - (\vartheta_z \partial_w \mathscr{T}_0)^* \rangle, \qquad q > 1.$$

(2.1) **Theorem** (Lieb-Range). *Let U be a boundary neighborhood of a bounded strongly pseudoconvex domain Ω. Suppose $u \in C^1_{(0,1),0}(U) \cap \operatorname{dom}(\overline{\partial^*})$, then*

$$u = T_q \overline{\partial} u + T^*_{q-1} \overline{\partial^*} u + E_q u + \mathscr{E}_{-2n-1} u + \mathscr{E}_{-2n} \overline{\partial} u + \mathscr{E}_{-2n} \overline{\partial^*} u + \textit{error terms}.$$

Here the operator $\mathscr{E}_j$ is an integral operator with kernel has size $|\mathscr{E}_j(z, w)| \approx C \cdot \|z - w\|_e^j$, where $\|\cdot\|$ means the Euclidean norm.

Since we want to find the integral representation for the kernel of $\overline{\partial^*}\mathbf{N}f$, where $f \in C^\infty_{(0,1),0}(U) \cap \operatorname{dom}(\overline{\partial^*})$. We just need to find the corresponding operators T_0^a, T_0^i and E_0. Let $\mathbf{D} = \{(z_1, \dots, z_{n+1}) \in \mathbf{C}^{n+1} : \mathfrak{I} z_{n+1} > |z_1|^2 + \cdots + |z_n|^2\}$ be the Siegel upper-half space with the boundary $\partial\mathbf{D} = \{\mathfrak{I} z_{n+1} = |z_1|^2 + \cdots + |z_n|^2\}$. It is a standard method to find the supporting function $\varphi(z, w)$ and the geodesic distance function $\psi^2(z, w)$. We omit all the computations.

Remarks. (1) Let us consider the standard Heisenberg vector fields on U, i.e.,

$$Z_k = \frac{\partial}{\partial z_k} + i\overline{z}_k \frac{\partial}{\partial_t}, \qquad k = 1, 2, \dots, n;$$

and

$$\overline{Z}_{n+1} = i\sqrt{2}\frac{\partial}{\partial \overline{z}_{n+1}}, \qquad T = \frac{1}{\sqrt{2}}(\overline{Z}_{n+1} - Z_{n+1}).$$

We also consider $\{\omega_1, \dots, \omega_{n+1}\}$ are the dual frame of $\{Z_1, \dots, Z_{n+1}\}$. In this case, $\omega_{n+1} = \sqrt{2}\partial\rho$. Here $\rho(z) = \operatorname{Im} z_{n+1} - \sum_{1\le j\le n} |z_j|^2$ is the "height" function defined on $\overline{U} \cap \overline{\mathbf{D}}$. Since we are solving $\overline{\partial u} = f$ with f is a given $(0, 1)$-form, then $u = \overline{\partial^*}\mathbf{N}f$ is a function in $\mathbf{D}$, hence we may choose the given Hermitian metric defined on $\overline{U} \cap \overline{\mathbf{D}}$ is the Levi metric! (i.e., $\{\omega_1, \dots, \omega_{n+1}\}$ is orthonormal). Once we obtain the same results for $\overline{\partial^*}\mathbf{N}f$ then we will have the same results for "all" other Hermitian metrics.

(2) Under the Levi metric assumption, we can get the parametrix $\Gamma_0(z, w)$ for the complex Laplacian $\square$ (which acts on $(0, 1)$-forms) is (see Phong-Stein

[27]):

$$\Gamma_0(z,w) = \mathscr{K}'((z',t)^{-1}\cdot(w',s);\mu-\rho) = c\tilde{\psi}^{-2n}$$

where $\mathscr{K}'(z',t;\rho) = 2^{n-1}\pi^{-(n+1)}\Gamma(n)[(2|z'|^2+t^2+\rho^2)]^{-n}$. Following the same idea of Henkin, we consider $\psi^2 = \tilde{\psi}^2+4\rho\mu$ as a new function which extend the kernel to $\overline{\mathbf{D}}\times\overline{\mathbf{D}}$ and nonzero on $\overline{\mathbf{D}}\times\overline{\mathbf{D}}-\Sigma = \{(z,z)\in\partial\mathbf{D}\}$. It is easy to see that $\psi^2(z,w) = c'\mathscr{K}((z',t)^{-1}\cdot(w',s);\rho+\mu)$ where $\mathscr{K}(z',t;\rho) = 2|z'|^2+t^2+\rho^2$.

(3) By the definition of ρ, we can see $\rho(z)\geq 0$ for all $z\in\overline{U}\cap\overline{\mathbf{D}}$. Also,

$$z = (z_1,\dots,z_n,z_{n+1}) = (z',z_{n+1}) = (z',t,\rho+|z'|^2)$$

where $t = \Re z_{n+1}$ and $(z',t)\in\partial U_1\subset\mathbf{H}^n$ is a point on the Heisenberg group, (i.e., when $\rho = 0$, $\Im z_{n+1} = |z'|^2\in\partial U_1\subset\mathbf{H}^n$). Use this coordinate system, we can write down the function φ precisely: suppose $z = (z',t,\rho+|z'|^2)$ and $w = (w',s,\mu+|w'|^2)$, then

$$\varphi(z,w) = \sum_{j=1}^{n}(-1)\overline{z}_j w_j + \frac{i}{2}(t-s) + \frac{1}{2}(\rho+\mu) + \frac{1}{2}(|z'|^2+|w'|^2).$$

Suppose we define $\mathscr{K}(z',t,\rho) = |z'|^2+\rho-it$, then

$$\varphi(z,w) = (1/2)\mathscr{K}((z',t)^{-1}\cdot(w',s);\rho+\mu).$$

To simplify the computations, w.o.l.g., we may assume $n=2$. In this situation:

$$\mathscr{T}_0 = -{}^*\overline{\mathscr{A}}_0$$

where

$$\mathscr{A}_0(\mathscr{W},\mathscr{B}) = (2\pi i)^{-3}\sum_{0\leq j\leq 1} a_j\mathscr{W}\wedge\mathscr{B}\wedge(\overline{\partial}_z\mathscr{W})^j\wedge(\overline{\partial}_z\mathscr{B})^{1-j}$$

and

$$\mathscr{W} = (\sqrt{2})^{-1}\omega_3/\varphi,\quad \mathscr{B} = \left\{\sum_{1\leq j\leq 3}(Z_j\psi^2)\omega_j\right\}/\psi^2.$$

Hence we have

$$\begin{aligned}\mathscr{T}_0(z,w) = c\{&a_1(\overline{Z}_1\psi^2)/\overline{\varphi^2}\psi^2\cdot\overline{\omega}_1\wedge\overline{\omega}_3 - a_1(\overline{Z}_2\psi^2)/\overline{\varphi^2}\psi^2\cdot\overline{\omega}_2\wedge\overline{\omega}_3\\ &+ a_0(\overline{Z}_1\psi^2)/\overline{\varphi}\psi^4\cdot\overline{\omega}_1\wedge\overline{\omega}_3 - a_0(\overline{Z}_2\psi^2)/\overline{\varphi}\psi^4\cdot\overline{\omega}_2\wedge\overline{\omega}_3\}\\ &+ \text{t.w.w.s.}\end{aligned}$$

Here we use t.w.w.s.=terms with weaker singularities and

(2.2)

$$
\begin{aligned}
\vartheta_0\mathcal{T}_0(z,w) &= -\sum_{1\le j\le 3}\varepsilon_{jk}Z_j(\mathcal{T}_0(z,w))\overline{\omega}_j \lrcorner\,\overline{\omega}_k \\
&= \{a_1 2\sqrt{2}(\overline{Z}_1\psi^2)/\overline{\varphi^3}\psi^2 + a_1\sqrt{2}\varphi^*(\overline{Z}_1\psi^2)/\overline{\varphi^2}\psi^4 \\
&\qquad + a_0\sqrt{2}(\overline{Z}_1\psi^2)/\overline{\varphi^2}\psi^4 + a_0 2\sqrt{2}\varphi^*(\overline{Z}_1\psi^2)/\overline{\varphi}\psi^6 + \text{t.w.w.s.}\}\overline{\omega}_1 \\
&\quad - \{a_1 2\sqrt{2}(\overline{Z}_2\psi^2)/\overline{\varphi^3}\psi^2 + a_1\sqrt{2}\varphi^*(\overline{Z}_2\psi^2)/\overline{\varphi^2}\psi^4 \\
&\qquad + a_0\sqrt{2}(\overline{Z}_2\psi^2)/\overline{\varphi^2}\psi^4 + a_0 2\sqrt{2}\varphi^*(\overline{Z}_2\psi^2)\overline{\varphi}\psi^6 + \text{t.w.w.s.}\}\overline{\omega}_2 \\
&\quad + \{a_1 2(Z_1\overline{\varphi})(\overline{Z}_1\psi^2)/\overline{\varphi^3}\psi^2 + a_1(Z_1\psi^2)(\overline{Z}_1\psi^2)/\overline{\varphi^2}\psi^4 \\
&\qquad - a_1 2(Z_2\overline{\varphi})(\overline{Z}_2\psi^2)/\overline{\varphi^3}\psi^2 - a_1(Z_2\psi^2)(\overline{Z}_2\psi^2)/\overline{\varphi^2}\psi^4 \\
&\qquad + a_0(Z_1\overline{\varphi})(\overline{Z}_1\psi^2)/\overline{\varphi^2}\psi^4 + a_0 2(Z_1\psi^2)(\overline{Z}_1\psi^2)/\overline{\varphi}\psi^6 \\
&\qquad - a_0(Z_2\overline{\varphi})(\overline{Z}_2\psi^2)/\overline{\varphi^2}\psi^4 - a_0 2(Z_2\psi^2)(\overline{Z}_2\psi^2)/\overline{\varphi}\psi^6 + \text{t.w.w.s.}\}\overline{\omega}_3 \\
&= K_1(z,w)\overline{\omega}_1 + K_2(z,w)\overline{\omega}_2 + K_3(z,w)\overline{\omega}_3 .
\end{aligned}
$$

Here we use the relations: $Z_3\overline{\varphi} = -\sqrt{2} + \mathcal{E}_1 + \mathcal{E}_0\gamma(z)$; $Z_3(\overline{Z}_j\psi^2) = \mathcal{E}_1$ for $j = 1,2$ and $Z_3\psi^2 = -\sqrt{2}\varphi^* + \mathcal{E}_0\gamma(z)^2 + \mathcal{E}_1\gamma(z) + \mathcal{E}_2$ which we can get by direct computations. Here γ is the defining function of $\mathbf{D}$. On the other hand, $K_0(z,w) = -\varphi^{-3}\omega_1\wedge\omega_2\wedge\omega_3\wedge\overline{\omega}_1\wedge\overline{\omega}_3$. Hence we have $-{}^*\overline{K}_0(z,w) = c'\overline{\varphi^{-3}}\cdot\omega_2$.

The operator E_0 is given by ${}^*\partial_z\overline{K}_0(z,w)$, we have

(2.3) $$E_0 = {}^*\,\partial_z\overline{K}_0(z,w) = 3(-\sqrt{2} + \mathcal{E}_1 + \mathcal{E}_0\gamma(z))/\overline{\varphi^4} + \text{t.w.w.s.}$$

Combine those results, we have the following theorem:

(2.4) **Proposition.** *Suppose $f \in C^\infty_{(0,1),0}(U) \cap \operatorname{dom}(\overline{\partial^*})$, then the integral representation of the Kohn solution $\overline{\partial^*}\mathbf{N}f$ for the Cauchy-Riemann equation $\overline{\partial u} = f$ is given by a convolution operator on the Siegel upper-half space*:

$$
\begin{aligned}
\overline{\partial^*}\mathbf{N}f &= T_0\overline{\partial}\overline{\partial^*}\mathbf{N}f + E_0\overline{\partial^*}\mathbf{N}f + \textit{more smoothing terms} \\
&= T_0 f + E_0\overline{\partial^*}\mathbf{N}f + \textit{more smoothing terms}
\end{aligned}
$$

where T_0 is an integral operator with vector-valued kernel

$$K = (K_1, K_2, K_3) + \textit{elliptic term} + \textit{t.w.w.s.}$$

The operators K_i, $i = 1,2,3$, are defined in (2.2) *and the elliptic term equals to $\overline{\partial_z}\Gamma_0(z,w)$. E_0 is defined in* (2.3).

Remark. (1) In this case, the principal part of the operator E_0 is exactly the Bergman (orthogonal) projection operator $\mathbf{P}_0\colon L^2(\Omega) \to H^2(\Omega)$ with respect to the metric on Ω.

3. Optimal L^p estimates for the kernels with mixed type homogeneities: Phong-Stein's theory

In this section we want to apply Phong-Stein's theory to discuss the L^p regularities of the kernel which we obtained in §2. As we have seen in §1, Phong [24] gave an explicit integral representation of the Neumann operator on the Siegel upper-half space $\mathbf{D}$ when the given metric on $\overline{\mathbf{D}}$ is Levi: The main term of $\mathbf{N}$ is a convolution operator defined by a kernel k where

(3.1)
$$[\mathrm{Op}(\mathscr{K})(f)](z',t;\rho) = \iint_{\mathbf{R}^+\mathbf{H}^n} \mathscr{K}((z',t)\cdot(w',s)^{-1};\rho+\mu)f(w',s;\mu)dV(w')\,ds\,d\mu$$

for $f \in C_0^\infty(U)$.

The kernel $\mathscr{K}$ can be written as

$$\mathscr{K}(z',t;\rho) = \sum_{k,l\geq 0} E_k(z',t;\rho)H_l(z',t;\rho),$$

with E_k and H_l homogeneous of degree $-k$ and $-l$ in the isotropic (Euclidean) and parabolic (Heisenberg) sense respectively, i.e.,

$$E_k(\lambda(z',t;\rho)) = E_k(\lambda z',\lambda t;\lambda\rho) = \lambda^{-k}E_k(z',t;\rho),$$
$$H_l(\lambda\cdot(z',t;\rho)) = H_l(\lambda z',\lambda^2 t;\lambda^2\rho) = \lambda^{-l}H_l(z',t;\rho), \qquad \forall\lambda > 0.$$

In order to study the optimal L^p estimates for the "transfer parametrix" of the $\bar\partial$-Neumann problem from the Siegel upper-half space to a bounded strongly pseudoconvex domain, Phong and Stein [26 and 27] investigated the deep theory about such operators via Hilbert integral operators, singular Radon transform and oscillatory integrals. In [26 and 27] they consider the kernel $E_k(z',t;\rho)H_l(z',t;\rho)$, where E_k has the form $\approx P/(2|z'|^2+\rho^2+t^2)^a$ which is homogeneous $-k$ in the isotropic dilation sense. The kernel H_l has the form $\approx Q/(|z'|^2+\rho-it)^b$ which is homogeneous $-l$ in the parabolic dilation sense. P and Q are polynomials. Here is one of the main theorems in [27]:

(3.2) **Theorem** (Phong-Stein). *Let $\mathscr{K}(z',t;\rho)\in C^\infty(U\backslash(0,0;0))$ be a kernel of the type $\mathscr{K}(z',t;\rho) = E_k(z',t;\rho)H_l(z',t;\rho)$ and consider the Hilbert integral operator $\mathrm{Op}\mathscr{K}$ associated to $\mathscr{K}$ by the formula* (3.1). *Then*

(i) $\mathrm{Op}(\mathscr{K})$ *can be extended as a bounded operator from $L^p(V_1)$ to $L^p(V_2)$, $1\leq p\leq\infty$, for any compact subsets $V_1, V_2\subset U$ when either*

$$k+l<2n+4,\quad k<2n \quad or \quad k+l/2<2n+2,\quad l<4.$$

(ii) $\mathrm{Op}(\mathscr{K})$ *can be extended as a bounded operator from $L^p(U)$ to $L^p(U)$, $1<p<\infty$ when either*

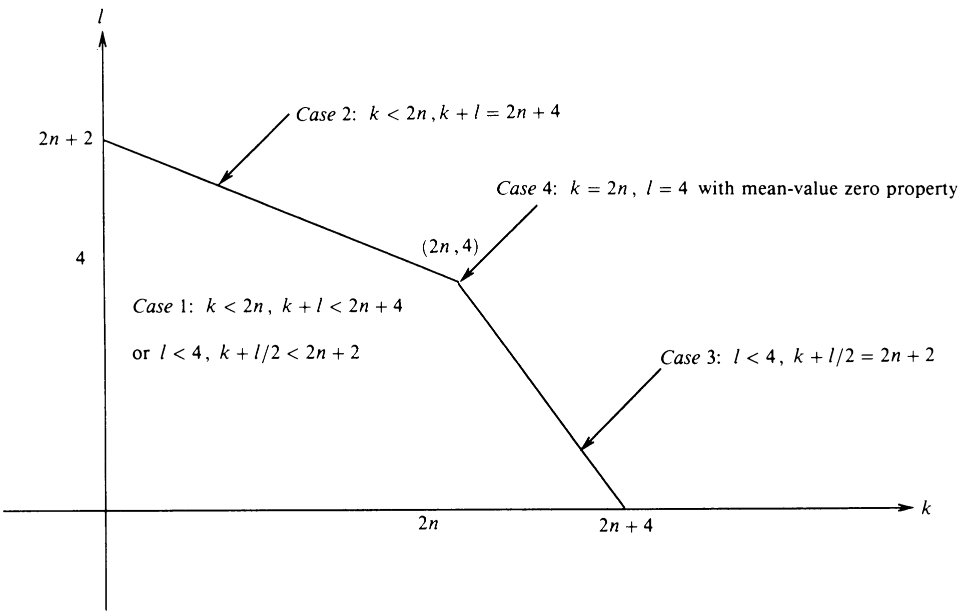
$$k+l=2n+4,\quad k<2n \quad or \quad k+l/2=2n+2,\quad l<4.$$

(iii) *Assume that $k=2n$, $l=4$, and k has the cancellation property*

(3.3)
$$\int_{a<|z'|<b}\mathscr{K}(z',t;\rho)dV(z') = 0$$

for any $(t;\rho) \in \mathbf{R}^1 \times \mathbf{R}^+$ *and any* $0 < a \le b < \infty$. *Then* $\mathrm{Op}(\mathscr{K})$ *extended as a bounded operator on* $L^p(U)$, $1 < p < \infty$.

(3.4) **Definition.** The operator $\mathrm{Op}(\mathscr{K})$ is defined by (3.1) where $\mathscr{K} = E_k H_l$. We say $\mathrm{Op}(\mathscr{K})$ belongs to the class $\mathscr{C}_1$ if either $k + l < 2n + 4$, $k < 2n$ or $k + l/2 < 2n + 2$, $l < 4$; $\mathrm{Op}(\mathscr{K})$ belongs to the class $\mathscr{C}_2$ if $k + l = 2n + 4$, $k < 2n$; $\mathrm{Op}(\mathscr{K})$ belongs to the class $\mathscr{C}_3$ if $k + l/2 = 2n + 2$, $l < 4$ and $\mathrm{Op}(\mathscr{K})$ belongs to the class $\mathscr{C}_4$ if $k = 2n$, $l = 4$ and $\mathscr{K}$ satisfies the cancellation property (3.3).



Now we go back to our integral kernel for $\overline{\partial}^*\mathbf{N}$, we have the following theorem:

(3.5) **Lemma.** *Suppose* U *is a boundary neighborhood of* $0 \in \partial\mathbf{D} \subset \mathbf{C}^3$. *The integral kernel for the Kohn solution* $\overline{\partial}^*\mathbf{N}$ *for the Cauchy-Riemann equation* $\overline{\partial u} = f$ *in* U *is given by*

$$\overline{\partial}^*\mathbf{N} = T_0 + E_0 + \textit{terms with weaker singularities}$$

where T_0 *is a vector-valued kernel, i.e.,* $T_0 = (K_1, K_2, K_3)+$ *elliptic term. The components of* T_0 *have the following representation*:

$$K_1 = c_1 E_3 H_4 + c_2 E_5 H_0 + c_3 E_1 H_6 + c_4 E_3 H_2,$$
$$K_2 = \tilde{\tilde{c}}_1 E_3 H_4 + \tilde{c}_2 E_5 H_0 + \tilde{c}_3 E_1 H_6 + \tilde{c}_4 E_3 H_2,$$
$$K_3 = c_1' E_4 H_2 + c_2' E_2 H_4 + c_3' E_0 H_6.$$

E_0 *is a scalar kernel and* $E_0 = c H_8$.

Proof. Since

$$\varphi = \tfrac{1}{2}\mathscr{H}((z',t)^{-1}\cdot(w',s);\rho+\mu)$$

with $\mathscr{H}(z',t;\rho) = c_3(|z'|^2+\rho-it)$ is homogeneous of degree 2 in the Heisenberg sense. On the other hand,

$$\psi^2 = \mathscr{K}((z',t)^{-1}\cdot(w',s);\mu+\rho)$$

with $\mathscr{K}(z',t;\rho) = \tilde{c}_3(2|z'|^2+\rho^2+t^2)$ is homogeneous of degree 2 in the Euclidean sense. We also know $|\overline{Z}_j\psi^2| \approx \mathscr{E}_1$ and $|\overline{Z}_j\overline{\varphi}(z,w)| \approx \mathscr{E}_1$ for $j=1,2$. Plug in these results in Theorem (2.2), it is easy to prove the theorem.

From the proof of Lemma (3.5), we can get the following result immediately by applying Phong-Stein's theorem:

(3.6) **Proposition.** *Suppose* U *is a boundary neighborhood of* $0 \in \partial\mathbf{D} \subset \mathbf{C}^3$, *then* $\overline{\partial}^*\mathbf{N}$ *is a bounded operator from* $L^p(U)$ *to* $L^p(U)$, $1<p<\infty$ *and is a bounded operator from* $L^p_{\mathrm{loc}}(U)$ *to* $L^p_{\mathrm{loc}}(U)$, $1\le p\le\infty$.

The importance of these operators T_0 and E_0 lies in their smoothing properties. They are not only bounded on $L^p(U)$, $1<p<\infty$, but also gain 1 in "good" directions. (Hence gain 1/2 in all directions.) From now on, we need to go through all calculations to look at what happen when we consider the derivatives of T_0 and E_0: First, let us consider the components K_1, K_2 of T_0. In fact, we just need to look at terms E_5H_0, E_3H_4 and E_1H_6 in K_1 and K_2. Suppose $j=1,2$

$$\begin{aligned}Z_j\{\varphi^*(\overline{Z}_k\psi^2)/\overline{\varphi}\psi^6\} &= 2\delta_{jk}\varphi^*/\overline{\varphi}\psi^6 + (Z_j\varphi^*)(\overline{Z}_k\psi^2)/\overline{\varphi}\psi^6\\ &\quad - \varphi^*(Z_j\overline{\varphi})(\overline{Z}_k\psi^2)/\overline{\varphi^2}\psi^6 - 3\varphi^*(Z_j\psi^2)(\overline{Z}_k\psi^2)/\overline{\varphi}\psi^8\\ &\quad + \text{t.w.w.s.}\end{aligned} \tag{3.7,a}$$

The critical terms of (3.7,a) are $2\delta_{jk}\varphi^*/\overline{\varphi}\psi^6$ and $\varphi^*(Z_j\psi^2)(\overline{Z}_k\psi^2)/\overline{\varphi}\psi^8$, but both of them belong to the class $\mathscr{E}_1$. The same situation will occur when we consider $\overline{Z}_j\{\varphi^*(\overline{Z}_k\psi^2)/\overline{\varphi}\psi^6\}$.

$$\begin{aligned}Z_j\{\overline{Z}_k\psi^2/\overline{\varphi^2}\psi^4\} &= 2\delta_{jk}/\overline{\varphi^2}\psi^4 + 2(Z_j\overline{\varphi})(\overline{Z}_k\psi^2)/\overline{\varphi^3}\psi^4\\ &\quad - 2(Z_j\psi^2)(\overline{Z}_k\psi^2)/\overline{\varphi^2}\psi^6 + \text{t.w.w.s.}\end{aligned} \tag{3.7,b}$$

The 2nd term of (3.7,b) belongs to the class $\mathscr{E}_2$. But the 1st and the 3rd term of (3.7,b) are the critical case, i.e., $k=4$, $l=4$. To control these two terms, we need the following lemma:

(3.8) **Lemma.** *The principal part of the kernel*

$$2\delta_{jk}/\overline{\varphi^2}\psi^4 - 2(Z_j\psi^2)(\overline{Z}_k\psi^2)/\overline{\varphi^2}\psi^6$$

satisfies the mean-value zero property:

$$\int_{a<(|z_1|^2+|z_2|^2)<b}\left\{\frac{2\delta_{jk}}{\overline{\varphi}^2\psi^4}-\frac{2(Z_j\psi^2)(\overline{Z}_k\psi^2)}{\overline{\varphi}^2\psi^6}\right\}dV(z')=0,$$

$$\forall 0<a\le b<\infty,\ \textit{and}\ \forall(t;\rho)\in\mathbf{R}^1\times\mathbf{R}^+.$$

Hence this kernel belongs to the class $\mathscr{C}_4$.

Proof. (i) $j=k$.

(3.9) $2\delta_{jk}/\overline{\varphi^2}\psi^4-2(Z_j\psi^2)(\overline{Z}_k\psi^2)/\overline{\varphi^2}\psi^6=2/\overline{\varphi^2}\psi^4-2(Z_j\psi^2)(\overline{Z}_j\psi^2)/\overline{\varphi^2}\psi^6$.

It is easy to see that both terms of (3.9) are satisfying $k=4$, $l=4$. We need to prove that this kernel has the mean-value zero property. To simplify the computation, we assume $\psi^2=\mathscr{K}(z',t;\rho)=2|z'|^2+t^2+\rho^2$ and $j=1$. Then we get

$$\begin{aligned}2/\overline{\varphi^2}\psi^4-2(Z_j\psi^2)(\overline{Z}_j\psi^2)/\overline{\varphi^2}\psi^6&=2\{\psi^2-(Z_1\psi^2)(\overline{Z}_1\psi^2)\}/\overline{\varphi^2}\psi^6\\&=4\{|z_2|^2-|z_1|^2\}/\overline{\varphi^2}\psi^6+\text{t.w.w.s.}\end{aligned}$$

(ii) $j\neq k$. We may assume $j=1, k=2$, then

$$2\delta_{jk}/\overline{\varphi^2}\psi^4-2(Z_j\psi^2)(\overline{Z}_k\psi^2)/\overline{\varphi^2}\psi^6=-8\overline{z}_1\cdot z_2/\overline{\varphi^2}\psi^6+\text{t.w.w.s.}$$

In cases (i) and (ii), we need to check whether the functions $|z_1|^2-|z_2|^2$ and $\overline{z}_1\cdot z_2$ satisfy the mean-value zero property. But both of them are harmonic polynomials of degree 2 in $\mathbf{R}^4$. When we restrict them to the unit sphere Σ_3, they become spherical harmonic polynomials of degree two. Since the subspaces (of $L^2(\Sigma_3))\mathfrak{P}_k$ and $\mathfrak{P}_l$ of all spherical harmonic polynomials of degree k and l are mutually orthogonal, it follows that the integration of an element in $\mathfrak{P}_2$ against one in $\mathfrak{P}_0$ over the unit sphere always yields zero. This proves the lemma and the kernel belongs to $\mathscr{C}_4$.

The same thing happens when we consider $\overline{Z}_j\{\overline{Z}_k\psi^2/\overline{\varphi^2}\psi^4\}$.

$$\begin{aligned}Z_j\{\varphi^*(\overline{Z}_k\psi^2)/\overline{\varphi^2}\psi^4\}&=2\delta_{jk}\varphi^*/\overline{\varphi^2}\psi^4+(Z_j\varphi^*)(\overline{Z}_k\psi^2)/\overline{\varphi^2}\psi^4\\&\quad-2\varphi^*(Z_j\overline{\varphi})(\overline{Z}_k\psi^2)/\overline{\varphi^3}\psi^4-2\varphi^*(Z_j\psi^2)(\overline{Z}_k\psi^2)/\overline{\varphi^2}\psi^6\\&\quad+\text{t.w.w.s.}\end{aligned}\tag{3.7,c}$$

The 1st and the 4th term of (3.7,c) belong to the class $\mathscr{C}_3$. The 2nd and the 3rd term of (3.7,c) belong to the class $\mathscr{C}_1$. We will get similar results when we consider $\overline{Z}_j\{\varphi^*(\overline{Z}_k\psi^2)/\overline{\varphi^2}\psi^4\}$.

Next, we consider the derivatives of the component K_3 of the operator T_0. By a straightforward computation, we know that when we differentiate (in all directions) the kernel of the terms E_2H_4 and E_0H_6 in K_3, it will produce operators belonging to the class $\mathscr{E}_1$ or the class $\mathscr{E}_2$. We just need to check the term E_4H_2:

(3.10)

$$\begin{aligned} &Z_j\{[(Z_1\psi^2)(\overline{Z}_1\psi^2) - (Z_2\psi^2)(\overline{Z}_2\psi^2)]/\overline{\varphi}\psi^6\} \\ &\quad = 2\{\delta_{j1}(Z_1\psi^2) - \delta_{j2}(Z_2\psi^2)\}/\overline{\varphi}\psi^6 \\ &\quad - \{(Z_j\overline{\varphi})(Z_1\psi^2)(\overline{Z}_1\psi^2) - (Z_j\overline{\varphi})(Z_2\psi^2)(\overline{Z}_2\psi^2)\}/\overline{\varphi^2}\psi^6 \\ &\quad - 3\{(Z_j\psi^2)(Z_1\psi^2)(\overline{Z}_1\psi^2) - (Z_j\psi^2)(Z_2\psi^2)(\overline{Z}_2\psi^2)\}/\overline{\varphi}\psi^8 + \text{t.w.w.s.} \end{aligned}$$

The 1st and the 3rd term of (3.10,a) always belong to the class $\mathscr{E}_3$. When $j = 1,2$, the 2nd term belongs to the class $\mathscr{E}_1$. But when $j = 3$, $Z_3\overline{\varphi} = -\sqrt{2} + \mathscr{E}_1 + \mathscr{E}_0\gamma(z)$. This tells us

$$\begin{aligned} &\{(Z_3\overline{\varphi})(Z_1\psi^2)(\overline{Z}_1\psi^2) - (Z_3\overline{\varphi})(Z_2\psi^2)(\overline{Z}_2\psi^2)\}/\overline{\varphi^2}\psi^6 \\ &\qquad = (-\sqrt{2})\{(Z_1\psi^2)(\overline{Z}_1\psi^2) - (Z_2\psi^2)(\overline{Z}_2\psi^2)\}/\overline{\varphi^2}\psi^6 + \text{t.w.w.s.} \end{aligned}$$

which is the critical case $k = 4$, $l = 4$. By the same kind of computation as we did in Lemma (3.8) or Phong-Stein theorem, we know that

$$\int_{a<(|z_1|^2+|z_2|^2)<b} \frac{(|Z_1\psi^2|^2 - |Z_2\psi^2|^2)}{\overline{\varphi}^2\psi^6} dV(z') = 0,$$

$$\forall 0 < a \le b < \infty, \text{ and } \forall (t;\rho) \in \mathbf{R}^1 \times \mathbf{R}^+.$$

Hence this kernel belongs to the class $\mathscr{E}_4$. We will have similar results when we consider $\overline{Z}_j\{[(Z_1\psi^2)(\overline{Z}_1\psi^2) - (Z_2\psi^2)(\overline{Z}_2\psi^2)]/\overline{\varphi}\psi^6\}$.

From the result of (3.10), we have the following lemma:

(3.11) **Lemma.** *The component K_3 of the operator T_0 is a bounded operator from $L^p(U)$ to $L_1^p(U)$, $1 < p < \infty$ (i.e., K_3 component of $\overline{\partial}^*\mathbf{N}$ gains one in all directions).*

This is not a surprising result. In fact, suppose we consider the $\overline{\partial}$-Neumann problem in $\mathbf{D} \subset \mathbf{C}^3$, i.e., $\Box u = f$ with boundary conditions $u_3|_{\partial\mathbf{D}} = 0$ and $\overline{Z}_3 u_j|_{\partial\mathbf{D}} = 0$, for $j = 1,2$. Then to solve u_3 we just need to consider the Dirichlet problem. By the classical Calderón-Zygmund theory, u_3 actually gains 2 in all directions which implies K_3 will gain one in all directions.

Now we go to the final step: Let us consider the operator E_0. First, we observe that when we apply the adjoint operator $(E_0)^*$ of E_0 to $\overline{\partial}^*\mathbf{N}f$, it is a smooth operator, i.e.,

$$\begin{aligned} (E_0)^*(\overline{\partial}^*\mathbf{N}f) &= \langle \overline{\partial}^*\mathbf{N}f, ({}^*{}_z\partial_z \overline{K_0(z,w)^*}) \rangle \\ &= \langle \mathbf{N}f, \overline{\partial}_z({}^*{}_w\overline{\partial}_w K_0(w,z)) \rangle = \mathscr{E}_\infty(\mathbf{N}f). \end{aligned}$$

Hence we can rewrite the integral representation of $\overline{\partial}^*\mathbf{N}f$ as follows:

$$\begin{aligned} \overline{\partial}^*\mathbf{N}f = T_0\overline{\partial}\overline{\partial}^*\mathbf{N}f + (E_0 - (E_0)^*)(\overline{\partial}^*\mathbf{N}f) + (E_0)^*(\overline{\partial}^*\mathbf{N}f) \\ + \text{more smoothing terms}. \end{aligned}$$

It remains to control the term $E_0 - (E_0)^*$. When we consider the model case $\mathbf{D}$, we know that $\overline{\varphi} = \overline{\varphi^*}$. It follows that $E_0 - (E_0)^* =$ smoothing operator. When Ω is a general strongly pseudoconvex domain, we can apply a result of Kerzman-Stein [12] to get $|\overline{\varphi} - \overline{\varphi^*}| \approx \mathscr{E}_3$. This allows us to prove that $E_0 - (E_0)^* =$ smoothing operator!

When the "normal" vector field Z_3 acts on the kernel, it will produce a bad situation. For example, let us consider

$$Z_3\{\overline{Z}_j\psi^2/\overline{\varphi^2}\psi^4\} = (Z_3\overline{Z}_j\psi^2)/\overline{\varphi^2}\psi^4 - 2(Z_3\overline{\varphi})(\overline{Z}_j\psi^2)/\overline{\varphi^3}\psi^4 - 2(Z_3\psi^2)(\overline{Z}_j\psi^2)/\overline{\varphi^2}\psi^6 + \text{t.w.w.s.}$$

We use the relation $Z_3\overline{\varphi} = -\sqrt{2} + \mathscr{E}_1 + \mathscr{E}_0\gamma(z)$ which gives us

$$(Z_3\overline{\varphi})(\overline{Z}_j\psi^2)/\overline{\varphi^3}\psi^4 \equiv E_3H_6$$

and E_3H_6 is the case $k = 3 < 4$, $l = 6 > 4$ but $k + l = 9 > 8$. This tells us $Z_3\{\overline{Z}_j\psi^2/\overline{\varphi^2}\psi^4\}$ will produce an unbounded operator from $L^p(U)$ to $L^p(U)$ for $1 \le p \le \infty$!

Summarizing all these results, we can prove the following theorem:

(3.12) **Theorem.** *Let U be a boundary neighborhood of $0 \in \partial\mathbf{D} \subset \mathbf{C}^{n+1}$, $f \in C^\infty_{(0,1),0}(U) \cap \operatorname{domain}(\overline{\partial^*})$ with $\overline{\partial f} = 0$. Then the Kohn solution $\overline{\partial^*}\mathbf{N}f$ of the Cauchy-Riemann equation $\overline{\partial u} = f$ satisfies: If $f \in L^p_k(U)$ then $\overline{\partial^*}\mathbf{N}f \in L^p_{k+1/2}(U) \cap S^p_{k+1}(U)$, $k = 0, 1, 2, \ldots$.*

Before we prove the theorem, we need some results about the left-invariant vector fields Z^L_j. (The standard Heisenberg vector fields), $j = 1, \ldots, n$ and right-invariant vector fields Z^R_j, $j = 1, \ldots, n$, where

$$Z^R_j = \frac{\partial}{\partial z_j} - \overline{z}_j\frac{\partial}{\partial t}, \qquad j = 1, \ldots, n.$$

By a result of Folland and Stein [7], we have

$$Z^L_j(f^*g) = f^*(Z^L_j g), \qquad Z^R_j(f^*g) = (Z^R_1 f)^*g, \qquad (Z^L_j f)^*g = f^*(Z^R_j g).$$

Moreover, if $I = (i_1, \ldots, i_n) \in \mathbf{N}$, then we have

$$Z^L_j = \sum_{|J|<|I|,d(J)>d(I)} P_{IJ}Z^R_J \quad \text{and} \quad Z^R_j = \sum_{|J|<|I|,d(J)>d(I)} Q_{IJ}Z^L_J.$$

where P_{IJ}, Q_{IJ} are homogeneous polynomials of degree $d(J) - d(I)$, $d(J)$ and $d(I)$ are homogeneous degree of the vector fields Z^R_J and Z^L_I.

Proof of the theorem. From Lemma (3.11), we know K_3 is a bounded operator from $L^p(U)$ to $L^p_1(U) \subset L^p_{1/2}(U)$ which proved the case $k = 0$ for K_3. By the computations (3.7,a) to (3.7,c) and Lemma (3.8), we know the kernels K_1 and K_2 are bounded operators from $L^p(U)$ to $S^P_1(U)$. Then we apply the inclusion result $S_1^p(U) \subset L_{1/2}^p(U)$ (see [6, 8, 14, 15, 28]) to prove the case $k=0$ for K_1 and K_2. For general $k \in \mathbf{Z}^+$, suppose $f \in L_k^p(U)$, then if $\mathfrak{P}(Z, \overline{Z})$ is a differential operator of degree k we have $k \in \mathbf{Z}^+$, suppose $f \in L_k^p(U)$, then if $\mathfrak{P}(Z, \overline{Z})$ is a differential operator of degree k we have $\|\mathfrak{P}(Z, \overline{Z})f\|_p \leq A$.

Let Z_j or $\overline{Z}_j$, $j=1,2$, be "allowable" vector fields, then

$$\begin{aligned}
Z_j^L \mathfrak{P}(Z^L, \overline{Z}^L)\overline{\partial}^* \mathbf{N} f &= Z_j^L \mathfrak{P}(Z^L, \overline{Z}^L)\Big(f^* \sum_{k,l} E_k H_l\Big) + \text{more smoothing terms} \\
&= Z_j^L\Big(f^* \mathfrak{P}(Z^L, \overline{Z}^L) \cdot \sum_{k,l} E_k H_l\Big) + \text{more smoothing terms} \\
&= Z_j^L\Big(\mathfrak{P}(Z^R, \overline{Z}^R) f^* \sum_{k,l} E_k H_l\Big) + \text{more smoothing terms} \\
&= \Big(\mathfrak{P}(Z^R, \overline{Z}^R) f^* \Big(Z_j^l \cdot \sum_{k,l} E_k H_l\Big)\Big) + \text{more smoothing terms}.
\end{aligned}$$

By the result of $k=0$ and $\mathfrak{P}(Z^R, \overline{Z}^R) f \in L^p(U)$, we can prove

$$\|Z_j^L \mathfrak{P}(Z^L, \overline{Z}^L)\overline{\partial}^* \mathbf{N} f\|_{L^p(U)} \leq A\|f\|_{L_k^p(U)}, \qquad 1 < p < \infty.$$

This tells us $\overline{\partial}^* \mathbf{N} f \in L_{k+1/2}^p(U)$.

Remark. Although $Z_3(\overline{\partial}^* \mathbf{N})$ raises an unbounded operator on $L^p(U)$ to itself, but $\overline{Z}_3(\overline{\partial}^* \mathbf{N})$ still define a bounded operator on $L^p(U)$. Let us consider the vector $\overline{Z}_3$ act on the kernels K_1 and K_2: Suppose $k = 1, 2$,

$$\begin{aligned}
\text{(a)} \quad \overline{Z}_3\{\varphi^*(\overline{Z}_k \psi^2)/\overline{\varphi}\psi^6\} &= (\overline{Z}_3 \varphi^*)(\overline{Z}_k \psi^2)/\overline{\varphi}\psi^6 - \varphi^*(\overline{Z}_3 \overline{\varphi})(\overline{Z}_k \psi^2)/\overline{\varphi^2}\psi^6 \\
&\quad - 3\varphi^*(\overline{Z}_3 \psi^2)(\overline{Z}_k \psi^2)/\overline{\varphi}\psi^8 + \text{t.w.w.s.}
\end{aligned}$$

the 1st term and the 3rd of (a) belong to the class $\mathscr{E}_3$. Since $|\overline{Z}_3 \overline{\varphi}| \approx \mathscr{E}_\infty$, so the 2nd term defines a more smoothing operator.

$$\begin{aligned}
\text{(b)} \qquad \overline{Z}_3\{\overline{Z}_k \psi^2/\overline{\varphi^2}\psi^4\} &= 2(\overline{Z}_3 \overline{\varphi})(\overline{Z}_k \psi^2)/\overline{\varphi^3}\psi^4 \\
&\quad - 2(\overline{Z}_3 \psi^2)(\overline{Z}_k \psi^2)/\overline{\varphi^2}\psi^6 + \text{t.w.w.s.}
\end{aligned}$$

The homogeneity of the crucial term in (b) (i.e., $(\overline{Z}_3 \psi^2)(\overline{Z}_k \psi^2)/\overline{\varphi^2}\psi^6$) is $k = 4$, $l = 4$. By the straightforward computation, we know that

$$\begin{aligned}
&\int_{a<(|z_1|^2+|z_2|^2)<b} \frac{(\overline{Z}_3 \psi^2)(\overline{Z}_k \psi^2)}{\overline{\varphi^2}\psi^6} dV(z') \\
&= \int_{a<(|z_1|^2+|z_2|^2)<b} \frac{(z_k - w_k)\{(t-s) - i(\rho + \mu)\} + \text{t.w.w.s.}}{\overline{\varphi^2}\psi^6} dV(z') \\
&= 0 + \text{t.w.w.s.} \quad \forall 0 < a \leq b < \infty.
\end{aligned}$$

So this term belongs to the class $\mathscr{C}_4$.

(c) $\overline{Z}_3\{\varphi^*(\overline{Z}_k\psi^2)/\overline{\varphi^2}\psi^4\} = (\overline{Z}_3\varphi^*)(\overline{Z}_k\psi^2)/\overline{\varphi^2}\psi^4$
$$- 2\varphi^*(\overline{Z}_3\overline{\varphi})(\overline{Z}_k\psi^2)/\overline{\varphi^3}\psi^4 - 2\varphi^*(\overline{Z}_3\psi^2)(\overline{Z}_k\psi^2)/\overline{\varphi^2}\psi^6 + \text{t.w.w.s.}$$

All the first three terms of (c) belong to the class $\mathscr{C}_1$.

(d)
$$\begin{aligned}\overline{Z}_3\{\overline{Z}_k\psi^2/\overline{\varphi^3}\psi^2\} = &- 3(\overline{Z}_3\overline{\varphi})(\overline{Z}_k\psi^2)/\overline{\varphi^4}\psi^2\\ &- (\overline{Z}_3\psi^2)(\overline{Z}_k\psi^2)/\overline{\varphi^3}\psi^4 + \text{t.w.w.s.}\end{aligned}$$

The 1st term of (d) belongs to the class $\mathscr{C}_1$ and the 2nd term of (d) belongs to the class $\mathscr{C}_2$.

Combine the computations (a) to (d) and Lemma (3.11), it is easy to see that the operator $\overline{Z}_3(K_i)$, $i = 1, 2, 3$, (and hence $\overline{Z}_3(\overline{\partial^*\mathbf{N}})$) define a bounded operator on $L^p(U)$ to itself. Hence we have the following corollary:

Corollary. *The Bergman projection* $\mathbf{P}_0 = \mathbf{I} - \overline{\partial^*}\mathbf{N}\overline{\partial}$ *can be extended to a bounded operator on* $L^p_k(U)$ *to itself, for all* $k = 0, 1, 2, \dots$, *i.e.,*
$$\mathbf{P}_0 : L^p_k(U) \to L^p_k(U), \qquad \forall k \in \mathbf{Z}^+.$$

Next, we want to discuss the optimal L^p and Hölder estimates of kernels with mixed type homogeneities. Before we go further, let us review some real variable results. Recall that a function $f: \mathbf{R}^n \to \mathbf{C}$ is said to be of weak type γ, $\gamma > 0$, if there exists a constant $A > 0$ such that
$$|\{x \in \mathbf{R}^n ; |f(x)| > s\}| \le A/s^\gamma, \quad \text{for all } s > 0.$$

Here $|\cdot|$ denotes the Lebesgue n-dimensional measure. By the results of Folland-Stein [6], Krantz [14, 15, 16], we have the following lemmas:

(3.13) **Lemma.** *Let* $\mathscr{K}: \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{C}$ *have the property that* $\mathscr{K}(x, \cdot)$ *is of weak type* γ, *as a function of y, uniformly in x, and* $\mathscr{K}(\cdot, y)$ *is weak type* γ *as a function of x, uniformly in y. Then the linear transformation*
$$f(x) \to Tf(x) = \int_{\mathbf{R}^m} \mathscr{K}(x, y) f(y)\, dy,$$
defined for $f: \mathbf{R}^m \to \mathbf{C}$, *satisfies* $\|Tf\|_{L^p} \le A_p \|f\|_{L^p}$, *whenever* $\gamma > 1$, $1 < p < q < \infty$, *and* $q^{-1} = p^{-1} + \gamma^{-1} - 1$. *Moreover,* $\|Tf\|_{L^{\gamma-\varepsilon}} \le A_\varepsilon \|f\|_{L^1}$ *for all* $\varepsilon > 0$.

(3.14) **Lemma.** *Let* $\mathscr{K}: \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{C}$ *have the property that*
$$\int_{\mathbf{R}^n} |\mathscr{K}(x, y)|^\gamma dx \le A^\gamma < \infty, \qquad \forall y \in \mathbf{R}^m,$$
and
$$\int_{\mathbf{R}^m} |\mathscr{K}(x, y)|^\gamma dy \le A^\gamma < \infty, \qquad \forall x \in \mathbf{R}^n.$$

Then the linear transformation

$$f(x) \to Tf(x) = \int_{\mathbf{R}^m} \mathscr{K}(x,y)f(y)\,dy,$$

defined for $f\colon \mathbf{R}^m \to \mathbf{C}$, *satisfies* $\|Tf\|_{L^q} \le A\|f\|_{L^p}$, *whenever* $\gamma \ge 1$, $1 \le p \le \infty$, *and* $q^{-1} = p^{-1} + \gamma^{-1} - 1$.

Remarks. (1) Observe that the hypotheses of Lemma (3.14) are stronger than those of Lemma (3.13), but they yield the stronger conclusions that the operator norm does not depend on p (which just depends on the $\max\{\|\mathscr{K}(x,\cdot)\|_{L^\gamma}, \|\mathscr{K}(\cdot,y)\|_{L^\gamma}\}$) and that we may allow $\gamma = 1$ and $p = 1,\infty$.

(2) Suppose $m = n$, Ω is a relatively compact subset of $\mathbf{R}^n$ and $\operatorname{supp}\mathscr{K} \subset \Omega \times \Omega$. Then the assumption that $\mathscr{K}(x,\cdot)$ and $\mathscr{K}(\cdot,y)$ are uniformly of weak type γ implies that they are uniformly in L^q for any fixed $1 \le q < \gamma$ (see Krantz [14, 15, 16]).

We also have

(3.15) **Lemma.** *Suppose* $\mathscr{K}$, γ, f *are as in Lemma* (3.13) *and* $f \in L^{\gamma'}$ *where* $\gamma^{-1} + \gamma'^{-1} = 1$. *Suppose* $n = m$, $\Omega \subset \mathbf{R}^n$ *and* $\operatorname{supp}\mathscr{K} \subset \overline{\Omega} \times \overline{\Omega}$. *Then*

$$\int_\Omega \exp\left(\left|\frac{\alpha Tf(x)}{\|f\|_{L^\gamma}}\right|^\gamma\right) dV \le M < \infty,$$

where α *is a sufficiently small constant and* M, a *do not depend on* f, *but on* γ *and* $|\Omega|$.

This lemma was first suggested by Stein, later on, Krantz [14] used it to prove the optimal L^p and Hölder estimates for the Henkin solution of the $\overline{\partial}$-equation.

Next, we will prove the kernel E_kH_l is a function of weak type $(2n+4)/(k+l)$ whenever $l \ge 4$, $k < 2n$, $k + l < 2n + 4$ and is a function of weak type $(4n+4)/(2k+l)$ whenever $l < 4$, $k + l/2 < 2n + 2$. Since we consider everything locally, we may assume the kernel is

$$\varphi(z',t;\rho)\mathscr{K}((z',t)\cdot(w,s)^{-1};\rho+\mu)\psi(w',s;\mu)$$

where $(z',t;\rho)$ and $(w',s;\mu)$ are two points in $\overline{\mathbf{D}}$, φ and ψ are two cut-off functions. Since the kernel has compact support, if $g\colon \operatorname{supp}\mathscr{K} \to \mathbf{C}$ is any measurable function, then for $0 < s \le 1$, we have

$$|\{x \in \operatorname{supp}\mathscr{K}, |g(x)| > s\}| \le |\operatorname{supp}\mathscr{K}| \le |\operatorname{supp}\mathscr{K}|/s^v \quad \text{for any } 0 < v < \infty.$$

Therefore, when we check

$$|\{(w',s;\mu) \in \overline{\mathbf{D}}\colon |\varphi(z',t;\rho)\mathscr{K}((z',t)(w',s)^{-1};\rho+\mu)\psi(w',s;\mu)| < s\}|$$

and

$$|\{(z',t;\rho) \in \overline{\mathbf{D}}\colon |\varphi(z',t;\rho)\mathscr{K}((z',t)\cdot(w',s)^{-1};\rho+\mu)\psi(w',s;\mu)| < s\}|$$

we just need to consider those large s, say $s > 1$.

Case (i): $l \geq 4$, $k < 2n$, $k + l < 2n + 4$, we define $\gamma = (2n+4)/(k+l)$. Fix $(z', t; \rho) \in \overline{\mathbf{D}}$,

$$
\begin{aligned}
&|\{(w', s; \mu) \in \overline{\mathbf{D}}: |\varphi(w', s; \mu)\mathscr{K}((z', t)\cdot(w', s)^{-1}; \rho+\mu)\psi(z', t; \rho)| < s\}| \\
&\quad \leq \left|\left\{(w', s; \mu) \in \mathbf{D}: \frac{\varphi(w', s; \mu)\cdot\psi(z', t; \rho)}{(|z'-w'|+|t-s|+(\rho+\mu))^k(|z'-w'|+|t-s|^{1/2}+(\rho+\mu)^{1/2})^l} < s\right\}\right| \\
&\quad \leq \left|\left\{(w', s; \mu) \in \mathbf{D}: \frac{\varphi(w', s; \mu)\cdot\psi(z', t; \rho)}{|z'-w'|^k(|z'-w'|+|t-s|^{1/2}+(\rho+\mu)^{1/2})^l} < s\right\}\right| \\
&\quad = |A| = \iint_{A\cap\mathbf{D}} dV(z')\,dt\,d\rho \\
&\quad \leq \iint_{A\cap\mathbf{D}} \left(\frac{s^{-1}}{|z'-w'|^k(|z'-w'|+|t-s|^{1/2}+(\rho+\mu)^{1/2})^l}\right)^{\gamma} dV(z')\,dt\,d\rho \\
&\quad = Cs^{-\gamma}.
\end{aligned}
$$

To prove the last inequality, we just need to count the homogeneities. Here the constants C will depend on the measure of the compact set $|\operatorname{supp}\varphi|$. We can get a similar result for

$$|\{(z', t; \rho) \in \overline{\mathbf{D}}: |\varphi(z', t; \rho)\mathscr{K}((z', t)\cdot(w', s)^{-1}; \rho+\mu)\psi(w', s; \mu)| < s\}| \leq Cs^{-\gamma}.$$

Case (ii): $l < 4$, $k + l/2 < 2n + 2$, we define $\tilde{\gamma} = (4n+4)/(2k+l)$. Fix $(z', t; \rho) \in \overline{\mathbf{D}}$, then we just need to apply the classical results of the Riesz potentials, then we can get

$$
\begin{aligned}
&|\{(z', t; \rho) \in \overline{\mathbf{D}}: |\varphi(z', t; \rho)\mathscr{K}((z', t)\cdot(w', s)^{-1}; \rho+\mu)\psi(w', s; \mu)| < s\}| \\
&\quad \approx |\{(w', s; \mu) \in \overline{\mathbf{D}}: |\varphi(z', t; \rho)\mathscr{K}((z', t)\cdot(w', s)^{-1}; \rho+\mu)\psi(w', s; \mu)| > s\}| \\
&\quad \leq C\left|\left\{(w', s; \mu) \in \mathbf{R}^{2n+2}: \frac{\varphi(w', s; \mu)\cdot(z', t; \rho)}{(|z'-w'|+|t-s|+(\rho+\mu))^k(|t-s|^{1/2}+(\rho+\mu)^{1/2})^l} > s\right\}\right| \\
&\quad = C|A| \leq C\iint_{A\cap\mathbf{D}} \left(\frac{s^{-1}}{(|z'-w'|+|t-s|+(\rho+\mu))^k(|t-s|^{1/2}+(\rho+\mu)^{1/2})^l}\right)^{\tilde{\gamma}} dV(z')\,dt\,d\rho \\
&\quad \leq \widetilde{C}s^{-\tilde{\gamma}}.
\end{aligned}
$$

From these arguments, we know the kernel

$$\varphi(z', t; \rho)\mathscr{K}((z', t)\cdot(w', s)^{-1}; \rho+\mu)\psi(w', s; \mu)$$

is weak type γ or $\tilde{\gamma}$, uniformly in $(z', t; \rho)$ and $(w', s; \mu)$. We also mention a result of Krantz [16]:

Theorem (Krantz). *Suppose that (X, μ) is a finite measure space and that $\mathscr{K}: X \times X \to \mathbf{C}$ is a measurable function satisfying the property that $\mathscr{K}(x, \cdot)$ is weak type γ uniformly in x (the hypothesis on $\mathscr{K}(\cdot, y)$ may be dropped), $1 < \gamma < \infty$. Let T be the integral operator*

$$Tf(x) = \int_X \mathscr{K}(x, y)f(y)\,d\mu(y).$$

Then T maps $L^{\gamma'}(\log^+ L)^{\gamma'-1+\varepsilon}$ to L^∞, any $\varepsilon > 0$. Here $\gamma' = \gamma/(\gamma-1)$ is the exponent conjugate to γ'.

Summarizing all this information, we have obtained the following propositions:

(3.16.1) **Proposition.** *Suppose the operator* $\mathrm{Op}(\mathscr{K})$ *defined in* (3.1) *and* $l \geq 4$, $k < 2n$, $k + l < 2n + 4$; *then we have the following*:

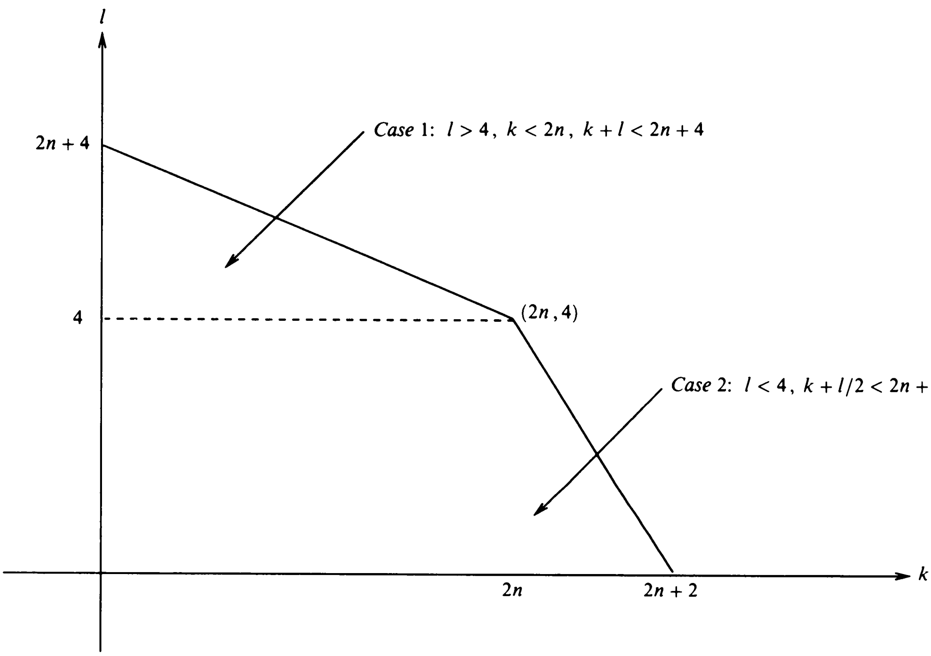
(i) $\mathrm{Op}(\mathscr{K})\colon L^p(U) \to L^q(U)$, $q^{-1} = p^{-1} - (2n+4-k-l)/(2n+4)$, $1 < p < (2n+4)/(2n+4-k-l)$;

(ii) $|\iint_U \exp(\alpha|\mathrm{Op}(\mathscr{K})(f)/\|f\|_{L^{(2n+4)/(2n+4-k-l)}}|^{(2n+4)/(k+l)})\,dV(w')\,ds\,d\mu| \leq C < \infty$, α *is a sufficiently small constant*; *i.e.*,

$$\mathrm{Op}(\mathscr{K})\colon L^{(2n+4)/(2n+4-k-l)}(U) \to L(\exp(\alpha|\cdot|^{(2n+4)/(k+l)}))(U).$$

(iii) $\mathrm{Op}(\mathscr{K})\colon L^1(U) \to L^{(2n+4)/(k+l)-\varepsilon}(U)$, $\forall \varepsilon > 0$;

(iv) $\mathrm{Op}(\mathscr{K})\colon L^{(2n+4)/(2n+4-k-l)}(\log^+ L)^{(2n+4)/(2n+4-k-l)-1+\varepsilon}(U) \to L^\infty(U)$, $\forall \varepsilon > 0$, *where* U *is a boundary neighborhood of* $0 \in \partial U_1 \subset \mathbf{H}^n$.



(3.16.2) **Proposition.** *Suppose the operator* $\mathrm{Op}(\mathscr{K})$ *defined in* (3.1) *and* $l < 4$, $k + l/2 < 2n + 2$; *then we have the following*:

(i′) $\mathrm{Op}(\mathscr{K})\colon L^p(U) \to L^q(U)$, $q^{-1} = p^{-1} - (4n+4-2k-l)/(4n+4)$, $1 < p < (4n+4-2k-l)/(4n+4)$;

(ii′) $|\iint_U \exp(\alpha|\mathrm{Op}(\mathscr{K})(f)/\|f\|_{L^{(4n+4)/(4n+4-2k-l)}}|^{(4n+4)/(2k+l)})\,d(w')\,ds\,d\mu| \leq C < \infty$, α *is a sufficiently small constant*; *i.e.*,

$$\mathrm{Op}(\mathscr{K})\colon L^{(4n+4)/(4n+4-2k-l)}(U) \to L(\exp(\alpha|\cdot|^{(4n+4)/(2k+l)}))(U).$$

(iii′) $\mathrm{Op}(\mathscr{K})\colon L^1(U) \to L^{(4n+4)/(2k+l)-\varepsilon}(U)$, $\forall \varepsilon > 0$;

(iv′) $\mathrm{Op}(\mathscr{K})\colon L^{(4n+4)/(4n+4-2k-l)}(\log^+ L)^{(4n+4)/(4n+4-2k-l)-1+\varepsilon}(U) \to L^\infty(U)$, $\forall\varepsilon > 0$, *where* U *is a boundary neighborhood of* $0 \in \partial U_1 \subset \mathbf{H}^n$.

Applying the results of these two propositions, we can prove the optimal L^p estimates for the Kohn solution of the $\overline{\partial}$-equation. By the result of §2, the crucial terms of the kernel $\overline{\partial^*\mathbf{N}} = (K_1, K_2, \dots, K_{n+1})$ have the form $E_{2n-1}H_4$ and $E_{2n-3}H_6$, hence we can write down the following theorem:

(3.17) **Theorem.**
(i) $\overline{\partial^*}\mathbf{N}\colon L^p(U) \to L^q(U)$, $1/q = 1/p - 1/(2n+4)$, *if* $1 < p < 2n+4$;
(ii) $|\iint_U \exp(\alpha|(\overline{\partial^*}\mathbf{N})(f)/\|f\|_{L^{2n+4}}|^{(2n+4)/(2n+3)})d(w')\,ds\,d\mu| \le C < \infty$, α *is a sufficiently small constant*; *i.e.*,

$$\mathrm{Op}(\overline{\partial^*}\mathbf{N})\colon L^{2n+4}(U) \to L(\exp(\alpha|\cdot|^{(2n+4)/(2n+3)}))(U).$$

(iii) $\overline{\partial^*}\mathbf{N}\colon L^1(U) \to L^{(2n+4)/(2n+3)-\varepsilon}(U)$, $\forall\varepsilon > 0$;
(iv) $\overline{\partial^*}\mathbf{N}\colon L^{2n+4}(\log^+ L)^{2n+3+\varepsilon}(U) \to L^\infty(U)$, $\forall\varepsilon > 0$.

4. Hölder estimates for the kernels with mixed type homogeneities

In this section, we want to discuss the Hölder estimation of the operator E_kH_l. Since we are considering the mixed type homogeneities, we define two different "norm functions" on $\overline{\mathbf{D}}$ as follows: (1) the "norm function" with respect to the Euclidean homogeneity: $\|(z', t; \rho)\|_e \approx |z'| + |t| + \rho$; (2) the "norm function" with respect to the Heisenberg homogeneity: $\|(z', t; \rho)\|_h \approx |z'| + |t|^{1/2} + \rho^{1/2}$. Note that the "norm functions $\|\cdot\|_e$ or $\|\cdot\|_h$ here are not the norm as usual, since

$$\|((z', t)\cdot(w', s)^{-1}; \rho+\mu)\|_e = 0, \quad \text{or} \quad \|((z', t)\cdot(w', s)^{-1}; \rho+\mu)\|_h = 0,$$

if and only if $\rho = \mu = 0$ and $(z', t) = (w', s)$. We use the notations $\|\cdot\|_e$ or $\|\cdot\|_h$ here just for convenience. We have the following obvious inequalities:

$$(4.1) \quad \|(z', t; \rho)\|_e \le \|(z', t; \rho)\|_h \le \|(z', t; \rho)\|_e^{1/2}, \quad \text{whenever } \|(z', t; \rho)\|_h \le 1.$$

First we establish two easy lemmas:

(4.2) **Lemma** (Triangle inequality). *There exists a constant* $C \ge 1$ *such that for all* $u = (z', t; \rho), v = (w', s; \mu) \in \overline{\mathbf{D}}$,

$$\|u+v\|_h \le C(\|u\|_h + \|v\|_h),$$
$$\|((z', t)\cdot(w', s); \rho+\mu)\|_h \le C(\|u\|_h + \|v\|_h).$$

Here $(z', t)\cdot(w', s)$ *denotes the multiplication on the Heisenberg group.*

Proof. By homogeneity we may assume that $\|u\|_h + \|v\|_h = 1$. The set $(u, v) \in \overline{\mathbf{D}} \times \overline{\mathbf{D}}$ satisfying this equation is compact, so we may take C to be the larger of the maximum values of $\|u+v\|_h$ and $\|((z', t)\cdot(w', s); \rho+\mu)\|_h$ on this set.

(4.3) **Lemma.** *If f is a homogeneous function of degree λ (in the Heisenberg dilations), $\lambda \in \mathbf{R}$ which is C^∞ away from the origin, there exists a constant $C > 0$ such that*

$$|f(u) - f(v)| \leq C\|u - v\|_h \cdot \|u\|_h^{\lambda-1}, \quad \text{whenever } \|u - v\|_h \leq \tfrac{1}{2}\|u\|_h,$$

$$|f((z',t)\cdot(w',s);\rho+\mu) - f(z',t;\rho)| \leq C\|v\|_h \cdot \|u\|_h^{\lambda-1}, \qquad \text{whenever } \|v\|_h \leq \tfrac{1}{2}\|u\|_h.$$

Proof. For the first case, we may assume, by homogeneity, that $\|u\|_h = 1$ and $\|u - v\|_h \leq 1/2$. But then v is bounded away from zero, so by the mean value theorem and (4.1) we have

$$|f(u) - f(v)| \leq C\|u - v\|_e \leq C\|u - v\|_h.$$

The same argument in the second case yields

$$|f((z',t)\cdot(w',s);\rho+\mu) - f(z',t;\rho)| \leq C\|((z',t)\cdot(w',s);\rho+\mu) - (z',t;\rho)\|_e.$$

But $g(w',s;\mu) = ((z',t)\cdot(w',s);\rho+\mu)$ is a smooth function, we have

$$\begin{aligned}\|((z',t)\cdot(w',s);\rho+\mu) - (z',t;\rho)\|_e \\ = \|g(v) - g(0)\|_e \leq C\|v - 0\|_e = C\|v\|_e \leq C\|v\|_h.\end{aligned}$$

We want to prove the Hölder estimates in nonisotropic sense first. Let us consider the following nonisotropic Lipschitz space: if $0 < \alpha < 1$, we define

$$\Gamma_\alpha = \left\{ f \in L^\infty \cap C^0 \colon \sup_{\substack{(z',t),(w',s)\in\mathbf{H}^n \\ \rho\in\mathbf{R}^+}} \frac{|f((z',t)\cdot(w',s);\rho) - f(z',t;\rho)|}{\|(w',s)\|_h^\alpha} < \infty \right\}.$$

when $\alpha = 1$, we define

$$\Gamma_1 = \left\{ f \in L^\infty \cap C^0 \colon \sup_{(z',t),(w',s)\in\mathbf{H}^n,\rho\in\mathbf{R}^+} \times \frac{|f((z',t)\cdot(w',s);\rho) + f((z',t)\cdot(w',s)^{-1};\rho) - 2f(z',t;\rho)|}{\|(w',s)\|_h^1} < \infty \right\}.$$

It is very easy to see these spaces contain all the functions which are Lipschitz α in the Heisenberg sense on each level set $\{\rho = \text{constant}\}$ on $\overline{\mathbf{D}}$. Using these notations, we can prove the following theorem:

(4.4) **Proposition.** *Let $\mathrm{Op}(\mathscr{K})$ be an operator defined by the kernel*

$$\mathscr{K} = \varphi(z',t;\rho)E_k H_l \psi(w',s;\mu) \in C^\infty(\overline{\mathbf{D}}\backslash(0,0;0))$$

on $\overline{\mathbf{D}}$. Suppose $k < 2n$, $l > 4$ and $k + l < 2n + 4$. Let f be a function of compact support. If $f \in L^p$ and $\beta = 2n + 4 - l - (2n+4)/p > 0$, then $g = \mathrm{Op}(\mathscr{K})(f) \in \Gamma_\beta(U)$.

Proof. *Case* 1: $\beta = 1$. Then $1 = 2n+4-k-l-(2n+4)/p$. We also need to fix ρ, w.l.o.g. we may assume $\rho = 0$. Then we have

$$\begin{aligned}
&|g((z',t)\cdot(w',s);0) + g((z',t)\cdot(w',s)^{-1};0) - 2g(z',t;0)| \\
&\quad \le \|f\|_{L^p}\left(\iint_{\mathbf{D}} |\mathscr{K}((u',r)^{-1}\cdot[(z',t)\cdot(w',s)];\nu)\right. \\
&\qquad\qquad + \mathscr{K}((u',r)^{-1}\cdot[(z',t)\cdot(w',s)^{-1}];\nu) \\
&\qquad\qquad\qquad \left. - 2\mathscr{K}((u',r)^{-1}\cdot(z',t);\nu)|^{p'}\,dV(u')\,dr\,d\nu\right)^{1/p'}
\end{aligned}$$

with $p^{-1} + p'^{-1} = 1$. By Lemma (4.3) we have

$$\begin{aligned}
&|\mathscr{K}((u',r)^{-1}\cdot[(z',t)\cdot(w',s)];\nu) + \mathscr{K}((u',r)^{-1}\cdot[(z',t)\cdot(w',s)^{-1}];\nu) \\
&\qquad\qquad - 2\mathscr{K}((u',r)^{-1}\cdot(z',t);\nu)| \\
&\qquad \le C\cdot\|(w',s;0)\|_h^2\cdot\|((u,r)^{-1}\cdot(z',t);\nu)\|_h^{-k-l-2}.
\end{aligned}$$

Now we estimate the integral,

$$\iint_{\mathbf{D}} = \iint_{\|(u',r)^{-1}\cdot(z',t);\nu)\|_h\le\|(w',s;0)\|_h} + \iint_{\|((u',r)^{-1}\cdot(z',t);\nu)\|_h>\|(w',s;0)\|_h} = \mathrm{I} + \mathrm{II}.$$

To estimate the integral I, we just need to estimate each term separately. We apply Lemma (4.2) to get

$$\|((u',r)^{-1}\cdot[(z',t)\cdot(w',s)];\nu)\|_h \le C(\|((u',r)^{-1}\cdot(z',t);\nu)\|_h + \|(w',s;0)\|_h)$$

when $\|((u',r)^{-1}\cdot(z',t);\nu)\|_h \le \|(w',s;0)\|_h$, we have

$$\|((u',r)^{-1}\cdot[(z',t)\cdot(w',s)];\nu)\|_h \le (1+C)\|(w',s;0)\|_h.$$

Then we have

$$\begin{aligned}
&\left(\iint_{\|((u',r)^{-1}\cdot(z',t);\nu)\|_h\le\|(w',s;0)\|_h^1}\right. \\
&\qquad\qquad \left.\times|\mathscr{K}((u',r)^{-1}\cdot[(z',t)\cdot(w',s)];\nu)|^{p'}\,dV(u')\,dr\,d\nu\right)^{1/p'} \\
&\qquad \le \left(\iint_{\|((u',r)^{-1}\cdot(z',t);\nu)\|_h\le\|(w',s;0)\|_h^1}\right. \\
&\qquad\qquad\qquad \left.\times\|((u',r)^{-1}\cdot(z',t);\nu)\|_h^{-p'(k+l)}\,dV(u')\,dr\,d\nu\right)^{1/p'} \\
&\qquad \le C\cdot\|(w',s;0)\|_h^{\{-p'(k+l)+2n+4\}/p'},
\end{aligned}$$

whenever $-(k+l)+(2n+4)/p'$ is positive, but $1 = 2n+4-k-l-(2n+4)/p$ hence we have $-(k+l)+(2n+4)/p' = -(k+l)+(2n+4)-(2n+4)/p = 1$. We have

$$\mathrm{I} \le C\cdot\|(w',s;0)\|_h^1.$$

To estimate II, we have

$$\left(\iint_{\|((u',r)^{-1}\cdot(z',t);\nu)\|_h > \|(w',s;0)\|_h^{\frac{1}{2}}} |\mathscr{K}((u',r)^{-1}\cdot[(z',t)\cdot(w',s)];\nu) + \mathscr{K}((u',r)^{-1}\cdot[(z',t)\cdot(w',s)^{-1}];\nu) - 2\mathscr{K}((u',r)^{-1}\cdot(z',t);\nu)|^{p'}\right)^{1/p'}$$

$$\leq C\cdot\|(w',s;0)\|_h^2\cdot\|(w',s;0)\|_h^{\{(-k-l-2)p'+2n+4\}/p'}.$$

whenever $(-k-l-2)+(2n+4)/p'$ is negative. However, because $1 = 2n + 4 - k - l - (2n+4)/p$, it follows that this number is -1, and thus

$$\mathrm{II} \leq C\cdot\|(w',s;0)\|_h^1.$$

Case 2: $0<\beta<1$, $\beta = -k-l+2n+4-(2n+4)/p$. It suffices to estimate

$$\left(\iint_{\mathbf{D}} |\mathscr{K}((u',r)^{-1}\cdot[(z',t)\cdot(w',s)];\nu) - \mathscr{K}((u',r)^{-1}\cdot(z',t);\nu)|^{p'}\,dV(u')\,dr\,d\nu\right)^{1/p'}.$$

Once again, we apply Lemma (4.2), (4.3) and the technique we used in Case 1 to show this integral is dominated by $C\cdot\|(w',s;0)\|_h^{\beta}$.

Next, we want to deal with the isotropic Lipschitz case. What we need to estimate is the term $\mathrm{Op}(\mathscr{K})(f)((z',t;\rho)+(w',s;\mu)) - \mathrm{Op}(\mathscr{K})(f)(z',t;\rho)$, where

$$\mathrm{Op}(\mathscr{K})(z',t;\rho) = \int_0^\infty\int_{\mathbf{H}^n} \mathscr{K}((u',r)^{-1}\cdot(z',t);\nu+\rho)f(u',r;\nu)\,dV(u')\,dr\,d\nu.$$

Suppose $x=(z',t)$, $y=(w',s)$ and $v=(u',r)$ are points on the Heisenberg group, then

$$\begin{aligned}\mathrm{Op}(\mathscr{K})(f)(x+y;\rho+\mu) &- \mathrm{Op}(\mathscr{K})(f)(x;\rho)\\ &= \mathrm{Op}(\mathscr{K})(f)(x\cdot\tilde{y};\rho+\mu) - \mathrm{Op}(\mathscr{K})(f)(x;\rho)\end{aligned}$$

where $\tilde{y} = (v, s - 2\mathfrak{I}z'\cdot\overline{w}')$ and $x\cdot\tilde{y}$ is the Heisenberg group multiplication of x and $\tilde{y}$. So we need to control the term $\mathscr{K}(v^{-1}\cdot(x\cdot\tilde{y});v+\rho+\mu) - \mathscr{K}(v^{-1}\cdot x;v+\rho)$, we prove the following lemmas first:

(4.5) **Lemma.** *Suppose* $l\geq 4$, $k<2n$, $k+l<2n+4$ *and* $0\leq\lambda_1 = 2n+4-k-l<2$, *then*

$$\int_0^\infty\int_{\mathbf{H}^n} |\mathscr{K}(\nu^{-1}\cdot(x\cdot\tilde{y});\nu+\rho+\mu) - \mathscr{K}(v^{-1}\cdot x;\nu+\rho)|\,dV(v)\,d\nu \leq C'\cdot\|(w',s;\mu)\|_e^{\lambda_2/2}.$$

Proof.

$$\begin{aligned}
&\int_0^\infty \int_{\mathbf{H}^n} |\mathscr{K}(v^{-1}\cdot(x\cdot\tilde{y});\nu+\rho+\mu)-\mathscr{K}(v^{-1}\cdot x;\nu+\rho)|dV(v)\,d\nu\\
&\quad=\int_0^\infty \int_{\mathbf{H}^n} |\mathscr{K}(\xi_1,\eta)-\mathscr{K}(\xi_2,\eta)|\,d\eta\\
&\quad=\iint_{\{\eta:\|\xi_1-\eta\|_h<d\|\xi_1-\xi_2\|_e^{1/2}\}} |\mathscr{K}(\xi_1,\eta)-\mathscr{K}(\xi_2,\eta)|d\eta\\
&\qquad+\iint_{\{\eta:\|\xi_1-\eta\|_h\le d\|\xi_1-\xi_2\|_e^{1/2}\}} |\mathscr{K}(\xi_1,\eta)-\mathscr{K}(\xi_2,\eta)|d\eta=\mathrm{I}+\mathrm{II}.
\end{aligned}$$

Here $\xi_1=(x\cdot\tilde{y};\rho+\mu)$, $\xi_2=(x;\rho)$ and $\eta=(v;\nu)$. Note that $\|\xi_1-\xi_2\|_h\le c\cdot\|\xi_1-\xi_2\|_e^{1/2}$ and $\|\xi_2-\eta\|_h\le C(\|\xi_1-\eta\|_h+\|\xi_1-\xi_2\|_h)$, this implies

$$\{\eta:\|\xi_1-\eta\|_h\le d\cdot\|\xi_1-\xi_2\|_e^{1/2}\}\subset\{\eta:\|\xi_2-\eta\|_h\le C\cdot(d+c)\cdot\|\xi_1-\xi_2\|_e^{1/2}\}.$$

Hence we have

$$\begin{aligned}
&\iint_{\{\eta:\|\xi_1-\eta\|_h<d\|\xi_1-\xi_2\|_e^{1/2}\}} |\mathscr{K}(\xi_1,\eta)-\mathscr{K}(\xi_2,\eta)|d\eta\\
&\quad\le\iint_{\{\eta:\|\xi_1-\eta\|_h<d\|\xi_1-\xi_2\|_e^{1/2}\}} |\mathscr{K}(\xi_1,\eta)|d\eta\\
&\qquad+\iint_{\{\eta:\|\xi_1-\eta\|_h<d\|\xi_1-\xi_2\|_e^{1/2}\}} |\mathscr{K}(\xi_2,\eta)|d\eta\\
&\quad\le\int_{\{0\le r\le d'\cdot\|\xi_1-\xi_2\|_e^{1/2}\}} r^{-k-l}\cdot r^{2n+3}dr\cdot\int_{\|\xi_2-\eta\|_h^1}\text{smooth function } d\sigma\\
&\quad\le C'\cdot\|\xi_1-\xi_2\|_3^{(2n+4-k-l)/2}=C'\cdot\|(w',s;\mu)\|_e^{\lambda_1/2}.
\end{aligned}$$

Next, we consider the term II:

$$\iint_{\{\eta:\|\xi_1-\eta\|_h\le d\|\xi_1-\xi_2\|_e^{1/2}\}} |\mathscr{K}(\xi_1,\eta)-\mathscr{K}(\xi_2,\eta)|d\eta.$$

We look at the integrand

$$\begin{aligned}
|\mathscr{K}(\xi_1,\eta)-\mathscr{K}(\xi_2,\eta)|&=|\mathscr{K}(v^{-1}\cdot(x\cdot\tilde{y});\nu+\rho+\mu)-\mathscr{K}(v^{-1}\cdot x;\nu+\rho)|\\
&=|\mathscr{K}(z'+w'-u',t+s-r-2\Im u'\cdot(\overline{z'}+\overline{w'});\nu+\rho+\mu)\\
&\qquad-\mathscr{K}(z'-u',t-r-2\Im u'\cdot\overline{z'};\nu+\rho)|\\
&\le C\cdot\{|w'|\cdot\|\xi_1-\eta\|_h^{-k-l-1}+|s-2\Im u'\cdot\overline{w'}|\\
&\qquad\cdot\|\xi_1-\eta\|_h^{-k-l-2}+\mu\|\xi_1-\eta\|_h^{-k-l-2}\}\\
&\le C\cdot\|\xi_1-\xi_2\|_e\cdot\|\xi_1-\eta\|_h^{-k-l-2}.
\end{aligned}$$

if $\|\xi_1-\eta\|_h \geq 2\|\xi_1-\xi_2\|_h$ and x, y, v are all in a small compact neighborhood of $0 \in \partial\mathbf{D}$. Hence we have

$$\begin{aligned}
&\iint_{\{\eta:\|\xi_1-\eta\|_h<d\cdot\|\xi_1-\xi_2\|_e^{1/2}\}} |\mathscr{K}(\xi_1, \eta) - \mathscr{K}(\xi_2, \eta)|d\eta \\
&\quad \leq \iint_{\{\eta:\|\xi_1-\eta\|_h<d\cdot\|\xi_1-\xi_2\|_e^{1/2}\}} \|\xi_1 - \xi_2\|_e \cdot \|\xi_1 - \eta\|_h^{-k-l-2} d\eta \\
&\quad \leq C \cdot \|\xi_1 - \xi_2\|_e \cdot \int_{\{r\geq d\cdot\|\xi_1-\xi_2\|_e^{1/2}\}} r^{-k-l-2} \cdot r^{2n+3} dr \\
&\qquad \cdot \int_{\|\xi_1-\eta\|_h=1} \text{smooth function } d\sigma \\
&\quad \leq C \cdot \|\xi_1 - \xi_2\|_e \cdot \|\xi_1 - \xi_2\|_e^{(2n+4-k-l-2)/2} = C' \cdot \|(w', s; \mu)\|_e^{\lambda_1/2},
\end{aligned}$$

whenever $2n + 4 - k - l - 2 < 0$ which coincides with our assumption and complete the proof.

(4.6) **Lemma.** *Suppose* $l < 4$, $k+l/2 < 2n+2$ *and* $0 < \lambda_2 = 2n+2-k-l/2 < 1$, *then*

$$\begin{aligned}
&\int_0^\infty \int_{\mathbf{H}^n} |\mathscr{K}(v^{-1} \cdot (x \cdot \tilde{y}); v + \rho + \mu) - \mathscr{K}(v^{-1} \cdot x; v + \rho)| dV(v)\, dv \\
&\quad \leq C' \cdot \|(w', s; \mu)\|_e^{\lambda_2}.
\end{aligned}$$

Proof. We use the same ideas as we did in Lemma (4.5) to prove this lemma except we use "Euclidean norm function" $\|\cdot\|_e$ to replace the "Heisenberg norm function" $\|\cdot\|_h$. Also, when we estimate $|\mathscr{K}(\xi_1, \eta) - \mathscr{K}(\xi_2, \eta)|$, we just gain 1 in $\|\xi_1 - \eta\|_e$.

(4.7) **Corollary.** *Suppose* $l < 4$, $k + l/2 = 2n + 1$, *then*

$$\begin{aligned}
&\int_0^\infty \int_{\mathbf{H}^n} |\mathscr{K}(v^{-1} \cdot (x \cdot \tilde{y}); \nu + \rho + \mu) - \mathscr{K}(v^{-1} \cdot x; \nu + \rho)| dV(v)\, d\nu \\
&\quad \leq C' \cdot \|(w', s; \mu)\|_e^1 \cdot \log\|(w', s; 0)\|_e
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\infty \int_{\mathbf{H}^n} &|\mathscr{K}(v^{-1} \cdot (x \cdot \tilde{y}); \nu + \rho + \mu) \\
&+ \mathscr{K}(v^{-1} \cdot (x \cdot \tilde{y}^{-1}); \nu + \rho - \mu) \\
&- 2\mathscr{K}(v^{-1} \cdot x; \nu + \rho)| dV(v)\, d\nu \leq C \cdot \|(w', s; \mu)\|_e^1.
\end{aligned}$$

Applying these lemmas, we may use the fact $\mathrm{Op}(\mathscr{K})$ is a convolution operator to prove:

(4.8) **Proposition.** *Suppose* $l < 4$, $k+l/2 < 2n+2$ *and* $\lambda_2 = (4n+4-2k-l)/2 > 0$, *then*

(i) $\mathrm{Op}(\mathscr{K})\colon \Lambda_\alpha(\overline{\mathbf{D}}) \to \Lambda_{\alpha+\lambda_2}(\overline{\mathbf{D}})$, *if* $0 < \alpha + \lambda_2 < 1$;

(ii) $\mathrm{Op}(\mathscr{K})\colon \Lambda_\alpha(\overline{\mathbf{D}}) \to \Gamma_{\alpha+\lambda_2}(\overline{\mathbf{D}})$, *if* $\alpha \geq 1$.

Suppose $l \geq 4$, $k < 2n$, $k + l < 2n + 4$ *and* $\lambda_1 = 2n + 4 - k - l > 0$, *then*

(i$'$) $\mathrm{Op}(\mathscr{K}): \Lambda_\alpha(\overline{\mathbf{D}})\Lambda_{\alpha+\lambda_1/2}(\overline{\mathbf{D}})$, *if* $0 < \alpha + \lambda_1 < 1$;

(ii$'$) $\mathrm{Op}(\mathscr{K}): \Lambda_\alpha(\overline{\mathbf{D}}) \to \Gamma_{\alpha+\lambda_1}(\overline{\mathbf{D}})$, *if* $\alpha \geq 1$.

Proof. The proof of this theorem is standard. Suppose $0 < \alpha < 1$. We just need to consider the Poisson extension $\{f_\varepsilon\}$ of $f \in \Lambda_\alpha(\overline{\mathbf{D}})$ and then consider the operator $\mathscr{K}$ acts on the family $\{f_\varepsilon\}$. The finite difference operator $\Delta^2_{(w',s;\mu)}f = \Delta^2_{(w',s;\mu)}(f - f_\varepsilon) + \Delta^2_{(w',s;\mu)}(f_\varepsilon)$ will show the results (i) or (i$'$). We will omit all the computations.

To prove the proposition when $m \leq \alpha < m + 1$, $m \in \mathbf{N}$, we can apply the arguments above m times (with f in place of f_ε) and reduce matters to the case $0 < \alpha < 1$. But we just can apply those "allowable" vector fields here. In this case, we need to switch the differentiation on the kernel to the differentiation on the function f. Since f has compact support on the Heisenberg group, it is easy to commute the differentiation. If we consider the normal derivative, the boundary integration will come into play which may make the estimation very bad. This tells us when $\alpha \geq 1$, we just know $\mathrm{Op}(\mathscr{K})$ maps $\Lambda_\alpha(\overline{\mathbf{D}})$ into $\Gamma_{\alpha+\lambda_2}(\overline{\mathbf{D}})$.

The case when α is an integer then follows by the standard interpolation theorem for Λ_α spaces.

Summarize all these results, we can state the following theorem:

(4.9.1) **Proposition.** *Suppose the operator* $\mathrm{Op}(\mathscr{K})$ *defined in* (3.1) *and* $l \geq 4$, $k < 2n$, $k + l < 2n + 4$; *we have the following*:

(i) $\mathrm{Op}(\mathscr{K}): L^p(U) \to \Gamma_{(n+2)(1-p^{-1})-(k+l)/2}(U)$,
$\mathrm{Op}(\mathscr{K}): L^p(U) \to \Gamma_{(2n+4)(1-p^{-1})-(k+l)}(U)$, *if* $(2n + 4)/(2n + 4 - k - l) < p < \infty$;

(ii) $\mathrm{Op}(\mathscr{K}): L^\infty(U) \to \Lambda_{(2n+4-k-l)/2}(U)$,
$\mathrm{Op}(\mathscr{K}): L^\infty(U) \to \Gamma_{2n+4-k-l}(U)$, *if* $0 < 2n + 4 - k - l < 2$;

(iii) $\mathrm{Op}(\mathscr{K}): \Lambda_\alpha(U) \to \Lambda_{\alpha+(2n+4-k-l)/2}(U)$, *if* $0 < \alpha+(2n+4-k-l)/2 < 1$,
$\mathrm{Op}(\mathscr{K}): \Lambda_\alpha(U) \to \Gamma_{\alpha+2n+4-k-l}(U)$, *if* $\alpha \geq 1$.

(4.9.2) **Proposition.** *Suppose the operator* $\mathrm{Op}(\mathscr{K})$ *defined in* (3.1) *and* $l < 4$, $k + l/2 < 2n + 2$; *we have the following*:

(i$'$) $\mathrm{Op}(\mathscr{K}): L^p(U) \to \Lambda_{(2n+2)(1-p^{-1})-(2k+l)/2}(U)$, *if* $(4n+4)/(4n+4-2k-l) < p < \infty$;

(ii$'$) $\mathrm{Op}(\mathscr{K}): L^\infty(U) \to \Lambda_{2n+2-k-l/2}(U)$, *if* $0 < 2n + 2 - k - l/2 < 1$;
$\mathrm{Op}(\mathscr{K}): L^\infty(U) \to \Lambda_1(U)$, *if* $k + l/2 = 2n + 1$.

(iii$'$) $\mathrm{Op}(\mathscr{K}): \Lambda_\alpha(U) \to \Lambda_{\alpha+(2n+2-k-l)/2}(U)$, *if* $0 < \alpha+(2n+2-k-l)/2 < 1$;
$\mathrm{Op}(\mathscr{K}): \Lambda_\alpha(U) \to \Gamma_{\alpha+2n+2-k-l/2}(U)$, *if* $\alpha \geq 1$.

Applying the results of these two propositions, we have the following:

(4.10) **Theorem.**

(i) $\overline{\partial}^*\mathbf{N}: L^p(U) \to \Lambda_{1/2-(2n+4)/2p}(U)$,
$\overline{\partial}^*\mathbf{N}: L^p(U) \to \Gamma_{1-(2n+4)/p}(U)$, *if* $2n+4 < p < \infty$;

(ii) $\overline{\partial}^*\mathbf{N}: L^\infty(U) \to \Lambda_{1/2} \cap \Gamma_1(U)$;

(iii) $\overline{\partial}^*\mathbf{N}: \Lambda_\alpha(U) \to \Lambda_{\alpha+1/2}(U)$, $\forall\alpha > 0$.

Remarks. (1) The results (i), (ii) of this theorem tell us the Kohn solution $\overline{\partial}^*\mathbf{N}$ map L^p functions to Λ_α, $\alpha = 1/2 - (2n+4)/2p$, in all directions but $\Lambda_{2\alpha}$ in the complex tangential directions. In fact, we may look at this result via a theorem of Krantz [16]:

Theorem (Krantz). *Let* $\Omega \subset \mathbf{C}^n$ *be strongly pseudoconvex with* C^k *boundary,* $k \geq 3$, *and let* $0 < \alpha < k$. *Let* $f: \Omega \to \mathbf{C}$ *be holomorphic and suppose* $f \in \Gamma_\alpha(\Omega)$. *Then* $f \in \Lambda_{\alpha/2}(\Omega)$ *hence* $f \in \Gamma_\alpha(\Omega) \cap \Lambda_{\alpha/2}(\Omega)$. *In other words*

$$\Gamma_\alpha(\Omega) \cap \{\textit{holomorphic functions}\} = \Lambda_{\alpha/2}(\Omega) \cap \{\textit{holomorphic functions}\}.$$

Using this theorem and the fact $\overline{\partial}^*\mathbf{N}(f) \in \Lambda_{\alpha/2}$, we may set $\overline{\partial}^*\mathbf{N}(f) = \mathbf{H}(f) + \{\overline{\partial}^*\mathbf{N}(f) - \mathbf{H}(f)\}$, where $\mathbf{H}(f)$ is the Henkin solution. From Krantz [15], we know $\mathbf{H}(f) \in \Gamma_\alpha(\Omega) \cap \Lambda_{\alpha/2}(\Omega)$. But $\overline{\partial}\{\overline{\partial}^*\mathbf{N}(f) - \mathbf{H}(f)\} = f - f = 0$ implies $\overline{\partial}^*\mathbf{N}(f) - \mathbf{H}(f)$ is holomorphic. This tells us $\overline{\partial}^*\mathbf{N}(f) \in \Gamma_\alpha(\Omega) \cap \Lambda_{\alpha/2}(\Omega)$.

(2) The result of (ii) is optimal due to a counter example of Kerzman and Stein [15, 16].

(3) The result (iii) of Theorem (4.10) is stronger than the result (iii) of Proposition (4.9.1), i.e., the operator $\overline{\partial}^*\mathbf{N}$ not only map $\Lambda_\alpha(U)$ to $\Gamma_{\alpha+1/2}(U)$ boundedly, in fact maps $\Lambda_\alpha(U)$ to $\Lambda_{\alpha+1/2}(U)$ boundedly! $\overline{\partial}^*\mathbf{N}$ is the kernel of the $\overline{\partial}$-equation and $\overline{\partial}$ is an elliptic operator, hence we can recover the normal direction by those admissible directions.

5. Transferred kernels on strongly pseudoconvex domains

From §2 to §4, we discussed the integral representation of Kohn solution $\overline{\partial}^*\mathbf{N}$ and optimal L^p and Hölder estimates for operators with mixed type homogeneities on the Siegel upper-half space. The final goal of this work is to discuss the problem on a bounded strongly pseudoconvex domain Ω. We will transfer the kernel $E_k H_l$ from the model to the domain Ω via a special Heisenberg coordinate system which was introduced by Phong and Stein [26, 27]. We just summarize Phong and Stein's results and compare them to the results obtained by Lieb and Range [18, 19]. We have mentioned standard Heisenberg coordinate system in §1 already. When we discuss the $\overline{\partial}$-Neumann problem, we cannot use standard Heisenberg coordinate system to transfer the kernel $\mathbf{N}$ from the model to general domain! (This is different from the parametrix of $\Box_b$). Phong and Stein [26, 27] constructed a "special Heisenberg coordinate system" to overcome this difficulty. A special Heisenberg coordinate system is any standard Heisenberg coordinate system $(z'^{\dagger}, t^{\dagger}; \rho^{\dagger}) \in \mathbf{H}^n \times \mathbf{R}^n$ for $\xi \in D_\eta$ (where D_η is a small neighborhood of η), depending smoothly on η, which satisfies the conditions

$$(5.1) \qquad |z'^{\dagger} - z'| = \overline{O}^2, \qquad |t^{\dagger} - t| = O^3, \qquad |\rho^{\dagger} - \rho| = O^3$$

where $(z', t; \rho)$ are standard Heisenberg coordinate system. Evidently such a system will also satisfy all the properties which standard Heisenberg coordinates have. We write $(z'^{\dagger}, t^{\dagger}) = \mathbf{\Theta}^{\dagger}(\xi, \eta)$. These coordinates for the $\overline{\partial}$-Neumann problem will satisfy the additional properties

$$(5.2) \qquad \overline{Z}_{n+1}(t^{\dagger} + i\rho^{\dagger}) = O^3, \qquad \overline{Z}_{n+1}(|z'^{\dagger}|^2) = O^3$$

which are crucial when we consider the differential operator $\Box$ and the boundary operator (i.e., $[\overline{Z}_{n+1}]_{\partial D}$) act on the "transferred kernel"! Here $\overline{O}^k$ and O^k denote respectively C^∞ functions $f(\xi, \eta)$ and $g(\xi, \eta)$ satisfying

$$|f(\xi, \eta)| \le (|z'| + |t| + \rho(\xi) + \rho(\eta))^k$$

and

$$|g(\xi, \eta)| \le (|z'| + |t|^{1/2} + \rho(\xi)^{1/2} + \rho(\eta)^{1/2})^k .$$

Given a function $\mathscr{K}(z'^{\dagger}, t^{\dagger}, \rho^{\dagger}) \in C^\infty(\mathbf{H}^n \times \mathbf{R}^+)$ define the "transferred kernels" $\mathscr{K}_D \in C^\infty(D \times D)$ by

$$(5.3) \qquad \mathscr{K}_D(\xi, \eta) = \psi(\xi)\mathscr{K}(\mathbf{\Theta}^{\dagger}(\xi, \eta); \rho^{\dagger}(\xi) + \rho^{\dagger}(\eta))\varphi(\eta)$$

and the Hilbert integral operator $\mathrm{Op}_D(\mathscr{K})$ associated to $\mathscr{K}_D$ is defined by

$$(5.4) \qquad \mathrm{Op}_D(\mathscr{K})(f)(\xi) = \int_D \mathscr{K}_D(\xi, \eta) f(\eta)\, d\eta$$

where $d\eta$ is any fixed measure on D which is C^∞ with respect to Lebesgue measure. Then we have the following theorems:

(5.5) **Theorem** [27]. *Let $\mathscr{K}(z'^{\dagger}, t^{\dagger}, \rho^{\dagger}) = E_k(z'^{\dagger}, t^{\dagger}, \rho^{\dagger})H_l(z'^{\dagger}, t^{\dagger}, \rho^{\dagger})$ be a kernel which belongs to any one of the cases $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3$, or $\mathscr{C}_4$ listed in Definition* (3.4). *Then $\mathrm{Op}_D(\mathscr{K})$ extends to a bounded operator on $L^p(D)$ to itself, for $1 < p < \infty$.*

By duality we immediately derive

(5.5$'$) **Theorem** [27]. *Under the same conditions as in Theorem* (5.5), *the operator $\mathrm{Op}_D(\mathscr{K})^*$ defined by*

$$(5.4') \qquad \mathrm{Op}^*_D(\mathscr{K})(f)(\xi) = \int_D \mathscr{K}^*_D(\xi, \eta) f(\eta)\, d\eta$$

extends to a bounded operator on $L^p(D)$ to itself, for $1 < p < \infty$.

Using the techniques in Folland-Stein [6], Greiner-Stein [8] and Rothschild-Stein [28], we also can prove the following propositions:

(5.6.1) **Proposition.** *Let* $\mathrm{Op}_D(\mathscr{K})$ *be an operator defined in* (5.4) *and* $l > 4$, $k < 2n$, $k + l < 2n + 4$. *Then* $\mathrm{Op}_D(\mathscr{K})$ *extends to a bounded operator in the following cases*:

(i) $\mathrm{Op}_D(\mathscr{K}): L^p(D) \to L^q(D)$, *if* $q^{-1} = p^{-1} - (2n + 4 - k - l)/(2n + 4)$, $1 < p < (2n + 4)/(2n + 4 - k - l)$;

(ii) $\mathrm{Op}_D(\mathscr{K}): L^p(D) \to \Lambda_{(n+2)(1-1/p)-1/2(k+l)}(D)$, *if* $(2n + 4)/(2n + 4 - k - l) < p < \infty$;

(iii) $\mathrm{Op}_D(\mathscr{K}): L^\infty(D) \to \Lambda_{(2n+4-k-l)/2}(D)$, *if* $0 < 2n + 4 - k - l < 2$;

(iv) $|\iint_D \exp(\alpha|\mathrm{Op}_D(\mathscr{K})(f)/\|f\|_{L^{(2n+4)/(2n+4-k-l)}}|^{(2n+4)/(k+l)})d\xi| \leq C < \infty$, α *is a sufficiently small constant*;

(v) $\mathrm{Op}_D(\mathscr{K}): L^1(D) \to L^{(2n+4)/(k+l)-\varepsilon}(D)$, $\forall \varepsilon > 0$;

(vi) $\mathrm{Op}_D(\mathscr{K}): L^{(2n+4)/(2n+4-k-l)}(\log^+ L)^{(2n+4)/(2n+4-k-l)-1+\varepsilon}(D) \to L^\infty(D)$, $\forall \varepsilon > 0$;

(vii) $\mathrm{Op}_D(\mathscr{K}): L^p(D) \to \Gamma_\alpha(D)$, $\alpha = 2n + 4 - k - l - (2n + 4)/p$, $(2n + 4)/(2n + 4 - k - l) < p < \infty$.

(5.6.2) **Proposition.** *Let* $\mathrm{Op}_D(\mathscr{K})$ *be an operator defined in* (5.4) *and* $l < 4$, $k + l/2 < 2n + 2$. *Then* $\mathrm{Op}_D(\mathscr{K})$ *extends to a bounded operator in the following cases*:

(i′) $\mathrm{Op}_D(\mathscr{K}): L^p(D) \to L^q(D)$, *if* $q^{-1} = p^{-1} - (4n + 4 - 2k - l)/(4n + 4)$, $1 < p < (4n + 4)/(4n + 4 - 2k - l)$;

(ii′) $\mathrm{Op}_D(\mathscr{K}): L^p(D) \to \Lambda_{(2n+2)(1-1/p)-1/2(2k+l)}(D)$, *if* $(4n + 4)/(4n + 4 - 2k - l) < p < \infty$;

(iii′) $\mathrm{Op}_D(\mathscr{K}): L^\infty(D) \to \Lambda_{2n+2-k-l/2}(D)$, *if* $0 < 2n + 2 - k - l/2 < 1$; $\mathrm{Op}_D(\mathscr{K}): L^\infty(D) \to \Lambda_1(D)$, *if* $k + l/2 = 2n + 1$;

(iv′) $|\iint_D \exp(\alpha|\mathrm{Op}_D(\mathscr{K})(f)/\|f\|_{L^{(4n+4)/(4n+4-2k-l)}}|^{(4n+4)/(2k+l)})d\xi| \leq C < \infty$, α *is a sufficiently small constant*;

(v′) $\mathrm{Op}_D(\mathscr{K}): L^1(D) \to L^{(4n+4)/(2k+l)-\varepsilon}(D)$, $\forall \varepsilon > 0$;

(vi′) $\mathrm{Op}_D(\mathscr{K}): L^{(4n+4)/(4n+4-2k-l)}(\log^+ L)^{(4n+4)/(4n+4-2k-l)-1+\varepsilon}(D) \to L^\infty(D)$, $\forall \varepsilon > 0$.

Applying the results of these two propositions, we can prove the optimal L^p and Hölder estimates for the Kohn solution of the Cauchy-Riemann equations on a bounded strongly pseudoconvex domain Ω. Using a partition of unity, we can transfer the kernel $\overline{\partial^*}\mathbf{N}$ from a small neighorhood U of $0 \in \mathbf{D}$ to small boundary neighborhood D of $\eta \in \partial\Omega = \mathscr{M}$. We call this "transferred kernel" $(\overline{\partial^*}\mathbf{N})_D$, then $(\overline{\partial^*}\mathbf{N})_D$ will be a vector $(K_D^1, K_D^2, \ldots, K_D^{n+1})$. Each K_D^j has the form $\sum_{k,l}(E_k)_D \cdot (H_l)_D$. The crucial terms of K_D^j's are the terms $(E_{2n-1})_D(H_4)_D$ and $(E_{2n-3})_D(H_6)_D$. Hence we have the following theorem:

(5.7) **Theorem.**

(i) $(\overline{\partial^*}\mathbf{N})_D: L^p(D) \to L^q(D)$, *if* $q^{-1} = p^{-1} - 1/(2n + 4)$, $1 < p < 2n + 4$;

(ii) $(\overline{\partial^* \mathbf{N}})_D: L^p(D) \to \Lambda_{1/2-(n+2)/p}(D)$, *if* $2n+4 < p < \infty$;
$(\overline{\partial^* \mathbf{N}})_D: L^p(D) \to \Gamma_{1-(2n+4)/p}(D)$, *if* $2n+4 < p < \infty$;

(iii) $(\overline{\partial^* \mathbf{N}})_D: L^\infty(D) \to \Lambda_{1/2}(D) \cap \Gamma_1(D)$;

(iv) $|\iint_U \exp(\alpha|(\overline{\partial^* \mathbf{N}})_D(f)/\|f\|_{L^{2n+4}}|^{(2n+4)/(2n+3)})d\xi| \le C < \infty$, α *is a sufficiently small constant; i.e.,*
$(\overline{\partial^* \mathbf{N}})_D: L^{2n+4}(D) \to L\{\exp(\alpha|\cdot|^{(2n+4)/(2n+3)})\}(D)$.

(v) $(\overline{\partial^* \mathbf{N}})_D: L^1(D) \to L^{(2n+4)/(2n+3)-\varepsilon}(D)$, $\forall \varepsilon > 0$;

(vi) $(\overline{\partial^* \mathbf{N}})_D: L^{2n+4}(\log^+ L)^{2n+3+\varepsilon}(D) \to L^\infty(D)$, $\forall \varepsilon > 0$.

On the other hand, using Theorems (5.5), (5.5′), we also have

(5.8) **Theorem.**

(i′) $(\overline{\partial^* \mathbf{N}})_D: L^p(D) \to L^p(D)$, $1 \le p \le \infty$;

(ii′) $(\overline{\partial^* \mathbf{N}})_D: L^p_k(D) \to L^p_{k+1/2}(D)$, $1 < p < \infty$;

(iii′) $(\overline{\partial^* \mathbf{N}})_D: S^p_k(D) \to S^p_{k+1}(D)$, $1 < p < \infty$.

Remark. It does not matter whether we use "standard Heisenberg coordinate system" or "special Heisenberg coordinate system" to consider the L^p and Hölder estimates for the "transferred kernels". When we consider the $\overline{\partial}$-Neumann problem, we have the Neumann operator $\mathbf{N}$ for the model case near $0 \in \mathbf{D} \cap U$. This gives a "transferred kernel" $\mathbf{N}_D$ on $\Omega \cap D$. Now the problem reduces to whether $\mathbf{N}_D$ satisfies $\square(\mathbf{N}_D) = \delta_0$ and the boundary condition $\overline{Z}_{n+1}(\mathbf{N}_D)|_{\partial\Omega} = 0$ or not! This is the reason why Phong and Stein need to use a special Heisenberg coordinate system instead of standard Heisenberg coordinates system. Even though they cannot get $\square(\mathbf{N}_D) = \delta_0$ and $\overline{Z}_{n+1}(\mathbf{N}_D)|_{\partial\Omega} = 0$; but by (5.1), they can get $\square(\mathbf{N}_D) = \delta_0 +$ at least type two operators (acceptable errors) and $\overline{Z}_{n+1}(\mathbf{N}_D)|_{\partial\Omega} = 0 +$ at least type two operators (acceptable errors). On the other hand, Lieb and Range [18, 19] just use standard Levi procedure to construct a coordinates system (i.e., standard Heisenberg coordinates system) in their kernel $(\overline{\partial^* \mathbf{N}})_D$ since they only solve the $\overline{\partial}$-equation. If we look at their papers carefully, they never plug in the second Neumann boundary condition!

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MATHEMATICAL SCIENCES RESEARCH INSTITUTE, 1000 CENTENNIAL DRIVE, BERKELEY, CALIFORNIA 94720

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF MARYLAND, COLLEGE PARK, MARYLAND 20742 (Current address)