Relations between $H^p_u$ and $L^p_u$

in a product space

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Abstract. Relations between $L^p_u$ and $H^p_u$ are studied for the product space $\mathbb{R}^1 \times \mathbb{R}^1$ in the case $1 < p < \infty$ and $u(x_1, x_2) = |Q_1(x_1)|^p |Q_2(x_2)|^p w(x_1, x_2),$ where $Q_1$ and $Q_2$ are polynomials and $w$ satisfies the $A_p$ condition for rectangles. A description of the distributions in $H^p_u$ is given. Questions about boundary values and about the existence of dense subsets of smooth functions satisfying appropriate moment conditions are also considered.

1. Introduction

In this paper we study in a product space setting the problem of identifying a weighted Hardy space $H^p_u$ with the corresponding weighted Lebesgue space $L^p_u,$ $1 < p < \infty.$ We restrict our attention to the special product $\mathbb{R}^1 \times \mathbb{R}^1$ and consider weight functions $u$ of the form

$$u(x_1, x_2) = |Q_1(x_1)|^p |Q_2(x_2)|^p w(x_1, x_2), \quad x_1, x_2 \in \mathbb{R}^1,$$

where $Q_1$ and $Q_2$ are polynomials in $x_1$ and $x_2,$ respectively, and $w$ satisfies the $A_p$ condition for rectangles, i.e., $w \geq 0$ and there is a finite constant $c$ such that

$$\left( \frac{1}{|R|} \iint_R w(x_1, x_2) \, dx_1 \, dx_2 \right)^{1/p} \leq c$$

for all rectangles $R$ with sides parallel to the coordinate axes.

Our main results are that, for such $u,$ $H^p_u$ and $L^p_u$ can be identified with equivalence of norms provided all the roots of $Q_1$ and $Q_2$ are real, while if there are complex roots, $H^p_u$ can be identified with a subspace of $L^p_u$ consisting of functions for which certain moments vanish. Similar results in the non-product case are given in [7 and 1]. These have had applications to fractional integrals and Sobolev embedding theorems [2], as well as to Fourier transform norm inequalities [5, 8]. Weights $u$ of the form above are also of interest since they do not satisfy the $A_p$ condition for rectangles; in fact, $u^{-1/(p-1)}$ is generally not locally integrable in either variable.

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By $L^p_u$, we mean the space

$$L^p_u = \left\{ f : \|f\|_{L^p_u} = \left( \int_{\mathbb{R}^2} |f(x_1, x_2)|^p u(x_1, x_2) \, dx_1 \, dx_2 \right)^{1/p} < \infty \right\}.$$  

We shall often write $\int \int$ for $\int \int_{\mathbb{R}^2}$.

To define $H^p_u$, let $\mathcal{S} = \mathcal{S}(\mathbb{R}^2)$ denote the Schwartz class of rapidly decreasing functions on $\mathbb{R}^2$, and let $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^2)$ be the dual class of tempered distributions. For $l \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$, let $N(l) = N_{\varphi}(l)$ denote the (product) nontangential maximal function defined by

$$N(l)(x_1, x_2) = \sup_{(\xi_1, \xi_2) \in \Gamma (x_1, x_2)} \{ |l(x_1, x_2)|, (x_1, x_2) \in \Gamma (x_1, x_2) \}$$

where for $i = 1, 2$, $\Gamma_i(x_i)$ denotes the “cone” in $\mathbb{R}^2_+$ of points $(\xi_i, t_i)$ with $|\xi_i - x_i| < \gamma_i t_i$, $\gamma_i > 0$, $\varphi_{t_1, t_2}(x_1, x_2) = t_1^{-1} t_2^{-1} \varphi(x_1/t_1, x_2/t_2)$, and $\langle l, \varphi \rangle$ denotes the action of $l$ on $\varphi$. Then, by definition, $H^p_u$ is the collection of all $l \in \mathcal{S}'$ such that $N(l) \in L^p_u$ for some $\gamma_1, \gamma_2 > 0$ and some $\varphi$ with $\int \int \varphi \neq 0$. All the weights $u$ which we will consider belong to the class $A_{\infty} = \bigcup_{p>1} A_p$. The condition that $l \in H^p_u$ is then independent of the particular choice of $\gamma_1, \gamma_2$ and $\varphi$; in fact, as we shall see in Lemma (2.6), this is true if $u$ merely satisfies the doubling condition in each variable uniformly in the other variable. We then set

$$\|l\|_{H^p_u} = \|N(l)\|_{L^p_u}$$

for some particular choice of $\gamma_1, \gamma_2 > 0$ and some $\varphi \in \mathcal{S}$ with $\int \int \varphi \neq 0$.

To describe how $H^p_u$ and $L^p_u$ are related, we consider a polynomial $Q(x)$, $x \in \mathbb{R}^1$, normalized so that

$$Q(x) = \prod_{k=1}^n (x - a_k)^{\mu_k},$$

where $\{a_k\}$ are the distinct roots of $Q$, $\mu_k$ is the multiplicity of $a_k$, and $Q$ has degree $N = \sum \mu_k$. Assuming that all the roots are real, we consider the partial fraction decomposition of $1/Q$ given by

$$\frac{1}{Q(x)} = \sum_{k=1}^n \sum_{l=1}^{\mu_k} \frac{A_{k,l}}{(x - a_k)^l}.$$  

With this decomposition, we associate the distribution

$$\mathcal{D}^Q = \sum_{k=1}^n \sum_{l=1}^{\mu_k} \frac{A_{k,l}}{(l-1)!} \delta_{a_k}^{(l-1)},$$

where $\delta_{a}^{(j)}$ denotes the $j$th derivative of the delta function at $a$. (See [7].) If $F(x, y)$ is a function of two variables, $\mathcal{D}^Q F(x, y)$ will denote the action of $\mathcal{D}^Q$ on $F$ as a function of $y$; the result is a function of $x$. 

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It is easy to check that
\[ D_y^Q \left( \frac{1}{x - y} \right) = \frac{1}{Q(x)}, \quad \text{i.e.,} \quad Q(x) D_y^Q \left( \frac{1}{x - y} \right) = 1. \]
If \( \varphi(x) \in \mathcal{S}(\mathbb{R}^1) \), let \( \mathcal{P}_\varphi^Q(x) \) be the interpolating polynomial defined by
\[ \mathcal{P}_\varphi^Q(x) = Q(x) D_y^Q \left( \frac{\varphi(y)}{x - y} \right), \quad x \in \mathbb{R}^1. \]
(See [7].) The degree of \( \mathcal{P}_\varphi^Q \) is \( N - 1 \), and the derivatives of \( \varphi \) and \( \mathcal{P}_\varphi^Q \) up to order \( \mu_k - 1 \) at \( a_k \) are the same for \( k = 1, \ldots, n \). One of the main results of [7] is that if \( 1 < p < \infty \) and \( u(x) = |Q(x)|^p w(x) \) with \( w \in A^p(\mathbb{R}^1) \), then \( H_u^p = L_u^p \); in fact, there is a unique correspondence between distributions \( l \in H_u^p \) and functions \( f \in L_u^p \) given by
\[ \langle l, \varphi \rangle = \int_{-\infty}^{\infty} f(x) \left[ \varphi(x) - \mathcal{P}_\varphi^Q(x) \right] dx, \quad \varphi \in \mathcal{S}(\mathbb{R}^1), \]
and \( ||l||_{H_u^p} \approx ||f||_{L_u^p} \).

Now let \( Q_1(x_1) \) and \( Q_2(x_2) \) by polynomials of degree \( N_1 \) and \( N_2 \) all of whose roots are real. If \( \varphi(x_1, x_2) \in \mathcal{S}(\mathbb{R}^2) \), let
\[ (1.1) \quad \mathcal{P}_\varphi^{Q_1, Q_2}(x_1, x_2) = Q_1(x_1) Q_2(x_2) D_{y_1}^{Q_1} D_{y_2}^{Q_2} \left[ \frac{\varphi(y_1, x_2) + \varphi(x_1, y_2) - \varphi(y_1, y_2)}{(x_1 - y_1)(x_2 - y_2)} \right]. \]
To see how this function arises naturally in the product space setting, suppose that \( \varphi \) is the product of one-dimensional functions: \( \varphi(x_1, x_2) = \varphi_1(x_1) \varphi_2(x_2) \). A simple computation then gives
\[ (1.2) \quad \varphi(x_1, x_2) - \mathcal{P}_\varphi^{Q_1, Q_2}(x_1, x_2) = [\varphi_1(x_1) - \mathcal{P}_\varphi^{Q_1}(x_1)] [\varphi_2(x_2) - \mathcal{P}_\varphi^{Q_2}(x_2)]. \]
Although this formula holds only when \( \varphi \) is a product, the form of the right-hand side together with the result mentioned above from [7] motivate the definition of \( \mathcal{P}_\varphi^{Q_1, Q_2} \) for general \( \varphi \). Note that \( \mathcal{P}_\varphi^{Q_1, Q_2} \) is not a polynomial even if \( \varphi \) is a product.

We can now state our main results. To simplify notation, it will be convenient to adopt a few conventions. We write
\[ x = (x_1, x_2), \quad z = (z_1, z_2), \quad \text{etc.,} \quad x_i, z_i \in \mathbb{R}, \quad i = 1, 2; \]
\[ t = (t_1, t_2), \quad t_1, t_2 > 0; \]
\[ (x, t) = (x_1, t_1; x_2, t_2) \in \mathbb{R}^2 \times \mathbb{R}^2_+; \]
\[ \Gamma(x) = \Gamma_1(x_1) \times \Gamma_2(x_2) = \{ (\xi, t) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+ : (\xi, t_1) \in \Gamma_1(x_1), \quad i = 1, 2 \}; \]
\[ f(x) = f(x_1, x_2), \quad \int f(x) dx = \int \int f(x_1, x_2) dx_1 dx_2; \]
\[ \varphi_t(x) = t_1^{-1} t_2^{-1} \varphi(x_1/t_1, x_2/t_2); \quad \mathcal{P}_\varphi^Q(x) = \mathcal{P}_\varphi^{Q_1, Q_2}(x_1, x_2); \]
\[ |Q(x)|^p = |Q_1(x_1) Q_2(x_2)|^p. \]
We will prove the following results.

**Theorem 1.** Let $1 < p < \infty$ and $u = |Q|^p w$ where $Q_1(x_1)$ and $Q_2(x_2)$ are polynomials of degrees $N_1$ and $N_2$, respectively, with all real roots, and $w \in A_p$ for rectangles. Let

$$f(x, t) = \int f(z)[\varphi(x - z) - \mathcal{R}_Q \varphi(x - ,)(z)] \, dz$$

where $\varphi$ has the property that

$$(1 + |x_1|)^{j_1+M}(1 + |x_2|)^{j_2+M} \left| \left( \frac{\partial}{\partial x_1} \right)^{j_1} \left( \frac{\partial}{\partial x_2} \right)^{j_2} \varphi(x_1, x_2) \right|$$

is bounded for $0 \leq j_1 \leq N_1$, $0 \leq j_2 \leq N_2$, and some $M > 1$. If

$$N(f)(x) = \sup_{(\xi, t) \in \Gamma(x)} |f(\xi, t)|,$$

then there is a constant $c$ independent of $f$ such that $\|N(f)\|_{L^p_u} \leq c\|f\|_{L^p_u}$.

**Theorem 2.** Let $1 < p < \infty$ and $u = |Q|^p w$ where $Q_1$ and $Q_2$ are polynomials with all real roots and $w \in A_p$ for rectangles. Then $H^p_u$ and $L^p_u$ can be identified in the following sense: there is a unique correspondence between distributions $l \in H^p_u$ and functions $f \in L^p_u$ given by

$$\langle l, \varphi \rangle = \int f(z)[\varphi(z) - \mathcal{R}^Q \varphi(z)] \, dz, \quad \varphi \in \mathcal{S}(\mathbb{R}^2).$$

Moreover, in this correspondence, $\|l\|_{H^p_u}$ and $\|f\|_{L^p_u}$ are equivalent in the sense that $c_1\|l\|_{L^p_u} \leq \|l\|_{H^p_u} \leq c_2\|f\|_{L^p_u}$ for positive constants $c_1$ and $c_2$ which are independent of $f$ and $l$.

In defining $\mathcal{R}^Q$ and stating Theorems 1 and 2, we have assumed tacitly that the degrees of both $Q_1$ and $Q_2$ are positive, i.e., that neither $Q_1$ nor $Q_2$ is identically 1. If, e.g., $Q_1 \equiv 1$ and $Q_2 \neq 1$, we simply set

$$\mathcal{R}^Q \varphi(x) = Q_2(x_2) \mathcal{R}^Q (\varphi(x_1, y_2)/x_2 - y_2).$$

Similarly, if $Q_1 \equiv Q_2 \equiv 1$, we set $\mathcal{R}^Q \equiv 0$.

Before stating a result which deals with the situation when $Q_1, Q_2$ have complex roots, we mention two facts which can be derived from the estimates used to prove Theorem 1.

**Theorem 3.** Let $p, u, \varphi, f$ and $f(x, t)$ be as in Theorem 1. Then for a.e. $x$,

$$f(\xi, t) \to f(x) \int \varphi$$

as $(\xi, t) \to x$ nontangentially, i.e., as $(\xi_i, t_i) \to (x_i, 0)$ in such a way that $(\xi, t_i) \in \Gamma_i(x_i)$, $i = 1, 2$.

**Theorem 4.** Let $1 < p < \infty$ and $u = |Q|^p w$ where $Q_1$ and $Q_2$ have only real roots and $w \in A_p$ for rectangles. Then the class of Schwartz functions whose
Fourier transforms have compact support not containing either axis is dense in $L^p_u$. In particular, the class of Schwartz functions $f$ satisfying
\[
\int_{-\infty}^{\infty} f(x_1, x_2) x_1^{k_1} \, dx_1 = \int_{-\infty}^{\infty} f(x_1, x_2) x_2^{k_2} \, dx_2
\]
\[
= \int \int f(x_1, x_2) x_1^{k_1} x_2^{k_2} \, dx_1 \, dx_2 = 0
\]
for $k_1, k_2 = 0, 1, 2, \ldots$ is dense in $L^p_u$.

For the case of complex roots, note that if $q_1(x_1)$ is a polynomial which has a total number $d$ of complex roots, then
\[
|q_1(x_1)| \approx (1 + |x_1|)^d |Q_1(x_1)|
\]
where $Q_1$ is a polynomial with only real roots.

**Theorem 5.** Let $1 < p < \infty$, $d_1$ and $d_2$ be positive integers and
\[
u = (1 + |x_1|)^{d_1 p} (1 + |x_2|)^{d_2 p} |Q|^p w
\]
where $Q = Q_1(x_1) Q_2(x_2)$, $Q_1$ and $Q_2$ are polynomials with only real roots, and $w \in A_p$ for rectangles. Then $H^p_u$ can be identified with the subspace of $L^p_u$ which consists of those $f \in L^p_u$ satisfying
\[
\int_{-\infty}^{\infty} f(x_1, x_2) Q_1(x_1) x_1^{k_1} \, dx_1 = \int_{-\infty}^{\infty} f(x_1, x_2) Q_2(x_2) x_2^{k_2} \, dx_2 = 0
\]
for a.e. $x_2$ and a.e. $x_1$, respectively, and for $k_1 = 0, 1, \ldots, d_1 - 1$ and $k_2 = 0, 1, \ldots, d_2 - 1$. The identification is given by $l \equiv f$, $l \in H^p_u$, $f \in L^p_u$, where
\[
\langle l, \varphi \rangle = \int f(z) [\varphi(z) - H^p_u \varphi] \, dz, \quad \varphi \in \mathcal{S}(\mathbb{R}^2),
\]
and $\|l\|_{H^p_u} \approx \|f\|_{L^p_u}$.

In case, e.g., $d_1 = 0$, the result remains true for the space of functions $f \in L^p_u$ with
\[
\int_{-\infty}^{\infty} f(x_1, x_2) Q_2(x_2) x_2^{k_2} \, dx_2 = 0, \quad k_2 = 0, 1, \ldots, d_2 - 1.
\]

The proofs of Theorems 1–5 are given in §§4–8, respectively. In §2, we list as lemmas a few facts which will be needed in the proofs. We also show why the definition of $H^p_u$ is independent of the cone apertures and of the convolution function $\varphi$. In §3, we derive the basic kernel estimates which are needed to prove Theorem 1.

2. Preliminaries

We shall use a few known facts about $A_p$ weights. For example, it is a familiar fact that $w(x_1, x_2) \in A_p$ for rectangles (with sides parallel to the axes)
if and only if \( w \in A_p(\mathbb{R}^1) \) in each variable uniformly in the other—i.e., iff for a.e. \( x_1 \) and every interval \([a, b] \subset \mathbb{R}^1\),

\[
\left( \frac{1}{b-a} \int_a^b w(x_1, x_2) \, dx_2 \right) \left( \frac{1}{b-a} \int_a^b w(x_1, x_2)^{-1/(p-1)} \, dx_2 \right)^{p-1} \leq C
\]

for a finite constant \( C \) which is independent of both \( x_1 \) and \([a, b]\), as well as a similar statement with the roles of \( x_1 \) and \( x_2 \) interchanged. The least constant \( C \) for which such an inequality holds is called the \( A_p \) constant of \( w \).

The following two lemmas are special cases of the one-dimensional Hardy’s inequality derived in [3].

**Lemma (2.1).** If \( 1 < p < \infty \), \( v \in A_p(\mathbb{R}^1) \) and \( a \) is real, then

\[
\int_{-\infty}^{\infty} \left( \int_{|r-a| \geq |s-a|} |f(r)| \, dr \right)^p v(s) \, ds \leq c \int_{-\infty}^{\infty} |f(s)|^p v(s) \, ds
\]

with \( c \) equal to a constant multiple independent of \( a, f \) and \( v \) of the \( A_p \) constant of \( v \).

**Lemma (2.2).** If \( 1 < p < \infty \), \( v \in A_p(\mathbb{R}^1) \) and \( a \) is real, then

\[
\int_{-\infty}^{\infty} \left( \frac{1}{|s-a|} \int_{|r-a| \leq |s-a|} |f(r)| \, dr \right)^p v(s) \, ds \leq c \int_{-\infty}^{\infty} |f(s)|^p v(s) \, ds
\]

with \( c \) equal to a constant multiple independent of \( a, f \) and \( v \) of the \( A_p \) constant of \( v \).

In this form, Lemma (2.2) also follows immediately from [4].

We shall use the following known fact (see Lemma (5.4) of [7]) about distributions on \( \mathbb{R}^1 \).

**Lemma (2.3).** Let \( l \in \mathcal{S}'(\mathbb{R}^1) \) be a distribution with compact support such that \( l \in H^p_{|Q|^{1/p} w} \) for some \( p \), \( 1 < p < \infty \), where \( Q \) is a polynomial on \( \mathbb{R}^1 \) of degree \( N \) with all real roots, and \( w \in A_p(\mathbb{R}^1) \). Then \( \langle l, x^k \rangle = 0 \) for \( k = 0, 1, \ldots, N-1 \). In particular, if \( \langle l, \varphi \rangle = \langle l, \varphi_Q \rangle \) for all \( \varphi \in \mathcal{S}(\mathbb{R}^1) \), then \( l \equiv 0 \).

We say that \( u(x_1, x_2) \) has doubling order \( \nu_1 \) in the \( x_1 \)-variable (uniformly in \( x_2 \)) if for any intervals \( I \subset J \subset \mathbb{R}^1 \) and any \( x_2 \),

\[
\int_J u(x_1, x_2) \, dx_1 \leq c \left( \frac{|J|}{|I|} \right)^{\nu_1} \int_I u(x_1, x_2) \, dx_1
\]

with \( c \) independent of \( I, J \) and \( x_2 \). A similar terminology applies to the other variable.

The purpose of the next three lemmas is to show that the definition of (product) \( H^p_u \) is independent of the cone apertures \( \gamma_1, \gamma_2 \) and of the particular convolution function \( \varphi \), provided that \( u \) is doubling in each variable uniformly
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in the other variable. We shall use the notation

\[ N_{\gamma_1, \gamma_2, \varphi}(l)(x_1, x_2) = \sup_{|\xi_1 - x_1| < \gamma_1 l, |\xi_2 - x_2| < \gamma_2 l} |(l * \varphi_{l_1, l_2})(\xi_1, \xi_2)|. \]

Lemma (2.4). Let \( u(x_1, x_2) \) satisfy the doubling condition in each variable uniformly in the other, and let \( \nu_1 \) and \( \nu_2 \) be the doubling orders of \( u \) in the \( x_1 \) and \( x_2 \) variables, respectively. Then for \( k_1, k_2 = 0, 1, 2, \ldots \) and \( 0 < p < \infty \),

\[ \|N_{2^{k_1} \gamma_1, 2^{k_2} \gamma_2, \varphi}(l)\|_{L_p^u} \leq c 2^{\nu_1 k_1 + \nu_2 k_2} \|N_{\gamma_1, \gamma_2, \varphi}(l)\|_{L_p^u} \]

with \( c \) independent of \( k_1, k_2, \gamma_1, \gamma_2, l \) and \( \varphi \).

Proof. For fixed \( \alpha > 0 \), let

\[ E = \{ x : N_{2^{k_1} \gamma_1, 2^{k_2} \gamma_2, \varphi}(l)(x) > \alpha \}, \]
\[ F = \{ x : N_{2^{k_1} \gamma_1, 2^{k_2} \gamma_2, \varphi}(l)(x) > \alpha \}, \]
\[ G = \{ x : N_{\gamma_1, \gamma_2, \varphi}(l)(x) > \alpha \}. \]

We shall use the notation \( u(E) \) for \( \int_E u \, dx \). It is enough to show that \( u(E) \leq c 2^{\nu_1 k_1 + \nu_2 k_2} u(G) \) with \( c \) independent of \( \alpha, k_1, k_2, \gamma_1, \gamma_2, l \) and \( \varphi \). We shall do this by showing that \( u(E) \leq c 2^{\nu_1 k_2} u(F) \) and \( u(F) \leq c 2^{\nu_1 k_1} u(G) \) for such \( c \).

If \( (x_1^0, x_2^0) \in E \), there is a rectangle

\[ R = (|\xi_1 - x_1^0| < 2^{k_1} \gamma_1 l, 1) \times (|\xi_2 - x_2^0| < 2^{k_2} \gamma_2 l, 2) = R_1 \times R_2 \]

such that \( |(l * \varphi_{l_1, l_2})(\xi_1^0, \xi_2^0)| > \alpha \) at some point \( (\xi_1^0, \xi_2^0) \in R \). By definition of \( N_{2^{k_1} \gamma_1, 2^{k_2} \gamma_2, \varphi}(l) \), it follows that \( N_{2^{k_1} \gamma_1, 2^{k_2} \gamma_2, \varphi}(l) > \alpha \) on the entire rectangle

\[ S = (|x_1 - x_1^0| < 2^{k_1} \gamma_1 l, 1) \times (|x_2 - x_2^0| < \gamma_2 l, 2) = S_1 \times S_2, \]

i.e., \( S \subset F \). Therefore, since \( x_1^0 \in S_1 \), we see that \( (x_1^0, x_2^0) \in F \) if \( x_2 \in S_2 \).

Note also that \( |S_2| = 2^{-k_2} |R_2| \) and consequently, for some \( c > 0 \),

\[ \int_{S_2} u(x_1^0, x_2) \, dx_2 \geq c 2^{-k_2} \int_{R_2} u(x_1^0, x_2) \, dx_2 \]

uniformly in \( x_1 \) by the doubling property of \( u \). Moreover, \( S_2 \subset 2R_2 \). Thus,

\[ \int_{2R_2} \chi_F(x_1^0, x_2) u(x_1^0, x_2) \, dx_2 \geq \int_{2R_2} u(x_1^0, x_2) \, dx_2 \]

\[ \geq c 2^{-k_2} \int_{S_2} u(x_1^0, x_2) \, dx_2 \]

\[ = 2^{-k_2} \int_{S_2} u(x_1^0, x_2) \, dx_2 \]

If we let \( M_u^{(2)} \) denote the one-dimensional Hardy-Littlewood maximal function with respect to \( u \) in the second variable, i.e.,

\[ M_u^{(2)}(g)(x_1, x_2) = \sup_{l_2} \int_{l_2} \int_{l_2} |g(x_1, x_2)| u(x_1, x_2) \, dx_2 / \int_{l_2} u(x_1, x_2) \, dx_2, \]
it follows since $x_2^0 \in 2R_2$ that
\[ M_{\nu}^{(2)}(\chi_F)(x_1^0, x_2^0) \geq c 2^{-k_2 \nu_2}. \]

Therefore, $E \subset \{(x_1, x_2) : M_{\nu}^{(2)}(\chi_F)(x_1, x_2) \geq c 2^{-k_2 \nu_2}\}$, and
\[ u(E) \leq \iint_{\{(x_1, x_2) : M_{\nu}^{(2)}(\chi_F)(x_1, x_2) \geq c 2^{-k_2 \nu_2}\}} u(x_1, x_2) \, dx_2 \, dx_1 \leq c 2^{k_2 \nu_2} \iint \chi_F(x_1, x_2) u(x_1, x_2) \, dx_2 \, dx_1 = c 2^{k_2 \nu_2} u(F) \]

by weak-type $(1, 1)$ for the one-dimensional maximal function. This proves the first of the desired inequalities. The argument showing that $F \subset \{(x_1, x_2) : M_{\nu}^{(1)}(\chi_F)(x_1, x_2) > c 2^{-k_1 \nu_1}\}$, and consequently that $u(F) \leq c 2^{k_1 \nu_1} u(G)$, is similar. This proves the lemma.

Lemma (2.5). Let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^2)$, $\int \varphi \neq 0$, and let $\lambda$ be a positive constant. Then
\[ N_{1,1,\psi}(l)(x) \]
\[ \leq c \sup_{(\xi, \epsilon) \in \mathbb{R}^2} |(l * \varphi, \epsilon)(\xi_1, \xi_2)| \left(1 + \frac{|x_1 - \xi_1|}{t_1}\right)^{-\lambda} \left(1 + \frac{|x_2 - \xi_2|}{t_2}\right)^{-\lambda} \]

with $c$ independent of $l$ and $x$.

Proof. If we add the restriction $t_1 = t_2$ to both sides above, the resulting inequality follows from the known inequality (see, e.g., [6])
\[ \sup_{(\xi, t) : |\xi-x|<2t} |(l * \psi, t)(\xi)| \leq c \sup_{(\xi, t) \in \mathbb{R}^2} |(l * \varphi, t)(\xi)| \left(1 + \frac{|x - \xi|}{t}\right)^{-2\lambda}, \]

$x = (x_1, x_2), \xi = (\xi_1, \xi_2)$, since $|x_1 - \xi_1| < t$ and $|x_2 - \xi_2| < t$ clearly implies that $|x - \xi| < 2t$, and for all $x, \xi$ and $\lambda > 0$,
\[ \left(1 + \frac{|x - \xi|}{t}\right)^{-2\lambda} \leq \left(1 + \frac{|x_1 - \xi_1|}{t}\right)^{-\lambda} \left(1 + \frac{|x_2 - \xi_2|}{t}\right)^{-\lambda} \leq \left(1 + \frac{|x - \xi|}{t}\right)^{-\lambda}. \]

The constant $c$ above is independent of $l$ and $x$.

Next, for $a > 0$, if we apply this special case at the point $(x_1/a, x_2)$ to the dilated distribution $l_a$ defined by
\[ \langle l_a, \theta \rangle = \left\langle l, \frac{1}{a} \theta \left(\frac{\cdot}{a}, \cdot\right)\right\rangle, \quad \theta \in \mathcal{S}(\mathbb{R}^2), \]
and note that \((l \ast \theta_{t, i})(\xi_1, \xi_2) = (l \ast \theta_{at, t})(a\xi_1, a\xi_2)\), we easily obtain

\[
\sup_{\xi_1, \xi_2} |(l \ast \psi_{at, t})(\xi_1, \xi_2)| \leq c \sup_{R^2} |(l \ast \varphi_{at, t})(\xi_1, \xi_2)| \left(1 + \frac{|\xi_1 - \xi_1|}{at}\right)^{-\lambda} \left(1 + \frac{|\xi_2 - \xi_2|}{t}\right)^{-\lambda}.
\]

Since \(c\) is independent of \(a\), this is equivalent to the desired inequality.

**Lemma (2.6).** Let \(u(x_1, x_2)\) satisfy the doubling condition in each variable uniformly in the other. Let \(\varphi, \psi \in \mathcal{S}(\mathbb{R}^2), \int \varphi \neq 0\), and let \(\gamma_1, \gamma_2, \alpha_1, \alpha_2 > 0\). Then

\[
\|N_{\gamma_1, \gamma_2, \varphi}(l)\|_{L^p_\alpha} \leq c \|N_{\alpha_1, \alpha_2, \varphi}(l)\|_{L^p_\alpha}, \quad 0 < p < \infty,
\]

with \(c\) independent of \(l\).

**Proof.** By Lemma (2.4), we may assume \(\gamma_1 = \gamma_2 = \alpha_1 = \alpha_2 = 1\). By Lemma (2.5), for \(\lambda > 0\),

\[
N_{1,1,\psi}(l)(x) \leq c \sum_{k_1, k_2=0}^{\infty} N_{2^k_1, 2^k_2, \varphi}(l)(x) \cdot 2^{-k_1+k_2}\lambda.
\]

Thus, by Lemma (2.4),

\[
\|N_{1,1,\psi}(l)\|_{L^p_\alpha} \leq c \sum_{k_1, k_2=0}^{\infty} 2^{(k_1+1+k_2+1)} 2^{-(k_1+k_2)} \|N_{1,1,\varphi}(l)\|_{L^p_\alpha} = c \|N_{1,1,\varphi}(l)\|_{L^p_\alpha}
\]

provided we choose \(\lambda\) sufficiently large. This completes the proof.

**Lemma (2.7).** If \(d_1\) and \(d_2\) are nonnegative integers, then

\[
\mathcal{R}_\varphi^{x_1^d_1}Q_1(x_1, x_2) = \mathcal{R}_\varphi^{Q_1}(x_1, x_2) + Q_1(x_1)Q_2(x_2)
\]

\[
\times \{P_{d_1-1}(x_1)G_{d_2-1}(x_2) + P_{d_2-1}(x_2)G_{d_1-1}(x_1)\}
\]

where \(P_{d_1-1}, P_{d_2-1}\) are polynomials of degrees \(d_1 - 1, d_2 - 1\), respectively, and \(G_{d_1-1}, G_{d_2-1}\) are functions satisfying

\[
|G_{d_1-1}(x_1)| \leq c(1 + |x_1|^{d_1-1}), \quad |G_{d_2-1}(x_2)| \leq c(1 + |x_2|^{d_2-1}).
\]

The polynomials \(P_{d_1-1}, P_{d_2-1}\) as well as the functions \(G_{d_1-1}, G_{d_2-1}\) may depend on \(\varphi, Q_1\) and \(Q_2\). In case either \(d_1\) or \(d_2\) is 0, we interpret the corresponding polynomial to be 0.

**Proof.** We shall use the following formula (see Lemma (2.7) of [7]) for polynomials of a single variable \(x \in \mathbb{R}^1:\)

\[
(2.8) \quad \mathcal{R}_\varphi^{xQ} = \mathcal{R}_\varphi^Q + \mathcal{R}_\varphi^{xQ}(x) \cdot Q.
\]
Consider first the case \( d_1 = 1, \ d_2 = 0 \). We shall assume that \( \deg Q_1 > 0 \).

By definition,

\[
\mathcal{Q}_{\varphi}^{x_i} Q_1, Q_2(x_1, x_2) = x_1 Q_1(x_1) Q_2(x_2) \mathcal{D}^{x_i} Q_1 \mathcal{D}^{x_i} Q_2 \left\{ \frac{\varphi(x_1, y_2) + \varphi(y_1, x_2) - \varphi(y_1, y_2)}{(x_1 - y_1)(x_2 - y_2)} \right\}
\]

\[
= Q_2(x_2) \mathcal{D}^{x_i} Q_2 \left[ x_1 Q_1(x_1) \mathcal{D}^{x_i} Q_1 \left\{ \frac{\varphi(x_1, y_2) + \varphi(y_1, x_2) - \varphi(y_1, y_2)}{(x_1 - y_1)(x_2 - y_2)} \right\} \right]
\]

Now applying (2.8) to the term in square brackets for the variable \( x_1 \) and the function \( \psi(z) = (\varphi(x_1, y_2) + \varphi(z, x_2) - \varphi(z, y_2))/(x_2 - y_2) \), we obtain

\[
\mathcal{Q}_{\varphi}^{x_i} Q_1, Q_2(x_1, x_2) = Q_2(x_2) \mathcal{D}^{x_i} Q_2 \left[ x_1 Q_1(x_1) \mathcal{D}^{x_i} Q_1 \left\{ \frac{\varphi(x_1, y_2) + \varphi(y_1, x_2) - \varphi(y_1, y_2)}{(x_1 - y_1)(x_2 - y_2)} \right\} \right]
\]

\[
+ \mathcal{D}^{x_i} Q_1 \left( \frac{\varphi(x_1, y_2) + \varphi(y_1, x_2) - \varphi(y_1, y_2)}{(x_2 - y_2)} \right) \cdot Q_1(x_1)
\]

\[
= \mathcal{Q}_{\varphi}^{x_i} Q_1, Q_2(x_1, x_2) + Q_1(x_1) Q_2(x_2) \mathcal{D}^{x_i} Q_1 \mathcal{D}^{x_i} Q_1
\]

In the last term on the right above we may drop the first of the three terms in the numerator since

\[
\mathcal{D}^{x_i} Q_1 \left( \frac{\varphi(x_1, y_2)}{x_2 - y_2} \right) = 0;
\]

in fact, \( \varphi(x_1, y_2)/(x_2 - y_2) \) is independent of \( y_1 \), while \( \mathcal{D}^{x_i} Q_1 \) is a linear combination of derivatives with respect to \( y_1 \) of order at least one (since \( x_1 Q_1 \) has degree at least \( 2 \)). Moreover, for the remaining two terms we can write

\[
\mathcal{Q}_{\varphi}^{x_i} Q_1, Q_2(x_1, x_2) = \mathcal{Q}_{\varphi}^{x_i} Q_1 (\varphi(y_1, z)) \cdot \psi(z) = \mathcal{Q}_{\varphi}^{x_i} Q_1 (\varphi(y_1, z)) \cdot (\varphi(z, y_2)/(x_2 - y_2))
\]

where \( \psi(z) = \mathcal{D}^{x_i} Q_1 (\varphi(y_1, z)) \). It is easy to see that (2.9) is a function of \( x_2 \) alone, and that it is bounded in absolute value by a multiple of \( (1 + |x_2|)^{-1} \).

Thus, we have shown that

\[
\mathcal{Q}_{\varphi}^{x_i} Q_1, Q_2(x_1, x_2) = \mathcal{Q}_{\varphi}^{x_i} Q_1 (\varphi(y_1, z)) \cdot (\varphi(z, y_2)/(x_2 - y_2))
\]

with \( |H_{-1}(x_2)| \leq c(1 + |x_2|)^{-1} \).

This formula was derived in case \( \deg Q_1 > 0 \).

If \( \deg Q_1 = 0 \), then \( Q_1 \equiv 1 \) and

\[
\mathcal{Q}_{\varphi}^{x_i} Q_1(x_1, x_2) = Q_2(x_2) \mathcal{D}^{x_i} Q_1 \mathcal{D}^{x_i} Q_1 \left( \frac{\varphi(x_1, y_2) + \varphi(y_1, x_2) - \varphi(y_1, y_2)}{x_2 - y_2} \right)
\]

\[
= Q_2(x_2) \mathcal{D}^{x_i} Q_1 \left( \frac{\varphi(x_1, y_2)}{x_2 - y_2} \right) + Q_2(x_2) \mathcal{D}^{x_i} Q_1 \left( \frac{\varphi(y_1, x_2)}{x_2 - y_2} \right)
\]
The first term on the right equals $\mathcal{D}_{\psi}^{Q_1}(x_2)$ and the second term can be treated as above.

Analogous results hold for $\mathcal{D}_{\psi}^{Q_1, x_2Q_2}$.

For the general case, we argue by induction on $d_1, d_2$. Write

$$\mathcal{D}_{\psi}^{x_1^{d_1}Q_1, x_2^{d_2}Q_2} = \mathcal{D}_{\psi}^{x_1^{d_1-1}Q_1, x_2^{d_2}Q_2}$$

$$= \mathcal{D}_{\psi}^{x_1^{d_1-1}Q_1, x_2^{d_2}Q_2} + (x_1^{d_1-1}Q_1)(x_2^{d_2}Q_2)H_{-1}(x_2) \quad \text{(by 2.10)}$$

$$= \mathcal{D}_{\psi}^{Q_1, Q_2} + Q_1Q_2\{P_{d_1-2}(x_1)G_{d_2-1}(x_2) + P_{d_2-1}(x_2)G_{d_1-2}(x_1)\}$$

$$+ (x_1^{d_1-1}Q_1)(x_2^{d_2}Q_2)H_{-1}(x_2)$$

by applying the induction assumption to $\mathcal{D}_{\psi}^{x_1^{d_1-1}Q_1, x_2^{d_2}Q_2}$. We may rewrite the third term on the right as

$$(x_1^{d_1-1}Q_1)(x_2^{d_2}Q_2)H_{-1}(x_2) = Q_1Q_2x_1^{d_1-1}F_{d_1-1}(x_2)$$

where

$$F_{d_1-1}(x_2) = x_2^{d_2}H_{-1}(x_2) = O((1 + |x_2|)^{d_1-1})$$

and then combine this term with the other terms above to complete the proof of the lemma.

3. Kernel estimates

In this section, we derive the kernel estimates on which our proofs are based. As usual, we use the notation $Q(x) = Q(x_1, x_2) = Q_1(x_1)Q_2(x_2)$ where $Q_1$ and $Q_2$ are polynomials of degree $N_1$ and $N_2$, respectively, with all real zeros. We also write $D_{x_1}^{j_1} = \partial^{j_1}\partial^{x_1^{j_1}}$, etc.

Lemma (3.1). Let $\phi(x)$ be a function on $\mathbb{R}^2$ for which $D_{x_1}^{j_1}D_{x_2}^{j_2}\phi$ is bounded for $j_1 \leq N_1$ and $j_2 \leq N_2$, and let

$$c_\phi = \sup_{(x_1, x_2) \in \mathbb{R}^2, j_1 \leq N_1, j_2 \leq N_2} |D_{x_1}^{j_1}D_{x_2}^{j_2}\phi(x_1, x_2)|.$$

Then

$$|\phi(x) - \mathcal{D}_{\psi}^{Q}(x)| \leq cc_\phi \frac{|Q(x)|}{(1 + |x_1|)(1 + |x_2|)}, \quad x = (x_1, x_2),$$

with $c$ independent of $\phi$ and $x$.

Proof. Let $R$ be a positive number which is larger than the absolute value of every root of $Q_1$ and every root of $Q_2$. We consider four cases. In case both $|x_1| > 2R$ and $|x_2| > 2R$, write $\phi(x) - \mathcal{D}_{\psi}^{Q}(x)$ as

$$Q(x)\mathcal{D}_{x_1}^{Q_1} \mathcal{D}_{x_2}^{Q_2}\left\{\frac{\phi(x_1, x_2) - \phi(y_1, x_2) - \phi(x_1, y_2) + \phi(y_1, y_2)}{(x_1 - y_1)(x_2 - y_2)}\right\}.$$

By simply applying Leibniz's differentiation rule term-by-term, we see this is bounded in absolute value by $cc_\phi |Q(x)|/|x_1x_2|$. 

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In case \(|x_1| \leq 2R\) and \(|x_2| > 2R\), we write the expression in the form

\[
Q(x)\mathcal{D}_{y_1}^Q \mathcal{D}_{y_2}^Q \left\{ \frac{[(\varphi(x_1, x_2) - \varphi(y_1, x_2)) - (\varphi(x_1, y_2) - \varphi(y_1, y_2))]}{x_2 - y_2} \right\} = Q(x)\mathcal{D}_{y_1}^Q \mathcal{D}_{y_2}^Q \times \left\{ \int_0^1 (D_{x_1} \varphi)(y_1 + s_1(x_1 - y_1), y_2 + s_2(x_2 - y_2)) ds_1 - \int_0^1 (D_{x_1} \varphi)(y_1 + s_1(x_1 - y_1), y_2) ds_1 \right\}.
\]

Now performing the indicated differentiations in \(y_1\) and \(y_2\) and using the fact that \(|x_2|\) is large, we see this is bounded in absolute value by \(cc_\varphi |Q(x)|/|x_2|\). Since \(|x_1|\) is bounded, we obtain the desired estimate. The case when \(|x_1| > 2R\) and \(|x_2| \leq 2R\) is similar.

Finally, in case both \(|x_1| \leq 2R\) and \(|x_2| \leq 2R\), we write the expression as

\[
Q(x)\mathcal{D}_{y_1}^Q \mathcal{D}_{y_2}^Q \left\{ \int_0^1 \int_0^1 (D_{x_1} D_{x_2} \varphi)(y_1 + s_1(x_1 - y_1), y_2 + s_2(x_2 - y_2)) ds_1 ds_2 \right\}.
\]

By Leibniz’s rule, this is bounded in absolute value by \(cc_\varphi |Q(x)|\), and the lemma follows.

By using Lemma (3.1), we can show that if \(f \in L_u^p\) for \(u = |Q|^p w\) with \(w \in A_p\) for rectangles, \(1 < p < \infty\), and if \(l_f\) is defined by

\[
\langle l_f, \varphi \rangle = \int_{\mathbb{R}^2} f(z)[\varphi(z) - \mathcal{P}_Q \varphi(z)] dz, \quad \varphi \in \mathcal{S}(\mathbb{R}^2),
\]

then \(l_f\) defines a tempered distribution. In fact, by Lemma (3.1),

\[
|\langle l_f, \varphi \rangle| \leq \int |f(z)||\varphi(z) - \mathcal{P}_Q \varphi(z)| dz \leq \int |f(z)| cc_\varphi \frac{|Q(z)|}{(1 + |z_1|)(1 + |z_2|)} dz.
\]

Thus, by Hölder’s inequality, \(|\langle l_f, \varphi \rangle| \leq cc_\varphi \|f\|_{L_u^p} \mathcal{F} \) where

\[
(3.2) \quad \mathcal{F} = \left( \int_{\mathbb{R}^2} \frac{w(z)^{-1/(p-1)}}{(1 + |z_1|)^{p'}(1 + |z_2|)^{p'}} dz \right)^{1/p'}, \quad 1/p + 1/p' = 1.
\]

If we show that \(\mathcal{F}\) is finite, it will follow easily from the definition of \(c_\varphi\) that \(l_f\) is a tempered distribution. To accomplish this, we first claim that if \(R_{1,1}\) is a square of edgelength 1 and \(R_{r_1,r_2}\) denotes the rectangle with the same center as \(R_{1,1}\) whose \(x_i\)-edgelength is \(r_i\), \(i = 1, 2\), then if \(w \in A_s\) for rectangles we have

\[
(3.3) \quad w(R_{r_1,r_2}) \leq cr_1^s r_2^s w(R_{1,1}), \quad r_1, r_2 > 1.
\]

In fact, if \(w \in A_s\) then

\[
\left( \frac{1}{r_1 r_2} \int_{R_{r_1,r_2}} w \right) \left( \frac{1}{r_1 r_2} \int_{R_{r_1,r_2}} w^{-1/(s-1)} \right)^{s-1} \leq c,
\]
and so since $R_{1,1} \subset R_{r_1,r_2}$,

$$\left( \frac{1}{r_1 r_2} \int_{R_{r_1,r_2}} w \right) \left( \frac{1}{r_1 r_2} \int_{R_{1,1}} w^{-1/(s-1)} \right)^{s-1} \leq c.$$  

Then, by Hölder’s inequality,

$$\left( \frac{1}{r_1 r_2} \int_{R_{r_1,r_2}} w \right) \left( \frac{1}{r_1 r_2} \int_{R_{1,1}} w \right)^{-1} \leq c,$$

which proves the claim.

Now let $R_{r_1,r_2}$ be centered at the origin, and write

$$\mathcal{F}^p' = \int (1 + |z_1|)^{-p'} (1 + |z_2|)^{-p'} w(z)^{-1/(p-1)} \, dz$$

$$= \int_{R_{1,1}} + \sum_{k_1,k_2=1}^{\infty} \int_{2^{k_1-1} < |z_1| < 2^{k_1}} \int_{2^{k_2-1} < |z_2| < 2^{k_2}} w^{-1/(p-1)} \, dz + \sum_{k_1,k_2=1}^{\infty} 2^{-p'(k_1+k_2)} \int_{R_{2^{k_1},2^{k_2}}} w^{-1/(p-1)} \, dz$$

Since $w \in A_p$ for rectangles, we have $w^{-1/(p-1)} \in A_{p'-\epsilon}$ for rectangles for some $\epsilon > 0$. Thus, by (3.3) applied to $w^{-1/(p-1)}$ with $s = p' - \epsilon$,

$$\mathcal{F}^p' \leq \left( \int_{R_{1,1}} w^{-1/(p-1)} \, dz \right) \left( 1 + c \sum_{k_1,k_2=1}^{\infty} 2^{-p'(k_1+k_2)} \int_{R_{2^{k_1},2^{k_2}}} w^{-1/(p-1)} \, dz \right)$$

$$\leq c \int_{R_{1,1}} w^{-1/(p-1)} \, dz < \infty.$$

**Lemma (3.4).** Let $\{a_1^{(i)}\}$ and $\{a_2^{(j)}\}$ denote the roots of $Q_1$ and $Q_2$, respectively, and let $E_{i,j}$ denote the set of points of $\mathbb{R}^2$ which are closer to $(a_1^{(i)}, a_2^{(j)})$ than to any other $(a_1^{(k)}, a_2^{(l)})$. Given a function $\varphi$ and a number $M \geq 1$, let

$$A_{\varphi,M} = \sup_{(z_1,z_2) \in \mathbb{R}^2} \left( (1 + |z_1|)^{j_1+M} (1 + |z_2|)^{j_2+M} |D_{z_1}^{j_1} D_{z_2}^{j_2} \varphi(z_1,z_2)| \right).$$

Then for $z \in (z_1, z_2) \in E_{i,j}$,

$$|\varphi(x - z) - \mathcal{P}^Q_{\varphi(x,z)}(z)|$$

$$\leq c A_{\varphi,M} \frac{|Q(z)|}{|Q(x)|} \left( \frac{1}{|x_1 - a_1^{(i)}| + |z_1 - a_1^{(i)}|} + \frac{1}{t_1(1 + |x_1 - z_1|/t_1)^M} \right) \times \left( \frac{1}{|x_2 - a_2^{(j)}| + |z_2 - a_2^{(j)}| + |z_2 - a_2^{(j)}|} + \frac{1}{t_2(1 + |x_2 - z_2|/t_2)^M} \right),$$
\[ x = (x_1, x_2), \ t = (t_1, t_2), \ t_1, t_2 > 0, \ \text{with} \ c \ \text{independent of} \ x, z, t, \varphi \ \text{and} \ M. \]

**Proof.** Note first that \( E^{i,j} = E^i_1 \times E^j_2 \) where

\[
E^i_1 = \left\{ z_1 : |z_1 - a^{(i)}_1| = \min_k |z_1 - a^{(k)}_1| \right\}
\]

and

\[
E^j_2 = \left\{ z_2 : |z_2 - a^{(j)}_2| = \min_l |z_2 - a^{(l)}_2| \right\}.
\]

The lemma will be proved by using the following one-dimensional estimate (see Lemma (3.3) of [7]): for a function \( \varphi(z), z \in \mathbb{R}^1 \), and a polynomial \( Q(z), z \in \mathbb{R}^1 \), with all real roots,

\[
|\varphi_t(x - z) - \varphi_t(y - z)| \leq c \frac{|Q(z)|}{|Q(x)|} \left[ \frac{n_\varphi}{|x - a| + |z - a|} + |\varphi_t(x - z)| \right], \quad t > 0,
\]

where \( a \) is the root of \( Q \) which is closest to \( z \), and

\[
n_\varphi = \sup_{z \in \mathbb{R}^1} (1 + |z|)^{j+1}|D_z^j \varphi(z)|.
\]

To see how the lemma follows from this estimate, let \((z_1, z_2) \in E^{i,j}, \) and define for fixed \( x_2, z_2 \) and \( t_2, \)

\[
\psi(z_1) = \psi(z_1; x_2, z_2, t_2)
\]

\[
= Q_2(z_2) Q_1(z_1) \left( \frac{1}{t_1} \varphi \left( z_1, \frac{x_2 - z_2}{t_2} \right) - \frac{1}{t_2} \varphi \left( z_1, \frac{x_1 - y_2}{t_2} \right) \right) .
\]

A computation gives

\[
\varphi_t(x - z) - \varphi_t(y - z) = Q_1(z_1) Q_1(z_1) \left( \frac{\psi_t(x_1 - z_1) - \psi_t(x_1 - y_1)}{z_1 - y_1} \right) .
\]

By applying (3.5) to the right-hand side, we obtain

\[
|\varphi_t(x - z) - \varphi_t(y - z)| \leq c \frac{|Q_1(z_1)|}{|Q_1(x_1)|} \left( \frac{n_\psi}{|x_1 - a^{(i)}_1| + |z_1 - a^{(i)}_1|} + |\psi_t(x_1 - z_1)| \right) \]

where

\[
n_\psi = \sup_{z \in \mathbb{R}^1} (1 + |z|)^{j+1}|D_z^j \psi(z)|.
\]

Of course, \( n_\psi \) depends on \( x_2, z_2 \) and \( t_2 \) since \( \psi \) does.
To estimate the right side of (3.6), first note from the definition of $\psi$ and (3.5) that

$$|\psi(z_1)| \leq c \frac{|Q_2(z_2)|}{|Q_2(x_2)|} \left( \frac{\hat{n}(z_1)}{|x_2 - a_2^{(j)}| + |z_2 - a_2^{(j)}|} + \frac{1}{t_2} \phi\left( z_1, \frac{x_2 - z_2}{t_2} \right) \right) ,$$

where

$$\hat{n}(z_1) = \sup_{z_2 \in \mathbb{R}^1 \setminus \mathcal{N}_2} (1 + |z_2|)^{j_2 + 1} |D_2^{j_2} \phi(z_1, z_2)| .$$

Clearly, by definition of $A_{\phi, M}$, we have $\hat{n}(z_1) \leq A_{\phi, M}(1 + |z_1|)^{-M}$, $M \geq 1$, uniformly in $z_1$. In particular, by dilation and translation,

$$|\psi_t(x_1 - z_1)| \leq c \frac{|Q_2(z_2)|}{|Q_2(x_2)|} A_{\phi, M} \frac{1}{t_1} \frac{1}{(1 + |x_1 - z_1|)^M} \frac{1}{|x_2 - a_2^{(j)}| + |z_2 - a_2^{(j)}|}$$

$$\quad + \frac{t_1}{t_2} \phi_t(x_1 - z_1, x_2 - z_2) .$$

To estimate $n_\psi$, write $(1 + |z_1|)^{j_1 + 1} \psi^{(j_1)}(z_1)$ as

$$(1 + |z_1|)^{j_1 + 1} Q_2(z_2) \mathcal{Q}_{y_2} \left\{ \frac{1}{t_2} D_2^{j_1} \phi\left( z_1, \frac{x_2 - z_2}{t_2} \right) - \frac{1}{t_2} D_2^{j_1} \phi\left( z_1, \frac{x_2 - z_2}{t_2} \right) \right\} .$$

By (3.5), this is bounded in absolute value by

$$|Q_2(z_2)| \left( \frac{\hat{n}(z_1)}{|x_2 - a_2^{(j)}| + |z_2 - a_2^{(j)}|} + \frac{1}{t_2} (1 + |z_1|)^{j_1 + 1} \left| D_2^{j_1} \phi\left( z_1, \frac{x_2 - z_2}{t_2} \right) \right| \right) ,$$

where

$$\hat{n}(z_1) = \sup_{z_2 \in \mathbb{R}^1 \setminus \mathcal{N}_2} (1 + |z_2|)^{j_2 + 1} (1 + |z_1|)^{j_1 + 1} |D_2^{j_2} D_2^{j_1} \phi(z_1, z_2)| .$$

Clearly, $\hat{n}(z_1) \leq A_{\phi, M}$ uniformly in $z_1$ by definition of $A_{\phi, M}$. Similarly, the second term inside the square brackets in (3.8) is bounded by

$$A_{\phi, M} \frac{1}{t_2} \frac{1}{(1 + |x_2 - z_2|)^M} .$$

Thus

$$n_\psi \leq c A_{\phi, M} \frac{|Q_2(z_2)|}{|Q_2(x_2)|} \left( \frac{1}{|x_2 - a_2^{(j)}| + |z_2 - a_2^{(j)}|} + \frac{1}{t_2} \frac{1}{(1 + |x_2 - z_2|)^M} \right) .$$

The lemma now follows by combining (3.7), (3.8) and (3.9), and by using in (3.8) the fact that

$$|\phi_{t_1, t_2}(x_1 - z_1, x_2 - z_2)| \leq A_{\phi, M} \frac{1}{t_1} \frac{1}{(1 + |x_1 - z_1|)^M} \frac{1}{t_2} \frac{1}{(1 + |x_2 - z_2|)^M} .$$
4. Proof of Theorem 1

Let \((\xi, t) = (\xi_1, t_1; \xi_2, t_2) \in \Gamma(x) = \Gamma_1(x_1) \times \Gamma_2(x_2)\), and write

\[
\begin{align*}
\mathcal{J}(\xi, t) &= \int \mathcal{J}(\xi - z) - \mathcal{R}^0_{\theta}(\xi - z) \, dz \\
&= \sum_{i,j} \int E_{i,j}
\end{align*}
\]

where \(E_{i,j}\) is defined as in Lemma (3.4). We will consider each term of the sum separately. Fix \(i\) and \(j\) and write \(a_1^{(i)} = a_1\), \(a_2^{(j)} = a_2\) and \(E_{i,j} = E\) for simplicity. Thus, for the rest of the argument, \(z \in E\) and the estimates of Lemma (3.4) hold. With \(x_1, \xi_1, t_1\) and \(x_2, \xi_2, t_2\) fixed, let

\[
\theta(z_1, z_2) = \phi \left( \frac{\xi_1 - x_1}{t_1} + z_1, \frac{\xi_2 - x_2}{t_2} + z_2 \right).
\]

Note that \(|\xi_1 - x_1|/t_1 < \gamma_1\) and \(|\xi_2 - x_2|/t_2 < \gamma_2\), where \(\gamma_1\) and \(\gamma_2\) are the apertures of \(\Gamma_1(x_1)\) and \(\Gamma_2(x_2)\). Moreover, note that

\[
\phi_t(\xi - z) - \mathcal{R}^0_{\theta}(\xi - z) = \theta_t(x - z) - \mathcal{R}^0_{\theta_t}(x - z).
\]

Thus, by applying Lemma (3.4) to the right-hand side, we obtain

\[
|\phi_t(\xi - z) - \mathcal{R}^0_{\theta_t}(\xi - z)| \leq c A_{\theta, M} \frac{|Q(z)|}{|Q(x)|} K(x, z, t)
\]

where

\[
K(x, z, t) = \left[ \frac{1}{|x_1 - a_1| + |z_1 - a_1|} + \frac{1}{t_1 \left( 1 + \frac{|x_1 - z_1|}{t_1} \right)^M} \right] \times \left[ \frac{1}{|x_2 - a_2| + |z_2 - a_2|} + \frac{1}{t_2 \left( 1 + \frac{|x_2 - z_2|}{t_2} \right)^M} \right].
\]

Note that in this estimate we have \(A_{\theta, M}\) (which also depends on \(x, \xi, t\)) rather than \(A_{\phi, M}\). However, \(A_{\theta, M} \leq c A_{\phi, M}\) uniformly in \(x, \xi, t\) if \((\xi, t) \in \Gamma(x)\); in fact,

\[
A_{\theta, M} = \sup_{|x_1 - a_1|, |x_2 - a_2|, |z_1 - a_1|, |z_2 - a_2|} \left( 1 + |z_1| \right)^{j_1 + M} \left( 1 + |z_2| \right)^{j_2 + M}
\]

\[
\times \left| D_{|z_1|^2}^{j_1} D_{|z_2|^2}^{j_2} \phi \left( \frac{\xi_1 - x_1}{t_1} + z_1, \frac{\xi_2 - x_2}{t_2} + z_2 \right) \right|,
\]

and since if \(|\xi_i - x_i|/t_i \leq \gamma_i, i = 1, 2\), we have

\[
\frac{1}{1 + \gamma_i} \leq \frac{1 + |z_i|}{1 + \left| \frac{\xi_i - x_i}{t_i} + z_i \right|} \leq 1 + \gamma_i,
\]

it follows easily that \(A_{\theta, M} \leq c A_{\phi, M}\) uniformly.
In particular

\[(4.1) \quad \sup_{(\xi, \eta) \in \Gamma(x)} \int_E |f(z)||\varphi_\xi(\xi - z) - \varphi_\eta(\eta - z)|\,dz \leq c A_{\varphi, M} \frac{1}{|Q(x)|} \sup_{t_1, t_2 > 0} \int |f(z)Q(z)|K(x, z, t)\,dz.\]

To prove the theorem, we must show that the $L^p_{|Q|^{\rho w}}$ norm of the expression of the left of (4.1) is bounded by a constant times the same norm of $f$. By (4.1), it is enough to show that (with $g = fQ$) the $L^p_w$ norm of

\[(4.2) \quad \sup_{t_1, t_2 > 0} \int |g(z)|K(x, z, t)\,dz\]

is at most $c\|g\|_{L^p_w}$. For fixed $x = (x_1, x_2)$, we divide the domain of integration in (4.2) into the following four regions:

I: $\{(z_1, z_2) : |z_1 - a_1| > |x_1 - a_1|, |z_2 - a_2| > |x_2 - a_2|\}$,
II: $\{(z_1, z_2) : |z_1 - a_1| > |x_1 - a_1|, |z_2 - a_2| < |x_2 - a_2|\}$,
III: $\{(z_1, z_2) : |z_1 - a_1| < |x_1 - a_1|, |z_2 - a_2| > |x_2 - a_2|\}$,
IV: $\{(z_1, z_2) : |z_1 - a_1| < |x_1 - a_1|, |z_2 - a_2| < |x_2 - a_2|\}$.

For the part of (4.2) with integration extended over region I, we have the bound

\[
\begin{aligned}
\sup_{t_1, t_2 > 0} \int_{|z_1 - a_1| > |x_1 - a_1|, |z_2 - a_2| > |x_2 - a_2|} & |g(z_1, z_2)| \left[ \frac{1}{|z_1 - a_1|} + \frac{1}{t_1 \left( 1 + \frac{|x_1 - z_1|}{t_1} \right)^M} \right] \\
\times & \left[ \frac{1}{|z_2 - a_2|} + \frac{1}{t_2 \left( 1 + \frac{|x_2 - z_2|}{t_2} \right)^M} \right] \,dz_1 \,dz_2 \\
\leq & \int_{|z_1 - a_1| > |x_1 - a_1|, |z_2 - a_2| > |x_2 - a_2|} |g(z_1, z_2)| \frac{dz_2}{|z_2 - a_2|} \frac{dz_1}{|z_1 - a_1|} \\
+ & c \int_{|z_1 - a_1| > |x_1 - a_1|} M^{(2)} g(z_1, x_2) \frac{dz_1}{|z_1 - a_1|} \\
+ & c \int_{|z_2 - a_2| > |x_2 - a_2|} M^{(1)} g(x_1, z_2) \frac{dz_2}{|z_2 - a_2|} + c M^{(1)} M^{(2)} g(x_1, x_2),
\end{aligned}
\]

where $M^{(1)}, M^{(2)}$ and $M^{(1)} M^{(2)}$ are respectively the classical one-dimensional Hardy-Littlewood maximal operator in the first variable, the same operator in the second variable, and the iterated Hardy-Littlewood maximal operator. To obtain the last inequality, we have chosen the constant $M > 1$ and used the standard majorization of an approximation of the identity by the Hardy-Littlewood maximal function. Each of the last four terms has $L^p_w$ norm bounded by $c\|g\|_{L^p_w}, 1 < p < \infty$, by repeated use of Lemma (2.1), the principal result of [4] in the one-dimensional case, and the previously mentioned fact.
that since \( w \in A_p \) for rectangles, \( w \in A_p(\mathbb{R}^1) \) on almost every line parallel to either axis uniformly in the other variable.

The arguments for the parts of (4.2) with the integration extended over regions II, III and IV are similar. For II, the bound is

\[
\sup_{t_1,t_2 > 0} \iint_{|z_1 - a_1| > |x_1 - a_1| \atop |z_2 - a_2| < |x_2 - a_2|} |g(z_1, z_2)| \times \left[ \frac{1}{|z_1 - a_1|} + \frac{1}{t_1 \left( 1 + \frac{|x_1 - z_1|}{t_1} \right)^M} \right] \times \left[ \frac{1}{|x_2 - a_2|} + \frac{1}{t_2 \left( 1 + \frac{|x_2 - z_2|}{t_2} \right)^M} \right] \, dz_1 \, dz_2
\]

\[
\leq \frac{1}{|x_2 - a_2|} \int_{|z_2 - a_2| < |x_2 - a_2|} \left[ \int_{|z_1 - a_1| > |x_1 - a_1|} |g(z_1, z_2)| \frac{dz_1}{|z_1 - a_1|} \right] \, dz_2
\]

\[
+ c \int_{|z_1 - a_1| > |x_1 - a_1|} M^{(2)} g(z_1, x_2) \, dz_1
\]

\[
+ c \frac{1}{|x_2 - a_2|} \int_{|z_2 - a_2| < |x_2 - a_2|} M^{(1)} g(x_1, z_2) \, dz_2 + cM^{(1)}M^{(2)}g(x_1, x_2).
\]

Note that the third term is actually less than the fourth. At any rate, each of the terms can again be treated by Hardy's inequalities (2.1) and (2.2), and [3].

Region III is similar to II by symmetry. Finally, for IV, the bound is

\[
\frac{1}{|x_1 - a_1|} \frac{1}{|x_2 - a_2|} \iint_{|z_1 - a_1| < |x_1 - a_1| \atop |z_2 - a_2| < |x_2 - a_2|} |g(z_1, z_2)| \, dz_1 \, dz_2
\]

\[
+ c \frac{1}{|x_1 - a_1|} \int_{|z_2 - a_2| < |x_2 - a_2|} M^{(2)} g(z_1, x_2) \, dz_1
\]

\[
+ c \frac{1}{|x_2 - a_2|} \int_{|z_2 - a_2| < |x_2 - a_2|} M^{(1)} g(x_1, z_2) \, dz_2 + cM^{(1)}M^{(2)}g(x_1, x_2).
\]

Each of these is majorized by the last, and Theorem 1 follows.

5. Proof of Theorem 2

Let \( Q_1 \) and \( Q_2 \) be polynomials on \( \mathbb{R}^1 \) with all real roots, let \( Q(x) = Q_1(x_1)Q_2(x_2) \), \( x = (x_1, x_2) \), and \( u(x) = |Q(x)|^p w(x) \) with \( w \in A_p \) for rectangles. We begin the proof of Theorem 2 by noting that \( L_u^p \) is continuously embedded in \( H_u^p \), \( 1 < p < \infty \), in the sense that if \( f \in L_u^p \) and \( l_f \) is defined by

\[
(l_f, \varphi) = \int_{\mathbb{R}^2} f(x)[\varphi(x) - \mathcal{P}_\varphi Q(x)] \, dx, \quad \varphi \in \mathcal{S}(\mathbb{R}^2),
\]

then \( l_f \) is a tempered distribution in \( H_u^p \) and \( \|l_f\|_{H_u^p} \leq c\|f\|_{L_u^p} \). In fact, this is a corollary of the discussion following Lemma (3.1) and of Theorem 1.

To prove the other half of Theorem 2, we will show that if \( l \in H_u^p \) there exists \( f \in L_u^p \) such that \( l = l_f \) and \( \|f\|_{L_u^p} \leq c\|l\|_{H_u^p} \). For fixed \( \psi \in \mathcal{S}(\mathbb{R}^2) \)
with \( \int \psi = 1 \), let

\[ l(x_1, s_1; x_2, s_2) = (l * \psi_{s_1, s_2})(x_1, x_2). \]

By hypothesis, \( \sup_{s_1, s_2 > 0} |l(x_1, s_1; x_2, s_2)| \in L^p_u \). Thus, taking \( s_1 = s_2 = s \), we see that \( \|l(\cdot, s; \cdot, s)\|_{L^p_u} \) is uniformly bounded in \( s > 0 \) by \( c\|l\|_{H^p_u} \). In particular, there is a sequence \( s^{(k)} \to 0 \) and a function \( f \in L^p_u \) such that \( l(\cdot, s^{(k)}; \cdot, s^{(k)}) \) converges weakly in \( L^p_u \) to \( f \), and \( \|f\|_{L^p_u} \leq c\|l\|_{H^p_u} \). Let \( l_f \) be the distribution induced by \( f \), i.e.,

\[ \langle l_f, \varphi \rangle = \int f(x)[\varphi(x) - \mathcal{P}_\varphi(x)] \, dx, \quad \varphi \in \mathcal{S}(\mathbb{R}^2), \]

and let \( l^{(1)}(s_1, s_2) \) be the distribution induced by \( l(x_1, s_1; x_2, s_2) \), i.e.,

\[ \langle l^{(1)}(s_1, s_2), \varphi \rangle = \int l(x_1, s_1; x_2, s_2)[\varphi(x) - \mathcal{P}_\varphi(x)] \, dx. \]

As shown by the discussion following Lemma (3.1), \( \varphi - \mathcal{P}_\varphi \in L^{p-1}_{u-1} \) (the dual space of \( L^p_u \)). It then follows from the weak convergence mentioned above that \( l^{(1)}(s^{(k)}, s^{(k)}) \to l_f \) as distributions.

Now define \( l^{(2)}(s_1, s_2) \in \mathcal{S}'(\mathbb{R}^2) \) by

\[ \langle l^{(2)}(s_1, s_2), \varphi \rangle = \int l(x_1, s_1; x_2, s_2)\varphi(x) \, dx, \quad \varphi \in \mathcal{S}(\mathbb{R}^2). \]

This is well-defined since \( l(x_1, s_1; x_2, s_2) \) is a locally bounded function of \( (x_1, x_2) \) which is also in \( L^p_u \) (see also (5.3)). By well-known facts about distributions, \( l^{(2)}(s_1, s_2) \to l \) as distributions when \( s_1, s_2 \to 0 \). Hence, in order to show that \( l = l_f \), and thus complete the proof, it is enough to show that \( l^{(1)}(s_1, s_2) = l^{(2)}(s_1, s_2) \). For \( \varphi \in \mathcal{S}(\mathbb{R}^2) \), we obtain by subtracting the two formulas above that

\[ \langle l^{(2)}(s_1, s_2) - l^{(1)}(s_1, s_2), \varphi \rangle = \int l(x_1, s_1; x_2, s_2)\mathcal{P}_\varphi(x) \, dx. \]

Note that by definition

\[ \mathcal{P}_\varphi(x) = Q_1(x_1)Q_2(x_2)Q_1^Q_2 \left[ \frac{\varphi(y_1, x_2) + \varphi(x_1, y_2) - \varphi(y_1, y_2)}{(x_1 - y_1)(x_2 - y_2)} \right] \]

\[ = P_{1,x_1}(x_1) + P_{2,x_2}(x_2) + P(x_1, x_2), \]

where \( P_{1,x_1}(x_1) \) is a polynomial in \( x_1 \) of degree \( N_1 - 1 \) whose coefficients are bounded functions of \( x_2 \), \( P_{2,x_2}(x_2) \) is a polynomial in \( x_2 \) of degree \( N_2 - 1 \) whose coefficients are bounded functions of \( x_1 \), and \( P(x_1, x_2) \) is a polynomial in \( x_1, x_2 \) of degree \( (N_1 - 1)(N_2 - 1) \). We claim that

(i) the resulting integrals in (5.2) converge absolutely;

(ii) for \( s_1, s_2 > 0 \), \( l(x_1, s_1; x_2, s_2) \) has one dimensional \( x_1 \)-moments of order \( \leq N_i - 1 \) equal to 0 for \( i = 1, 2 \).
Once these claims are established, it will follow immediately from Fubini's theorem and (5.2) that \( I^{(2)}(s_1, s_2) = I^{(1)}(s_1, s_2) \), as desired, and the proof of Theorem 2 will be complete.

To prove claim (i), we will show that

\[
\int |l(x_1, s_1; x_2, s_2)| (1 + |x_1|)^{N_1-1} (1 + |x_2|)^{N_2-1} \, dx < \infty.
\]

By Hölder's inequality, the integral in (5.3) is at most

\[
(5.4) \left( \int |l(x_1, s_1; x_2, s_2)|^p (1 + |x_1|)^{N_1p} (1 + |x_2|)^{N_2p} w(x) \, dx \right)^{\frac{1}{p}} \times \left( \int \frac{w(x)^{-1/(p-1)}}{(1 + |x_1|)^{p'}(1 + |x_2|)^{p'}} \, dx' \right)^{1/p'}.
\]

The second factor is finite by the argument given after (3.2). To show that the first factor is finite, given \( s_1, s_1 > 0 \), let \( \Omega_{s_1, s_2} \) be the complement of

\[
(x_1, x_2); \left\{ \begin{array}{l}
|a_1^{(i)} - x_1| < s_1/4 \text{ for some } i, \text{ or } \\
|a_2^{(j)} - x_2| < s_2/4 \text{ for some } j;
\end{array} \right.
\]

i.e., \( \Omega_{s_1, s_2} \) is the points not belonging to any of the strips of width \( \frac{1}{2}s_1 \) centered around the lines \( x_1 = a_1^{(i)} \), or to any of the strips of width \( \frac{1}{2}s_2 \) centered around \( x_2 = a_2^{(j)} \), where \( \{a_1^{(i)}\} \) and \( \{a_2^{(j)}\} \) are the zeros of \( Q_1 \) and \( Q_2 \), respectively.

Let

\[
R_{s_1, s_2}(x_1, x_2) = \{(z_1, z_2) : |z_i - x_i| < s_i, \ i = 1,2\}.
\]

If \( s_1 \) and \( s_2 \) are small, there is a constant \( c > 0 \) such that for any \( (x_1, x_2) \),

\[
|R_{s_1, s_2}(x_1, x_2) \cap \Omega_{s_1, s_2}| > c |R_{s_1, s_2}(x_1, x_2)|.
\]

Also, since \( (x_1, s_1; x_2, s_2) \) lies in the product cone at any point of \( R_{s_1, s_2}(x_1, x_2) \), we have

\[
|l(x_1, s_1; x_2, s_2)| \leq N(l)(z_1, z_2) \quad \text{if (}z_1, z_2) \in R_{s_1, s_2}(x_1, x_2).
\]

Moreover, for such \( (z_1, z_2) \),

\[
1 + |x_i| \leq 1 + |z_i| + |x_i - z_i| \leq 1 + |z_i| + s_i
\]

\[
\leq c(1 + |z_i|), \quad i = 1, 2.
\]

Hence,

\[
|l(x_1, s_1; x_2, s_2)| (1 + |x_1|)^{N_1p} (1 + |x_2|)^{N_2p} \leq c \frac{1}{w(R_{s_1, s_2}(x_1, x_2))}
\]

\[
\times \iint_{R_{s_1, s_2}(x_1, x_2) \cap \Omega_{s_1, s_2}} N(l)(z_1, z_2)^p (1 + |z_1|)^{N_1p} (1 + |z_2|)^{N_2p} w(z_1, z_2) \, dz_1 \, dz_2.
\]
It follows by integration that the $p$th power of the first factor in (5.4) is at most
\[
c \int_{\Omega_{1,2}} N(l)(z)^p \left(1 + |z_1| \right)^{N_1p} \left(1 + |z_2| \right)^{N_2p} w(z)
\times \left( \int_{R_{1,2}(z_1,z_2)} \frac{w(x)}{w(R_{1,2}(x_1,x_2))} \, dx \right) \, dz
\leq c \int_{\Omega_{1,2}} N(l)(z)^p \left(1 + |z_1| \right)^{N_1p} \left(1 + |z_2| \right)^{N_2p} w(z) \, dz,
\]
since for $(x_1, x_2) \in R_{1,2}(z_1, z_2)$, we have $w(R_{1,2}(x_1,x_2)) \approx w(R_{1,2}(z_1,z_2))$ and therefore
\[
\int_{R_{1,2}(z_1,z_2)} \frac{w(x)}{w(R_{1,2}(x_1,x_2))} \, dx \approx 1.
\]
Finally, if $(z_1, z_2) \in \Omega_{1,2}$, then
\[
(1 + |z_1|)^{N_1p} (1 + |z_2|)^{N_2p} \leq c_{s_1,s_2} |Q_1(z_1)|^{N_1} |Q_2(z_2)|^{N_2}
= c_{s_1,s_2} |Q(z)|^p,
\]
and consequently the integral above is at most
\[
c_{s_1,s_2} \int_{\Omega_{1,2}} N(l)(z)^p |Q(z)|^p w(z) \, dz \leq c_{s_1,s_2} \|l\|_{H^p_Q,w}^p.
\]
In particular, (5.3) is true for $s_1, s_2$ small. The argument for $s_1, s_2 > c > 0$ is similar and simpler. This completes the proof of the first claim (i).

To prove the second claim (ii), let us first show that for fixed $s_1, s_2 > 0$,
\[
l^{(2)}(s_1, s_2) \in H^p_Q(w) \quad (\text{see (5.1) for the definition of } l^{(2)}(s_1, s_2)).
\]
Since $l \in H^p_Q(w)$ and
\[
l^{(2)}(s_1, s_2) \ast \varphi_{t_1,t_2} = (l \ast \psi_{s_1,s_2}) \ast \varphi_{t_1,t_2} = l \ast (\psi_{s_1,s_2} \ast \varphi_{t_1,t_2}),
\]
it suffices to show that
\[
\sup_{(\xi_1, \xi_2) \in \Gamma_1(x_1)} |(l \ast (\psi_{s_1,s_2} \ast \varphi_{t_1,t_2}))(\xi_1, \xi_2)| \leq c l^* (x_1, x_2),
\]
with $c$ independent of $(s_1, s_2)$, where $l^*$ denotes the "grand" maximal function defined by
\[
l^*(x_1, x_2) = \sup_{(\xi_1, \xi_2) \in \Gamma_1(x_1)} |(l \ast \theta_{t_1,t_2})(\xi_1, \xi_2)|,
\]
$\mathcal{A}$ being the collection of Schwartz functions $\theta$ for which a sufficiently large number of Schwartz seminorms are bounded by a fixed constant $A$.

We will show that
\[
\psi_{s_1,s_2} \ast \varphi_{t_1,t_2} = \theta_{u_1,u_2}, \quad u_i = \max\{s_i, t_i\}, \quad i = 1,2,
\]
where $\theta \in \mathcal{A}$ (although $\theta$ may depend on $t_i, s_i$). This will suffice since $(y_i, t_i) \in \Gamma_j(x_i)$ implies that $(y_i, u_i) \in \Gamma_j(x_i)$. By checking Fourier transforms, it is easy to see that
\[
\psi_{s_1,s_2} \ast \varphi_{t_1,t_2} = (\psi_{s_1/u_1,s_2/u_2} \ast \varphi_{t_1/u_1,t_2/u_2})_{u_1,u_2} = \theta_{u_1,u_2}.
\]
where

\[ \theta = \psi_{s_1,t_1} \ast \phi_{t_1,s_2} , \quad \bar{s}_i = s_i/u_i , \quad \bar{t}_i = t_i/u_i , \quad i = 1,2 . \]

Note that \( \max \{ \bar{s}_i, \bar{t}_i \} = 1 \) for \( i = 1,2 \). Thus, letting \( \| | \cdot | |_m \) denote the \( m \)th Schwartz norm, i.e,

\[
\| | \theta | |_m = \sup_{(x_1,x_2)} \left( 1 + |x_1| \right)^m \left( 1 + |x_2| \right)^m \left[ \left( \frac{\partial}{\partial x_1} \right)^j \left( \frac{\partial}{\partial x_2} \right)^k \theta(x_1,x_2) \right] ,
\]

it is enough to show that

\[
\| | \psi_{s_1,t_2} \ast \phi_{t_1,s_2} | |_m \leq A \quad \text{when} \quad \max \{ s_i, t_i \} = 1 , \quad i = 1,2 ,
\]

with \( A \) independent of \( s_i, t_i \).

Letting \( r_i = \min \{ s_i, t_i \} , \quad i = 1,2 , \) we see that there are four cases:

1. \( r_1 = t_1 \leq s_1 = 1 \) and \( r_2 = t_2 \leq s_2 = 1 \);
2. \( r_1 = t_1 \leq s_1 = 1 \) and \( r_2 = s_2 \leq t_2 = 1 \);
3. \( r_1 = s_1 \leq t_1 = 1 \) and \( r_2 = t_2 \leq s_2 = 1 \);
4. \( r_1 = s_1 \leq t_1 = 1 \) and \( r_2 = s_2 \leq t_2 = 1 \).

Since (1) and (4) are similar and (2) and (3) are similar, we shall consider only (1) and (2). In case (1),

\[
(\psi \ast \phi_{r_1,r_2})(x_1,x_2) = \int \psi(x_1 - \xi_1,x_2 - \xi_2) \phi_{r_1,r_2}(\xi_1,\xi_2) d\xi_1 d\xi_2 ,
\]

and since \( \| | \psi(\cdot - \xi_1, \cdot - \xi_2) | |_m \leq \| | \psi | |_m (1 + |\xi_1|)^m (1 + |\xi_2|)^m \), we have

\[
\| | \psi \ast \phi_{r_1,r_2} | |_m \leq \| | \psi | |_m \int (1 + |\xi_1|)^m (1 + |\xi_2|)^m |\phi_{r_1,r_2}(\xi_1,\xi_2)| d\xi_1 d\xi_2 .
\]

Changing variables in the integral and using \( 0 < r_1, r_2 \leq 1 \) we obtain

\[
\| | \psi \ast \phi_{r_1,r_2} | |_m \leq c \| | \psi | |_m \text{ with } c = c_{\phi,m} \text{ independent of } r_1, r_2 .
\]

In case 2, write

\[
(\psi_{s_1,t_2} \ast \phi_{t_1,s_2})(x_1,x_2) = \int \psi_{s_1,t_2}(x_1 - \xi_1, x_2 - \xi_2) \phi_{t_1,s_2}(\xi_1,\xi_2) d\xi_1 d\xi_2 .
\]

Thus,

\[
\| | \psi_{s_1,t_2} \ast \phi_{t_1,s_2} | |_m \leq \int \| | \psi_{s_1,t_2} (\cdot - \xi_1, \cdot - \xi_2) | |_{m,R_1} \| | \phi_{t_1,s_2} (\xi_1, \cdot - \xi_2) | |_{m,R_1} d\xi_1 d\xi_2 ,
\]

where \( \| | \cdot | |_{m,R_1} \) denotes the \( m \)th Schwartz seminorm of a function of one variable. In particular, for any \( N \), there is a constant \( c \) independent of \( r_1 \) and \( r_2 \) such that the last integral is bounded by

\[
c \| | \psi | |_m \| | \phi | |_m \times \int \left[ (1 + |\xi_1|)^m \left( 1 + \frac{|\xi_2|}{r_2} \right)^{-N} \right] \left[ (1 + |\xi_2|)^m \left( 1 + \frac{|\xi_1|}{r_1} \right)^{-N} \right] d\xi_1 d\xi_2
\]

by changing variables, using \( 0 < r_1, r_2 \leq 1 \) and choosing \( N \geq m + 2 \). This shows that \( f^{(2)}(s_1,s_2) \in H^{p}_{Q \infty \sigma} \) for each \( s_1, s_2 > 0 \).
It follows that the one-dimensional distribution defined by
\[(5.5) \quad \langle l_1, \varphi \rangle = \int l(x_1, s_1; x_2, s_2)\varphi(x_1) \, dx_1, \quad \varphi \in \mathcal{S}(\mathbb{R}^1),\]
belongs to $H^p_{\lfloor Q_1^p w(x_2)\rfloor}(\mathbb{R}^1)$ for each $s_1, s_2 > 0$ and a.e. $x_2$, as we now show. Pick $\varphi \in \mathcal{S}(\mathbb{R}^1)$ with $\varphi \geq 0$ and let $\overline{\varphi}(x_1, x_2) = \varphi(x_1) \varphi(x_2)$. Then $\overline{\varphi} \in \mathcal{S}(\mathbb{R}^2)$ and since $l(\overline{\varphi})(s_1, s_2) \in H^p_{\lfloor Q_1^p w(x_2)\rfloor}(\mathbb{R}^1)$, we have $N_{\overline{\varphi}}(l(\overline{\varphi})(s_1, s_2)) \in L^p_{\lfloor Q_1^p w(x_2)\rfloor}$, i.e.,
\[\sup_{(\xi_1, \xi_2) \in \Gamma_1(x_1)} \left| \int l(z_1, s_1; z_2, s_2)\varphi_{t_1}(\xi_1 - z_1) \, dz_1 \right| \sup_{(\xi_2, \xi_2) \in \Gamma_2(x_2)} \left| \int l(z_2, s_2; x_2, s_2)\varphi_{t_2}(\xi_2 - z_2) \, dz_2 \right| \in L^p_{\lfloor Q_1^p w(x_2)\rfloor}(\mathbb{R}^2).\]
However, this double supremum exceeds a positive constant times
\[\sup_{(\xi_1, \xi_2) \in \Gamma_1(x_1)} \left| \int l(z_1, s_1; x_2, s_2)\varphi_{t_1}(\xi_1 - z_1) \, dz_1 \right| \sup_{(\xi_2, \xi_2) \in \Gamma_2(x_2)} \left| \int l(z_2, s_2; x_2, s_2)\varphi_{t_2}(\xi_2 - z_2) \, dz_2 \right|.

Thus, by Fubini's theorem, the last expression belongs to $L^p_{\lfloor Q_1^p w(x_2)\rfloor}(\mathbb{R}^1)$ for a.e. $x_2$, as desired. A similar fact holds for the other variable.

We can now complete the proof of the second claim (ii) made earlier. By above, the distribution $l_1$ defined in (5.5) belongs to $H^p_{\lfloor Q_1^p w(x_2)\rfloor}(\mathbb{R}^1)$ for a.e. $x_2$. Also, since $l(\cdot, s_1; x_2, s_2) \in L^p_{\lfloor Q_1^p w(x_2)\rfloor}(\mathbb{R}^1)$ for a.e. $x_2$, we have from the one-dimensional theory in [7] that the distribution defined by
\[\langle l_2, \varphi \rangle = \int l(x_1, s_1; x_2, s_2)[\varphi(x_1) - \mathcal{R}_{\varphi}(x_1)] \, dx_1, \quad \varphi \in \mathcal{S}(\mathbb{R}^1),\]
belongs to $H^p_{\lfloor Q_1^p w(x_2)\rfloor}(\mathbb{R}^1)$, and therefore, so does $l_1 - l_2$. Since
\[\langle l_1 - l_2, \varphi \rangle = \int l(x_1, s_1; x_2, s_2)\mathcal{R}_{\varphi}(x_1) \, dx_1,\]
we see that $l_1 - l_2$ has compact support, and it follows from Lemma (2.3) that $l_1 - l_2 = 0$. Now choosing $\varphi_k \in \mathcal{S}(\mathbb{R}^1)$ such that $\mathcal{R}_\varphi(x_1) = x_1^k$ for each $k = 0, \ldots, N_1 - 1$ (e.g., let $\varphi_k(x_1) = x_1^k \rho(x_1)$ where $\rho \in \mathcal{S}(\mathbb{R}^1)$ and $\rho = 1$ on a set containing all the roots of $Q_1(x_1)$), we obtain
\[\int l(x_1, s_1; x_2, s_2)x_1^k \, dx_1 = 0, \quad k = 0, 1, \ldots, N_1 - 1,\]
for a.e. $x_2$, as desired. A similar fact holds for the variable $x_2$. This completes the proof of claim (ii), and so also of Theorem 2.

6. Proof of Theorem 3
Let $f$ and $\varphi$ satisfy the hypothesis of Theorem 3, and let
\[f(\xi, t) = \int f(z)[\varphi(z - z) - \mathcal{R}_{\varphi}(z)] \, dz,\]
\[ \xi = (\xi_1, \xi_2), \ t = (t_1, t_2). \] Suppose first that \( \varphi \) has compact support. Note that if \( z = (z_1, z_2) \) then

\[ \mathcal{P}_{\varphi_i, t_i}(z) = Q_1(z_1)Q_2(z_2)D_1^{Q_1}D_2^{Q_2} \]

\[ \times \left\{ \phi_{t_1,t_2}(\xi_1 - y_1, \xi_2 - z_2) - \phi_{t_1,t_2}(\xi_1 - z_1, \xi_2 - y_2) - \phi_{t_1,t_2}(\xi_1 - y_1, \xi_2 - y_2) \right\} \]

For such \( x = (x_1, x_2) \) does not lie on any line \( x_1 = a_1 \) or \( x_2 = a_2 \), where \( a_1 \) and \( a_2 \) denote roots of \( Q_1 \) and \( Q_2 \) respectively, then \( |\xi_i - a_i|/t_i \to \infty \) as \( (\xi_i, t_i) \to x_i, i = 1, 2, \) for each root \( a_i \). Therefore, since \( \varphi \) has compact support, we see that \( \mathcal{P}_{\varphi_i, t_i}(z) = 0 \) for \( (\xi_i, t_i) \) close to \( x_i, i = 1, 2, \) for such \( x \) (in fact, each of the three terms of \( \mathcal{P} \) is zero), and consequently,

\[ f(\xi, t) = \int f(z) \varphi_i(\xi - z) \, dz \]

for such \( x \) and \( (\xi, t) \). This expression converges nontangentially in the product sense to \( f(x) \int \varphi \) a.e. by standard facts about "strong" differentiability since \( \exists q > 1 \) so that \( f \) is locally in \( L^q \) away from the lines \( x_i = a_i \); in fact, away from these lines \( |Q_1| \) and \( |Q_2| \) are bounded below away from zero, so that \( f \in L^p_w \) there; thus, \( f \) is locally in \( L^q \) there for some \( q > 1 \) since \( w \in A_{p-\varepsilon} \) for some \( \varepsilon > 0 \).

In case \( \varphi \) does not have compact support, let \( r > 0 \) and write

\[ \varphi(x) = \rho(rx)\varphi(x) + (1 - \rho(rx))\varphi(x) = \tilde{\varphi}(x) + \hat{\varphi}(x), \]

where \( \rho \) is a smooth truncation with \( \rho(x) = 1 \) for \( |x| < 1 \), \( \rho(x) = 0 \) for \( |x| > 2 \), and \( \rho \in C^\infty \). For fixed \( r, \tilde{\varphi} \) has compact support and the corresponding extension \( \tilde{f}(\xi, t) \) converges nontangentially to \( f(x) \int \tilde{\varphi} \) a.e. by the previous case. Moreover, as \( r \to 0 \) we see that \( \int \tilde{\varphi} \to \int \varphi \) and that the constant \( A_{\tilde{\varphi}, M} \) defined in Lemma (3.4) tends to zero. Hence, to prove the theorem, it is enough to show that for \( M > 1 \) and a.e. \( x \), there is a finite number \( c_{x, f, M} \) such that \( N(f)(x) \leq c_{x, f, M}A_{\varphi, M} \). However, from the proof of Theorem 1, \( N(f)(x) \) is bounded by a sum of terms of the type in (4.1), one term for each pair \( a_1, a_2 \) of roots. Note that \( A_{\varphi, M} \) is a factor on the right in (4.1). Moreover, as shown in the argument following (4.1), the remaining factor in (4.1) belongs to \( L^p_u \) if \( M > 1 \); in particular, this factor is finite a.e., and the proof is complete.

7. Proof of Theorem 4

Let \( f \in L^p_u \) and let \( f(x, t) \) be the extension formed by using a convolver \( \varphi \) with \( \int \varphi = 1 \). By the proof of Theorem 2 (see (ii) in §5), \( f(x, t) \) has one-dimensional \( x_i \)-moments of order \( \leq N_i - 1 \) equal to zero for \( i = 1, 2 \). Moreover, \( f(x, t) \in L^1 \) as a function of \( x \) by (5.3). Also, by Theorem 3, \( f(x, t) \) converges pointwise a.e. to \( f(x) \) as \( t \to 0 \), and since \( \sup_{t>0} |f(x, t)| \in L^p_u \) by Theorem 1, we see from the dominated convergence theorem that \( f(x, t) \to f(x) \) in \( L^p_u \).
as \( t \to 0 \). Thus, we may assume from the start that \( f \in L^p_u \cap L^1 \) and that \( f \) satisfies
\[
\int_{-\infty}^{\infty} f(x_1, x_2) x_1^{k_1} \, dx_1 = \int_{-\infty}^{\infty} f(x_1, x_2) x_2^{k_2} \, dx_2
\]
\[
= \iint_{R^2} f(x_1, x_2) x_1^{k_1} x_2^{k_2} \, dx_1 \, dx_2 = 0
\]
for \( k_1 = 0, 1, \ldots, N_1 \) and \( k_2 = 0, 1, \ldots, N_2 \).

Let us show that if \( t = (t_1, t_2) \) and if either \( t_1 \) or \( t_2 \to \infty \), then \( f(x, t) \to 0 \) in \( L^p_u \). By definition of \( N(f) \), we have
\[
|f(x, t)| \leq \frac{1}{\left( \iint_{|z_1 - x_1| < t_1, |z_2 - x_2| < t_2} u(z_1, z_2) \, dz_1 \, dz_2 \right)^{1/p}} \iint_{|z_1 - x_1| < t_1, |z_2 - x_2| < t_2} N(f)^p u(z_1, z_2) \, dz_1 \, dz_2
\]
\[
\leq c \left( \iint_{|z_1 - x_1| < t_1, |z_2 - x_2| < t_2} u(z_1, z_2) \, dz_1 \, dz_2 \right)^{1/p} \|f\|_{L^p_u} \to 0
\]
as either \( t_1 \) or \( t_2 \to \infty \) since \( u \) satisfies the doubling condition in each variable. This shows pointwise convergence; norm convergence follows from the dominated convergence theorem since \( \sup_{t_1, t_2 > 0} |f(x, t)| \leq N(f)(x) \).

Now suppose in addition that the convolution function \( \phi \) is a product \( \phi_1(x_1) \phi_2(x_2) \) of one-dimensional functions with \( \phi_i(x_i) \) compactly supported and equal to 1 near \( x_i = 0 \). Let us write \( f(x, t) = F_{t_1, t_2}(x_1, x_2) \). Note that due to the moment conditions on \( f \),
\[
F_{t_1, t_2}(x_1, x_2) = \iint f(z_1, z_2) \phi_{t_1, t_2}(x_1 - z_1, x_2 - z_2) \, dz_1 \, dz_2,
\]
and therefore, since \( f \in L^1 \),
\[
\hat{F}_{t_1, t_2}(x_1, x_2) = \hat{f}(x_1, x_2) \hat{\phi}_1(t_1 x_1) \hat{\phi}_2(t_2 x_2).
\]

Now, for \( t_1, t_2 \) small and \( T_1, T_2 \) large, let
\[
G = F_{t_1, t_2} - F_{t_1, T_2} - F_{T_1, t_2} + F_{T_1, T_2}.
\]
Note that
\[
\hat{G}(x_1, x_2) = \hat{f}(x_1, x_2) [\hat{\phi}_1(t_1 x_1) \hat{\phi}_2(t_2 x_2) - \hat{\phi}_1(t_1 x_1) \hat{\phi}_2(T_2 x_2) - \hat{\phi}_1(T_1 x_1) \hat{\phi}_2(t_2 x_2) + \hat{\phi}_1(T_1 x_1) \hat{\phi}_2(T_2 x_2)].
\]
Since \( \hat{\phi}_i = 1 \) near \( x_i = 0 \) and \( \hat{\phi}_i \) has compact support, it follows that \( \hat{G} \) vanishes near both axes and that \( \hat{G} \) has compact support. Also,
\[
\|G - f\|_{L^p_u} \leq \|F_{t_1, t_2} - f\|_{L^p_u} + \|F_{t_1, T_2}\|_{L^p_u} + \|F_{T_1, t_2}\|_{L^p_u} + \|F_{T_1, T_2}\|_{L^p_u}.
\]
The first term on the right is small if \( t_1, t_2 \) are small. With \( t_1, t_2 \) fixed, each of the remaining three terms on the right is small if \( T_1, T_2 \) are large. Thus, \( G \)
has all the required properties except that $G$ and its derivatives may not decay at $\infty$. Note that $G \in C^\infty$. Let $\hat{\psi} \in C_0^\infty$ and $\psi(0,0) = 1$. Define $H_t$ by

$$
\hat{H}_t = \hat{G} \ast t^{-2} \hat{\psi} \left( \frac{\cdot}{t}, \frac{\cdot}{t} \right).
$$

Then $\hat{H}_t$ has compact support and vanishes near the axes when $t$ is small. Also, $\hat{H}_t \in C^\infty$, so that $\hat{H}_t \in \mathcal{S}$ and, consequently, $H_t \in \mathcal{S}$. Finally, since

$$
H_t(x_1, x_2) = G(x_1, x_2)\psi(tx_1, tx_2)
$$

and $\psi$ is bounded and $\psi(tx_1, tx_2) \to \psi(0,0) = 1$ as $t \to 0$, it follows that $H_t \to G$ in $L^p_u$. This completes the proof of Theorem 4.

8. Proof of Theorem 5

To prove the first half of Theorem 5, suppose that $f \in L^p_u$ with $u = (1 + |x_1|)^{d_1p}(1 + |x_2|)^{d_2p}|Q|^{p'} w$, $Q = Q_1(x_1)Q_2(x_2)$ and $w \in A_p$ for rectangles. Suppose also that $f$ satisfies

$$
(8.1) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2)Q_1(x_1)x_1^{k_1} dx_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2)Q_2(x_2)x_2^{k_2} dx_2 = 0
$$

for a.e. $x_2$ and a.e. $x_1$, respectively, and for $k_1 = 0, 1, \ldots, d_1 - 1$ and $k_2 = 0, 1, \ldots, d_2 - 1$. We wish to show that the distribution $l$ defined by

$$
\langle l, \varphi \rangle = \int \int f(z)[\varphi(z) - \mathcal{R}_\varphi(z)] \, dz, \quad \varphi \in \mathcal{S}(\mathbb{R}^2),
$$

satisfies $l \in H^p_u$ with $\|l\|_{H^p_u} \leq c\|f\|_{L^p_u}$.

Let us first show that the integrals in (8.1) are well-defined a.e. It is easy to see that

$$
(8.2) \quad \int \int |f(x_1, x_2)|Q_1(x_1)|Q_2(x_2)|((1 + |x_1|)^{d_1-1}(1 + |x_2|)^{d_2-1} \, dx_1 dx_2 < \infty
$$

if $f \in L^p_u$ since by Hölder's inequality the integral is at most

$$
\|f\|_{L^p_u} \left( \int \int \frac{w^{-1/(p-1)}}{(1 + |x_1|)^{p'}}(1 + |x_2|)^{p'} \, dx_1 dx_2 \right)^{1/p'},
$$

which is finite by (3.2). It then follows from Fubini's theorem that the integrals in (8.1) converge absolutely a.e.

Since

$$
(1 + |x_1|)^{d_1p}(1 + |x_2|)^{d_2p} \approx 1 + |x_1|^{d_1p} + |x_2|^{d_2p} + |x_1|^{d_1}x_2^{d_2}|^p,
$$

the fact that $f \in L^p_u$ implies that $f$ also belongs to each of $L^p_{u_j}$, $j = 1, 2, 3, 4$, where $u_1 = |Q_1Q_2|^p w$, $u_2 = |x_1^{d_1}Q_1|Q_2|^p w$, $u_3 = |Q_1|^p|x_2^{d_2}Q_2|^p w$ and $u_4 = |x_1^{d_1}Q_1|^p|x_2^{d_2}Q_2|^p w$. Moreover,

$$
\|f\|_{L^p_u} \approx \sum_{j=1}^{4} \|f\|_{L^p_{u_j}}.
$$
From Theorem 2, \( f \) defines four distributions \( l_j, j = 1, 2, 3, 4 \), by

\[
\begin{align*}
\langle l_1, \varphi \rangle &= \int f[\varphi - \mathcal{P}_Q_1 \cdot Q_2] \, dx, \\
\langle l_2, \varphi \rangle &= \int f[\varphi - \mathcal{Q}_1^Q \cdot Q_2] \, dx, \\
\langle l_3, \varphi \rangle &= \int f[\varphi - \mathcal{Q}_1 \cdot Q_2^Q] \, dx, \\
\langle l_4, \varphi \rangle &= \int f[\varphi - \mathcal{Q}_1^Q \cdot Q_2^Q] \, dx,
\end{align*}
\]

and \( \|l_j\|_{H^p_u} \leq c \|f\|_{L^p_u} \) for each \( j \). In particular, \( \|l_j\|_{H^p_u} \leq c \|f\|_{L^p_u} \) for each \( j \).

We claim that \( l_1 = l_2 = l_3 = l_4 \). Taking this momentarily for granted and calling the common value \( l \), it follows that \( l \in H^p_u \) and \( \|l\|_{H^p_u} \approx \sum \|l_j\|_{H^p_u} \leq c \|f\|_{L^p_u} \), as desired.

To verify the claim, we will show that \( l_1 = l_4 \); the proofs that \( l_1 = l_2 \) and \( l_1 = l_3 \) are similar. We have

\[
\begin{align*}
\langle l_1, \varphi \rangle - \langle l_4, \varphi \rangle &= \int f[\mathcal{Q}_1^Q \cdot Q_2^Q - \mathcal{Q}_1^Q \cdot Q_2] \, dx \\
&= \int \int fQ_1Q_2(P_{d_1-1}(x_1)G_{d_2-1}(x_2) + P_{d_2-1}(x_2)G_{d_1-1}(x_1)) \, dx_1 \, dx_2
\end{align*}
\]

by Lemma (2.7), where \( P_{d_1-1} \) and \( P_{d_2-1} \) are polynomials of degrees \( d_1 - 1 \) and \( d_2 - 1 \), and

\[
\begin{align*}
|G_{d_1-1}(x_1)| &\leq c(1 + |x_1|)^{d_1-1}, \\
|G_{d_2-1}(x_2)| &\leq c(1 + |x_2|)^{d_2-1}.
\end{align*}
\]

The last integral converges absolutely by (8.2), and we may rewrite the integral by using Fubini's theorem as

\[
\int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(x_1, x_2)Q_1(x_1)P_{d_1-1}(x_1) \, dx_1 \right) Q_2(x_2)G_{d_2-1}(x_2) \, dx_2 \\
+ \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(x_1, x_2)Q_2(x_2)P_{d_2-1}(x_2) \, dx_2 \right) Q_1(x_1)G_{d_1-1}(x_1) \, dx_1.
\]

The inner integrals in both these terms vanish for a.e. \( x_2 \) and for a.e. \( x_1 \), respectively, by (8.1), and the claim follows. This completes half of the proof of Theorem 5.

To prove the other half, let \( l \) be any element of \( H^p_u \). Then, with \( u_j \) defined as above, \( l \in \bigcap_{j=1}^{4} H^p_{u_j} \). By Theorem 2, there exist \( f_j \in L^p_{u_j} \) for each \( j \) such
that

\[ \langle l, \varphi \rangle = \int f_1[\varphi - \mathcal{P}_{\varphi}^{Q_1,Q_2}] \, dx = \int f_2[\varphi - \mathcal{P}_{\varphi}^{x_1^i Q_1,Q_2}] \, dx = \int f_3[\varphi - \mathcal{P}_{\varphi}^{Q_1,x_2^i Q_2}] \, dx = \int f_4[\varphi - \mathcal{P}_{\varphi}^{x_1^i Q_1,x_2^i Q_2}] \, dx \]

and \[ \|f_j\|_{L^p_{\mathbf{H}}} \leq c\|l\|_{H^p_{\mathbf{L}}} \leq c\|l\|_{H^p_{\mathbf{L}}} \] for each \( j \). Choosing \( \varphi \) to be supported away from the lines corresponding to zeros of \( x_i^d Q_i \), \( i = 1,2 \), we see that

\[ \mathcal{P}_{\varphi}^{Q_1,Q_2} = \mathcal{P}_{\varphi}^{x_1^i Q_1, Q_2} = \mathcal{P}_{\varphi}^{Q_1, x_2^i Q_2} = \mathcal{P}_{\varphi}^{x_1^i Q_1, x_2^i Q_2} = 0 \]

and that

\[ \int f_1 \varphi = \int f_2 \varphi = \int f_3 \varphi = \int f_4 \varphi. \]

Therefore, \( f_1 = f_2 = f_3 = f_4 \) a.e. If we call this common value \( f \), we see that

\[ \|f\|_{L^p_{\mathbf{H}}} \leq c\sum_{j=1}^4 \|f_j\|_{L^p_{\mathbf{H}}} \leq c\|l\|_{H^p_{\mathbf{L}}}, \]

It remains to show that \( f \) satisfies the moment conditions (8.1). Pick \( \varphi(x_1,x_2) \) to be a product of one-dimensional functions:

\[ \varphi(x_1,x_2) = \varphi_1(x_1)\varphi_2(x_2). \]

In this case, by (1.2),

\[ \varphi - \mathcal{P}_{\varphi}^{Q_1,Q_2} = [\varphi_1(x_1) - \mathcal{P}_{\varphi_1}^{Q_1}(x_1)][\varphi_2(x_2) - \mathcal{P}_{\varphi_2}^{Q_2}(x_2)], \]

and similar formulas hold for \( \varphi - \mathcal{P}_{\varphi}^{x_1^i Q_1, Q_2} \), etc. Thus, by subtracting the first and second representations of \( \langle l, \varphi \rangle \) above, and also the first and third representations, we obtain both

\[ \iint f(x_1, x_2)[\mathcal{P}_{\varphi_1}^{x_1^i Q_1}(x_1) - \mathcal{P}_{\varphi_1}^{Q_1}(x_1)][\varphi_2(x_2) - \mathcal{P}_{\varphi_2}^{Q_2}(x_2)] \, dx_1 \, dx_2 = 0 \]

and

\[ \iint f(x_1, x_2)[\varphi_1(x_1) - \mathcal{P}_{\varphi_1}^{Q_1}(x_1)][\mathcal{P}_{\varphi_2}^{x_2^i Q_2}(x_2) - \mathcal{P}_{\varphi_2}^{Q_2}(x_2)] \, dx_1 \, dx_2 = 0. \]

Consider the first of these with \( \varphi_1(x_1) = x_1^k Q_1(x_1) \rho(x_1) \) for \( k = 0,1, \ldots, d_1 - 1 \) and \( \rho \in C_0^\infty \) with \( \rho = 1 \) near the zeros of \( x_1^k Q_1(x_1) \). Then, by Lemma (2.5) of [7],

\[ \mathcal{P}_{\varphi_1}^{x_1^i Q_1} = \mathcal{P}_{\varphi_1}^{x_1^i Q_1} = x_1^k Q_1 \quad \text{and} \quad \mathcal{P}_{\varphi_1}^{Q_1} = \mathcal{P}_{\varphi_1}^{Q_1} = 0. \]
Consequently,
\[ \int\int f(x_1, x_2) x_1^{k_1} Q_1(x_1)[\varphi_2(x_2) - \mathcal{P}^{Q_2}(x_2)] \, dx_1 \, dx_2 = 0. \]

Since \( \varphi_2 \) is arbitrary and \( \mathcal{P}^{Q_2}(x_2) = 0 \) if \( \varphi_2 \) is supported away from the zeros of \( Q_2 \), it follows that
\[ \int_{-\infty}^{\infty} f(x_1, x_2) x_1^{k_1} Q_1(x_1) \, dx_1 = 0 \quad \text{a.e. } x_2. \]

Similarly,
\[ \int_{-\infty}^{\infty} f(x_1, x_2) x_2^{k_2} Q_2(x_2) \, dx_2 = 0 \quad \text{a.e. } x_1 \]
if \( k_2 = 0, 1, \ldots, d_2 - 1 \). This completes the proof of Theorem 5.

REFERENCES


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