

DEVIATIONS OF TRAJECTORY AVERAGES AND THE DEFECT IN PESIN'S FORMULA FOR ANOSOV DIFFEOMORPHISMS

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ABSTRACT. A large deviation theorem at the Donsker Varadhan level three is obtained for the convergence of trajectory averages of Anosov diffeomorphisms. It is possible to provide an explicit description of the rate function.

1. INTRODUCTION

This paper is concerned with the convergence of trajectory averages of certain smooth dynamical systems. If $f: M \mapsto M$ is a transitive Anosov diffeomorphism of a manifold M , the results of Bowen [B] imply the following.

For each continuous function $\varphi: M \mapsto \mathbf{R}$ the trajectory averages of φ , that is the sequence $\frac{1}{n} \sum_0^{n-1} \varphi(f^j(x))$, converge to a constant for Lebesgue almost every initial point $x \in M$. This constant is $\int \varphi d\mu$, where μ is the Bowen-Ruelle-Sinai measure. An equivalent assertion is that the measures $\frac{1}{n} \sum_0^{n-1} \delta_{f^j(x)}$ converge weakly to μ for Lebesgue almost all x .

We are concerned with estimating the probability that the n th trajectory average differs from $\int \varphi d\mu$; and in particular, with estimating the rate of exponential decay of the numbers

$$m \left\{ x: \frac{1}{n} \sum_0^{n-1} \varphi(f^j(x)) \in A \right\} \quad \text{as } n \mapsto \infty.$$

Here m is a normalized Lebesgue measure and $A \subset \mathbf{R}$ is some Borel set.

Our first conclusion, Corollary 4.7, is that for each continuous function φ there is a lower semicontinuous $K^\varphi: \mathbf{R} \mapsto [0, \infty]$ which describes these decay rates in the sense that

$$\limsup_{n \rightarrow \infty} n^{-1} \log m \left\{ x: \frac{1}{n} \sum_0^{n-1} \varphi(f^j(x)) \in A \right\} \leq -\inf\{K^\varphi(t): t \in A\}$$

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if $A \subset \mathbf{R}$ is closed and

$$\liminf_{n \rightarrow \infty} n^{-1} \log m \left\{ x: \frac{1}{n} \sum_0^{n-1} \varphi(f^j(x)) \in A \right\} \geq -\inf\{K^\varphi(t): t \in A\}$$

if $A \subset \mathbf{R}$ is open.

The second conclusion presented is that there is a lower semicontinuous function K defined on the set $\mathcal{M}(M)$ of probability measures on M , and taking values in $[0, \infty]$ so that

$$(i) \quad K^\varphi(t) = \inf \left\{ K(\nu): \nu \in \mathcal{M}(M) \text{ and } \int \varphi d\nu = t \right\}.$$

In addition, the function K has an explicit representation

$$(ii) \quad K(\nu) = \begin{cases} -h_\nu(f) - \int \varphi^u d\nu & \text{if } \nu \text{ is invariant under } f, \\ +\infty & \text{otherwise.} \end{cases}$$

Here $h_\nu(f)$ is the metric entropy of f and $\varphi^u(x) = -\log|\det Df|_{E^u}|$ is the negative logarithm of the determinant of Df restricted to the unstable subspace $E^u \subset TM$ determined by f .

Recognizing $-\int \varphi^u d\nu$ as the sum of the positive ν -Lyapunov exponents χ_i^ν of f , (ii) can be rewritten as

$$(ii') \quad h_\nu(f) + K(\nu) = \sum [\chi_i^\nu]^+.$$

In this formulation the “rate function” K can be seen to measure the defect in “Pesin’s formula” ([P]), apparently due originally to G. A. Margolis, which says,

$$h_\nu(f) \leq \sum [\chi_i^\nu]^+.$$

In the language of the theory of large deviations as set forth in [V], what we show is that the sequence $\Phi_n(\cdot) = m\{x: \frac{1}{n} \sum_0^{n-1} \delta_{f^j(x)} \in \cdot\}$, $n = 1, 2, \dots$, of measures on $\mathcal{M}(M)$ satisfies a large deviation principle with rate function given by (ii). This result was suggested to us by remarks in [T].

Some concepts of large deviation theory are summarized in §2. In §3, the facts required concerning hyperbolic dynamical systems are recalled, and §4 is devoted to the proofs of our main results.

2. PRELIMINARIES ON RATE FUNCTIONS AND SHIFTS

Rate functions. A generalized dynamical system is (X, T, μ) where X is a topological space, $T: X \mapsto X$ is a homeomorphism, and μ is a Borel probability measure. Note that we do not require that μ be T -invariant. The probability measures on X will be denoted $\mathcal{M}(X)$ and those invariant under T by $\mathcal{M}_T(X)$.

If $\varphi: X \mapsto V$ is a map of X into a convex subset of a topological linear space, set $S_n(\varphi)(x) = \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ T^j(x)$. For a subset $A \subset V$ define

$$\lambda_*(A) = \liminf_{n \rightarrow \infty} n^{-1} \log \mu(\{x: S_n(\varphi)(x) \in A\})$$

and

$$\lambda^*(A) = \limsup_{n \rightarrow \infty} n^{-1} \log \mu([x: S_n(\varphi)(x) \in A]).$$

A function $K: V \mapsto [0, \infty]$ is a *rate function* for (X, T, μ) and φ if the following conditions hold:

- (1) $-\inf\{K(a): a \in A\} \leq \lambda_*(A)$ for each open set $A \subset V$,
- (2) $-\inf\{K(a): a \in A\} \geq \lambda^*(A)$ for each closed set $A \subset V$,
- (3) K is lower semicontinuous,
- (4) $\{a: K(a) \leq t\}$ is compact for each t .

If K is a rate function for (X, T, μ^γ) and φ for every γ in some index set Γ , it is a *uniform rate function* for the family if the limits in the inequalities (1) and (2) above hold uniformly for $\gamma \in \Gamma$.

When a rate function exists, it is unique. In our work below, V will be compact and condition (4) will be trivially true.

We are particularly interested in establishing the existence of rate functions where $\varphi: X \mapsto \mathbf{R}$ is a continuous "observation". One way to establish the existence of rate functions for all $\varphi \in C(X, \mathbf{R})$ simultaneously is to show that there is a rate function for the system (X, T, μ) and the function $\delta: X \rightarrow M(X)$ defined by $\delta(x) = \delta_x = \text{point mass at } x$, and then apply Proposition 2.1 below. This proposition is a variant of the "contraction principle" of [V], see [O, Proposition 1.2].

2.1. Proposition. *For $i = 1, 2$ let $(X_i, T_i, \mu_i^{x_i})$, $x_i \in X'_i$, be a family of dynamical systems, where $X'_i \subset X_i$. Suppose $\phi_i: X_i \mapsto V_i$ are maps where V_i is a convex subset of a metrizable linear space. Suppose X_1 is compact and that there exists a measurable map $L: X_1 \mapsto X_2$ so that $\mu^{x_i} \circ L^{-1} = \mu^{Lx_i}$ for all $x_i \in X'_i$ and $L(X'_1) = X'_2$, $L \circ T_1 = T_2 \circ L$. Also suppose there is a bounded linear map $M: V_1 \mapsto V_2$ so that $\phi_2 \circ L = M \circ \phi_1$. Then, if K_1 is a uniform rate function for $(X_1, T_1, \mu_1^{x_1})$, and δ_1 (for $x_1 \in X'_1$), the function K_2 defined by*

$$K_2(y) = \begin{cases} \inf\{K_1(x): Mx = y\} \\ +\infty \text{ if } y \notin \text{range}(M) \end{cases}$$

is a uniform rate function for $(X_2, T_2, \mu_2^{x_2})$, and δ_2 (for $x_2 \in X'_2$).

In case (X, T, μ) and the function $x \mapsto \delta_x$ has a rate function, we will simply say that the system has a rate function. In this case the rate function can be shown to have the value $+\infty$ off of the set $\mathcal{M}_T(X)$.

2.2. Corollary. *Suppose $N > 0$ and that the system (X, T^N, μ) has a rate function K_1 . Then the system (X, T^N, μ) and the function*

$$x \mapsto N^{-1}(\delta_x + \delta_{Tx} + \dots + \delta_{T^{N-1}x})$$

has a rate function K_2 , and $K_2(\nu) = K_1(\nu)$ if $\nu \in \mathcal{M}_{T^N}(X)$.

Proof. Apply Proposition 2.1 to the following diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{T^N} & X & \xrightarrow{x \rightarrow \delta_x} & \mathcal{M}(X) \\
 \text{id} \Big| & & \text{id} \Big| & & \Big| M(\nu) = N^{-1}(\nu + T^* \nu + \dots + (T^*)^{N-1} \nu) \\
 X & \xrightarrow{T^N} & X & \xrightarrow{x \rightarrow N^{-1}(\delta_x + \dots + \delta_{T^{N-1}x})} & \mathcal{M}(X)
 \end{array}$$

Note that $M(\nu) = \nu$ on $\mathcal{M}_{T^N}(X)$.

For future reference we record the following, which is a consequence of the lower semicontinuity of rate functions and the fact that the average of the function $\phi(x) = N^{-1}(\delta_x + \dots + \delta_{T^{N-1}x})$ along trajectories of T^N is also an average of the function $x \rightarrow \delta_x$ along trajectories of T . That is, for each set A , $\{x: n^{-1} \sum_0^{n-1} \phi(T^{Nj}(x)) \in A\} = \{x: (Nn)^{-1} \sum_0^{Nn-1} \delta_{T^j(x)} \in A\}$.

2.3. Proposition. *If (X, T^N, μ) has a rate function K_1 for*

$$\phi(x) = N^{-1}(\delta_x + \dots + \delta_{T^{N-1}x}),$$

then (X, T, μ) has a rate function K_2 for the function δ_x , and $K_2 = N^{-1}K_1$.

Shifts and measures. When $M = \{1, 2, \dots, n\}$ is a finite set given the discrete topology, form $\Omega = \prod_{-\infty}^{\infty} M$, and give Ω the product topology and product Borel algebra \mathcal{F} . Denote by $\mathcal{F}_{m,n}$ for $m < n$ and $m, n \in \mathbf{Z} \cup \{-\infty, \infty\}$ the smallest sub- σ -algebra of \mathcal{F} with respect to which the coordinate functions $\omega \mapsto \omega_i$ for $m \leq i \leq n$ are measurable. The shift homeomorphism on Ω is $\sigma: \Omega \mapsto \Omega$ defined by $[\sigma(\omega)]_n = \omega_{n+1}$.

If A is an $n \times n$ matrix with each entry equal to 0 or 1, define a closed σ -invariant subset of Ω by $\Sigma_A = \{\omega \in \Omega: A_{\omega_i, \omega_{i+1}} = 1 \text{ for all } i \in \mathbf{Z}\}$. Such subsets, equipped with the relative σ -algebras $\mathcal{F}_{m,n}$ are called subshifts of finite type.

For $\mu \in \mathcal{M}(\Omega)$, denote by $\mu(\cdot | \mathcal{F}_{m,n})(\omega)$ the conditional probabilities of μ given the σ -algebra $\mathcal{F}_{m,n}$. A version $\mu^*(\cdot)(\omega)$ of $\mu(\cdot | \mathcal{F}_{m,n})(\omega)$ is called a family of regular conditional probabilities given $\mathcal{F}_{m,n}$ if for each $\omega \in \Omega$ the function $A \mapsto \mu^*(A)(\omega)$ is a probability measure on \mathcal{F} . We will always use the notation $\nu(\cdot | G)(x)$ for conditional probabilities and $\nu^*(\cdot)(x)$ or $\nu_x^*(\cdot)$ for regular conditional probabilities—the σ -algebra being understood.

Gibbs measures. Equip Ω (or Σ_A) with a metric by defining $d(\omega, \eta) = \sum_{-\infty}^{\infty} 2^{-|i|} \delta_{\omega_i}^{\eta_i}$. If $\phi: \Omega \mapsto \mathbf{R}$ is Hölder continuous in this metric, there is a Gibbs measure μ_ϕ associated with ϕ . It is the unique σ -invariant measure which satisfies the property:

There exist positive constants C_1 and C_2 and a number $P(\phi)$ so that for each $\omega \in \Omega$

$$C_1 \leq \frac{\mu_\phi(C^n(\omega))}{\exp\left(-P(\phi)n + \sum_0^{n-1} \phi(\sigma^j(\omega))\right)} \leq C_2$$

where $C^n(\omega) = \{\eta \in \Omega: \eta_i = \omega_i \text{ for } i = 0, 1, 2, \dots, n - 1\}$. For a reference on Gibbs measures, see [B].

We will require the following result:

2.2. Theorem [OP, Theorem 3.2]. *Let (Σ_A, σ, μ) be a subshift of finite type with μ a Gibbs measure for some Hölder continuous function $\phi: \Sigma_A \mapsto \mathbf{R}$. There exists a family $\mu^*(\cdot)(\omega)$, $\omega \in \Sigma_A$, of regular conditional probabilities of μ given $\mathcal{F}_{-\infty, 0}$ such that the function $K: \mathcal{M}(\Sigma_A) \mapsto [0, \infty]$ given by*

$$K(\nu) = \begin{cases} -h_\nu(\sigma) - \int \phi d\nu + P(\phi), & \nu \in \mathcal{M}_\sigma(\Sigma_A), \\ +\infty & \text{otherwise} \end{cases}$$

is a uniform rate function for the family of systems $(\Sigma_A, \sigma, \mu_\omega^*(\cdot))$, $\omega \in \Sigma_A$.

It is the purpose of the next two sections to use the methods of symbolic dynamics to “transfer” this result to the setting of Anosov diffeomorphisms.

3. ANOSOV DIFFEOMORPHISMS

Let M be a compact Riemannian manifold and $f: M \mapsto M$ a transitive C^2 Anosov diffeomorphism. That is, suppose the tangent bundle to M splits in a continuous sum $TM = E^s + E^u$ where there exist constants $C > 0$ and $\lambda \in (0, 1)$ so that

- (i) $\|Df^n(x) \cdot v\| \leq C\lambda^n \|v\|$ for all $n \geq 0$ and $v \in E_x^s$,
- (ii) $\|Df^n(x) \cdot v\| \leq C\lambda^n \|v\|$ for all $n \leq 0$ and $v \in E_x^u$.

To fix notation we record some facts concerning Anosov diffeomorphisms.

(1) *Stable manifolds.* Through each $x \in M$ pass two C^1 submanifolds W_x^s and W_x^u tangent at x to the subspaces E_x^s and E_x^u . These submanifolds are the collections of points whose forward (backward) orbit under f approaches the forward (backward) orbit of x . The foliations of M given by these manifolds are invariant under f (leaves are mapped to leaves), and transverse.

(2) *Local product structure.* The local stable and unstable manifolds of x are

$$W_\epsilon^s(x) = \{y \in W_x^s: d(f^j x, f^j y) \leq \epsilon \text{ for all } j = 0, 1, 2, \dots\}$$

and

$$W^u(x) = \{y \in W^u: d(f^j x, f^j y) \leq \epsilon \text{ for all } j = 0, -1, -2, \dots\}.$$

There exist ϵ and δ so that if $x, y \in M$ satisfy $d(x, y) < \delta$, then $W_\epsilon^u(x) \cap W_\epsilon^s(x)$ consists of exactly one point, denoted $[x, y]$. Thus one obtains a homeomorphism from a neighborhood of x in $W^u(x)$ to a neighborhood of y in $W^u(y)$ for y in $W^s(x)$ near enough x , by setting

$$h_{x,y}(p) = W_\epsilon^s(p) \cap W_\epsilon^u(y) = [p, y].$$

A theorem of Anosov [A] states that this homeomorphism is absolutely continuous with respect to Lebesgue measure (induced on $W_\epsilon^u(x)$ and $W_\epsilon^u(y)$ by restricting the metric of M to the tangent spaces of these manifolds and extending the resulting volume form to a measure). If m_x and m_y are the Lebesgue

measures on $W^u(x)$ and $W^u(y)$, define a new measure \bar{m} on a neighborhood N of x in $W^u(x)$ by $\bar{m}(A) = m_y(h_{x,y}(A))$. Anosov's theorem says that \bar{m} and m_x are equivalent, and that in fact, the densities of one with respect to the other are bounded and depend continuously on the point y .

(3) *Markov partitions.* A set $R \subset M$ is called a rectangle if $[x, y] \in R$ whenever x and y are in R and if, in addition, $R = \text{closure}(\text{interior}(R))$. If R is a rectangle, the boundary of R comes in two pieces

$$\begin{aligned} \partial^e R &= \{x \in R : x \notin \text{int}(W_\epsilon^s(x) \cap R)\}, \\ \partial^c R &= \{x \in R : x \notin \text{int}(W_\epsilon^u(x) \cap R)\}. \end{aligned}$$

A Markov partition of M is a cover of M by rectangle R_1, R_2, \dots, R_n so that

- (i) $\text{int}(R_i) \cap \text{int}(R_j)$ is empty if $i \neq j$,
- (ii) $f(\partial^c R_i) \subset \bigcup_j \partial^c R_j$, for all i ,
- (iii) $f^{-1}(\partial^e R_i) \subset \bigcup_j \partial^e R_j$, for all i .

For a detailed account of the existence of stable manifolds, local coordinates and Markov partitions, consult [B] and the references found there. The theory of Markov partitions provides, for each sufficiently fine Markov partition $\mathcal{R} = \{R_1, \dots, R_n\}$ of M , a subshift of finite type Σ_A , and a continuous map $\pi: \Sigma_A \rightarrow M$ (onto) so that $\pi \circ \sigma = f \circ \pi$. The subshift and map are, roughly, defined as follows. For A take the $n \times n$ matrix defined by $a_{ij} = 0$ or 1 according to whether $\text{int}(R_j) \cap \text{int}(f(R_i))$ is empty or not. For $\omega \in \Sigma_A$, the set $\bigcap_{j \in \mathbb{Z}} f^{-j}(R_{\omega_j})$ contains exactly one point x , and one defines $\pi(\omega) = x$. One can show π is well defined, continuous, and onto. In addition, π is 1-1 except at those points ω which map to the first category set $\bigcup_{i \in \mathbb{Z}} f^i(\bigcup_j \partial R_j)$.

Denote by ξ the "partition" of M into sets having the form $R_i \cap W_\epsilon^u(x)$, for $R_i \in \mathcal{R} = \{R_1, \dots, R_n\}$ a member of a generating Markov partition. Then ξ is a partition of M modulo a set having zero Lebesgue measure: different sets in ξ can only intersect at a subset of $\bigcup_i \partial R_i$, which has Lebesgue measure zero. This fact was established by Sinai [S1] and by Ratner [Ra]. We can convert ξ into a true partition by selecting a scheme for assigning points in the common boundaries of two sets R_i and R_j to one of these sets, and removing them from the other. In what follows, we will consider conditional probabilities given ξ . The theory of conditional probabilities given a partition (see [ME]) corresponds very closely to the theory of conditional probabilities given a σ -algebra. It will be convenient to continue to use the notation ν_x^ξ for a version of regular conditional probabilities given ξ even when ξ is a partition. This should cause no confusion. Under the map $\pi: \Sigma_A \rightarrow M$, the algebra generated by ξ correspond to $\mathcal{F}_{-\infty,0}$.

(4) *Conditional Lebesgue measure.* If m is (normalized) Lebesgue measure on M , the family of conditional measures of m given ξ , denoted $m(\cdot|\xi)(x)$, is defined m -a.e. There is a second means by which we might try to construct

such a family of conditional measures explicitly: restrict the metric of M to $TW^u(x)$, and determine a volume form on $W^u(x)$, extend this volume to a measure, and finally restrict this measure to $C_\xi(x) =$ the set of ξ containing x , and normalize it. Denote the result of this construction by m_x^* . To see that, for m -a.e. x , the measures m_x^* and $m(\cdot|\xi)(x)$ restricted to $C_\xi(x)$ are equivalent, it suffices to work locally (that is, in one particular rectangle R_i), and to verify that there is a constant $K > 1$ so that

$$\int_R m_x^*(C_\xi(x) \cap A) dm(x) \in \int_R m(C_\xi(x) \cap A | \xi)(x) dm(x) \cdot [K^{-1}, K],$$

independent of x , where $A \subset R_i$ is an arbitrary rectangle in R .

This would be immediate if the partition ξ were a partition of a neighborhood of 0 in \mathbf{R}^n into a family of k -dimensional "horizontal" planes. The general case can be reduced to this one by constructing a "rectifying" homeomorphism as follows. Pick $p \in R$ and let $\phi_1: \Delta^1 \mapsto W^s(p)$, $\phi_2: \Delta^2 \mapsto W^u(p)$ be embeddings of disks $\Delta^1 \subset \mathbf{R}^k$, $\Delta^2 \subset \mathbf{R}^{n-k}$, where $k = \dim(W^s(p))$, $n = \dim(M)$, so that $\text{Range}(\phi_1) = W^s(p) \cap R_i$ and $\text{Range}(\phi_2) = W^u(p) \cap R_i$. Extend these functions to $\Delta^1 \times \Delta^2$ by setting $\phi(x, y) = [\phi_2(y), \phi_1(x)]$. Then Anosov's theorem says that for each fixed x (or y) the map $y \mapsto \phi(x, y)$ (or $x \mapsto \phi(x, y)$) is absolutely continuous. This observation and the uniform hyperbolicity of f suffice to complete the proof.

4. THE MAIN RESULTS

Let $f: M \mapsto M$ be a transitive C^2 Anosov diffeomorphism, $\mathcal{R} = \{R_1, \dots, R_n\}$ a generating Markov partition, and $\pi: \Sigma_{A(\mathcal{R})} \mapsto M$ be as in §3. We require that each set R_i of the partition be so small that it lies entirely in one local coordinate neighborhood. Define $\varphi^u(x) = -\log |\det Df(x)|_{E^u(x)}$ and set $\bar{\varphi} = \varphi^u \circ \pi$.

The function φ^u is Hölder continuous [B, Lemma 4.6] and $\bar{\varphi}$ inherits this property, since there exist constants C and γ so that for every $\omega, \eta \in \Sigma_A$,

$$d(\pi(\omega), \pi(\eta)) \leq C(d(\omega, \eta))^\gamma.$$

Thus $\bar{\varphi}$ determines a Gibbs measure $\bar{\mu}$. This measure is characterized as the unique σ -invariant measure on Σ_A at which the supremum (over σ -invariant measures ν) of the function $\nu \mapsto h_\nu(\sigma) + \int \bar{\varphi} d\nu$ is attained.

Similarly, φ^u determines a Gibbs measure μ characterized by

$$\begin{aligned} P(\varphi^u) &= \sup \left\{ h_\eta(f) + \int \varphi^u d\eta : \eta \in \mathcal{M}_f(M) \right\} \\ &= h_\mu(f) + \int \varphi^u d\mu. \end{aligned}$$

One may refer to [B, §10 and Theorem 4.1]. The measure μ (denoted by μ^+ in [B]) is the Bowen-Ruelle-Sinai measure. In fact, $\mu = \pi^* \bar{\mu}$, $\bar{\mu}(\bar{\varphi}) = \mu(\varphi^u)$, and

$h_{\bar{\mu}}(\sigma) = h_{\mu}(f)$, so that $P(\bar{\varphi}) = P(\varphi^u)$. For a proof see [R, §7.8]. Furthermore, $P(\varphi^u) = 0$. (See [B, Proposition 4.8].)

In this section we prove the following:

4.1. Theorem. *Let m be normalized Lebesgue measure on M . The generalized dynamical system (M, f, m) has a rate function $K: \mathcal{M}(M) \mapsto [0, \infty]$ given by*

$$K(\nu) = \begin{cases} -h_{\nu}(f) - \int \varphi^u d\nu & \text{if } \nu \in \mathcal{M}_f(M), \\ +\infty & \text{otherwise.} \end{cases}$$

The proof proceeds in a number of steps. We begin with an overview. By Theorem 2.2, there is a rate function which works uniformly for the measures $\mu^*(\cdot)(\omega)$, where μ^* is a family of regular conditional probabilities of $\bar{\mu}$ given $\mathcal{F}_{-\infty,0}$.

We can construct a version of $\mu(\cdot|\xi)(x)$ by defining $\mu(A|\xi)(x)$ to be $\mu^*(\pi^{-1}(A))(\pi^{-1}(x))$ for μ almost every $x \in M$. This provides a rate function for $\mu(\cdot|\xi)(x)$ as in Lemma 4.2. A result known to Anosov [A] (see also [L]) says that $\mu(\cdot|\xi)(x)$ and m_x^* are equivalent. Using the absolute continuity of the maps h_{xy} with respect to m_x^* and m_y^* , we obtain estimates on $m_x^*\{y: \frac{1}{n} \sum_0^{n-1} \delta_{f^j(x)} \in A\}$ for m -almost every x in M . Finally, the equivalence of m_x^* and $m(\cdot|\xi)(x)$ is used to compute $m\{y: \frac{1}{n} \sum_0^{n-1} \delta_{f^j(x)} \in A\}$, which completes the proof.

To simplify notation, write $L_n(x)$ for $\frac{1}{n} \sum_0^{n-1} \delta_{f^j(x)}$. Let $D \subset M$ be the set of points $x \in M$ for which $\pi^{-1}(x)$ is not a singleton. That is, $D = \bigcup_{\mathcal{Z}} f^j(\partial \mathcal{R})$, where $\partial \mathcal{R} = \bigcup_i \partial R_i$. Recall that $\mu(D) = \bar{\mu}(\pi^{-1}(D)) = 0$, as shown in [B, Theorem 4.1], and that $m(D) = 0$, since $m(\partial \mathcal{R}) = 0$, as remarked above. Fix regular conditional probabilities $\mu^*(\cdot)(\omega)$ for $\bar{\mu}$ given $\mathcal{F}_{-\infty,0}$, and some version of $\mu(\cdot|\xi)(x)$. For $B \subset M$, set

$$\hat{\mu}_x^{\xi}(B) = \begin{cases} \mu(B|\xi)(x) & \text{if } x \in D, \\ \mu^*(\pi^{-1}B)(\pi^{-1}x) & \text{if } x \notin D. \end{cases}$$

Then $\hat{\mu}_x^{\xi}$ is a version of $\mu(\cdot|\xi)(x)$.

4.2. Lemma. *Let (μ_x^{ξ}) be a family of regular conditional probabilities of μ given ξ . There exists a set $V \subset M$ with $\mu(V) = 1$ such that there is a uniform rate function K for the systems (M, f, μ_x^{ξ}) , $x \in V$. If $\nu \in \mathcal{M}(M)$ satisfies $\nu(D) = 0$, then*

$$(1) \quad K(\nu) = \begin{cases} -h_{\nu}(f) - \int \varphi^u d\nu & \text{if } \nu \in \mathcal{M}_f(M), \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. Since the Borel σ -field on M is generated by a countable set and for any Borel set A , $\hat{\mu}_x^{\xi}(A) = \mu_x^{\xi}(A)$ for μ -a.e. x , the measures $\hat{\mu}_x^{\xi}$ and μ_x^{ξ} coincide for all x outside a μ -null set F . It will suffice to prove the lemma

for the family $\hat{\mu}_x^\xi$ with $V = M - D$. The result for μ_x^ξ then follows with $V = M - (D \cup F)$.

First apply Theorem 2.2 to the family $(\Sigma_A, \sigma, \mu_\omega^*(\cdot))$ with μ_ω^* as in Theorem 2.2. We have already noted that $P(\bar{\phi}) = 0$, and so we obtain the uniform rate function

$$(2) \quad K_1(\nu) = \begin{cases} -h_\nu(\sigma) - \int \bar{\phi} d\nu & \text{if } \nu \in \mathcal{M}_\sigma(\Sigma_A), \\ +\infty & \text{otherwise.} \end{cases}$$

Now apply Proposition 2.1 to the families $(\Sigma_A, \sigma, \mu_\omega^*(\cdot))$, $\omega \in \pi^{-1}(M - D)$ and $(M, f, \hat{\mu}_x^\xi)$, $x \in M - D$, where $V_1 = \mathcal{M}(\Sigma_A)$, $V_2 = \mathcal{M}(M)$, ϕ_1 is the map $\delta: \Sigma_A \mapsto \mathcal{M}(\Sigma_A)$, (that is $\delta(\omega) = \delta_\omega$), and ϕ_2 is the map $x \mapsto \delta_x$. The function $L: \Sigma_A \rightarrow M$ is π . The function $M: V_1 \mapsto V_2$ is π^* , defined by $\pi^*(\bar{\nu}) = \nu \circ \pi^{-1}$. Since π is continuous, so is π^* . Since K_1 is uniform on Σ_A for $\omega \in \Sigma_A - \pi^{-1}(D)$, and $\pi^*(\hat{\mu}_\omega^*) = \mu_{\pi(\omega)}^\xi$, we obtain a uniform rate function K for $(M, f, \hat{\mu}_x^\xi)$, $x \in M - D$. For $\nu \in \mathcal{M}_f(M)$, $\bar{\nu} \in \mathcal{M}_\sigma(M)$ with $\pi^*\bar{\nu} = \nu$, one has $\int \bar{\phi} d\bar{\nu} = \int \phi d\nu$. Suppose that $\nu(D) = 0$. Then π provides a measure theoretical isomorphism between the systems $(M - D, f, \nu)$ and $(\Sigma_A - \pi^{-1}(D), \sigma, \bar{\nu})$. Consequently, $h_\nu(f) = h_{\bar{\nu}}(\sigma)$, and the expression for K given in the statement of the lemma follows from the expression for K_1 .

4.3. **Lemma.** *Relation (1) holds for any $\nu \in \mathcal{M}_f$.*

Proof. If $\nu \in \mathcal{M}_f$ but $\nu(D) = 0$ is not assumed, the argument used in Lemma 4.2 gives

$$(3) \quad K(\nu) = -\bar{h}_\nu(f) - \int \phi^u d\nu$$

where $\bar{h}_\nu(f) := \sup\{h_{\bar{\nu}}(\sigma) : \pi^*(\bar{\nu}) = \nu\}$. The definition of $\bar{h}_\nu(f)$ involves the mapping π and hence the Markov partition \mathcal{R} ; however, formula (3) shows that in fact it is independent of the choice of \mathcal{R} . To prove the lemma we must show that $h_\nu(f) = \bar{h}_\nu(f)$. If $\nu \in \mathcal{M}_f$ and $\bar{\nu} \in \mathcal{M}_\sigma$ with $\pi^*(\bar{\nu}) = \nu$, one sees easily that $h_{\bar{\nu}}(\sigma) \geq h_\nu(f)$, so that only the inequality

$$(4) \quad h_\nu(f) \geq \bar{h}_\nu(f)$$

must be verified.

Using the fact that $h_\nu(f)$ and $h_{\bar{\nu}}(\sigma)$ are affine functions of ν and $\bar{\nu}$ respectively, we see that the set of ν for which (1) holds is convex. We exploit this observation repeatedly. In addition, either because of Proposition 2.3 or because of the property of entropy that says if $n > 0$ then $h_\nu(f^n) = nh_\nu(f)$, we may verify the lemma with f replaced by some iterate f^m .

Any $\nu \in \mathcal{M}_f$ can be written as a convex combination of ν_1 and ν_2 in \mathcal{M}_f with $\nu_1(D) = 0$ and $\nu_2(D) = 1$. In fact, we may assume that $\nu_2(\partial\mathcal{R}) = 1$. To see this, let $S_0 = \partial^s(\mathcal{R})$ and $U_0 = \partial(\mathcal{R}) \setminus S_0$. Set $S_j = f^j(S_0)$, so that $S_{j+1} \subset S_j$. The invariance of ν_2 implies that $\nu_2(S_{j+1} \setminus S_j) = 0$. It follows that

$\bigcup_{j \in \mathbb{Z}} S_j$ differs from $S_\infty = \bigcap_{j \in \mathbb{Z}} S_j$ by a ν_2 -null set. After a similar argument concerning U_0 we see that $D = S_{-\infty} \cup U_\infty$ differs from $S_\infty \cup U_{-\infty} \subset \partial \mathcal{R}$ by a ν_2 -null set. Since ν_1 satisfies (1) by Lemma 4.2, we need only show that (4), or equivalently that (1), holds for ν_2 .

To verify that (1) holds for ν_2 we require a finite collection of generating Markov partitions $\mathcal{R} = \mathcal{R}_0, \mathcal{R}_1, \dots, \mathcal{R}_k$ with the property that the intersection of the boundaries of all the partitions is empty: $\bigcap (\partial \mathcal{R}_i) = \emptyset$. Such a collection can be constructed using a slight modification of the construction of Markov partitions given by Sinai [S2] in which a cover of M by rectangles is modified an arbitrarily small amount to obtain a cover by rectangles whose boundaries map in the way required of a Markov partition, and then these rectangles are divided to produce a partition. Select a collection of covers of M by small rectangles the boundaries of which are smooth, and so that the intersection of the boundaries of the different covers is empty. (That a finite number of covers suffices is seen by an argument about the dimension of the intersections of the boundaries. In fact, since the final step of Sinai's construction is a "cutting up", which adds extra boundary, it is necessary to require that slightly "extended" boundaries of the covers have intersections disjoint.) Perform Sinai's construction on the various covers moving them so little that their boundaries still do not intersect.

Now ν_2 can be written as a convex combination of measures ν_α each supported on the (disjoint) sets obtained as intersections of the form $\partial \mathcal{R}_0 \cap \partial \mathcal{R}_1^{e_1} \cap \partial \mathcal{R}_2^{e_2} \cap \dots \cap \partial \mathcal{R}_k^{e_k}$, where the exponents e_i take values $+1$ and -1 and indicate whether the set or its complement is to be used. Since the intersection with all the $e_i = +1$ is empty, each ν_α gives zero mass to one of the sets $\partial \mathcal{R}_i$. Using this partition \mathcal{R}_i and the argument of Lemma 4.2 to show that (1) holds for ν_α completes the proof.

4.4. Theorem. *The system (M, f, μ) has a rate function given by*

$$K(\nu) = \begin{cases} -h_\nu(f) - \int \phi^u d\nu & \text{if } \nu \in \mathcal{M}_f(M), \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. Since $\mu[L_n \in A] = \int \mu_z^\xi[L_n \in A] d\mu(z)$, and for A closed (open) Lemma 4.2 gives upper (lower) bounds for $\mu_z^\xi[L_n \in A]$ which hold uniformly for z outside a μ -null set, the existence of the rate function follows. Also, the formula for $K(\nu)$ is a consequence of the two preceding lemmas.

4.5. Lemma. *m_x^* and $\mu(\cdot|\xi)(x)$ are equivalent, uniformly for x in a set of full μ -measure.*

Proof. A proof of this lemma appears in Ledrappier [L], where weaker hypotheses than ours are assumed. One need only verify that Ledrappier's proof provides a uniform equivalence in the case of Anosov diffeomorphism.

Define a metric on $\mathcal{M}(M)$ by selecting a sequence $\{\phi_j\}$ in the unit ball of $C(M, \mathbf{R})$ and setting, for μ, ν in $\mathcal{M}(M)$,

$$\rho(\mu, \nu) = \sum_{j=0}^{\infty} 2^{-j} |\mu(\phi_j) - \nu(\phi_j)|.$$

If $A \subset \mathcal{M}(M)$ is closed, write, for $\delta > 0$, $A^\delta = \{\nu : \text{dist}(\nu, A) \leq \delta\}$. If $A \subset \mathcal{M}(M)$ is open and $\delta > 0$, set $A_\delta = \{\nu : \text{dist}(\nu, (\mathcal{M}(M) \setminus A)) > \delta\}$.

Recall that $C_\xi(x)$ denotes the set in ξ containing x , and is uniquely defined for $x \notin \partial\mathcal{R}$. If x, x' lie in the same rectangle R_i of \mathcal{R} then the map $h_{x,x'}$ takes $C_\xi(x)$ to $C_\xi(x')$ and is defined by $h_{x,x'}(y) = [y, x']$.

4.6. Lemma. *For each $\delta > 0$ there is an N so that if x and x' belong to $\text{int}(R_i)$ for some i , and $n > N$ then:*

- (i) $h_{x',x}([z : L_n(z) \in A^\delta]) \cap C_\xi(x') \supset [z : L_n(z) \in A] \cap C_\xi(x)$ and
- (ii) $h_{x',x}([z : L_n(z) \in A_\delta] \cap C_\xi(x')) \subset [z : L_n(z) \in A] \cap C_\xi(x)$

where in (i) A is closed and in (ii) A is open.

Proof. Postponed to the end of this section.

PROOF OF THEOREM 4.1. Since $\mu(R_i) > 0$ for each i , the sets $R_i \cap V$ are all nonempty. (The set V is defined in Proposition 4.2.) For each closed set $A \subset \mathcal{M}(M)$ and $x \in M$, select a point $x' \in V$ which is in the interior of some rectangle R_i containing x . Denoting $\{y : L_n(y) \in A\}$ by $[L_n \in A]$, we have for $\delta > 0$ and n sufficiently large,

$$\begin{aligned} m([L_n \in A] | \xi)(x) &\leq K_1 m_x^*[L_n \in A], \quad \text{by equivalence of } m | \xi \text{ and } m^*, \\ &\leq K_1 m_x^*(h_{x',x}[L_n \in A^\delta]), \quad \text{by Lemma 4.6(i),} \\ &\leq K_1 K_2 m_{x'}^*[L_n \in A^\delta], \quad \text{by Anosov's Theorem,} \\ &\leq K_1 K_2 K_3 \mu([L_n \in A^\delta] | \xi)(x'), \quad \text{by Lemma 4.5.} \end{aligned}$$

Here the constants K_i are independent of x and x' . Because $x' \in V$ and K is a rate function for $(M, f, \mu(\cdot | \xi)(x'))$ we have, for each $\delta, \varepsilon > 0$ and n sufficiently large, that

$$m([L_n \in A] | \xi)(x) \leq \exp(-n(\inf\{K(\nu) : \nu \in A^\delta\} + \varepsilon)).$$

Thus, for each $\delta, \varepsilon > 0$,

$$\begin{aligned} \limsup n^{-1} \log m[L_n \in A] &= \limsup n^{-1} \log \int m_x^\xi[L_n \in A] dm(x) \\ &\leq -\inf\{K(\nu) : \nu \in A^\delta\} + \varepsilon. \end{aligned}$$

Since $\inf_{\delta > 0} \{-\inf\{K(\nu) : \nu \in A^\delta\}\} \leq -\inf\{K(\nu) : \nu \in A\}$, by the lower semi-continuity of K , this last inequality shows that K is a lower deviation function for the system (M, f, m) .

If $B \subset \mathcal{M}(M)$ is open, we argue as above: If $\delta > 0$ and n is sufficiently large,

$$\begin{aligned} m([L_n \in B] \mid \xi)(x) &\geq K_1^{-1} m_x^*[L_n \in B] \\ &\geq K_1^{-1} m_x^*(h_{x',x}[L_n \in B_\delta]) \\ &\geq K_1^{-1} K_2^{-1} m_{x'}^*[L_n \in B_\delta] \\ &\geq (K_1 K_2 K_3)^{-1} \mu([L_n \in B_\delta] \mid \xi)(x'). \end{aligned}$$

Take $x' \in V$ and integrate $m(\cdot \mid \xi)$ as above to conclude that for each $\delta > 0$, $\liminf n^{-1} \log m[L_n \in B] \geq -\inf\{K(\nu) : \nu \in B_\delta\}$. Since $\inf_{\delta>0} \inf_{B_\delta}(K(\nu)) \leq \inf_B(K(\nu))$, this last inequality shows that K is an upper deviation function for (M, f, m) as well, completing the proof of Theorem 4.1.

PROOF OF LEMMA 4.6. The proof depends on the following estimate (see [B]): There exist constants $C > 0$ and $\gamma \in (0, 1)$ so that if $y \in W_\epsilon^s(x)$ and $j > 0$, then

$$d(f^j(y), f^j(x)) < \gamma^j C d(y, x).$$

The proof of the assertions of the lemma reduce to proving that $\rho(L_n(x), L_n(y))$ goes to zero uniformly for $x \in M$ and $y \in W_\epsilon^s(x)$. Using the triangle inequality one obtains the estimate:

$$\begin{aligned} \rho(L_n(x), L_n(y)) &\leq n^{-1} \sum_{k=0}^T \sum_{j=0}^M 2^{-j} |\phi_j(f^k(x)) - \phi_j(f^k(y))| \\ &\quad + n^{-1} \sum_{k=T+1}^{n-1} \sum_{j=0}^M 2^{-j} |\phi_j(f^k(x)) - \phi_j(f^k(y))| \\ &\quad + n^{-1} \sum_{k=0}^{n-1} \sum_{j=M+1}^\infty 2^{-j} |\phi_j(f^k(x)) - \phi_j(f^k(y))|. \end{aligned}$$

First select M so the last term is small. Then select δ so that $|f_j(t) - \phi_j(s)|$ is small when $d(s, t) < \delta$ for $j = 0, 1, 2, \dots, M$. Finally pick T so that $d(f^j(x), f^j(y)) < \delta$ when $x \in M$, $y \in W_\epsilon^s(x)$ and $j > T$. This makes the second term small. Then pick n so the first term is small.

4.7. Corollary. *For each continuous $\phi: M \mapsto \mathbf{R}$ there is a function $K^\phi: \mathbf{R} \mapsto [0, \infty]$ which is a rate function for the convergence of the trajectory averages $n^{-1} \sum_0^{n-1} \phi(f^j(x))$. In particular, for each ϕ and $\epsilon > 0$,*

$$m \left\{ x: \left| n^{-1} \sum_0^{n-1} \phi(f^j(x)) - \int \phi d\mu \right| > \epsilon \right\} \text{ decays exponentially as } n \mapsto \infty.$$

Proof. Theorem 4.1 and the ‘‘homomorphism theorem,’’ Proposition 2.1, followed by an appropriate selection of the set A in the definition of deviation function.

Remark. Theorem 4.1 provides more evidence in favor of a conjecture by Takahashi [T] which states that a system is structurally stable if it has a rate function which is an affine map.

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