PIXLEY-ROY HYPERSPACES OF ω-GRAPHS

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Abstract. The techniques developed by Wage and Norden are used to show that the Pixley-Roy hyperspaces of any two ω-graphs are homeomorphic. The Pixley-Roy hyperspaces of several subsets of $\mathbb{R}^n$ are also shown to be homeomorphic.

I. Introduction

Since it was introduced in 1969, the Pixley-Roy hyperspace, $\text{PR}[X]$, of a topological space $X$ has been intensely studied with the hope of establishing how the properties of $X$ affect those of $\text{PR}[X]$. This study has met with some success, especially in the area of cardinal functions. However, there is a class of questions which, until recently, eluded investigators: For which spaces $X$ and $Y$ will $\text{PR}[X]$ be homeomorphic to $\text{PR}[Y]$? For several years the only results in this area were some embedding results obtained by van Douwen [vD] and Lutzer [L]. In 1985 Wage [W] achieved a breakthrough by developing a technique for breaking up neighborhoods around points in certain spaces which allowed him to define homeomorphisms between those neighborhoods. Using this technique he was able to show that Pixley-Roy hyperspaces of spaces like $\mathbb{R}$ or $[0, 1]$ are homogeneous. In 1986 Norden [N] extended Wage's technique to one which broke up an entire space. With this he was able to show that the Pixley-Roy hyperspaces of any two $P$-graphs (one-dimensional polyhedra with a finite number of points removed) are homeomorphic. It follows that the Pixley-Roy hyperspaces of spaces like $\mathbb{R}$, $[0, 1]$, and the circle are all homeomorphic. It is the purpose of this paper to use Norden's technique to show that Pixley-Roy hyperspaces of infinite, as well as finite, graphs are all the same.

Definition. A $T_2$ space $X$ with no isolated points is an ω-graph if there is a countable discrete subset $D$ of $X$ and a countable collection $I$ of pairwise disjoint copies of $(0, 1)$ such that $X \setminus D = \bigcup I$, $I$ is locally finite on $X$, and for every $x \in D$, $\{x\} \cup \bigcup\{I \in I: x \in I\}$ is a neighborhood of $x$ which can be embedded in $\mathbb{R}^2$. The set $D$ is called a dividing set for $X$.

The main result of this paper can be stated as follows.

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Theorem 1. If $X$ and $Y$ are $\omega$-graphs then $\text{PR}[X]$ is homeomorphic to $\text{PR}[Y]$.

§II will consist of preliminary definitions, notation, and observations necessary for the proof of the Theorem 1. Theorem 1 will be proved in §III, and §IV will contain some related results.

We will use $\text{PR}[X]$ to denote the Pixley-Roy hyperspace of $X$. Our notation for the open subsets of $\text{PR}[X]$ will be standard. We will use $F[A]$ to denote the set of nonempty finite subsets of a set $A$, and $F'[A]$ to denote the set of all finite subsets of $A$. The notation “$X \approx Y$” will mean that $X$ is homeomorphic to $Y$.

II. Preliminary matters

Let $X$ be an $\omega$-graph and let $X_0$ be a dividing set for $X$. Enumerate $X_0$ as $\{x_n : n < \omega\}$. Let $I_0$ be the countable collection of pairwise disjoint copies of $(0,1)$ whose union makes up $X \setminus X_0$. We may assume that every element of $I_0$ has at least one endpoint in $X_0$. For each $n < \omega$ let $\mu(n)$ be the number of elements of $X \setminus X_0$ having $x_n$ as an endpoint. For each $I \in I_0$, fix a linear structure and orientation for $I$. Let $Q_0$ be the set of all midpoints of elements of $I_0$, and for each $p \in X_0$, let $O_p$ be the component of $X \setminus Q_0$ containing $p$. Then $Q_0$ is a discrete subset of $X$ and $O_p \cap O_q = \emptyset$ if $p \neq q$.

For each $p \in X_0$ and each $I \in I_0$ having $p$ as an endpoint, choose a sequence of points in $I \cap O_p$ converging monotonically to $p$. This can be done because each element of $X_0$ is the endpoint of at least one element of $I_0$. Let $Q_1$ be the set of all points of $X$ which are elements either of $Q_0$ or of the sequences just chosen. Call $Q_1$ the 1st cut-set of $X$. Set $\hat{Q}_0 = Q_1$. Let $I_1$ be the countable collection of pairwise disjoint copies of $(0,1)$ whose union makes up $X \setminus (\hat{Q}_1 \cup X_0)$. Call $I_1$ the set of intervals in $X$ derived from $\hat{Q}_1$.

Assume that $n < \omega$, that $Q_n$ is a discrete subset of $X \setminus X_0$, and that $I_n$ is a countable collection of pairwise disjoint intervals in $X$. Let $Q_{n+1}$, the $(n+1)$th cut-set of $X$, be the set of midpoints of elements of $I_n$ and let $\hat{Q}_{n+1} = \hat{Q}_n \cup Q_{n+1}$. Let $I_{n+1}$, the set of intervals in $X$ derived from $\hat{Q}_{n+1}$, be the countable collection of pairwise disjoint copies of $(0,1)$ whose union makes up $X \setminus (\hat{Q}_{n+1} \cup X_0)$. Set $Q = \bigcup_{n<\omega} Q_n$.

For every $1 \leq m < \omega$ and every $n < \omega$, let $I_{m,n} = \{I \in I_m : I \subset O_{x_n}\}$. This is the set of those elements of $I_m$ which "cluster" around $x_n$.

For every $1 \leq n < \omega$ let $\Sigma(n)$ be the set of sequences, $\sigma$, defined on $n+1$ such that $\sigma(0), \sigma(1) \in \omega$ and $\sigma(m) \in \{0,1\}$ for all $1 < m \leq n$. Let $m < \omega$. Since $I_{1,m}$ is countable, it can be enumerated as $\{I_{(m,n)} : n < \omega\}$. In this way the set $I_1$ is indexed by $\Sigma(1)$. Assume that the elements of $\Sigma(n)$ have been used to index the elements of $I_n$. Let $I \in I_{n+1}$. There is a unique $\sigma \in \Sigma(n)$ such that $I \subset I_{\sigma}$. If $I$ is the left-hand half of $I_{\sigma}$, then let $\tau$ be the element of $\Sigma(n+1)$ such that $\tau(n+1) = 1$ and $\tau(n+1) = 0$ and set $I_{\tau} = I$. If $I$ is the right-hand half of $I_{\sigma}$, then let $\tau$ be the element of $\Sigma(n+1)$ such that $\tau(n+1) = \sigma$ and $\tau(n+1) = 1$ and set $I_{\tau} = I$. Let $\Sigma = \bigcup_{1 \leq n < \omega} \Sigma(n)$.
The following lemma consists of observations which are immediate consequences of the previous definitions and its proof is omitted.

**Lemma 2.** Let $1 \leq m \leq n < \omega$.

1. If $I \in I_n$ then $I \cap Q_m \neq \emptyset$.
2. If $p \in Q_m$ then there are exactly two elements, $I_1$ and $I_2$, of $I_n$ such that $p$ is an endpoint of both $I_1$ and $I_2$. Furthermore, $I_1 \cup I_2 \cup \{p\}$ is open in $X$.
3. If $I \in I_m$ then there are exactly two elements of $I_{m+1}$ that are subintervals of $I$.
4. If $I_{\sigma} \in I_n$ then there is exactly one element, $I_{\sigma 1 m+1}$, of $I_m$ that contains $I_{\sigma}$.
5. If $\sigma \in \Sigma(1)$, $\sigma(0) = k$, and $\sigma(1) = 1$, then $I_{\sigma}$ is the $l$th element of $I_{k}$.
6. If $I_{\sigma} \in I_{n,k}$ then $\sigma \in \Sigma(n)$ and $\sigma(0) = k$.
7. For any $n, k < \omega$, $\{\text{Int}[\text{Cl}(\{I_{\sigma} \in I_{n,k} : \sigma(1) > a\})] : a < \omega\}$ forms a local base for $x_k$.

For each $p \in X$ and each $1 \leq n < \omega$ let $A_n(p) = \{I \in I_n : p \in \overline{I}\}$ and let $A_n^*(p) = \bigcup A_n(p)$. If $p \in Q_n$ then $A(p)$ and $A^*(p)$ will denote $A_{n+1}(p)$ and $A_{n+1}^*(p)$ respectively. If $B \in PR[X]$ then set $A_n(B) = \bigcup_{p \in B} A_n(p)$ and $A_n^*(B) = \bigcup_{p \in B} A_n^*(p)$. If $B \in F[Q_n]$ then set $A(B) = \bigcup_{p \in B} A(p)$ and $A^*(B) = \bigcup_{p \in B} A^*(p)$.

Set $M_0 = \{\emptyset\}$ and, for each $1 \leq n < \omega$, let $M_n = \{E \in F(\widehat{Q}_n) : E \cap Q_m \neq \emptyset$ for all $1 \leq m \leq n\}$. For $1 \leq n < \omega$ call $M_n$ the set of elements of $PR[X]$ compatible with $\widehat{Q}_n$. Note that if $m > n$ and $E \in M_n$ then $E \cap Q_m = \emptyset$. Also, if $k \neq l$ then $M_k \cap M_l = \emptyset$. For each $n < \omega$ and each $E \in M_n$, let $S_E = \{A \in PR[X] : A \cap \widehat{Q}_{n+1} = E\}$. Thus, if $A \in S_E$ and $E \in M_n$, then $A \cap Q_{n+1} = \emptyset$. The set $\{S_E : E \in M\}$ where $M = \bigcup_{n<\omega} M_n$ is a partition of $PR[X]$ and is called the fundamental partition of $PR[X]$ based on $M$. If $E \in M_n$ then $S_E$ can be written as $\{A \cup B \cup E : A \in F'[X_0]$ and $B \in F'[X \setminus \widehat{Q}_{n+1} \cup X_0]\}$.

Recall that $X \setminus (\widehat{Q}_{n+1} \cup X_0) = \bigcup I_{n+1}$.

For each $E \in M_n$ let $\widehat{F}_E = \{I \in I_{n+1} : I \subset A^*(E)\}$. If $n \geq 2$, let $E' = E \setminus Q_n = E \cap \widehat{Q}_{n-1}$. If $n \geq 3$ then $E'' = E \cap \widehat{Q}_{n-2}$. If $n = 2$ then set $E'' = \emptyset$.

Now let $Y$ be another $\omega$-graph and let $Y_0$ be a dividing set for $Y$. Enumerate $Y_0$ as $\{y_n : n < \omega\}$. Then the function $\lambda : X_0 \rightarrow Y_0$ given by $\lambda(x_n) = y_n$ is a bijection. Let $J_0$ be a countable collection of pairwise disjoint copies of $(0, 1)$ whose union is $Y \setminus Y_0$. We may again assume that every element of $J_0$ has at least one endpoint in $Y_0$. Let $R_0$ be the set of midpoints of elements of $J_0$. Let $\{R_n : 1 \leq n < \omega\}$ be the collection of cut-sets for $Y$ and set $R = \bigcup_{n<\omega} R_n$. Let $P_n$ be the component of $Y \setminus R_0$ that contains $y_n$. For each $0 < n < \omega$ let $J_n$ be the set of intervals of $PR[Y]$ derived from $R_n$, each indexed as before by the elements of $\Sigma$. Let $\{N_k : k < \omega\}$ be the collection of
sets of elements of $\text{PR}[Y]$ compatible with $\{R_k : k < \omega\}$ and let $\{T_E : E \in \mathcal{N}\}$ be the fundamental partition of $\text{PR}[Y]$ based on $N = \bigcup_{k<\omega} N_k$. If $E \subseteq Q$ and $f : E \to R$, then $f$ is level preserving if $f(E \cap Q_n) \subseteq R_n$ for all $n < \omega$.

For each $I \in I_n$ and $J \in J_n$ there is a unique linear homeomorphism between $I$ and $J$ that preserves orientation. Denote this homeomorphism by $\eta_{I,J}$. If $\sigma, \tau \in \Sigma(n)$, $I = I_{\sigma|m+1}$, and $J = J_{\tau|m+1}$ for some $m < n$, then $\eta_{I,J}(I_\sigma) = J_\tau$ if and only if $\sigma(k) = \tau(k)$ for all $m < k \leq n$. If $\Gamma : I_n \to J_n$ is a bijection, then $\Gamma^* : \bigcup I_n \to \bigcup J_n$ is the function $\bigcup_{I \in I_n} \eta_{I,J(I)}$. $\Gamma^*$ is a homeomorphism that is linear and orientation preserving on each element of $I_n$.

Now order each $I_n$ and $J_n$ lexicographically using the indices of their elements. These collections then have order-type $\omega^2$. Let $F \subseteq I_n$ and $G \subseteq J_n$ be equipotent finite sets and let $\gamma : F \to G$ be a bijection. Then $I_n \setminus F$ and $J_n \setminus G$ still have order-type $\omega^2$, so there is a unique order isomorphism $\Delta_F : I_n \setminus F \to J_n \setminus G$. Define $\Gamma : I_n \to J_n$ by $\Gamma = \gamma \cup \Delta_F$. Then $\Gamma$ is a bijection.

In those situations where more than one $F$ is being considered and subscripts are used to distinguish the various set, the same subscripts will be used to distinguish the corresponding $\gamma$, $\Delta$, and $\Gamma$ functions. For example, the functions associated with $F_1$ will be $\gamma_1$, $\Delta_1$, and $\Gamma_1$.

It will be necessary in what follows to compare the index of $I_\sigma$ with that of $\gamma(I_\sigma)$ or $\Gamma(I_\sigma)$. In order to facilitate this, we will use $\gamma(\sigma)$ and $\Gamma(\sigma)$ to denote the indices of $\gamma(I_\sigma)$ and $\Gamma(I_\sigma)$ respectively.

The next lemma is obvious and its proof is omitted.

**Lemma 3.** Let $m < n < \omega$ and let $F_1 \subseteq I_m$ and $F_2 \subseteq I_n$ with $\{I \in I_n : I \subseteq F_1\} \subseteq \bigcup F_2$. If $\gamma_1 : F_1 \to J_m$ is a one-to-one function and $\gamma_2 : F_2 \to J_n$ is defined by $\gamma_2(I) = \Gamma^*(I)$, then $\Gamma^*(I) = \Gamma^*(I)$ for all $I \in I_n$.

**Lemma 4.** Let $F \subseteq I_k$ be finite and let $\gamma : F \to J_k$ be a one-to-one function. Assume that there are $b, c, m < \omega$ such that

1. $c - m > b$;
2. If $I_\sigma \in F$ then either $\sigma(1) \leq b$ or $\sigma(1) > c$;
3. if $I_\sigma \in F \cap I_{k,n}$ and $m \leq \sigma(1) \leq b$ then $\gamma(I_\sigma) \in J_{k,n}$ and $\gamma(\sigma)(1) \leq b$; and
4. if $I_\sigma \in F \cap I_{k,n}$ and $\sigma(1) > c$ then $\gamma(I_\sigma) \in J_{k,n}$ and $\gamma(\sigma)(1) > b$.

Then $\Gamma(I_\sigma) \in J_{k,n}$ and $\Gamma(\sigma)(1) > b$ for all $I_\sigma \in I_{k,n}$ with $\sigma(1) > c$.

**Proof.** Let $n < \omega$. The elements of $J_{k,n} \setminus \gamma(F)$ are the images under $\Delta_F$ of $I_{k,n} \setminus F$. By conditions 2 and 3,

$$|F \cap \{I_\sigma \in I_{k,n} : m \leq \sigma(1) \leq c\}| = |\{I_\sigma \in I_{k,n} : m \leq \sigma(1) \leq b\}|$$

$$\leq |\{J_\sigma \in J_{k,n} : J_\sigma \in \gamma(F) \text{ and } \sigma(1) \leq b\}|$$

$$= |\gamma(F) \cap \{J_\sigma \in J_{k,n} : \sigma(1) \leq b\}|.$$
Also, $|\{I_\sigma \in I_{k,n} : m \leq \sigma(1) \leq c\}| \geq |\{J_\sigma \in J_{k,n} : \sigma(1) \leq b\}|$ because $c - m > b$.

Therefore,

$$|\{I_\sigma \in I_{k,n} : m \leq \sigma(1) \leq c\}| \cap F = |\{I_\sigma \in I_{k,n} : m \leq \sigma(1) \leq c\}| \cap (F \cap \{I_\sigma \in I_{k,n} : m \leq \sigma(1) \leq c\}) \geq |\{J_\sigma \in J_{k,n} : \sigma(1) \leq b\}| \cap (\gamma(F) \cap \{J_\sigma \in J_{k,n} : \sigma(1) \leq b\})| = |\{J_\sigma \in J_{k,n} : \sigma(1) \leq b\}| \gamma(F)|.
$$

Thus, if $J_\tau \in J_{k,n}$ and $\tau(1) \leq b$ then there is $I_\alpha \in I_k$ such that either $I_\sigma \in F$ or $I_\sigma \in I_{k,n}$ and $\sigma(1) \leq c$, and $\Gamma(I_\sigma) = J_\tau$. It follows from this and condition 4 that if $I_\sigma \in I_{k,n}$ and $\sigma(1) > c$, then $\Gamma(I_\sigma) \in J_{k,n}$ and $\Gamma(\sigma)(1) > b$.

**Lemma 5.** Let $F_1, F_2 \subset I_k$ be finite and let $\gamma_1 : F_1 \rightarrow J_k$ and $\gamma_2 : F_2 \rightarrow J_k$ be one-to-one functions. Let $a, b, m < \omega$ such that

1. $b - a > m$;
2. $\{I_\sigma \in F_1 : \sigma(1) < a\} = \{I_\sigma \in F_2 : \sigma(1) < a\} = G$; and
3. $\gamma_1(I_\sigma) = \gamma_2(I_\sigma)$ for all $I_\sigma \in G$;

and that for $i = 1$ or 2,

4. if $I_\sigma \in \gamma_i(F_i)$ then either $\sigma(1) \leq a$ or $\sigma(1) > b$;
5. if $I_\sigma \in F_i$ and $\sigma(1) > b$ then $\gamma_i(\sigma)(1) > a$; and
6. for all $n < \omega$, if $I_\sigma \in \gamma_i(F_i) \cap J_{k,n}$ and $\gamma_i^{-1}(J_\sigma) \notin I_{k,n}$ then $\sigma(1) < m$.

Then $\Gamma_1(I_\sigma) = \Gamma_2(I_\sigma)$ for all $I_\sigma \in I_n$ with $\sigma(1) \leq a$.

**Proof.** Let $n < \omega$. By condition 2,

$$\{I_\sigma \in I_{k,n} : \sigma(1) \leq a\} \cap F_1 = I_{k,n} \cap G = \{I_\sigma \in I_{k,n} : \sigma(1) \leq a\} \cap F_2$$

and

$$\{I_\sigma \in I_{k,n} : \sigma(1) \leq a\} \cap F_1 = \{I_\sigma \in I_{k,n} : \sigma(1) \leq a\} \cap G$$

$$= \{I_\sigma \in I_{k,n} : \sigma(1) \leq a\} \cap F_2.$$

By conditions 2, 3, and 4,

$$\{J_\sigma \in J_{k,n} : \sigma(1) \leq b\} \gamma_1(F_1) = \{J_\sigma \in J_{k,n} : \sigma(1) \leq b\} \gamma_1(G)$$

$$= \{J_\sigma \in J_{k,n} : \sigma(1) \leq b\} \gamma_2(F_2).$$

If $I_\sigma \in I_{k,n} \cap G$ then $\Gamma_1(I_\sigma) = \gamma_1(I_\sigma) = \gamma_2(I_\sigma) = \Gamma_2(I_\sigma)$. The values of $\Gamma_1$ and $\Gamma_2$ on $\{I_\sigma \in I_{k,n} : \sigma(1) \leq a\} \cap G$ are determined by $\Delta_1$ and $\Delta_2$ respectively. We can establish the equality of $\Gamma_1$ and $\Gamma_2$ on $\{I_\sigma \in I_{k,n} : \sigma(1) \leq a\} \cap G$ by showing that this set is no larger than $\{J_\sigma \in J_{k,n} : \sigma(1) \leq b\} \gamma_1(G)$. Then, since both $\Delta_1$ and $\Delta_2$ take the $\alpha$th element of $\{I_\sigma \in I_{k,n} : \sigma(1) \leq a\} \cap G$ to the
\[\{J_\sigma \in J_{k,n} : \sigma(1) \leq b\}\gamma_1(G)\]

\[= \{J_\sigma \in J_{k,n} : \sigma(1) \leq a\}\gamma_1(G) + \{J_\sigma \in J_{k,n} : a < \sigma(1) \leq b\}\]

(by condition 4)

\[= \{J_\sigma \in J_{k,n} : \sigma(1) \leq a\}\gamma_1(I_\sigma) + \{J_\sigma \in J_{k,n} : a < \sigma(1) \leq b\}\gamma_1(I_\sigma)\]

(by conditions 1 and 6)

III. Proof of Theorem 1

Let \(X\) and \(Y\) be \(\omega\)-graphs with dividing sets \(X_0\) and \(Y_0\). We will use the structures and definitions developed in §II. Let \(g : Q_1 \to R_1\) be a bijection such that \(g(Q_1 \cap O_n) = R_1 \cap P_n\) for all \(n < \omega\). Then \(g(Q_0) = R_0\). For our convenience later in the proof, we will assume that the first \(\mu(n)\) elements of any \(I_{m,n}\) are those elements of \(I_{m,n}\) having an element of \(Q_0\) as an endpoint.

The homeomorphism we will define is essentially that defined by Norden in [N].

Define \(r_\phi : I_1 \to J_1\) by \(r_\phi(I_\sigma) = J_\sigma\), and \(h_\phi : \bigcup I_1 \to \bigcup J_1\) by \(h_\phi = \Gamma_\phi^*\). Then \(h_\phi\) is a homeomorphism. Set \(\theta(\phi) = \phi\).

Let \(E \in M_1\). Set \(f_E = g \mid E\) and \(\theta(E) = f_E\). Let \(F_E = \Gamma_\phi\) and \(F_{\theta(E)} = \Gamma_{\theta(E)}\). Each \(I \in F_E\) is adjacent to exactly one element of \(E\) and each element of \(E\) is the endpoint of exactly two elements of \(F_E\). Similarly, each element of \(F_{\theta(E)}\) is adjacent to exactly one element of \(\theta(E)\) and each element of \(\theta(E)\) is the endpoint of exactly two elements of \(F_{\theta(E)}\). Define \(\gamma_E : F_E \to F_{\theta(E)}\) as follows. Let \(I \in F_E\) and let \(p \in E\) be an endpoint of \(I\). If \(p\) is the right-hand endpoint of \(I\), then set \(\gamma_E(I)\) equal to the element of \(F_{\theta(E)}\) which has \(g(p)\) for its right-hand endpoint. If \(p\) is the left-hand endpoint of \(I\), then set \(\gamma_E(I)\) equal to the element of \(F_{\theta(E)}\) which has \(g(p)\) for its left-hand endpoint. Then \(\gamma_E\) is a bijection. Define \(h_E : (\bigcup I_1) \cup E \to (\bigcup J_2) \cup \theta(E)\) by \(h_E = \Gamma_{\phi^*} \cup f_E\). Both \(\Gamma_{\phi^*}\) and \(f_E\) are bijections so \(h_E\) is a bijection. It is also a homeomorphism on \(\bigcup I_2\) because \(\Gamma_{\phi^*}\) is. Let \(x \in E\) and let \(V\) be a neighborhood of \(f_E(x)\) in \(Y\). By the definition of \(\gamma_E\) there is a neighborhood \(U\) of \(x\) in \(A^*(x) \cup \{x\}\) such that \(h_E(U) \subseteq V\). Thus \(h_E\) is continuous at \(x\). A similar argument shows that \(h_E^{-1}\) is continuous at \(h_E(x)\), so \(h_E\) is a homeomorphism.
Let $2 \leq 1 < \omega$ and assume that for all $k < 1$ and all $E \in M_k$,

1. $f_E: E \to \hat{R}_k$ is a level preserving one-to-one function and $\theta(E) = f_E(E)$;

2. $F_E \subset I_{k+1}$ and $F_{\theta(E)} \subset J_{k+1}$ are finite and $\gamma_E: F_E \to F_{\theta(E)}$ is a bijection; and

3. the function $h_E: (\bigcup I_{k+1}) \cup E \to (\bigcup J_{k+1}) \cup \theta(E)$ given by $h_E = \Gamma_E \cup f_E$ is a homeomorphism.

Fix $E \in M_k$. Each element of $E \cap Q_l$ is the midpoint of some element of $I_{l-1}$ and $h_E^{'''}$, which is defined on $\bigcup I_{l-1}$, takes midpoints to midpoints. Thus $h_E^{'''}(p) \in R_l$ for all $p \in E \cap Q_l$. Define $f_E: E \to \hat{R}_l$ by

$$f_E(p) = \begin{cases} h_E^{'}(p) & \text{if } p \in E \cap \hat{Q}_{l-1}, \\ h_E^{'''}(p) & \text{if } p \in E \cap Q_l. \end{cases}$$

Then $f_E$ is a one-to-one level preserving function. Note that if $p \in E \cap \hat{Q}_{l-1}$ then $f_E(p) = h_{E^{'}}(p) = f_{E^{'}}(p)$. Extending this backward, we can see that if $1 \leq k < l$ and $p \in E \cap \hat{Q}_k$ then $f_E(p) = f_{E \cap \hat{Q}_k}(p)$.

Let $F_{E_1} = A(E \cap Q_l)$ and $F_{\theta(E_1)} = A(\theta(E) \cap R_l)$. Let $I \in F_{E_1}$ and let $p \in E \cap Q_l$ be an endpoint of $I$. Then $f_E(p) = h_E^{'''}(p) \in R_l$ and $h_E^{'''}(p)$ is an endpoint of $h_E^{'''}(I)$ because $h_E^{'''}$ is continuous. Thus $h_E^{'''}(I) \in F_{\theta(E_1)}$. A similar argument shows that if $h_E^{'''}(I) \in F_{\theta(E_1)}$ then $I \in F_{E_1}$.

Let $F_{E_2} = \{I \in \hat{F}_E \setminus F_{E_1}: h_{E^{'}}(I) \in \hat{F}_{\theta(E)} \setminus F_{\theta(E)} \}$ and let $F_{\theta(E_2)} = \{J \in \hat{F}_{\theta(E)} \setminus F_{\theta(E)}: h_{E^{''}}^{-1}(J) \in \hat{F}_E \setminus F_{E} \}$. Clearly $I \in F_{E_2}$ if and only if $h_{E^{'}}(I) \in F_{\theta(E_2)}$. Set $F_E = F_{E_1} \cup F_{E_2}$ and $F_{\theta(E)} = F_{\theta(E_1)} \cup F_{\theta(E_2)}$. Define $\gamma_E: F_E \to F_{\theta(E)}$ by

$$\gamma_E(I) = \begin{cases} h_{E^{''}}(I) & \text{if } I \in F_{E_1}, \\ h_{E^{'}}(I) & \text{if } I \in F_{E_2}. \end{cases}$$

Then $\gamma_E$ is a bijection.

Define $h_E: (\bigcup I_{l+1}) \cup E \to (\bigcup J_{l+1}) \cup \theta(E)$ by $h_E = \Gamma_E \cup f_E$. The function $h_E$ is a bijection because $\Gamma_E$ and $f_E$ are bijections and is a homeomorphism on $\bigcup I_{l+1}$ because $\Gamma_E$ is. If $p \in E \cap Q_l$ then $A(p) \subset F_{E_1}$ and $h_E(A^*(p) \cup \{p\}) = h_E^{'''}(A^*(p) \cup \{p\})$. Now let $p \in E'$. If $I \in A_{l+1}(p)$ then $I \in \hat{F}_E$. Since $p$ is an endpoint of $I$ and $p \in \hat{Q}_{l-1}$, the other endpoint of $I$ must be an element of $Q_{l+1}$. Hence $I \not\in F_{E_1}$. To show that $h_{E^{'}}(I) \in \hat{F}_{\theta(E)} \setminus F_{\theta(E)}$, note that $p \in E'$ and $h_{E^{'}}$ is continuous on $(\bigcup I_l) \cup E'$. So $f_{E^{'}}(p) = F_{E^{'}}(p)$ is an endpoint of $h_{E^{'}}(I)$. But $f_{E^{'}}$ is level preserving, so $f_{E^{'}}(p) \in \hat{R}_{l+1}$. Again, the other endpoint of $h_{E^{'}}(I)$ must be an element of $R_{l+1}$. Hence $h_{E^{'}}(I) \in \hat{F}_{\theta(E)} \setminus F_{\theta(E)}$. It follows that $A_{l+1}(p) \subset F_{E_2}$ and $h_E(A_{l+1}^*(p) \cup \{p\}) = h_{E^{'}}(A_{l+1}^*(p) \cup \{p\})$. But $h_{E^{'}}$ is a homeomorphism on $(\bigcup I_l) \cup E'$ and $h_E^{'''}$ is a homeomorphism on $\bigcup I_{l-1}$, so $h_E$ is a homeomorphism on $(\bigcup I_{l+1}) \cup E$.
Notice that for any \( k < \omega \), \( E \subseteq M_k \), \( x_n \in X_0 \), and \( I_\sigma \in I_{k,n} \), if \( \Gamma_{E}(I_\sigma) \notin J_{k,n} \) then \( \sigma(1) < \mu(n) \) because only the first \( \mu(n) \) elements of \( I_{1,n} \) have endpoints in \( Q_0 \).

For all \( n < \omega \) and all \( E \subseteq M_n \), define \( H_{E} : S_{E} \rightarrow T_{\theta(E)} \) by \( H_{E}(A) = \lambda(A \cap X_0) \cup h_{E}(A \setminus X_0) \). Finally, define \( H : \text{PR}[X] \rightarrow \text{PR}[Y] \) by \( H = \bigcup_{E \in M} H_{E} \).

To show that \( H \) is a bijection it is sufficient to show that \( \theta \) is a bijection. Let \( E, D \subseteq M \) and \( E \neq D \). Then \( \theta(E) = f_{E}(E) \) and \( \theta(D) = f_{D}(D) \). Both \( f_{E} \) and \( f_{D} \) are level-preserving one-to-one functions, so \( \theta(E) \neq \theta(D) \) if \( E \subseteq M_k \) and \( D \subseteq M_l \) and \( k \neq l \). Assume that \( E, D \subseteq M_k \). Then \( \theta(E) = g(E) \neq g(D) = \theta(D) \) since \( g \) is a bijection. Assume that \( E, D \subseteq M_k \) for some \( k > 1 \). Either \( E \cap Q_k \neq D \cap Q_k \) or \( E' \neq D' \). But the functions \( h_{E'}, h_{E''} \), \( h_{D'}, h_{D''} \) are all one-to-one, so either \( h_{E''}(E \cap Q_k) \neq h_{D''}(D \cap Q_k) \) or \( h_{E'}(E') \neq h_{D'}(D') \). In either case, \( \theta(E) \neq \theta(D) \).

Let \( A \subseteq S_{E} \) where \( E \subseteq M_k \) and let \( V \) be a neighborhood of \( H(A) \) in \( Y \). Pick \( a < \omega \) such that if \( I_{\alpha} \in A_{1}(A) \) then \( \sigma(1) < a \) and if \( J_{\alpha} \in A_{1}(H(A)) \) then \( \sigma(1) < a \). Let \( m = \max \{ \mu(n) : A_{1}(A) \cap I_{1,n} \neq \emptyset \} \). Let \( b \in \omega \) such that \( b - m > a \) and

\[
\text{Int} \left[ \text{Cl} \left( \bigcup \{ J_{\alpha} \in J_{1,n} : \sigma(1) > b \} \right) \right] \subseteq V
\]

for all \( y_n \in H(A) \cap Y_0 \). Set

\[
V_{y_n} = \text{Int} \left[ \text{Cl} \left( \bigcup \{ J_{\alpha} \in J_{1,n} : \sigma(1) > b \} \right) \right]
\]

and set \( V'_{y} = \bigcup_{p \in H(A) \cap Y_0} V_{p} \). Pick \( c \in \omega \) such that \( c - m > b \) and if \( x_n \in A \cap X_0 \) and \( p \in Q_{1} \cap \text{Int}[\text{Cl}(\bigcup \{ I_{\alpha} \in I_{1,n} : \sigma(1) > c \})] \), then \( g(p) \in V_{y_n} \). For each \( x_n \in A \cap X_0 \) set \( U_{x} = \text{Int}[\text{Cl}(\bigcup \{ I_{\alpha} \in I_{1,n} : \sigma(1) > c \})] \). Let \( U_{0} = \bigcup_{p \in A \cap X_0} U_{p} \).

If \( A \cap X_{0} = \emptyset \) then set \( U_{0} = \emptyset \). Pick \( r \geq k + 1 \) such that \( h_{E}(A_{r}(p)) \subseteq V \) for all \( p \in A \cap X_{0} \). Set \( U_{p} = A_{r}(p) \cup \{ p \} \) for \( p \in A \cap X_{0} \) and set \( U_{1} = \bigcup_{p \in A \cap X_{0}} U_{p} \).

Let \( U = U_{0} \cup U_{1} \). Note that:

1. if \( I_{\sigma} \cap U_{1} \neq \emptyset \) then \( \sigma(1) \leq a \);  
2. if \( J_{\sigma} \cap (H(A) \cap Y_{0}) \neq \emptyset \) then \( \sigma(1) \leq a \);  
3. if \( I_{\sigma} \cap U_{x} \neq \emptyset \) for some \( x_n \in A \cap X_0 \) then \( I_{\sigma} \cap I_{1,n} \neq \emptyset \);  
4. if \( J_{\sigma} \cap V_{y} \neq \emptyset \) for some \( y_n \in H(A) \cap Y_{0} \) then \( J_{\sigma} \cap J_{1,n} \neq \emptyset \);  
5. if \( p \in A \cap X_{0} \) then \( U_{p} \cap \hat{Q}_{k+1} \subseteq \{ p \} \).  
6. \( a, b, c \) and \( m \) satisfy condition 1 in Lemmas 4 and 5; and  
7. if \( I_{\sigma} \in I_{k,n} \) and \( m \leq \sigma(1) \) then \( H_{D}(I_{\sigma}) \subseteq \bigcup J_{1,n} \) for any \( 0 < 1 < \omega, n < \omega \), and \( D \in M \).

The heart of the proof that \( H([A, U]) \subseteq [H(A), V] \) is contained in Lemmas 6 and 7.

**Lemma 6.** Let \( D \subseteq M_j \) where \( 1 \leq j \leq k \), \( D \subseteq U \), and \( D \cap U_{1} = E \cap \hat{Q}_{j} \). Let \( C = E \cap \hat{Q}_{j} \). Then

1. if \( p \in D \cap U_{q} \) for some \( q \in A \cap X_{0} \) then \( f_{D}(p) \in V_{q} \);
2. If \( p \in D \cap U_q \) for some \( q \in E \) then \( p = q \) and \( f_D(p) = f_E(p) \);
3. If \( I_\sigma \in J_{j+1} \) and \( \sigma(1) \leq a \) then \( \Gamma_C(I_\sigma) = \Gamma_D(I_\sigma) \); and
4. If \( I_\sigma \in J_{j+1,n} \), \( x_n \in A \cap X_0 \), and \( \sigma(1) > c \), then \( \Gamma_D(I_\sigma) \in J_{j+1,n} \) and \( \Gamma_D(\sigma)(1) > b \).

Proof. To begin with, let us take note of three useful facts. First, since \( \Gamma_D(I_\sigma) = J_\sigma \) for all \( I_\sigma \in I_1 \), if \( I_\sigma \in J_{j+1,n} \) and \( \sigma(1) > c \), then \( \Gamma_D(I_\sigma) = J_\sigma = J_{j+1,n} \) and \( \Gamma_D(\sigma)(1) = \sigma(1) > c > b \). Also, for any \( j \), if \( p \in C \) then \( f_C(p) = f_E(p) \).

Furthermore, if \( I_\sigma \in F_D \) then either \( \sigma(1) \leq a < b \) or \( \sigma(1) > c \).

Let \( j = 1 \). Then \( D \subseteq Q_1 \) and \( D \cap U_1 = E \cap Q_1 \). Let \( p \in D \). If \( p \in U_q \) for some \( q \in A \setminus X_0 \), then \( f_D(p) = g(p) \in V_{\beta(q)} \). If \( p \in U_q \) for some \( q \in A \setminus X_0 \), then \( q = p \) and \( f_D(p) = g(p) = f_C(p) \).

Let \( n < \omega \) and let \( I_\sigma \in J_{j,n} \cap F_D \) with \( \sigma(1) > c \). Let \( p \in D \) be an endpoint of \( I_\sigma \). Since \( \sigma(1) > c \), \( p \) must be in \( U_x \). Then \( f_D(p) \), which is an endpoint of \( \gamma_D(I_\sigma) \), is in \( V_{\gamma_x} \). Thus \( \gamma_D(I_\sigma) \in J_{j,n} \) and \( \gamma_D(\sigma)(1) > b > a \).

It follows from \( D \cap U_1 = C \) that \( F_C = \{ I_\sigma \in F_D : \sigma(1) \leq a \} \). Let \( I_\sigma \in F_C \). Let \( p \in D \) be an endpoint of \( I_\sigma \). Then \( p \) must be an element of \( U_x \), so \( f_D(p) = f_E(p) = f_C(p) \). Thus \( f_E(p) \) is an endpoint for both \( \gamma_C(I_\sigma) \) and \( \gamma_D(I_\sigma) \). Since both \( \gamma_C \) and \( \gamma_D \) preserve orientation, it must be true that \( \gamma_C(I_\sigma) = \gamma_D(I_\sigma) \).

Also, \( \gamma_D(\sigma)(1) \leq a < b \) because \( f_D(p) \in H(A) \cap Y_0 \).

By Lemma 4, if \( I_\sigma \in J_{j,n} \) and \( \sigma(1) > c \), then \( \Gamma_D(I_\sigma) \in J_{j,n} \) and \( \Gamma_D(\sigma)(1) > b > a \). By Lemma 5, if \( I_\sigma \in J_{j,n} \) and \( \sigma(1) \leq a \), then \( \Gamma_D(I_\sigma) = \Gamma_C(I_\sigma) \).

Let \( 2 \leq j \leq k \) and assume that the lemma is valid for all \( 1 \leq i < j \) and all \( D \in M_i \) with \( D \subset U \) and \( D \cap U_1 = E \cap Q_{\hat{i}} \). Let \( D \in M_j \) with \( D \subset U \) and \( D \cap U_1 = E \cap Q_{\hat{j}} \). Then \( D' \in M_{j-1} \), \( D' \subset U \), and \( D' \cap U_1 = E \cap Q_{\hat{j}-1} = C' \), so the lemma is valid for \( D' \). If \( j = 2 \), then \( D'' = C'' = \emptyset \). If \( j > 2 \), then \( D'' \in M_{j-2} \), \( D'' \subset U \), and \( D'' \cap U_1 = E \cap Q_{\hat{j}-2} = C'' \). Thus the lemma is valid for \( D'' \).

Let \( p \in D \cap U_\sigma \) for some \( x_n \in A \cap X_0 \). If \( p \in \hat{j}_1 \) then \( f_D(p) = f_D'(p) \in V_{\gamma_x} \). If \( p \in C \) then \( f_D(p) = h_D''(p) \). Now \( p \) is the midpoint of some element \( I_\sigma \) of \( J_{j-1,n} \) where \( \sigma(1) > c \). But \( \Gamma_D''(I_\sigma) \in J_{j-1,n} \) and \( \Gamma_D''(\sigma)(1) > b > a \) and \( h_D''(I_\sigma) \) is the midpoint of \( \Gamma_D''(I_\sigma) \). Hence \( f_D(p) \in V_{\gamma_x} \).

Let \( p \in D \cap U_q \) for some \( q \in A \setminus X_0 \). Then \( q \in E \) and \( q = p \). If \( p \in \hat{j}_1 \) then \( f_D(p) = f_D'(p) \). If \( p \in C \) then \( f_D(p) = f_D''(p) \). If \( p \in \hat{j}_q \) then

\[
\begin{align*}
\gamma_D(\sigma)(1) \leq a < b
\end{align*}
\]

Let \( n < \omega \) and let \( I_\sigma \in F_D \cap J_{j+1,n} \) with \( \sigma(1) > c \). Either \( \gamma_D(I_\sigma) = \Gamma_D(I_\sigma) \) or \( \gamma_D(I_\sigma) = \Gamma_D''(I_\sigma) \). In either case, \( \gamma_D(I_\sigma) \in J_{j+1,n} \) and \( \gamma_D(J_\sigma)(1) > b > a \).

It follows from the inductive hypotheses that \( F_C = \{ I_\sigma \in F_D : \sigma(1) \leq a \} \) and \( F_C = \{ I_\sigma \in F_D : \sigma(1) \leq a \} \). Let \( I_\sigma \in F_C \). If \( I_\sigma \in F_D \) then \( \gamma_D(I_\sigma) = \Gamma_D''(I_\sigma) \). But \( \Gamma_D''(I_\sigma) = \Gamma_D''(I_\sigma) \) so \( \gamma_D(I_\sigma) = \gamma_D(I_\sigma) \). If \( I_\sigma \in F_D \) then \( \gamma_D(I_\sigma) = \Gamma_D''(I_\sigma) \). But \( \Gamma_D''(I_\sigma) = \Gamma_D''(I_\sigma) \) so \( \gamma_D(I_\sigma) = \gamma_C(I_\sigma) \). In either case, \( \gamma_D(I_\sigma) \leq a < b \).
By Lemma 4, if \( I_\sigma \in I_{j+1,n} \) and \( \sigma(1) > c \), then \( \Gamma_D(I_\sigma) \in J_{j+1,n} \) and \( \Gamma_D(\sigma)(1) > b \). By Lemma 5, if \( I_\sigma \in I_{j+1} \) and \( \sigma(1) \leq a \), then \( \Gamma_D(I_\sigma) = \Gamma^*_E(I_\sigma) \).

Lemma 7. If \( k < l \), \( D \in M_l \), and \( E \subset D \subset U \), then

1. if \( p \in D \cap U_q \) for some \( q \in A \cap X_0 \) then \( f_D(p) \in V_{A(q)} \);
2. if \( p \in D \cap U_q \) for some \( q \in A \setminus X_0 \) then \( f_D(p) \in V \);
3. if \( I_\sigma \in I_{l+1,n} \) for some \( x_n \in A \cap X_0 \) and \( \sigma(1) > c \), then \( \Gamma_D(I_\sigma) \in J_{l+1,n} \) and \( \Gamma_D(\sigma)(1) > b \); and
4. if \( I_\sigma \in I_{l+1} \) and \( \sigma(1) \leq a \) then \( \Gamma_D(I_\sigma) = \Gamma^*_E(I_\sigma) \).

Note that condition 4 implies that \( \gamma_D(\sigma)(1) \leq a \) for all \( I_\sigma \in F_D \) with \( \sigma(1) \leq a \).

Proof. The case \( k = 1 \) is given by Lemma 6.

Assume that \( l = k + 1 \). Then \( D' \in M_k \), \( D' \subset U \), and \( D' \cap U_1 = E \). Also, \( D'' \in M_{k-1} \), \( D'' \subset U \), and \( D'' \cap U_1 = E' \). So Lemma 6 holds for \( D' \) and \( D'' \).

Let \( p \in D \cap U_q \) for some \( q \in A \cap X_0 \) if \( p \in \hat{Q}_k \) then \( f_D(p) = f_{D'}(p) \in U_q \). Let \( p \in Q_k \). Then \( p \) is the midpoint of some element \( I_\sigma \) of \( I_{k,n} \) where \( \sigma(1) > c \). Also, \( f_D(p) = h_{D''}(p) \) and \( h_{D''}(p) \) is the midpoint of \( \Gamma_{D''}(I_\sigma) \). But \( \Gamma_{D''}(I_\sigma) \in J_{k,n} \) and \( \Gamma_{D''}(\sigma)(1) > b \). Thus \( f_D(p) \in U_q \).

Let \( p \in D \cap U_q \) for some \( q \in A \setminus X_0 \). Now \( U_q \cap \hat{Q}_1 \subset \{q\} \) so \( p = q \) and \( p \in \hat{Q}_k \). Thus \( f_D(p) = f_{D'}(p) = f_E(p) \in V \).

Let \( I_\sigma \in F_D \cap I_{l+1,n} \) for some \( x_n \in A \cap X_0 \) and let \( \sigma(1) > c \). Either \( \gamma_D(I_\sigma) = \Gamma^*_{D'}(I_\sigma) \) or \( \gamma_D(I_\sigma) = \Gamma^*_{D''}(I_\sigma) \). In either case, \( \gamma_D(I_\sigma) \in J_{l+1,n} \) and \( \gamma_D(\sigma)(1) > b > a \).

To show that conditions 3 and 4 hold, consider the sets \( F = \{I \in I_{l+1} : I \subset \bigcup F_E\} \) and \( G = \{I_\sigma \in F_D : \sigma(1) \leq a\} \). Define \( \gamma \) on \( G \) by \( \gamma(I) = \Gamma^*_E(I) \). We will show that \( F \subset G \). Let \( I_\sigma \in F \). Then \( \sigma(1) \leq a \) and \( I_{\sigma|k+1} \in \hat{F}_E \). Now \( A(E) \subset A(D) \) because \( E \subset D \). Also, \( A(\theta(E)) \subset A(\theta(D)) \). Thus \( I_\sigma \in \hat{F}_D \) and \( h_{D'}(I_\sigma) = h_{E}(I_\sigma) = \Gamma^*_E(I) \). If \( I_\sigma \in F_{D1} \) then there is \( p \in D \cap Q_l \) such that \( p \) is an endpoint of \( I_\sigma \). Then, since \( \sigma(1) \leq a \), \( p \in U_1 \). But \( U_1 \cap Q_l = \emptyset \), so \( I_\sigma \notin F_{D1} \). If \( p \in D \cap Q_l \), then \( p \in U_0 \) and \( f_D(p) \in V_0 \). But \( \Gamma_{D'}(\sigma)(1) \leq a \), so \( h_{D'}(I_\sigma) \) cannot have an endpoint in \( \theta(D) \cap R_l \). Therefore \( h_{D'}(I_\sigma) \in \hat{F}_{\theta(D)} \), and \( I_{\sigma} \in G \). By Lemma 3, \( \Gamma(I) = \Gamma^*_E(I) \) for all \( I \in I_{l+1} \). If \( I \in G \) then \( I \in F_{D2} \) so \( \gamma_D(I) = \Gamma^*_E(I) = \Gamma^*_E(I) = \gamma(I) \). Thus \( \gamma_D(I_\sigma) \in J_{l+1,n} \) and \( \gamma_D(\sigma)(1) \leq a < b \) for all \( I_\sigma \in F_D \cap I_{l+1,n} \) with \( m \leq \sigma(1) \leq b \). By Lemma 4, if \( I_\sigma \in I_{l+1,n} \) for some \( x_n \in A \cap X_0 \) and \( \sigma(1) > c \), then \( \Gamma_D(I_\sigma) \in J_{l+1,n} \) and \( \Gamma_D(\sigma)(1) > b \). By Lemma 5, \( \Gamma_D(I_\sigma) = \Gamma(I_\sigma) = \Gamma^*_E(I_\sigma) \) for all \( I_\sigma \in I_{l+1} \) with \( \sigma(1) \leq a \).

Let \( l \geq k + 2 \) and assume that if \( j = l - 1 \) or \( j = l - 2 \), \( C \in M_j \), and \( E \subset C \subset U \), then the lemma holds for \( C \). Let \( D \in M_l \) with \( E \subset D \subset U \). Then \( D \cap U_q \cap \hat{Q}_{k+1} = \emptyset \). Furthermore \( D' \in M_{l-1} \), \( E \subset D' \subset U \), \( D'' \in M_{l-2} \), and \( E \subset D'' \subset U \). Thus the lemma holds for \( D' \) and \( D'' \).
Let \( p \in D \cap U_x \) for some \( x_n \in A \cap X_0 \). If \( p \in Q_{l-1} \) then \( f_D(p) \subseteq f_D'(p) \subseteq V_n \). If \( p \in Q_l \) then \( p \) is the midpoint of some \( I_a \in J_{l-1,n} \) with \( \sigma(1) > c \). But \( f_D'(p) = h_D''(p) \) is the midpoint of \( \Gamma_D''(I_a) \) and \( \Gamma_D''(I_a) \in J_{l-1,n} \) with \( \Gamma_D''(\sigma)(1) > b \). Hence \( f_D(p) \in V_n \).

Let \( p \in D \cap U_q \) for some \( q \in A \setminus X_0 \). If \( p \in Q_{l-1} \) then \( f_D(p) = f_D'(p) \subseteq V \).

If \( p \in Q_l \) then \( f_D(p) = h_D''(p) = \Gamma_D''(p) = \Gamma_E(p) \subseteq V \) because \( h_E(U_q) \subseteq V \).

Let \( I_a \in F_D \cap I_{l+1,n} \) for some \( x_n \in A \cap X_0 \) and let \( \sigma(1) > c \). Either \( \gamma_D(I_a) = \Gamma_D'(I_a) \) or \( \gamma_D(I_a) = \Gamma_D''(I_a) \). In either case, \( \gamma_D(I_a) \in J_{l+1,n} \) and \( \gamma_D(\sigma)(1) > b > a \).

To show that conditions 3 and 4 hold, consider the sets \( F = \{ I \in I_{l+1}: I \subseteq \bigcup F_E \} \) and \( G = \{ I \in F_D: \sigma(1) \leq a \} \). Define \( \gamma \) on \( G \) by \( \gamma(I) = \Gamma_E(I) \).

Let \( I_\sigma \in F \). Then \( I_\sigma \in F_D \) because \( E \in D \) and \( h_D'(I_\sigma) = h_E(I_\sigma) \in F_{\theta(D)} \) because \( \theta(E) \subseteq \theta(D) \). Assume that \( I_\sigma \notin F_D \). Let \( p \in D \cap Q_1 \). We will show that \( f_D(p) \) cannot be an endpoint of \( h_D'(I_\sigma) \).

If \( p \in U_0 \), then \( f_D(p) \subseteq V_0 \). But \( \Gamma_D'(\sigma)(1) \leq a \) so \( f_D(p) \) is not an endpoint of \( h_D'(I_\sigma) \).

If \( p \in U_1 \) then \( p \in I_{k+2} \) for some \( I_{k+2} \in I_{k+2,n} \) with \( \tau(1) \leq a \). By the induction hypotheses, \( f_D(p) = h_D'(p) = h_E(p) \subseteq h_E(I_{k+2}) \).

If \( \sigma \in \Gamma_D'(I_{k+2}) \) then \( \gamma_D(I_{k+2}) \in F_{\theta(D)} \) and \( \sigma(1) < b \).

By Lemma 3, \( \gamma(I) = \Gamma_E(I) \) for all \( I \in I_{l+1} \). If \( I \in G \) then either \( \gamma_D(I) = \Gamma_D'(I) \) or \( \gamma_D(I) = \Gamma_D''(I) \). In either case, \( \gamma_D(I) = \Gamma_E(I) = \gamma(I) \).

Thus \( \gamma_D(I_\sigma) \in J_{l+1,n} \) and \( \gamma_D(\sigma)(1) \leq b \) for all \( I_\sigma \in F_D \cap I_{l+1,n} \) with \( m \leq \sigma(1) \leq b \).

By Lemma 4, if \( I_\sigma \in I_{l+1,n} \) for some \( x_n \in A \cap X_0 \) and \( \sigma(1) > c \), then \( \Gamma_D(I_\sigma) \in J_{l+1,n} \) and \( \Gamma_D(\sigma)(1) > b \). By Lemma 5, if \( I_\sigma \in I_{l+1,n} \) and \( \sigma(1) \leq a \), then \( \Gamma_D(I_\sigma) = \Gamma_E(I_\sigma) \).

Now let \( B \in [A, U] \) and let \( B \in S_D \). Then \( D \in M_I \) for some \( l \geq k \) and \( E \subseteq D \subseteq U \). Also, \( B \cap X_0 = A \cap X_0 \) so \( h(B \cap X_0) = h(A \cap X_0) \subseteq V \). Let \( p \in B \setminus X_0 \). If \( p \in D \), then \( f_D(p) \subseteq V \) by Lemma 7. Assume that \( p \notin D \).

There is \( I_\sigma \in I_{l+1} \) such that \( p \in I_\sigma \). If \( p \in U_{x_n} \) for some \( x_n \in A \cap X_0 \) then \( I_\sigma \in I_{l+1,n} \) and \( \sigma(1) > c \). By Lemma 7, \( h_D(I_\sigma) = \Gamma_D'(I_\sigma) \subseteq J_{l+1,n} \) and \( \Gamma_D'(\sigma)(1) > b \).

Thus \( h_D(p) \subseteq V \). If \( p \in U_q \) for some \( q \in A \setminus X_0 \) then \( \sigma(1) \leq a \). By Lemma 7, \( h_D(I_\sigma) = \Gamma_D'(I_\sigma) = \Gamma_E(I_\sigma) \). Thus \( h_D(p) \subseteq V \) because \( h_E(U_q) \subseteq V \).

Therefore \( H(B) \in [H(A), V] \) and \( H \) is continuous. A similar argument shows that \( H^{-1} \) is continuous.

IV. Related results

Corollary 8. If \( X \) and \( Y \) are \( \omega \)-graphs and \( D \) and \( E \) are equipotent discrete subsets of \( X \) and \( Y \) respectively, then \( \bigcup_{p \in D}[p, X] \) is homeomorphic to \( \bigcup_{p \in E}[p, Y] \).
Proof. Extend $D$ and $E$ to dividing sets $X_0$ and $Y_0$ of $X$ and $Y$. Order the sets $X_0$ and $Y_0$ so that $\lambda(D) = E$. Then the homeomorphism defined in the proof of Theorem 1 takes $\bigcup_{p \in D} [p, X]$ to $\bigcup_{p \in E} [p, Y]$, so these two sets are homeomorphic.

The finally results are about spaces other than graphs or $\omega$-graphs. Theorem 2 of [N] shows that points may be removed from certain $T_1$ spaces without affecting its Pixley-Roy hyperspace. The next three lemmas generalize this result. Theorem 12 applies this procedure to $\mathbb{R}^n$.

**Lemma 9.** If $(Z_n : n < \omega)$ is a sequence of disjoint homeomorphic open and closed subsets of $\mathcal{PR}[X]$ such that $\bigcup_{n < \omega} Z_n$ is open and closed in $\mathcal{PR}[X]$, then $\mathcal{PR}[X] \setminus Z_0 \cong \mathcal{PR}[X]$.

**Proof.** For each $n < \omega$ let $H_n : Z_n \to Z_{n+1}$ be a homeomorphism. Define $H : \mathcal{PR}[X] \to \mathcal{PR}[X] \setminus Z_0$ by

$$H(A) = \begin{cases} A & \text{if } A \not\subseteq \bigcup_{n < \omega} Z_n, \\ H_n(A) & \text{if } A \in Z_n. \end{cases}$$

Then $H$ is a homeomorphism.

**Lemma 10.** If $U$ is an open subset of space $X$ and $C$ is closed in $U$ then $\bigcup_{p \in C} [p, U]$ is open and closed in $\mathcal{PR}[X]$.

**Proof.** Clearly $\bigcup_{p \in C} [p, U]$ is an open subset of $\mathcal{PR}[X]$. Let

$$A \in U \setminus \bigcup_{p \in C} [p, U].$$

If $A \not\subseteq U$ then $[A, X]$ is a neighborhood of $A$ that misses $\bigcup_{p \in C} [p, U]$. If $A \subseteq U$ then $A \cap C = \emptyset$, so $[A, U \setminus C]$ is a neighborhood of $A$ in $\mathcal{PR}[X]$ that misses $\bigcup_{p \in C} [p, U]$.

**Lemma 11.** Let $(U_n : n < \omega)$ be a sequence of disjoint open subsets of a space $X$ and let $(C_n : n < \omega)$ be a sequence of subsets of $X$ such that $C_n \subseteq U_n$ and $C_n$ is closed in $U_n$ for all $n < \omega$. Then $\bigcup_{n < \omega} \bigcup_{p \in C_n} [p, U_n]$ is open and closed in $\mathcal{PR}[X]$.

**Proof.** It is clear that $\bigcup_{n < \omega} \bigcup_{p \in C_n} [p, U_n]$ is open in $\mathcal{PR}[X]$. By Lemma 10, each $\bigcup_{p \in C_n} [p, U_n]$ is closed in $\mathcal{PR}[X]$. Let $A \in \mathcal{PR}[X]$. Since $A$ is finite and the $U_n$'s are disjoint, there is a finite subset $B$ of $\omega$ such that $A \cap U_n \neq \emptyset$ if and only if $n \in B$. Then $(\bigcup_{m \in B} [A, U_m]) \cap (\bigcup_{p \in U_n} [p, U_n]) \neq \emptyset$ only if $n \in B$. Thus $\{\bigcup_{p \in C_n} [p, U_n] : n < \omega\}$ is locally finite, and $\bigcup_{n < \omega} \bigcup_{p \in C_n} [p, U_n]$ is closed.

**Theorem 12.** Let $0 < n < \omega$ and let $X = \{\overline{x} \in \mathbb{R}^n : 0 < |\overline{x}| < 1\}$ where $|\overline{x}|$ denotes the Euclidean norm. For any $0 < m < \omega$,

$$\mathcal{PR}[\mathbb{R}^n] \cong \mathcal{PR}[m \times \mathbb{R}^n] \cong \mathcal{PR}[\omega \times \mathbb{R}^n] \cong \mathcal{PR}[m \times X] \cong \mathcal{PR}[\omega \times X].$$

**Proof.** We will show that each of these spaces is homeomorphic to $\mathcal{PR}[\mathbb{R}^n]$. Let $D$ be a discrete subset of $\{x \in \mathbb{R} : x \geq 0\}$ which contains 0 and let $\pi : \mathbb{R}^n \to \mathbb{R}$
be the projection onto the first coordinate. Let \( L = \{ \vec{x} \in \mathbb{R}^n : \pi(\vec{x}) \in D \} \) and let \( C = \{ \vec{x} \in \mathbb{R}^n : |\vec{x}| \in D \}. \) If \( D \) is finite then \( \mathbb{R}^n \setminus L = (|D| + 1) \times \mathbb{R}^n \) and \( \mathbb{R}^n \setminus C = |D| \times X. \) If \( D \) is infinite then \( \mathbb{R}^n \setminus L \approx \omega \times \mathbb{R}^n \) and \( \mathbb{R}^n \setminus C \approx \omega \times X. \)

Let \( U_0 = \mathbb{R}^n \) and let \( \langle U_k : 0 < k < \omega \rangle \) be a sequence of disjoint open balls in \( \mathbb{R}^n, \) each of which has empty intersection with \( L \) and \( C. \)

Set \( C_0 = L. \) For every \( 0 < k < \omega \) let \( C_k \) be a subset of \( U_k \) which is homeomorphic to \( L. \) Then \( C_k \) is closed in \( U_k \) for all \( k < \omega. \) For each \( k < \omega \) set \( Z_k = \bigcup_{p \in C_k} [p, U_k). \) By Lemma 10, each \( Z_k \) is open and closed in \( \text{PR}[\mathbb{R}^n]. \) By Lemma 11, \( \bigcup_{0 < k < \omega} Z_k \) is open and closed in \( \text{PR}[\mathbb{R}^n], \) so \( \bigcup_{k < \omega} Z_k \) is open and closed in \( \text{PR}[\mathbb{R}^n]. \) Clearly each \( Z_k \) is homeomorphic to every other \( Z_k, \) so \( \text{PR}[\mathbb{R}^n] \approx \text{PR}[\mathbb{R}^n] \setminus Z_0 \approx \text{PR}[\mathbb{R}^n \setminus L]. \) If \( D \) is finite then \( \text{PR}[\mathbb{R}^n] \approx \text{PR}[|D| + 1] \times \mathbb{R}^n]. \) If \( D \) is infinite then \( \text{PR}[\mathbb{R}^n] \approx \text{PR}[\omega \times \mathbb{R}^n]. \)

Now let \( C_0 = C \) and for every \( k < \omega \) let \( C_k \) be a subset of \( U_k \) homeomorphic to \( C. \) Set \( Z_k = \bigcup_{p \in C_k} [p, U_k] \) for all \( k < \omega. \) Again, \( \langle Z_k : k < \omega \rangle \) is a sequence of disjoint homeomorphic open and closed subsets of \( \text{PR}[\mathbb{R}^n] \) so \( \text{PR}[\mathbb{R}^n] \approx \text{PR}[\mathbb{R}^n] \setminus Z_0 \approx \text{PR}[\mathbb{R}^n \setminus C]. \) If \( D \) is finite then \( \text{PR}[\mathbb{R}^n] \approx \text{PR}[|D| \times X]. \) If \( D \) is infinite then \( \text{PR}[\mathbb{R}^n] \approx \text{PR}[\omega \times X]. \)

**Bibliography**


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