TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 315, Number 2, October 1989

A MANDELBROT SET WHOSE BOUNDARY IS PIECEWISE SMOOTH

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ABSTRACT. It is proved that the Mandelbrot set associated with the pair of maps $w_{1,2}: \mathbf{C} \to \mathbf{C}$, $w_1(z) = sz + 1$, $w_2(z) = s^* z - 1$, with parameter $s \in \mathbf{C}$, is connected and has piecewise smooth boundary.

INTRODUCTION

The discovery [1] of the Mandelbrot set M for the iterated complex polynomial $z^2 + s$ has generated considerable research activity [2, 3], especially because of its relation to cascades of bifurcations and universal phenomena [4].

The Mandelbrot set M consists of those values of $s \in \mathbf{C}$ such that the Julia set $J(s)$ for $z^2 - s$ is connected. Barnsley and Harrington [5] considered an analogous Mandelbrot set D associated with the two affine maps $T_{1,2}: \mathbf{C} \to \mathbf{C}$ defined by

$$T_1(z) = sz + 1, \quad T_2(z) = sz - 1$$

for $s \in \mathbf{C}$ and $|s| < 1$. There is a unique nonempty compact set $A(s)$ which is invariant under T_1 and T_2 (i.e., $T_1(A(s)) \cup T_2(A(s) = A(s))$ [5, 6]. Generically, $A(s)$ is a fractal. D is defined to be the set of $s \in \mathbf{C}$, $|s| < 1$ for which $A(s)$ is disconnected. The boundary of D contains self-similar structures (see Figure 2) and appears to be a fractal. It is not known whether D is connected; however, new pictures of this set presented here indicate that it is.

In this paper we study the Mandelbrot set G associated with the two affine maps $w_{1,2}: \mathbf{C} \to \mathbf{C}$ defined by

$$w_1(z) = sz + 1, \quad w_2(z) = s^* z - 1$$

for $s \in \mathbf{C}$ with $|s| < 1$. (Here s^* denotes the conjugate of s.) As in the previous case, there is a unique invariant compact set $A(s)$ which is generically a fractal. Despite the apparent similarity between the two pairs of maps, G is easier to analyze than D. We will show among other things that G is connected and, remarkably, has a piecewise smooth boundary. Pictures of the associated fractals as one travels around the boundary of G are given.

Received by the editors November 23, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 40A99, 51M99.

Research by M. F. Barnsley supported in part by DARPA "Image Compression Project," ONR contract N00014-86-C-0446.

1. Preliminaries

Let (X, d) be a compact metric space or $\mathbf{R}^n$ and let H denote the set of all nonempty compact subsets of X. If B, $C \subset X$, then define

$$d(B, C) = \inf\{d(b, c) \mid b \in B,\ c \in C\}$$

and define the Hausdorff metric h on H by

$$h(B, C) = \sup\{d(\{b\}, C), d(\{c\}, B) \mid b \in B,\ c \in C\}.$$

It is known that (H, h) is a complete metric space [6].

Let $0 \leq c < 1$ and let the mappings $w_i \colon X \to X$, $i = 1, \ldots, N$, be such that $d(w_i(x), w_i(y)) \leq c d(x, y)$ for all x, $y \in X$. Following Barnsley and Demko [7] we call $\{X, w_i \colon i = 1, \ldots, N\}$ a hyperbolic iterated function system (HIFS). Define $\underline{w} \colon H \to H$ by

$$\underline{w}(B) = \bigcup_{i=1}^{N} w_i(B) = \bigcup \{w_i(x) \mid x \in B,\ i \in [1, \ldots, N]\}$$

for all $B \in H$. From the definition of h it is immediate that $\underline{w}$ is a contraction on H with $h(\underline{w}(B_1), \underline{w}(B_2)) \leq c h(B_1, B_2)$ for B_1, $B_2 \in H$. Since H is complete, the Banach fixed point theorem implies

Theorem 1. (1) *$\underline{w}$ has a unique fixed point $A \in H$. (A is called the attractor for the HIFS $(X, \underline{w})$.)*

(2) *$\lim_{n\to\infty} \underline{w}^{\circ n}(B) = A$ (i.e., $\lim_{n\to\infty} h(\underline{w}^{\circ n}(B), A) = 0$) for any $B \in H$, where we define $\underline{w}^{\circ 0}(B) = B$ and $\underline{w}^{\circ n}(B) = \underline{w}(\underline{w}^{\circ (n-1)}(B))$ for $n \in \mathbf{N}$.*

We will need the following lemma.

Lemma 2. *If $B \in H$ and $B \supset \underline{w}(B)$ then $\underline{w}^{\circ n}(B) \supset A$ for all $n \in \mathbf{N}$, where A is the attractor for $\{X, \underline{w}\}$. If $\underline{w}^{\circ n}(B)$ is connected for all $n \in \mathbf{N}$ and some $B \subset H$ then A is connected.*

Proof. If $\underline{w}(B) \subset B$, then $\underline{w}^{\circ n}(B) \subset \underline{w}^{\circ (n-1)}(B)$ for all $n \in \mathbf{N}$. Thus $A = \lim_{n\to\infty} \underline{w}^{\circ n}(B) = \bigcap_{n=1}^{\infty} \underline{w}^{\circ n}(B)$, because the sequence of compact sets $\{\underline{w}^{\circ n}(B)\}$ is decreasing.

Suppose A is disconnected; then $A = B_1 \cup B_2$ with B_1, $B_2 \in H$ and $B_1 \cap B_2 = \varnothing$. Thus $d(B_1, B_2) > 0$ and so for any set C such that $h(A, C) < d(B_1, B_2)/2$ then C is also disconnected. Since $\lim_{n\to\infty} h(A, \underline{w}^{\circ n}(B)) = 0$, $\underline{w}^{\circ n}(B)$ is eventually disconnected for any $B \in H$. □

The following corollary generalizes a result of Barnsley and Harrington [5].

Corollary 3. *Let (X, w_1, w_2) be an HIFS with attractor A such that there exists a nonempty connected $B \in H$ with $\underline{w}(B) \subset B$. A is disconnected if and only if $w_1(A) \cap w_2(A) = \varnothing$.*

Proof. If $w_1(A) \cap w_2(A) = \varnothing$ then $w_1(A)$ and $w_2(A)$ form a disconnection of A.

Suppose $w_1(A) \cap w_2(A) \neq \varnothing$. By Lemma 2, $\underline{w}^{\circ n}(B) \supset A$. Suppose $\underline{w}^{\circ n}(B)$ is connected; then $w_1(\underline{w}^{\circ n}(B)) \cap w_2(\underline{w}^{\circ n}(B)) \supset w_1(A) \cap w_2(A) \neq \varnothing$. By continuity $w_1(\underline{w}^{\circ n}(B))$ and $w_2(\underline{w}^{\circ n}(B))$ are connected so $\underline{w}^{\circ(n+1)}(B)$ is connected. Since B is connected, $\underline{w}^{\circ(n)}(B)$ is connected for all $n \in \mathbf{N}$ by induction. By Theorem 1, $\lim_{n\to\infty} h(\underline{w}^{\circ n}(B), A) = 0$ and so by Lemma 2, A is connected. $\square$

Note that if $X = \mathbf{R}^n$ then we can always find a nonempty connected $B \in H$ such that $\underline{w}(B) \subset B$; for instance, if we pick the radius large enough we can take B to be a closed ball centered at the origin.

If $(X, \underline{w}(\lambda, \cdot))$ is an HIFS for each λ in an index set Λ, then we define the Mandelbrot set for the family $\{(X, \underline{w}(\lambda, \cdot)) \mid \lambda \in \Lambda\}$ to be the set of $\lambda \in \Lambda$ for which $A(\lambda)$ (i.e., the attractor for $(X, \underline{w}(\lambda, \cdot))$) is disconnected.

2. A previously considered Mandelbrot set

Consider the family of pairs of maps $T_i : \mathbf{C} \to \mathbf{C}$, $i = 1, 2$, defined by

$$T_1(s, \cdot): z \to sz + 1, \qquad T_2(s, \cdot): z \to sz - 1$$

for $s \in \mathbf{C}$ and $|s| < 1$. Note that T_1 and T_2 are similitudes. Let $\theta(s) = \arg(s)$ and suppose $B \subset K$. Geometrically, $\underline{T}(s, B) = T_1(s, B) \cup T_2(s, B)$ is generated by shrinking B by $|s|$ toward 0, rotating by $\theta(s)$ about 0, and translating one such copy by $1 + i0$ and another by $-1 + i0$.

Since $|T_i(s, a) - T_i(s, b)| = |s||a - b|$ for $i = 1, 2$ and a, $b \in \mathbf{C}$, we see that $(\mathbf{C}, T_1(s, \cdot),\ T_2(s, \cdot))$ is an HIFS for $|s| < 1$. Let $A(s)$ denote the attractor for this HIFS. Figure 1 shows $A(s)$ for several values of s. It is instructive to identify $T_1(s, A(s))$ and $T_2(s, A(s))$ and to note that $A(s)$ is indeed the fixed point of $\underline{T}(s, \cdot)$.

Barnsley and Demko [7] investigated the Mandelbrot set for the family of HIFSs $\{(\mathbf{C}, \underline{T}(s, \cdot)) \mid s \in \mathbf{C},\ |s| < 1\}$. We will denote this Mandelbrot set by D. Figure 2 shows a computer-generated picture of D (from [7]) along with several blowups of portions of the boundary of D. They hypothesized that D may be disconnected; however, Figure 2 suggests that the opposite may be true.

They found inner and outer bounds for D using the fact that if $s \in D$ then the Hausdorff dimension d of $A(s)$ is given by

$$d = \log(\tfrac{1}{2}) / \log(|s|).$$

We will prove the same bounds for D using the results we developed in the previous section. In the following, we will suppress the s dependence of T_1 and T_2.

Proposition 4. *If* $|s| < .5$ *then* $s \in D$.

Proof. Let $R_s = 1/(1 - |s|)$ and $B(x, r) = \{z \in \mathbf{C} \mid |z - x| \leq r\}$. Then $T_1(B(0, R_s)) = B(1, |s|R_s) \subset B(0, R_s)$ and $T_2(B(0, R_s)) = B(-1, |s|R_s)$ (see Figure 3). Thus, $\underline{T}(B(0, R_s)) \subset B(0, R_s)$ and, by Lemma 2, $A(s) \subset B(0, R_s)$. If $|s| < .5$ then $|s|R_s < 1$ and so $B(-1, |s|R_s) \cap B(1, |s|R_s) = \varnothing$. Thus $T_1(A(s)) \cap T_2(A(s)) = \varnothing$ and $A(s)$ is disconnected. $\square$

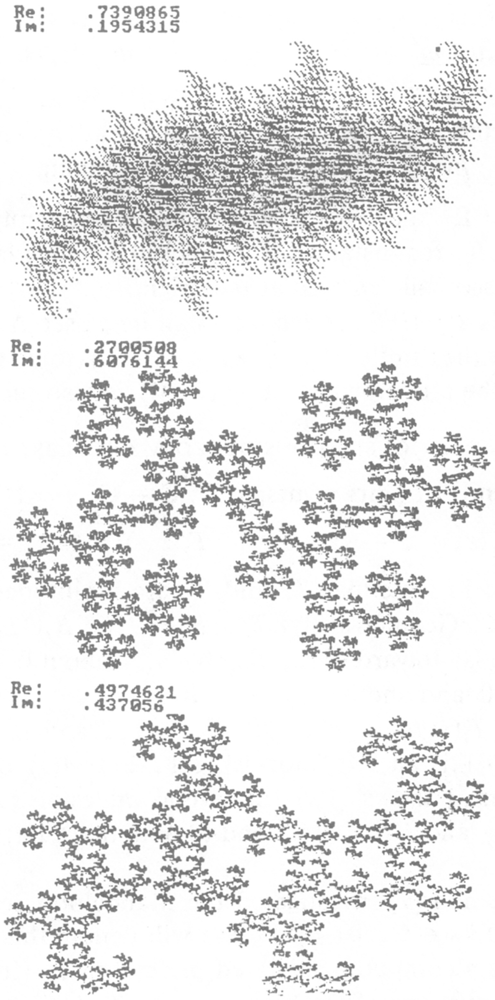


FIGURE 1. The attractor $A(s)$ for I shown for various values of s

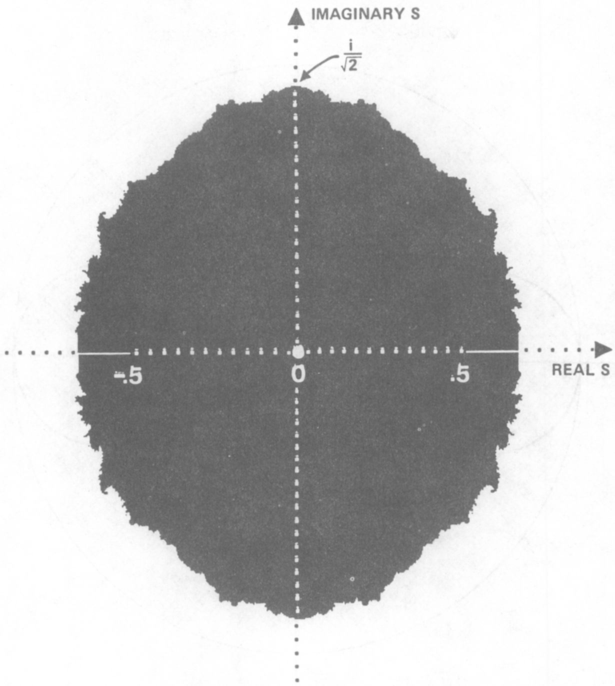
We can calculate successively better inner bounds for D by determining the values of s for which it is true that $T_1(\underline{T}^{\circ n}(B(0,R_s)))\cap T_2(\underline{T}^{\circ n}(B(0,R_s)))=\varnothing$ for successively larger values of n. In fact, all of D can be calculated in this manner.

Theorem 5. *$s\in D$ if and only if $T_1(\underline{T}^{\circ n}(B(0,R_s)))\cap T_2(\underline{T}^{\circ n}(B(0,R_s)))=\varnothing$ for some $n\in\mathbf{N}$.*

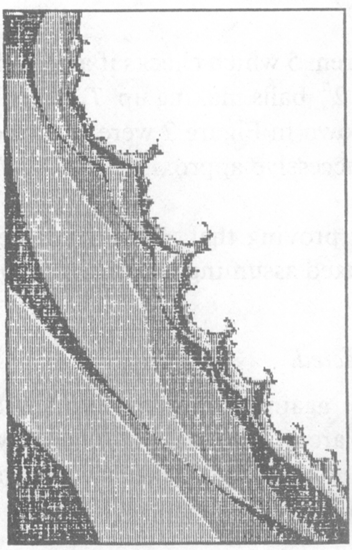
Proof. Let $B=B(0,R_s)$ and $B_n=\underline{T}^{\circ n}(B(0,R_s))$ for $n\in\mathbf{N}$.

By Lemma 2, $A(s)\subset B_n$ for all $n\in\mathbf{N}$, so $A(s)$ is disconnected if $T_1(B_n)\cap T_2(B_n)=\varnothing$ for some $n\in\mathbf{N}$.

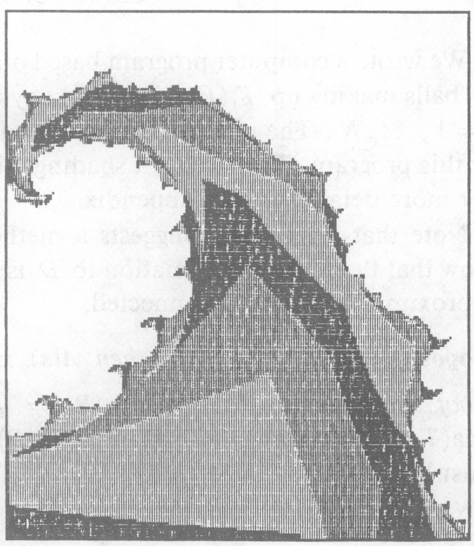
If $T_1(B_n)\cap T_2(B_n)\neq\varnothing$ for all $n\in\mathbf{N}$, then since T_1 and T_2 are continuous and B is connected, we get, via an induction, that B_n is connected. By Lemma 2, $A(s)$ is connected. $\square$



(a) The Mandelbrot set D



(b) Blowup of part of D, where $.49 \leq \mathrm{Re}[s] \leq .55$ and $.35 \leq \mathrm{Im}[s] \leq .45$



(c) Blowup of part of D, where $.572 \leq \mathrm{Re}[s] \leq .593$ and $.352 \leq \mathrm{Im}[s] \leq .378$

FIGURE 2

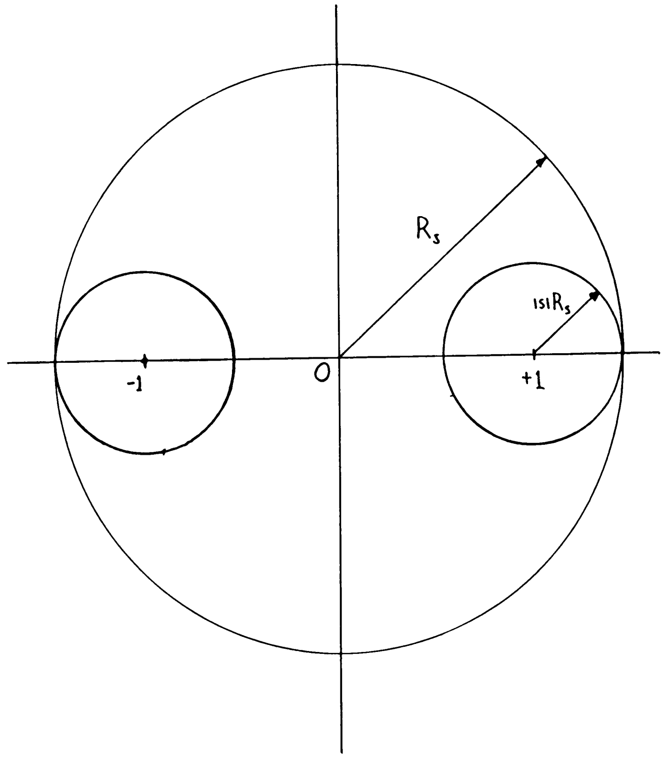


FIGURE 3. $B(0, R_s)$

We wrote a computer program based on Theorem 5 which checks if any of the 2^n balls making up $T_1(B_n)$ intersect any of the 2^n balls making up $T_2(B_n)$ for $n = 1, \ldots, N$. The pictures of the boundary shown in Figure 2 were generated by this program. The different shadings show successive approximations to D. For more details, see the appendix.

Note that Theorem 5 suggests a method for proving that D is connected: show that the nth approximation to D is connected assuming that the $(n-1)$th approximation to D is connected.

Proposition 6. *If* $|s| < 1/\sqrt{2}$ *then* $A(s)$ *is connected.*

Proof. Let $B = B(0, R_s)$ and $B_n = \underline{T}^{\circ n}(B)$ again. If $|s| > 1/\sqrt{2}$ then $\text{area}(T_1(B)) > .5\,\text{area}(B)$ and $\text{area}(T_2(B)) > .5\,\text{area}(B)$. Since $\underline{T}(B) \subset B$ we must have $T_1(B) \cap T_2(B) \neq \varnothing$. Since $\underline{T}(B_n) \subset B_n$ we must have, in the same way, that $T_1(B_n) \cap T_2(B_n) \neq \varnothing$. By Proposition 5, $A(s)$ is connected. □

Proposition 6 can be generalized to get

Proposition 7. *If* $(\mathbf{R}^n, w_1, w_2)$ *is an HIFS such that* $\text{vol}_n(w_1(B)) > \frac{1}{2}\text{vol}_n(B)$ *for every set* $B \in \mathbf{R}^n$ *with finite and nonzero n-dimensional volume* $\text{vol}_n(B)$, *then the attractor* A *is connected.*

3. A piecewise smooth Mandelbrot set

Now consider the family of HIFSs arising from the pair of maps $w_i: \mathbf{C} \to \mathbf{C}$, $i = 1, 2$, defined by

$$w_1(s, \cdot): z \to sz + 1, \qquad w_2(s, \cdot): z \to s^* z - 1$$

for $s \in \mathbf{C}$ and $|s| < 1$.

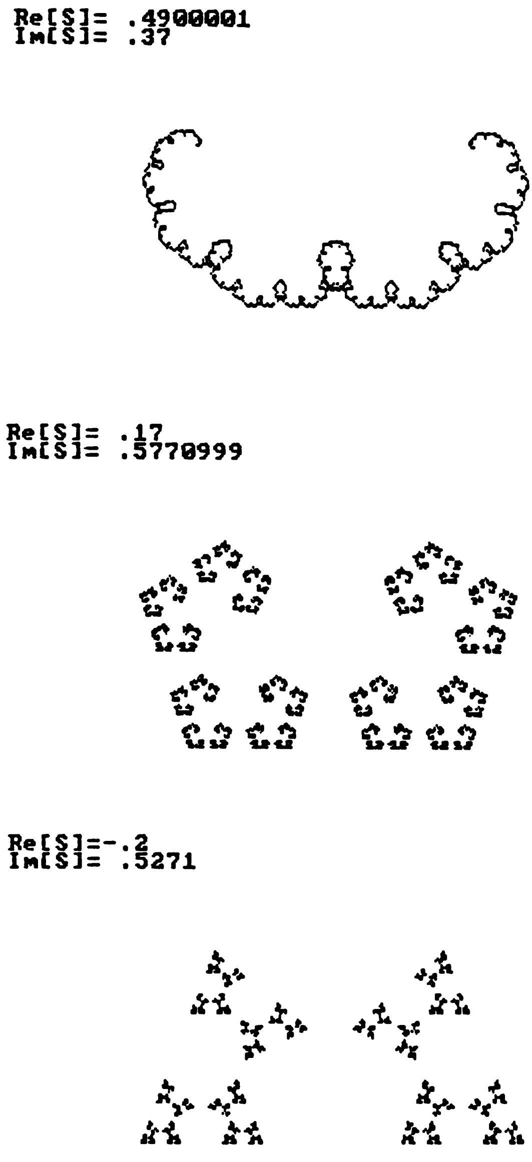


FIGURE 4. $A(s)$ for various values of s

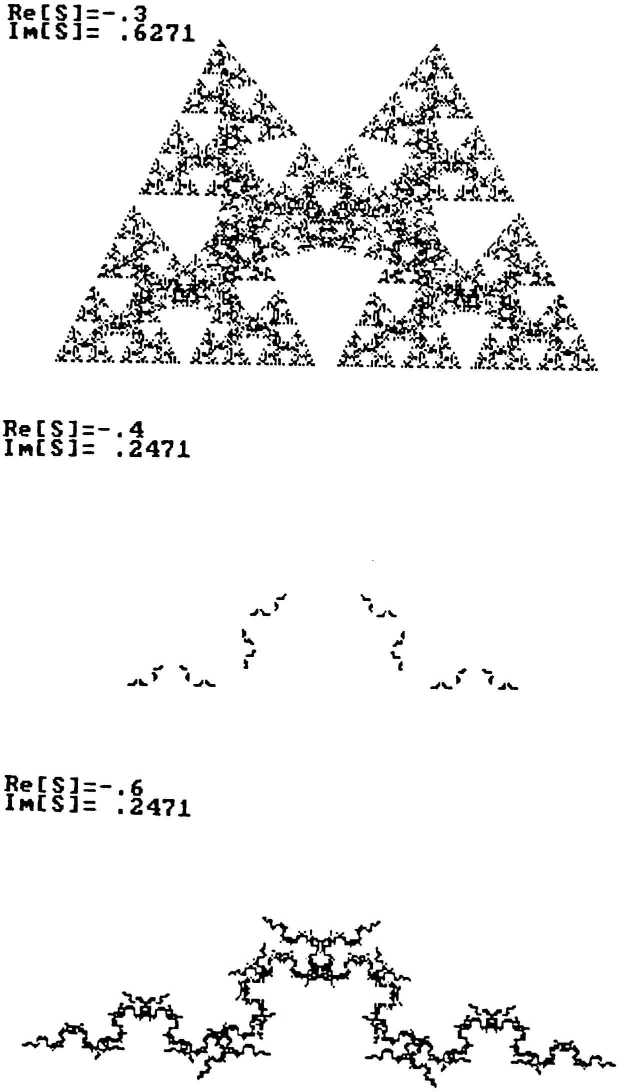


FIGURE 4 (continued)

Geometrically, $\underline{w}$ acts on a set B in almost the same way as $\underline{T}$, the difference being that one of the shrunken copies is rotated by $-\theta(s)$ where $\theta(s) = \arg(s)$ again. Figure 4 shows $A(s)$ for various values of s.

Let G denote the Mandelbrot set for $\{(\mathbf{C}, \underline{w}(s, \cdot)) | s \in \mathbf{C},\ |s| < 1\}$. As we shall see, in contrast to all other known cases, G can be completely described in an elementary way. We will show that G is connected and that the boundary of G is a countable collection of pieces of polynomial curves in $x = \mathrm{Re}[s]$ and $y = \mathrm{Im}[s]$. Figure 5 shows a picture of G. Note that the inner and outer bounds for D are also applicable to G by exactly the same arguments.

First we will prove that G is symmetric about the real axis.

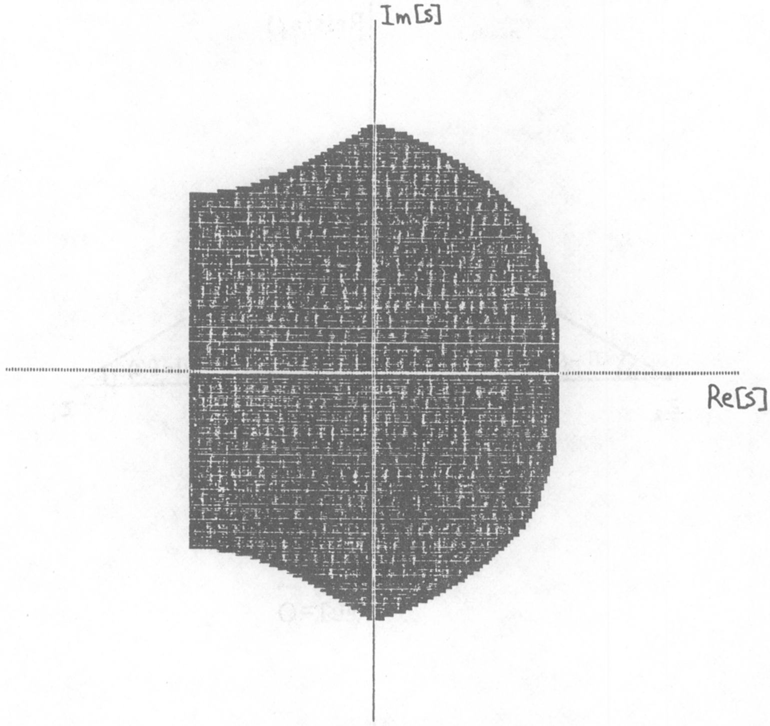


FIGURE 5. The Mandelbrot set G

Proposition 8. *$A(s^*) = -A(s)$ so $s \in G$ if and only if $s^* \in G$.*

Proof. $A(s)$ satisfies $\underline{w}(s, A(s)) = (sA(s)+1) \cup (s^*A(s)-1) = A(s)$. Thus, $-A(s) = (s(-A(s))-1) \cup (s^*(-A(s))+1) = w_2(s^*, -A(s)) \cup w_1(s^*, -A(s)) = \underline{w}(s^*, -A(s))$. Since $\underline{w}(s^*, \cdot)$ has a unique fixed point, $-A(s) = A(s^*)$. □

Let z_1 be the fixed point of the contraction $w_1 \circ w_2$ and z_2 be the fixed point of $w_2 \circ w_1$. We will need the following collection of facts, which follow directly from the definitions of w_1 and w_2. Hereafter we will suppress the s dependence of w_1 and w_2.

Lemma 9.

(a) $z_1 = (1-s)/(1-|s|^2) = -z_2^* = w_1(z_2)$.

(b) $z_2 = (s^*-1)/(1-|s|^2) = -z_1^* = w_2(z_1)$.

(c) $-w_1(x)^* = w_2(-z^*)$ *for* $z \in \mathbf{C}$.

(d) *From the above we get*

(i) $w_2(z_2) = -w_1(z_1)^*$ *and*

(ii) $w_2 \circ w_2(z_2) = -(w_1 \circ w_1(z_1))^*$.

Proposition 10. *If $3\pi/4 \le \theta(s) \le \pi$ and $\mathrm{Re}[w_1(z_1)] > 0$ then $A(s)$ is disconnected.*

Proof. Since $|s| < 1$, it is clear that $\mathrm{Re}[z_1] = \mathrm{Re}[(1-s)/(1-|s|^2)] > 0$. Let B denote the closed convex hull of $\{z_1, z_2, w_1(z_1), w_2(z_2)\}$. The idea of

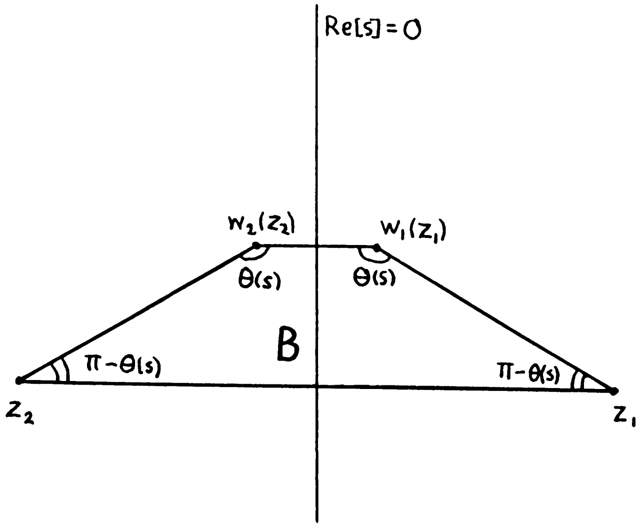


FIGURE 6

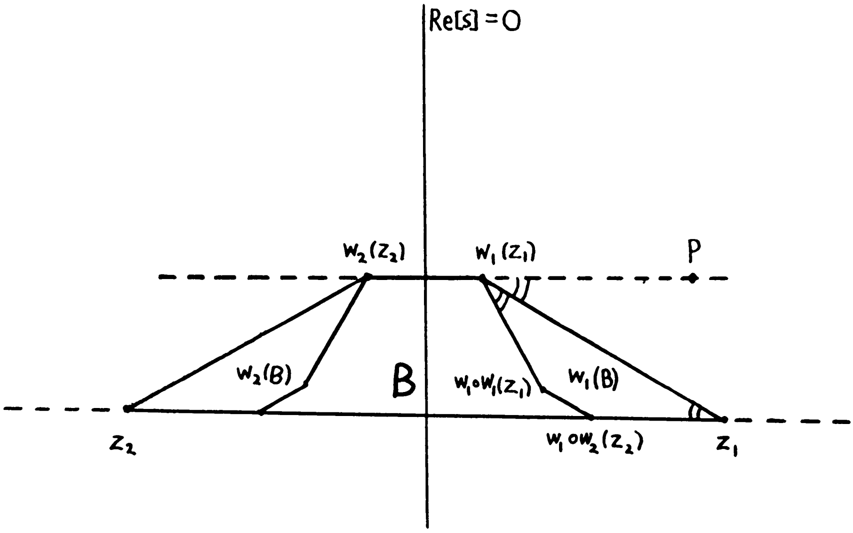


FIGURE 7

the proof is to show that $\underline{w}(B) \subset B$ so that $A \subset B$ and then to show that $w_1(B) \cap w_2(B) = \varnothing$.

Lemma 9 gave $z_2 = -z_1^*$ and $w_2(z_2) = -(w_1(z_1))^*$ so B is a trapezoid as shown in Figure 6. Since $w_1(\overline{z_2 z_1}) = \overline{z_1 w_1(z_1)}$, then from the definition of w_1 we see that the vertex angle at $w_1(z_1)$ is $\theta(s)$ and the vertex angle at z_1 is $\pi - \theta(s)$. By symmetry the vertex angle at z_2 is $\pi - \theta(s)$ and the vertex angle at $w_2(z_2)$ is $\theta(s)$.

Consider Figure 7. Since the angle $\angle z_1 z_2 w_2(z_2)$ measured from $\overline{z_2 z_1}$ in a counterclockwise direction is $\pi - \theta(s)$ (i.e., the vertex angle at z_2) we see that

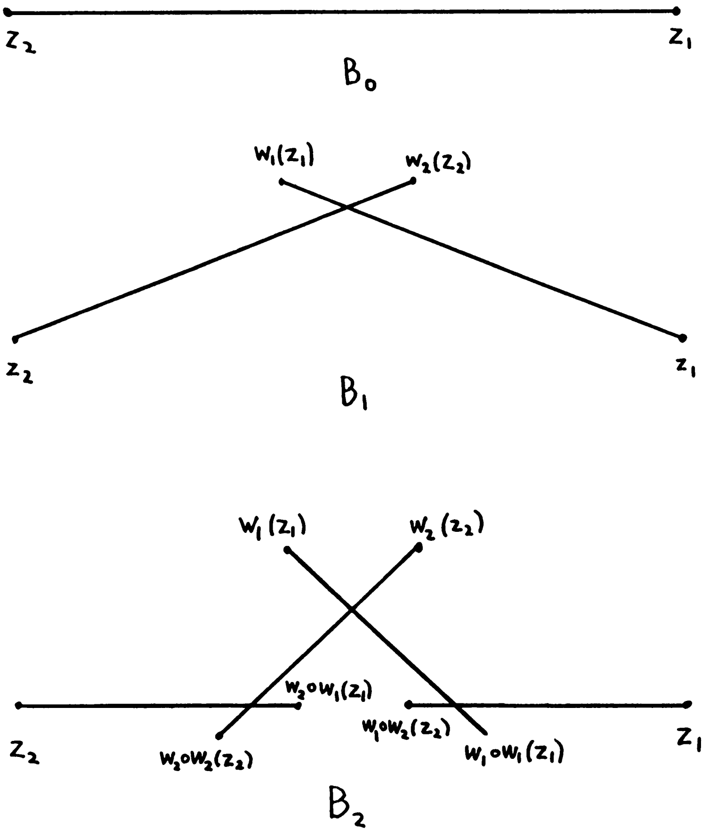


FIGURE 8

the angle $\angle w_1(z_1)z_1w_1 \circ w_2(z_2)$ measured from $\overline{z_1w_1(z_1)}$ in a ccw direction is $\pi - \theta(s)$. Thus $w_1 \circ w_2(z_2)$ lies on $\overline{z_1z_2}$. Similarly, $\angle w_1w_1(z_1)w_1(z_1)z_1$ measured from $\overline{z_1w_1(z_1)}$ in a ccw direction is $\pi - \theta(s)$. Let P be a point on the line through $w_1(z_1)$ and $w_2(z_2)$ with $\mathrm{Re}[P] > \mathrm{Re}[w_1(z_1)]$; then $\angle Pw_1(z_1)z_1$ measured from $\overline{Pw_1(z_1)}$ in a ccw direction is $\pi - \theta(s)$. Thus $\angle Pw_1(z_1)w_1w_1(z_1)$ is $2(\pi - \theta(s))$, which is between 0 and $\pi/2$. Thus, $w_1 \circ w_1(z_1) \in B$ and $\mathrm{Re}[w_1(z_1)] \le \mathrm{Re}[w_1 \circ w_1(z_1)] \le \mathrm{Re}[w_1 \circ w_2(z_2)] \le \mathrm{Re}[z_1]$. Now $w_1(B)$ is the trapezoid with vertices $\{z_1, w_1(z_1), w_1 \circ w_1(z_1),\ w_1 \circ w_2(z_2)\}$, all of which we have shown to lie in B. Thus, $w_1(B) \subset B$. Furthermore, if $z \in w_1(B)$ then $\mathrm{Re}[z] \ge \mathrm{Re}[w_1(z_1)] > 0$. Lemma 9 implies $w_2(B) = -(w_1(B))^*$, so if $z \in w_2(B)$ then $\mathrm{Re}[z] < 0$. Thus $w_2(B) \subset B$ and so $A(s)$ is disconnected. □

The converse is also true.

Proposition 11. *If* $3\pi/4 \le \theta(s) \le \pi$ *and* $\mathrm{Re}[w_1(z_1)] \le 0$ (*equivalent* $\mathrm{Re}[s] \le -.5$) *then* $A(s)$ *is connected.*

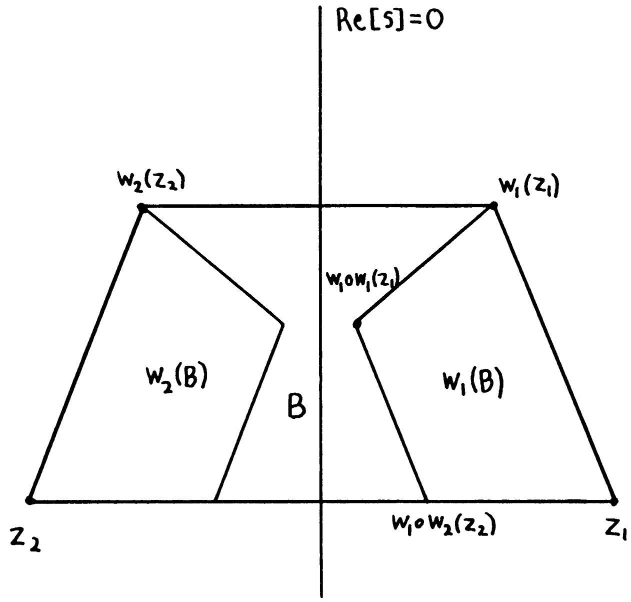


FIGURE 9

Proof. Let $B_0 = \overline{z_1 z_2}$ and $B_n = \underline{w}^{\circ n}(B_0)$ for $n \in \mathbf{N}$. Figure 8 shows B_0, B_1, B_2 for a typical s. We will first show that B_n is connected.

Since $w_2(z_1) = z_2$, $w_1(z_2) = z_1$, and z_1, $z_2 \in B_0$ then z_1, $z_2 \in B_n$ and thus $w_1(z_1)$, $z_1 \in w_1(B_n)$ for $n \in \mathbf{N}$.

Note that $-B_0^* = B_0$. Suppose $-B_n^* = B_n$; then $-B_{n+1}^* = (-w_1(B_n))^* \cup (-w_2(B_n))^* = w_2(-B_n^*) \cup w_1(-B_n^*) = \underline{w}(B_n) = B_{n+1}$. By induction $-B_n^* = B_n$ for $n \in \mathbf{N}$. Thus $-(w_1(B_n))^* = w_2(-B_n^*) = w_2(B_n)$, so if $x \in w_1(B_n)$ and $\mathrm{Re}[x] = 0$ then $x \in w_2(B_n)$.

Note that B_0 is connected. If B_n is connected then $w_1(B_n)$ and $w_2(B_n)$ are connected. Recall that $w_1(z_1)$, $z_1 \in w_1(B_n)$, and that $\mathrm{Re}[z_1] > 0$ and by hypothesis $\mathrm{Re}[w_1(z_1)] \leq 0$. By the intermediate value theorem there must be some $a \in w_1(B_n)$ with $\mathrm{Re}[a] = 0$. But then $a \in w_2(B_n)$ so $w_1(B_n) \cap w_2(B_n) \neq \varnothing$ and $\underline{w}(B_n)$ is connected. The proposition then follows from Lemma 2. $\square$

Figures 9 and 10 illustrate the case for $\theta(s) \in [\pi/2, 3\pi/4]$. Now $w_1 \circ w_1(z_1)$ plays the role that $w_1(z_1)$ played for $\theta(s) \in [3\pi/4, \pi]$.

Proposition 12. *If* $3\pi/4 \leq \theta(s) \leq \pi/2$ *then* $A(s)$ *is disconnected if and only if* $\mathrm{Re}[w_1 \circ w_1(z_1)] > 0$.

Proof. Suppose $\mathrm{Re}[w_1 \circ w_1(z_1)] > 0$. Again let B be the trapezoid with vertices $\{z_1, z_2, w_1(z_1), w_2(z_2)\}$. From the proof of Proposition 11 it still follows that $\underline{w}(B) \subset B$. Since $\angle P w_1(z_1) w_1 \circ w_1(z_1) = 2(\pi - \theta(s)) \in [\pi/2, \pi]$ we see that $\mathrm{Re}[w_1 \circ w_1(z_1)] \leq \mathrm{Re}[w_1(z_1)]$. Since $\angle w_1 \circ w_1(z), w_1 \circ w_2(z_2) z_1 = \theta(s) \in [3\pi/4, \pi/2]$ we see that $\mathrm{Re}[w_1 \circ w_1(z_1)] \leq \mathrm{Re}[w_1 \circ w_2(z_2)] \leq \mathrm{Re}[z_1]$. Thus $0 < \mathrm{Re}[w_1 \circ w_1(z_1)] \leq \mathrm{Re}[z]$ for all $z \in w_1(B)$ and so $w_1(B) \cap w_2(B) = \varnothing$.

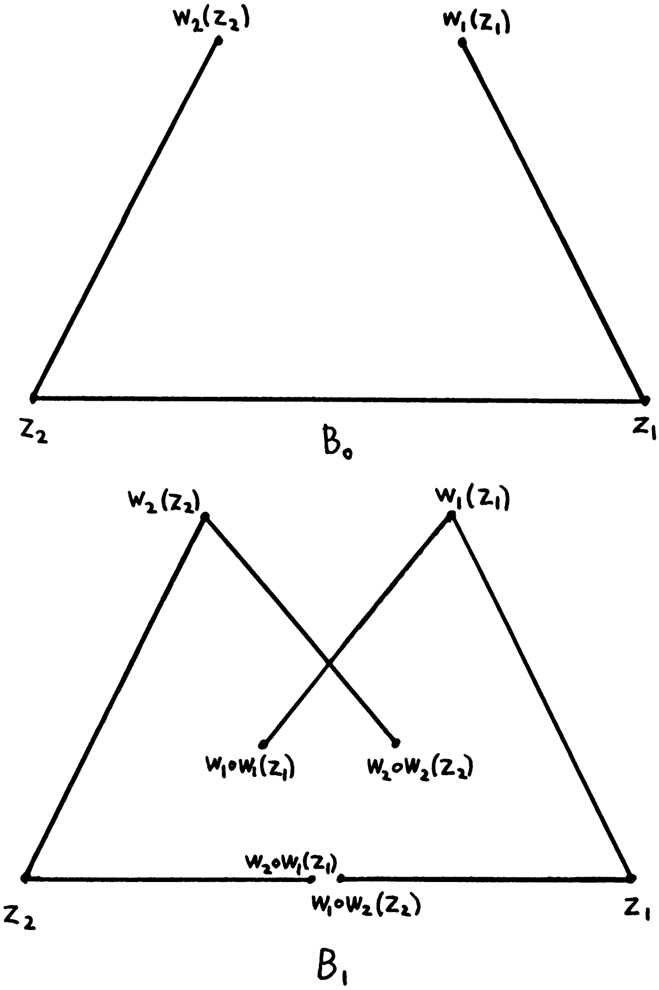


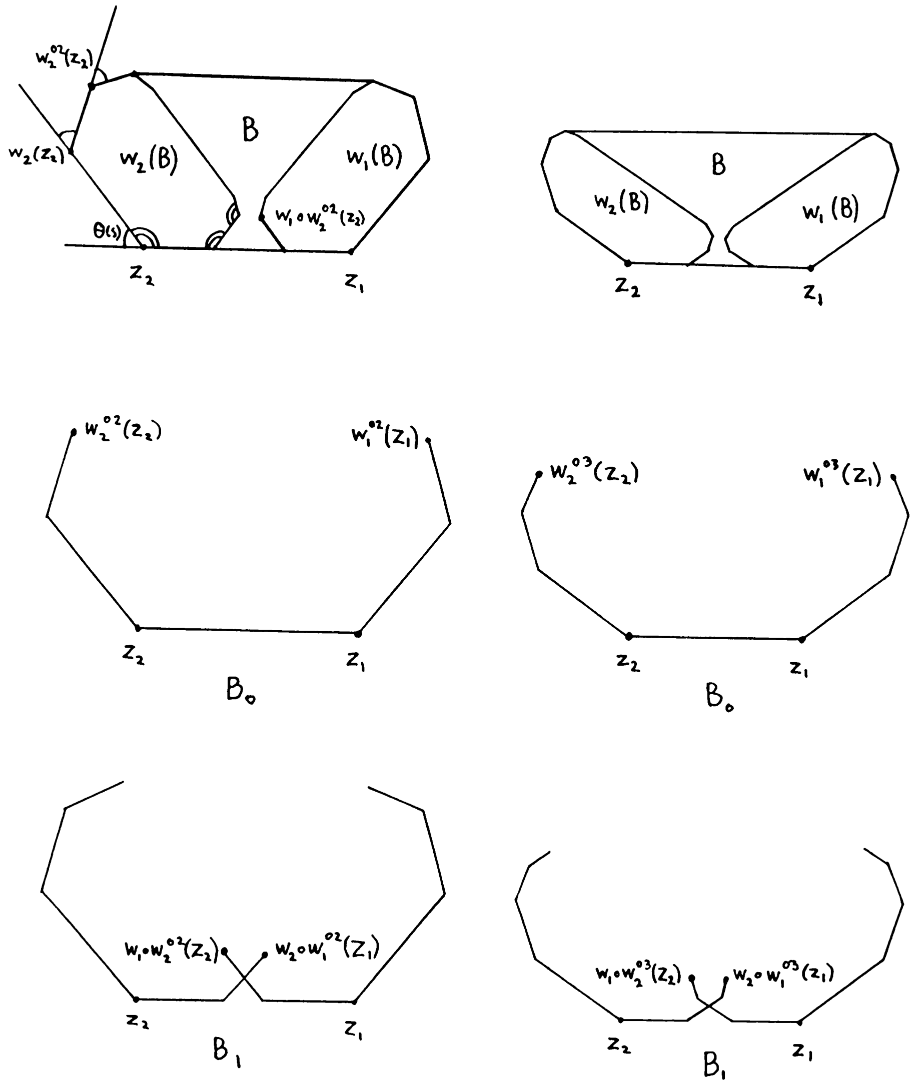
FIGURE 10

Now suppose $\mathrm{Re}[w_1 \circ w_1(z_1)] \leq 0$. Let $B_0 = \overline{w_1(z_1)z_1} \cup \overline{z_1 z_2} \cup \overline{z_2 w_2(z_2)}$ and let $B_n = \underline{w}^{\circ n}(B_0)$. By an induction B_n is connected and so by Lemma 2 $A(s)$ is connected. □

Proposition 13. *If* $n \in \mathbf{N}$ *and* $\pi/(2n+2) \leq \theta(s) \leq \pi/(2n)$ *then* $A(s)$ *is disconnected if and only if* $\mathrm{Re}[w_1 \circ w_2^{\circ(n+1)}(z_2)] > 0$.

Proof. Since the method of proof should be familiar by now, we will only outline the proof of this proposition. Figure 11 illustrates the case for $n = 1$ and $n = 2$.

Suppose $\mathrm{Re}[w_1 \circ w_2^{\circ(n+1)}(z_2)] > 0$. Let B be the closed convex hull of $\{z_1, z_2, w_1(z_1), w_2(z_2), \ldots, w_1^{\circ(2n+1)}(z_1),\ w_2^{\circ(2n+1)}(z_2)\}$; then $\underline{w}(B) \subset B$ and if $z \in w_1(B)$ then $\mathrm{Re}(z) \geq \mathrm{Re}(w_1 \circ w_2^{\circ(n+1)}(z_2)) > 0$. Thus $w_1(B) \cap w_2(B) = \varnothing$ and so $A(s)$ is disconnected.



$\pi/4 \leq \theta(s) \leq \pi/2$ $\qquad$ $\pi/6 \leq \theta(s) \leq \pi/4$

FIGURE 11

Suppose $\operatorname{Re}[w_1 \circ w_2^{\circ(n+1)}(z_2)] \leq 0$. Let

$$B_0 = \overline{z_1 z_2} \cup \left\{ \bigcup_{i=0}^{n} \left(\overline{w_1^{\circ(i+1)}(z_1) w_1^{\circ(i)}(z_1)} \cup \overline{w_2^{\circ(i+1)}(z_2) w_2^{\circ(i)}(z_2)} \right) \right\}.$$

An induction shows that $\underline{w}^{\circ k}(B_0)$ is connected for all $k \in \mathbf{N}$, so $A(s)$ is connected. $\square$

The conditions given in Propositions 10–13 that s be on ∂G can be expressed as polynomial curves in $x = \operatorname{Re}[s]$ and $y = \operatorname{Im}[s]$. For $|s| < 1$,

$\mathrm{Re}[w_1(z_1)] = 0$ if and only if $x = -.5$, and $\mathrm{Re}[w_1 \circ w_1(z_1)] = 0$ if and only if $2x + 2x + 1 - 2y^2 = 0$. Note that $\mathrm{Re}[w_1 \circ w_2^{\circ(n+1)}(z_2)] = 0$ if and only if

$$\mathrm{Re}\left[|s|^2(s^n - s^{n+1}) + (1 - |s|^2)\left(|s|^2 \sum_{p=0}^{n} s^{p-1} - 1\right)\right] = 0,$$

which describes a polynomial curve for each $n \in \mathbf{N}$. We will now use these conditions to prove our main result.

Theorem 14. *G is connected.*

Proof. We will show that for each $\theta(s) \in [0, \pi]$ there is an $r^* \in (0, 1)$ such that $s \in G$ if and only if $|s| < r^*$. Recall that we already know that $s \in G$ if $|s| < .5$ and that $s \notin G$ if $|s| > 1/\sqrt{2}$. Thus we need only show that the appropriate function (for instance $\mathrm{Re}[w_1 \circ w_1(z_1)]$ for $\theta(s) \in [\pi/2, 3\pi/4]$) can be zero at most once in the interval $(.5, 1/\sqrt{2}]$.

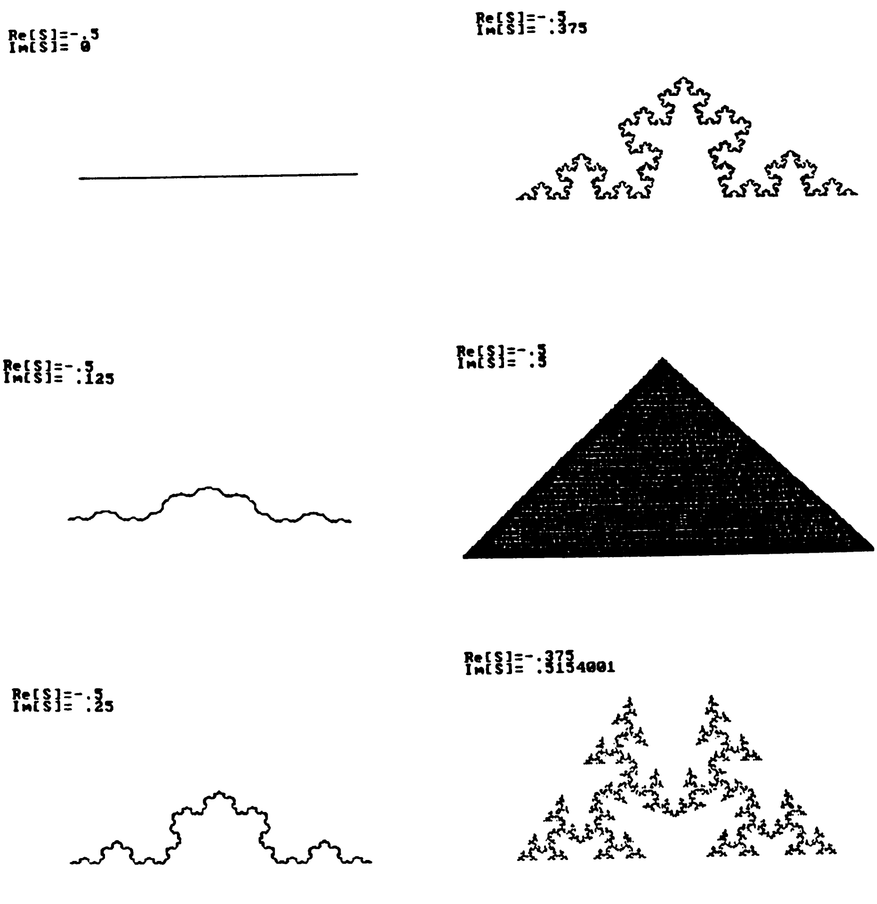


FIGURE 12. $A(s)$ as s varies along ∂G

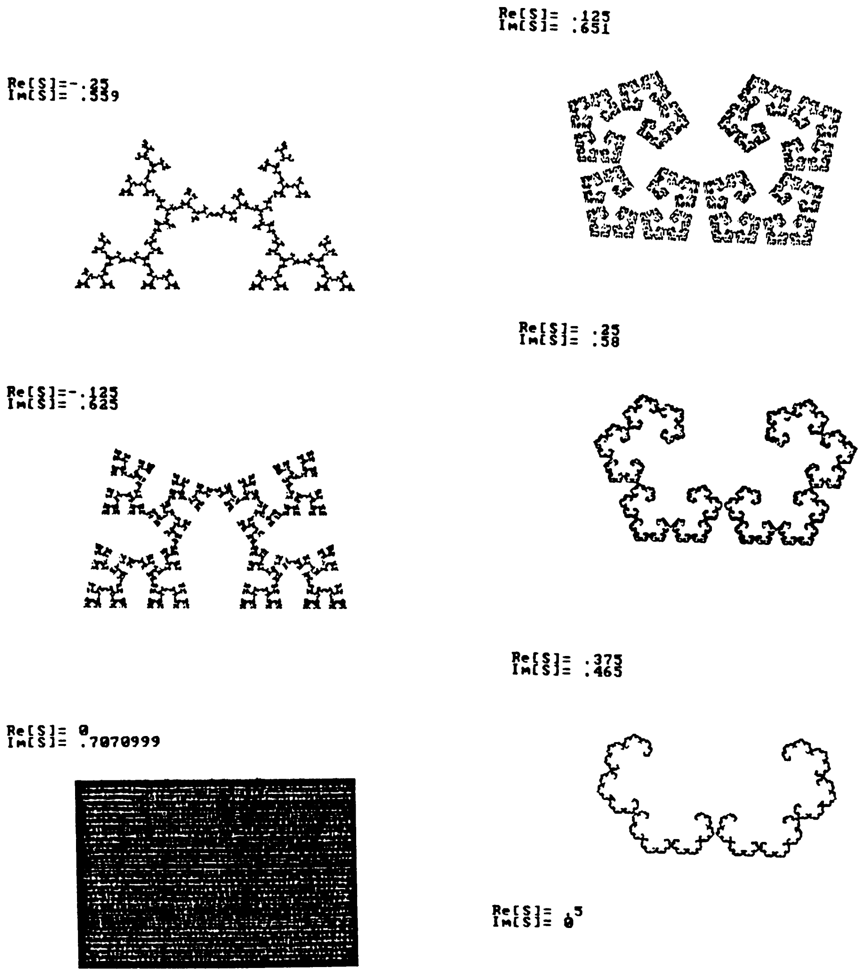


FIGURE 12 (continued)

Case 1. $\theta(s) \in [3\pi/4, \pi]$: Propositions 10 and 11 imply $s \in G$ if and only if $\mathrm{Re}[s] < .5$; however, $\mathrm{Re}[s] < .5$ if and only if $|s| < .5|\sec\theta(s)| \equiv r^*$.

Case 2. $\theta(s) \in [\pi/2, 3\pi/4]$: By Proposition 12, $s \in G$ if and only if $\mathrm{Re}[w_1 \circ w_1(z_1)] = 0$. Define

$$f(r) = \mathrm{Re}[w_1 \circ w_1(z_1)] = [r^2/(1-r^2)][\cos 2\theta(s) - r\cos 3\theta(s)] + r\cos\theta(s) + 1,$$

where $r = |s|$. Since $\cos\theta(s) < 0$, $\cos 2\theta(s) < 0$, and $\cos 3\theta(s) > 0$, it is clear that $f(r)$ is a decreasing function for $r \in (0, 1)$ and thus can be zero at most once in the interval $(.5, 1/\sqrt{2}]$.

Case 3. $\theta(s) \in (0, \pi/2]$: Let n be such that $\pi/2^{n+1} \leq \theta(s) \leq \pi/2^n$. Now $s \in G$ if and only if $\mathrm{Re}[w_1 \circ w_2^{\circ(n+1)}(z_2)] > 0$. Define

$$\begin{aligned} f(r) &= \mathrm{Re}[w_1 \circ w_2^{\circ(n+1)}(z_2)] \\ &= (r^{(n+3)} \cos[(n+1)\theta(s)] - r^{(n+2)} \cos[n\theta(s)])/(1-r^2) \\ &\quad - \left(\sum_{p=0}^{n} r^{p+1} \cos[(p-1)\theta(s)] \right) + 1. \end{aligned}$$

It is a short exercise in freshman calculus to show that $f(r)$ is decreasing on $(.5, 1/\sqrt{2})$.

Case 4. $\theta(s) = 0$: $A(s)$ is an interval if $|s| \geq .5$ and a Cantor set if $|s| < .5$, so $s \in G$ if and only if $|s| < .5$.

Since G is symmetric about the real axis, we see that G is connected. □

A tour around the boundary of G. The evolution of $A(s)$ as s varies along ∂G is rather interesting. Figure 12 shows $A(s)$ at various values of s on ∂G. Note that $A(s)$ consists of a family of Koch [8] curves as s varies from $-.5$ to $-.5 + i.5$, at which point $A(s)$ is a right triangle. The other interesting point is $s = i/\sqrt{2}$, where $A(s)$ is a rectangle.

The family of attractors for $\theta(s) \in (0, \pi/2)$ includes fractals which arise as natural boundaries in the complex t-plane for nonintegrable dynamical systems [9, 10]. In fact, these fractals provided our original motivation for studying this particular family.

Appendix

In this appendix we present a computer program which generates computer images of D. The program can be used with minor modifications to find the Mandelbrot set for any family of pairs of similitudes on $\mathbf{R}^2$.

The program runs on the IBM PC microcomputer in compiled BASIC. A typical picture is produced in approximately 12 hours when the number of iterations is between 10 and 15. The program is much slower in regions which are near ∂D and which are near the real axis.

```
10 DIM AX (4),AY(4),X(2,2,30),Y(2,2,30),P(2,30),RSC(30),LN(30)
20 INPUT "window in parameter space a<Re[S]<b;c<Im[S]<d";AA,BB,CC,DD
30 INPUT "pixel window; px1,px2,py1,py2 where 0<=px1<px2<320
          and 0<=py1<py2<200 (e.g. 40,279,0,199 gives a square)";PX1,PX2,PY1,PY2
40 INPUT "file name for picutre";PICFILE$
50 INPUT "number of iterations<=30";NUMIT
60 HX=(BB-AA)/(PX2-PX1) : HY=(DD-CC)/(PY2-PY1)
70 SCREEN 1,0:KEY OFF: CLS
80 P(1,0)=1:P(2,0)=2
90 FOR SY=CC TO DD STEP HY
100 FOR SX=AA TO BB STEP HX
110 PSX=(PX2-PX1)*(SX-AA)/(BB-AA)+PX1 : PSY=(PY2-PY1)*(DD-SY)/(DD-CC)+PY1
120 SC=SX*SX+SY*SY
130 IF SC>.5 THEN GOTO 300
140 RSC(0)=SC/(1-SQR(SC))^2
150 FOR K=1 TO NUMIT
160 RSC(K)=RSC(K-1)*SC
170 NEXT K
180 IF RSC(0)<1 THEN COLCODE=3:GOTO 370
190 N=1:COLCODE=1
200 LN(N)=1:WP=1:P(1,N)=1:GOSUB 380
210 IF COLCODE<N THEN COLCODE=N
220 P(2,N)=1:WP=2:GOSUB 380
230 A=1:B=1:GOSUB 520: IF DST<=RSC(N) THEN GOTO 290
240 LN(N)=2:WP=2:P(2,N)=2:GOSUB 380
250 A=1:B=2:GOSUB 520: IF DST<=RSC(N) THEN GOTO 290
260 LN(N)=3:WP=1:P(1,N)=2:GOSUB 380
270 A=2:B=1:P(2,N)=1:GOSUB 520:IF DST<=RSC(N) THEN GOTO 290
280 LN(N)=4:A=2:B=2:P(2,N)=2:GOSUB 520:IF DST>RSC(N) THEN GOTO 550
290 IF N<NUMIT THEN N=N+1 : GOTO 200
300 NEXT SX
310 NEXT SY
320 DEF SEG = &HB800 :BSAVE PICFILE$,0,&H4000
330 INPUT WONT
340 IF WONT THEN GOTO 20
350 END
360 COLCODE = COLCODE MOD 3 +1
370 PSET (PSX,PSY),COLCODE : GOTO 300
380 XX=0:YY=0
390 FOR K=0 TO N
400 ON P(WP,N-K) GOSUB 440,480
410 NEXT K
420 X(WP,P(WP,N),N)=XX:Y(WP,P(WP,N),N)=YY
430 RETURN
440 XN=SX*XX-SY*YY+1
450 YY=SX*YY+SY*XX
460 XX=XN
470 RETURN
480 XN=SX*XX-SY*YY-1
490 YY=SX*YY+SY*XX
500 XX=XN
510 RETURN
520 DELX=X(1,A,N)-X(2,B,N):DELY=Y(1,A,N)-Y(2,B,N)
530 DST=.25*(DELX*DELX+DELY*DELY)
540 RETURN
550 IF N=1 THEN GOTO 360
560 N=N-1 : ON LN(N) GOTO 240,260,280,550
```

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