DOUBLE SHOCK FRONTS FOR HYPERBOLIC SYSTEMS
OF CONSERVATION LAWS IN MULTIDIMENSIONAL SPACE

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Abstract. The existence of a unique double shock front for hyperbolic systems
of conservation laws in several space variables is established, extending an earlier
result of Metivier. An example of a double shock wave arising from physical
applications is given.

The purpose of this paper is to show the existence of a unique double shock
front for hyperbolic systems of conservation laws in several space variables
describing physical phenomena.

Unlike the one-dimensional case, it is only recently that Majda [7], using
microlocal analysis, established the existence of a unique stable shock front for
hyperbolic systems. For double shock fronts the only known result is due to
Metivier [8, 9], where $2 \times 2$ systems of conservation laws in two-dimensional
space are considered.

In this paper the existence of a double shock front for hyperbolic systems of
conservation laws in several space variables is established. When restricted to
$2 \times 2$ systems in two-dimensional space, our result is similar to that of Metivier
but our approach is somewhat different and perhaps simpler.

The notation and the basic assumptions are given in §1. The free-boundary
problem is transformed in §2 into one with fixed boundaries. The linearized
problem is considered in §3 and the nonlinear one is studied in §4, using the iter-
ation method. Finally, in §5 an application to isentropic gas in two-dimensional
space is given.

1

Let $G$ be an open subset of $R^n$ with a smooth boundary, $\partial G$, and let
$x = (x_1, \ldots, x_n)$ be the generic point of $G$. For each $n$-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$
of nonnegative integers we write

$$D^\alpha = \prod_{j=1}^n D_j^{\alpha_j}, \quad D_j = \partial / \partial x_j, \quad |\alpha| = \sum_{j=1}^n \alpha_j.$$
We denote by $H^k(G)$ the Sobolev space
\[ H^k(G) = \{ u ; D^n u \text{ in } L^2(G) , \, |\alpha| \leq k \} . \]
It is a Hilbert space with the norm
\[ \| u \|_{H^k(G)} = \left\{ \sum_{|\alpha| \leq k} \| D^n u \|_{L^2(G)}^2 \right\}^{1/2} \]
and the corresponding inner product.

$H^{k-1/2}(\partial G)$ is the usual space of trace functions.

Let $\eta > 1$, $S$ be a subset of $R_i$ and denote by $H^k_\eta(\partial G \times S)$ the hyperbolic $\eta$-weighted Sobolev space $H^k_\eta(\partial G \times S) = \{ u(x,t) : \eta^{k_i}(D^n D^{k_2-\alpha} u) \text{exp}(-\eta t) \text{ in } L^2(\partial G \times S), \, |\alpha| \leq k_2, \, k_1 + k_2 \leq k \} .$

It is a Hilbert space with the norm
\[ \| u ; \partial G \times S \|_{k_1, \eta} = \| u \|_{H^k_\eta(\partial G \times S)} \]
\[ = \left\{ \sum_{k_1 + k_2 \leq k} \int_{\partial G \times S} \eta^{2k_i}(D^n D^{k_2-\alpha} u)^2 \text{exp}(-2\eta t) \, dx \, dt \right\}^{1/2} . \]
$H^k_\eta(\partial G \times S)$ is similarly defined with the norm
\[ |u ; \partial G \times S|_{k_1, \eta} = \| u \|_{H^k_\eta(\partial G \times S)} . \]

In this paper we shall consider the system
\[ (1.1) \quad D_t \{ F_0(u) \} + \sum_{j=1}^n D_j \{ F_j(u) \} = 0 \text{ in } R^n \times R_i^+ , \]
where $u = (u_1 , \ldots , u_m)$. The system (1.1) may be rewritten as
\[ (1.2) \quad D_t u + \sum_{j=1}^n A_j(u)D_j u = 0 \text{ in } R^n \times R_i^+ . \]

Throughout the paper it is implicitly assumed that the system (1.2) is symmetric hyperbolic; i.e., there is a smoothly varying positive matrix $A_0(u)$ with $A_0 A_j$ symmetric and $A_j \in C^\infty(Q)$ for $1 \leq j \leq n$, where $Q$ is the value domain of $u$ in $R^m$.

Let $\Gamma_0$ be a smooth hypersurface containing the origin and consider the Cauchy problem
\[ (1.3) \quad \begin{cases} D_t \{ F_0(u) \} + \sum_{j=1}^n D_j \{ F_j(u) \} = 0 \text{ in } R^n \times R_i^+ , \\ u(x,0) = u_0(x) \text{ in } R^n , \end{cases} \]
where $u_0(x)$ is a piecewise smooth function with a discontinuity of the first kind along $\Gamma_0$.
Since the speed of propagation of a hyperbolic system is finite and we are seeking a local solution of (1.3), there is no loss of generality in assuming that outside a neighborhood of the origin, \( u_0 = \text{constant} \) and \( \Gamma_0 \) is represented by \( x_1 = 0 \). In the neighborhood \( G \) of the origin, we assume that \( \Gamma_0 \) is given by
\[
x_1 = \phi_0(x_2, \ldots, x_n) = \phi_0(x').
\]
Set
\[
\Omega_0^+ = \{x: x \in G, \phi_0(x') < x_1\}, \quad \Omega_0^- = \{x: x \in G, x_1 < \phi_0(x')\}.
\]
Let \( \Gamma_j(t), \, j = 1, 2, \) be two hypersurfaces given by \( x_1 = \phi_j(x', t) \) and let
\[
\Omega_j(t) = \{x: x_1 < \phi_j(x', t)\}, \quad \Omega_2(t) = \{x: \phi_1 < x_1 < \phi_2\}, \quad \text{and}
\]
\[
\Omega_3(t) = \{x: \phi_2(x', t) < x_1\}.
\]
Denote by
\[
\Omega_T^i = \bigcup_{0 \leq i \leq T} \Omega_i(t), \quad 1 \leq i \leq 3.
\]

**Definition 1.1.** Let \( \phi_0, u_0^\pm \) be \( C^\infty \)-functions with \( \phi_0 = 0, \ u_0^\pm = \text{const. for } |x| \) large. Then \( \{u; \Gamma_1(t), \Gamma_2(t)\} \) is said to be an \( {\mathcal{A}}^k_T \)-double shock wave solution of (1.3) if
\[
\begin{align*}
(\text{i}) & \quad \Gamma_1(0) = \Gamma_2(0) = \Gamma_0, \quad u(x, 0) = u_0 = \{u_0^+, u_0^-\}; \\
(\text{ii}) & \quad \Gamma_j(t) \text{ is given by} \\
& \quad x_1 = \phi_j(t, x'), \quad j = 1, 2,
\end{align*}
\]
with \( \phi_j \) in \( H^{k+1}(R^{n-1} \times (0, T)) \) for some \( T > 0 \);
\[
(\text{iii}) \quad u \text{ is a solution of (1.3) with } u|\Omega_T^i \text{ in } H^k(\Omega_T^i), \ 1 \leq i \leq 3;
\]
and the Rankine-Hugoniot conditions are satisfied:
\[
N_0^j[F_0(u)]^j + \sum_{i=1}^n N_i^j[F_i(u)]^j = 0 \quad \text{on } \Gamma_j(t), \ j = 1, 2,
\]
where \( (N_0^j, N_1^j, \ldots, N_n^j) \) is the unit exterior normal vector to \( \Gamma_j(t) \) and \( \lfloor f \rfloor^j \) is the jump of \( f \) across \( \Gamma_j(t) \).

In this paper we shall establish the existence of an \( {\mathcal{A}}^k_T \)-double shock wave solution of (1.3). We shall need the following assumptions.

**Assumption (I).** (i) \( F_j \) is in \( C^\infty \) for \( j = 1, \ldots, n \) and the system (1.2) satisfies the block structure of Majda [6].

(ii) There are two scalar functions \( n_0^1, n_0^2 \) with \( n_0^2 > n_0^1 \) and a state function \( \hat{u} \) defined on \( \Gamma_0 \), all in \( H^\infty_{ul} \) (the uniformly local Sobolev space of Kato [3]) such that
\[
n_0^j[F_0(\hat{u})]^j - [F_j(\hat{u})]^j + \sum_{i=2}^n (F_j(\hat{u}))^i D_j \phi_0 = 0 \quad \text{at } t = 0, \quad i = 1, 2,
\]
where \( \hat{u} = (u_0^-, \hat{u}, u_0^+) \).
(iii) Two constant state shock fronts \((u_0^-, \tilde{u}, n_0^1)\) and \((\tilde{u}, u_0^+, n_0^2)\) are separately uniformly stable in the sense of Majda [6].

A compatibility condition is required throughout the paper.

**Assumption (II).** The compatibility conditions arising from
\[
D_t\{F_0(u)\} + \sum_{j=1}^{n} D_j\{F_j(u)\} = 0
\]
and from
\[
[F_0(u)]^i D_i \phi_i - [F_i(u)]^i + \sum_{j=2}^{n} [F_j(u)]^i D_j \phi_i = 0, \quad i = 1, 2,
\]
are satisfied up to order \(r\) with \(r\) large.

**Remark 1.1.** (1) The compatibility condition (II) is equivalent to the requirement that \(\lambda \geq \lambda_0\) in Theorem 6.11 of [9].

(2) For an \(m \times m\) system of conservation laws, if we are seeking an \(m\)-shock wave solution, then the compatibility condition is not needed. One has \(m \times m\) boundary conditions with \(m \times m\) unknowns. If a smooth solution exists, then the compatibility condition is automatically satisfied.

(3) In this paper, with the Euler equations of polytropic gases in mind, Assumption (II) is needed. A calculation shows that the stable multi-shock front solution for polytropic gases can have at most two shocks emanating from a single discontinuity.

2

In this section we shall transform the problem of finding a double shock wave solution of (1.3) into one with fixed boundaries. It will be carried out in three steps.

**Step 1.** Set
\[
t = \tau, \quad y' = x', \quad y_1 = t\{x_1 - \phi_1(x', \tau)\}/(\phi_2 - \phi_1).
\]
The transformation (2.1) has been used in [2] to treat rarefaction waves; see also [10].

With the transformation (2.1) the domains \(\Omega'(t)\) become respectively
\[
Y_1 = R^{n-1} \times R_1^+ \times R^+ = \{(y, t) : y_1 < 0, t > 0\},
\]
\[
Y_2 = R^{n-1} \times R_1^+ \times \{y_1 : 0 < y_1 < t\}, \quad \text{and}
\]
\[
Y_3 = R^{n-1} \times R_1^+ \times \{y_1 : 0 < t < y_1\}.
\]
Moreover, \(\partial / \partial t = \partial / \partial \tau + a_1(y, \tau) \partial / \partial y_1\) with
\[
a_1(y, \tau) = D_1\{t(x_1 - \phi_1)(\phi_2 - \phi_1)^{-1}\}
\]
\[
= -t(\phi_2 - \phi_1)^{-1}D_1\phi_1 + y_1 t^{-1}(\phi_2 - \phi_1)^{-1}
\]
\[
\times \{\phi_2 - \phi_1 - tD_1(\phi_2 - \phi_1)\}.
\]
**Proposition 2.1.** Suppose that Assumption (I) is satisfied and that \( \phi_j \) is in \( \mathcal{H}_{\text{loc}}^{k+1}(\mathbb{R}^{n-1} \times \mathbb{R}^+_{\tau}) \). Then \( a_i \) is in \( \mathcal{H}_{\text{loc}}^{k}(\mathbb{R}^n \times \mathbb{R}^+_{\tau}) \) for any \( k \) with \( k \geq k_0 > 1 + [(n + 1)/2] \).

**Proof.** For \( k > 1 + [(n + 1)/2] \), \( \mathcal{H}^k \) is an algebra. It is clear that we have only to consider the case when \( t \to 0^+ \). Since \( \phi_2(x', 0) - \phi_1(x', 0) = 0 \), we have

\[
\phi_2(x', t) - \phi_1(x', t) = t \int_0^1 D_i \{ \phi_2 - \phi_1 \}(x', st) \, ds.
\]

So

\[
(2.3) \quad \lim_{t \to 0^+} \{ (\phi_2(x', t) - \phi_1(x', t))t^{-1} \} = \int_0^1 D_i \{ \phi_2 - \phi_1 \}(x', 0) \, ds
\]

by Assumption (I). Rewriting \( a_1(y, t) \) as

\[
a_1(y, t) = -D_j \phi_1 \{ t(\phi_2 - \phi_1)^{-1} \} + y_j (\phi_2 - \phi_1) t^{-1} D_i \{ t(\phi_2 - \phi_1)^{-1} \}
\]

we get the proposition by taking into account (2.3) and the Banach algebra property of \( \mathcal{H}_{\text{loc}}^k \).

With the transformation (2.1) we have

\[
(2.4) \quad \frac{\partial}{\partial y_j}(y_j) = a_j(y, t) = -t(\phi_2 - \phi_1)^{-1} D_j \phi_1 - y_j (\phi_2 - \phi_1) t^{-1} D_i \{ t^{-1}(\phi_2 - \phi_1) \}
\]

for \( 2 \leq j \leq n \).

**Proposition 2.2.** Suppose the hypothesis of Proposition 2.1 are satisfied. Then \( a_j \) is in \( \mathcal{H}_{\text{loc}}^{k}(\mathbb{R}^n \times \mathbb{R}^+_{\tau}) \), \( 2 \leq j \leq n \).

**Proof.** Same as that of Proposition 2.1.

We now rewrite equation (1.3) as

\[
(2.5) \quad D_i u^i + \tilde{A}_1(u^i; \phi_1, \phi_2) D_j u^j + \sum_{j=2}^n A_j(u^i) D_j u^i = 0 \quad \text{in} \, \mathcal{Y}_i, \quad 1 \leq i \leq 3,
\]

where

\[
(2.6) \quad \tilde{A}_1(u^i; \phi_1, \phi_2) = A_1(u^i) + a_1(\phi_1, \phi_2) - \sum_{j=2}^n A_j(u^i) a_j(\phi_1, \phi_2).
\]

**Step 2.** We shall now use the compatibility Assumption (II) to transform the problem into one with homogeneous initial data.

Let \( \tilde{u}_i, n_i^j, \, i = 1, 2 \), be as in Assumption (I). We can construct \( u_0^i, \phi_0^i \) in \( \mathcal{H}^r \), \( r \) large, such that

\[
(2.7) \quad u^i(y, t) = u^i - u_0^i, \quad \psi^i(y, t) = \phi_i - \phi_0^i
\]
have zero initial data. Equation (2.5) becomes

\[ D_t v^i + A_1(v^i + u_0^i; \phi_0^i + \psi^i)D_t v^i + \sum_{k=2}^{n} A_k(v^i + u_0^i)D_k(v^i + u_0^i) \]
\[ + A_1(v^i + u_0^i; \phi_0^i + \psi^i)D_t u_0^i = -D_t u_0^i, \quad 1 \leq j \leq 3, \quad i = 1, 2. \]

Using a Taylor's expansion for \( A_1, A_k \) we obtain

\[ D_t v^i + A_1(v^i + u_0^i; \phi_0^i + \psi^i)D_t v^i + \sum_{k=2}^{n} A_k(v^i + u_0^i)D_k(v^i + u_0^i) \]
\[ + M_j(v^i + u_0^i; \phi_0^i + \psi^i)v^i + \sum_{i=1}^{2} \{ N_{ij}^1(v^i + u_0^i; \phi_0^i + \psi^i)\psi^i + N_{ij}^2(v^i + u_0^i; \phi_0^i + \psi^i)\nabla \psi^i \} = f^j, \]

where

\[ f^j(y, t) = D_t u_0^i + A_1(u_0^i; \phi_0^i)D_t u_0^i + \sum_{k=2}^{n} A_k(u_0^i)D_k u_0^i \]

and \( M_j, N_{ij}^1, N_{ij}^2 \) can be determined by Cauchy integral remainder, the explicit form of which is of no consequence in the following discussion.

Let us note that \( f^j \) depends only on \( u_0^i, \phi_0^i \), i.e., on the given data. With \( u_0^i \) in \( H^{r+1}(R^n \times R_t^+) \), \( \phi_0^i \) in \( H^{r+2}(R^{n-1} \times R_t^+) \) for large \( r > 0 \), we have \( f^j \) in \( H^r \) \( _{loc} \).

From Assumption (II) we obtain

\[ D_s f^j(y, 0) = 0, \quad 0 \leq s \leq r - 1, \quad 1 \leq j \leq 3, \]

with \( r \) large.

From the Rankine-Hugoniot condition we have

\[ [F_0(v^i + u_0^i)]D_t \psi^i + [F_0(v^i + u_0^i)]D_t \psi^i - [F_1(v^i + u_0^i)] \]
\[ + \sum_{k=2}^{n} \{ D_k \psi^i \}[F_k(v^i + u_0^i)] + \sum_{k=2}^{n} D_k \phi_0^i[F_k(v^i + u_0^i)] = 0 \]

for \( y_i = (i - 1)t, \quad i = 1, 2. \)

Using the Taylor's expansion we rewrite (2.11) as

\[ b_0^i(v^i, v^{i+1}; u_0^i, u_0^{i+1})D_t \psi^i + \sum_{k=2}^{n} b_k^i(v^i, v^{i+1}; u_0^i, u_0^{i+1})D_k \psi^i \]
\[ + B_i(v^i, v^{i+1}; u_0^i, u_0^{i+1}, \phi_0^i) = g^i, \]

where \( B_i \) is a linear function,

\[ b_0^i = [F_0(v^i + u_0^i)], \quad b_k^i = [F_k(v^i + u_0^i)], \quad \text{and} \]
\[ g^i = D_t \phi_0^i[F_0(u_0^i)] - [F_1(u_0^i)] + \sum_{k=2}^{n} [F_k(u_0^i)]D_k \phi_0^i. \]
It is clear that \( q^i \) depend only on the given data and by Assumption (II):
\[
(2.13) \quad D^i_r g^i(\cdot, 0) = 0, \quad 0 \leq s \leq r - 1,
\]
for large \( r \).

**Step 3.** We shall now use a singular transformation and reduce the original problem to that of fixed boundaries. Let
\[
(2.14) \quad \tau = \log t; \quad x' = y'; \quad x_i = y_i \quad \text{for} \quad y_i < 0;
\]
\[
x_i = y_i t^{-1} \quad \text{for} \quad 0 \leq y_i < t;
\]
\[
x_i = y_i - t + 1 \quad \text{if} \quad t \leq y_i.
\]

Then
\[
Y_1 = R^{n-1} \times R^+_t \times R^- \quad \text{becomes} \quad X_1 = R^{n-1} \times R_t \times R^-,
\]
\[
Y_2 = R^{n-1} \times R^+_t \times \{y_i : 0 < y_i < t\} \quad \text{is mapped into} \quad X_2 = R^{n-1} \times R_t \times (0, 1),
\]
\[
Y_3 = R^{n-1} \times R^+_t \times \{y_i : t < y_i\} \quad \text{is transformed into} \quad X_3 = R^{n-1} \times R_t \times (1, \infty).
\]

To simplify the notation we shall write \( t \) for \( \tau \) and set \( v = (v^1, v^2, v^3) \) with \( v^j \) defined on \( X_j \).

From (2.8) we get
\[
(2.15) \quad D_t v + a(v, \psi) D_t v + \sum_{j=2}^n a_j(v) D_j v + M(v, \psi) v + N(v, \psi) = e^t f,
\]
where
\[
a(v, \psi) = \begin{pmatrix}
e^t \tilde{A}_1(v^1; \psi^1, \psi^2) & 0 \\
0 & \tilde{A}_1 - x_1 I
\end{pmatrix}, \quad e^t (\tilde{A}_1 - I),
\]
\[
a_j = \begin{pmatrix}
e^t A_j(v^1) & 0 \\
e^t A_j(v^2) & 0 \\
0 & e^t A_j(v^3)
\end{pmatrix}, \quad 2 \leq j \leq n,
\]
\[
M(v, \psi) = \begin{pmatrix}
e^t M_1 & 0 \\
e^t M_2 & 0
\end{pmatrix},
\]
with
\[
N(v, \psi) = \begin{pmatrix}e^t \sum_{i=1}^2 (N^1_i \psi^i + N^2_i \nabla \psi^i) & 0 \\
0 & e^t \sum_{i=1}^2 (N^2_i \psi^i + N^3_i \nabla \psi^i)
\end{pmatrix}.
\]

The boundary conditions are now
\[
(2.17) \quad b^i_0(v^i, v^{i+1}) D_t \psi^i + e^t \sum_{j=2}^n b^i_j(v^i, v^{i+1}) D_j \psi^i + e^t B_i(v^i, v^{i+1}) = e^t g^i
\]
on \( S_i = \{(x, t) : x_i = i - 1\} \quad \text{for} \quad i = 1, 2.\)
The problem (2.15), (2.17) will be studied in the next two sections. The crucial observation is that the boundaries are fixed and are uncoupled.

3

In this section we shall consider a linearized form of equations (2.15) and (2.17). Let $w^j$ be in $H^k(x_j)$, $\chi^i$ be in $H^{k+1}(S_i)$ for $1 \leq j \leq 3$ and $i = 1, 2$.

We shall study the linear problem

\[ \begin{align*} 
D_t v + a(w, \chi) D_1 v + \sum_{j=2}^{n} a_j(w) D_j v + M(w, \chi) v + N(v; \psi) &= e^i f, \\
 b_0^i(w^j, w^{i+1}) D_1 \psi^i + e_i \sum_{j=2}^{n} b_j^i(w^j, w^{i+1}) D_j \psi^i + e^i B_i(v^i, v^{i+1}) &= e^i g^i 
\end{align*} \]

on $S_i$, $i = 1, 2$.

Since the speed of propagation of a hyperbolic system is finite, we may assume without loss of generality that the coefficients of the matrices $a$, $a_j$, $M$, $N$, $b_k^j$, $B_i$ are constants for large $|x|$.

First we consider the simpler problem where the lower-order terms in (3.1) are omitted:

\[ \begin{align*} 
D_t v + a(w; \chi) D_1 v + \sum_{j=2}^{n} a_j(w) D_j v &= e^i f, \\
 b_0^i(w^j, w^{i+1}) D_1 \psi^i + e_i \sum_{j=2}^{n} b_j^i(w^j, w^{i+1}) D_j \psi^i + e^i B_i(v^i, v^{i+1}, \chi^i) &= e^i g^i 
\end{align*} \]

on $S_i$, $i = 1, 2$.

Proposition 3.1. Let $f^i$, $g^i$ be given by (2.9)–(2.10) and by (2.13), respectively. Then:

(i) $e^i f^1$ is in $H^k_\eta(R^{n-1} \times (-c, 0) \times (-\infty, T))$,

(ii) $e^i f^2$ belongs to $H^k_\eta(R^{n-1} \times (0, 1) \times (-\infty, T))$,

(iii) $e^i f^3$ is in $H^k_\eta(R^{n-1} \times (1, c) \times (-\infty, T))$,

(iv) $e^i g$ is in $H^k_\eta(R^{n-1} \times (-\infty, T))$

for $0 < c < \infty$.

Proof. With $\tau = \log t$, we have from (2.10) by writing $t$ for $\tau$:

\[ e^{-\eta t} D_t f^i(y, t) \in L^2(X_j). \]

A simple calculation shows that $e^i f^j$ belongs to the stated spaces if $r = \eta + k$. Similarly for $e^i q^i$.

By using a partition of unity we shall now assume that $e^i f^j$, $e^i q^i$ are zero and the coefficients $a$, $a_j$, $M$, $N$, $b_k^j$, $B_i$ in (3.2) are all constant for large $|x|$.

Definition 3.1. The $H^k_\eta$-double shock wave solution of (3.1) is said to be uniformly linearly stable if the linear problem (3.2) is well posed and its solution
satisfies the energy estimate
\[ \eta \left| v \right|_{0,\eta}^2 + \left| v \right|_{0,\eta}^2 + \left| \psi \right|_{1,\eta}^2 \leq C \left\{ \eta^{-1} \left| e^t f \right|_{0,\eta}^2 + \sum_{i=1}^{2} \left| e^t g_i \right|_{0,\eta}^2 \right\} \]

for \( \eta > \eta_0 > 0 \) where \( v = (v^1, v^2, v^3), \psi = (\psi^1, \psi^2), \) and

\[ \left| v \right|_{0,\eta}^2 = \sum_{j=1}^{3} \int_{X_j} (v^j)^2 \exp(-2\eta t) \, dx \, dt, \]

\[ \left| v \right|_{0,\eta}^2 = \sum_{i=1}^{2} \int_{S_i} \left\{ (v^i)^2 + (v^{i+1})^2 \right\} \exp(-2\eta t) \, dx' \, dt \]

with

\[ \left| \psi \right|_{1,\eta}^2 = \sum_{i=1}^{2} \sum_{k_1 + k_2 \leq 1} \int_{S_i} \eta^2 k_1 (D_x^\alpha D_t^{k_2-\alpha} \psi^i)^2 \exp(-2\eta t) \, dx' \, dt. \]

The boundaries in (3.2) are uncoupled; we now consider the problem separately near \( S_1 \) and near \( S_2 \). Set

\[ \tilde{v} = (v^1, v^2), \quad \tilde{f} = (f^1, f^2), \]

\[ \tilde{a}(x, t) = \begin{pmatrix} e^t A \tilde{A}_1 & 0 \\ 0 & A_1 - x_1 t \end{pmatrix}, \quad \tilde{a}_j = \begin{pmatrix} e^t A_j & 0 \\ 0 & e^t A_j \end{pmatrix}. \]

We have

(3.3)
\[ \begin{cases} D_t \tilde{v} + \tilde{a} D_1 \tilde{v} + \sum_{j=2}^{n} \tilde{a}_j D_j \tilde{v} = e^t \tilde{f}, \\ b_0^1 D_t \psi^1 + e^t \sum_{j=2}^{n} b_j^1 D_j \psi^1 + e^t B_1 (v^1, v^2) = e^t g^1 \quad \text{on} \ S_1. \end{cases} \]

Extending the coefficients to be constant for large \( |x_1| \) and making the change \( x_1 \mapsto -x_1 \) in \( X_2 \), we obtain

(3.4)
\[ \begin{cases} D_t \tilde{v} + \tilde{a} D_1 \tilde{v} + \sum_{j=2}^{n} \tilde{a}_j D_j \tilde{v} = e^t \tilde{f}, \quad \text{in} \ R^{n-1} \times R^- \times R_1, \\ b_0^1 D_t \psi^1 + \sum_{j=2}^{n} e^t b_j^1 D_j \psi^1 + e^t B_1 (\tilde{v}, \tilde{v}^2) = e^t g^1 \quad \text{on} \ S_1. \end{cases} \]

Here \( \tilde{v} = \tilde{t}(\tilde{v}^1, \tilde{v}^2), \quad \tilde{v}^1 = v^1, \quad \tilde{v}^2(x_1, x', t) = v^2(-x_1, x', t) \).

For simplicity of notation we shall drop \( \tilde{\ } \) when no confusion is possible.

In a similar fashion, we have

(3.5)
\[ \begin{cases} D_t \tilde{v} + \tilde{a} D_1 \tilde{v} + \sum_{j=2}^{n} \tilde{a}_j D_j \tilde{v} = e^t \tilde{f}, \\ b_0^1 D_t \psi^1 + \sum_{j=2}^{n} e^t b_j^1 D_j \psi^1 + e^t B_2 (v^2, v^3) = e^t g^2 \quad \text{on} \ S_2. \end{cases} \]

The main result of the section is the following theorem.
Theorem 3.1. Suppose that Assumptions (I), (II) are satisfied. Let \( w^j \) be in \( H^k(X_j) \), \( \chi^i \) be in \( H^{k+1}(S_i) \) for \( k \geq k_0 > 1 + \left[ (n+1)/2 \right] \), and \( \|w^2\|_{H^k} + \|\chi^i\|_{H^{k+1}} < \varepsilon_0 \), for \( \varepsilon_0 \ll 1 \). Then there exists a unique solution \( \{\psi, \nu\} \) of (3.1), satisfying
\[
\eta((\psi))_{k,n} + |\nu|_{k,n}^2 + \eta|e^{-t}\psi|_{k,n}^2 + (\nabla \psi)_{k,n}^2 \\
\leq C \left\{ \sum_{i=1}^{2} |g_i|_{k,n}^2 + \eta^{-1} \sum_{j=1}^{3} |f_j|_{k,n}^2 \right\},
\]
where
\[
((\psi))_{k,n}^2 = |e^{-t/2}v^1|_{k,n}^2 + |v^2|_{k,n}^2 + |e^{-t/2}v^3|_{k,n}^2,
\]
\[
(\nabla \psi)_{k,n}^2 = |e^{-t}D_\psi|_{k,n}^2 + |D\psi|_{k,n}^2.
\]
\( C \) is a constant depending only on \( \|A_j(\cdot, w, \chi)\|_{H^k} \), \( \|b_j(\cdot, w)\|_{H^k} \), and \( \eta \) is a sufficiently large number.

First we shall study the problem (3.4).

Theorem 3.2. Suppose that the conditions in Theorem 3.1 are satisfied. Then there exist a unique \( \{\tilde{\psi}, \psi^1\} \), solution of (3.4). Moreover
\[
\eta((\tilde{\psi}))_{k,n} + |\tilde{\psi}|_{k,n}^2 + \eta|e^{-t}\psi^1|_{k,n}^2 + (\nabla \psi^1)_{k,n}^2 \\
\leq C \left\{ |g|^1_{k,n} + \eta^{-1}(|f_j^1|_{k,n}^2 + |f_j^1|_{k,n}^2) \right\},
\]
for \( k \geq k_0 > 1 + \left[ (n+1)/2 \right] \).

\( C \) depends on \( \|A_j(\cdot, w, \chi)\|_{H^k} \), \( \|b_j(\cdot, w)\|_{H^k} \), and \( \eta \) is a sufficiently large number.

Proof. (1) By hypothesis of separate stability at \( (w, \chi) = 0 \), we have for \( \varepsilon_0 \ll 1 \):
\[
|b_0^1s + e^t e \sum_{j=2}^{n} b_j^1 \omega_j| \geq \gamma > 0
\]
for \( |s|^2 + |\omega|^2 = 1 \) with \( s = i \xi + \eta \), \( \eta > 0 \), and \( \omega_j = e^t \omega_j \), \( t \) negative.

Let \( \Pi(D_1, e^t D_j) \) be the projection operator with symbol \( \pi(x^t, t; s, e^t \omega) \) given by
\[
\pi(x^t, t; s, e^t \omega)w = w - \left( b_0^1s + e^t \sum_{j=2}^{n} b_j^1 \omega_j \right) \frac{\langle b_0^1s + e^t \sum_{j=2}^{n} \omega_j, w \rangle}{|b_0^1s + e^t \sum_{j=2}^{n} b_j^1 \omega_j|^2}.
\]

The symbol is clearly well defined for all \((x, t, \omega, s)\).

The boundary condition in (3.4) may be rewritten as
\[
(3.6) \quad \Pi(D_1 e^t D_j)B_1(v) = \Pi(g_1) - \Pi \left( e^t b_0^1 D_1 \psi + \sum_{j=2}^{n} b_j^1 D_j \psi \right).
\]
We shall now construct the Kreiss symmetrizer for (3.4), (3.6) by modifying Majda’s proof [6]. Corresponding to $\tilde{A}_1$, $A_j e^t$, and $e^t b_j^1$, $2 \leq j \leq n$, we construct a matrix-valued function $R(x, t; s, e^t \omega)$ such that

1. $R$ is Hermitian,
2. 
3. 

for some $\delta > 0$ and for $\eta \geq \eta_0 > 0$, 

(iii) 

The positive constants $\delta$ and $\delta_*$ depend only on

\[ \|A_r(\cdot, w, x)\|_{H^k}, \quad \|b^1_j(\cdot, w)\|_{H^k}. \]

Remark. Here the Kreiss symmetrizer $R$ is exactly the same as in [6], except that we have put $e^t \omega$ in the place of $\omega$ in [6]. This means that instead of constructing symmetrizer $R$ on the unit sphere $|s|^2 + |\omega|^2 = 1$ and extending it as a homogeneous function of 0-degree outside $|s|^2 + |\omega|^2 = 1$ as usual, now $R$ is homogeneous in $(s, \omega)$ only outside a larger and larger set $|s|^2 + |e^t \omega|^2 \leq 1$ as $t \to -\infty$. Consequently, the lower-order remainder terms coming from the composite or transpose of pseudodifferential operators may have norms which are uniformly bounded but no longer bounded by $C/\eta$.

This difficulty can be eliminated by noticing that in the following energy estimate, the operators $D_j$ ($j = 2, \ldots, n$) are always accompanied by the factor $e^t$. Thus, the norm of the lower-order remainder terms can again be controlled by a small constant for $-t$ sufficiently large.

We now establish the energy estimate. Consider

\[ \left( v, Ra^{-1}\left(D_t + e^t \sum_{j=2}^n a_j D_j\right)v \right) = (v, Ra^{-1}(e^t f)) - (v, RD_1 v). \]

A simple calculation gives

\[ -2 \Re(v, RD_1 v) = \Re(v, (R - R^*)D_1 v) + (v, D_1 R v) - \langle v, R v \rangle, \]

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(R^{n-1})$ with $x_1 = 0$. 

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Therefore:

\begin{align*}
(3.7) \quad & \Re \left\{ \left( v, R a^{-1}(D_t + e^t \sum_{j=2}^{n} a_j D_j) v \right) \right\} \\
& = \Re \left\{ \frac{1}{2} (v, (R - R^*) D_t v) \right\} \\
& \quad + \frac{1}{2} (v, (D_t R) v) - \frac{1}{2} (v, R v) + (v, R a^{-1}(e^t f)).
\end{align*}

By Lemma 4.2 of [6] and the techniques in [1, 4, 5], \( R a^{-1} \) is an operator of order 0. So

\[ |\Re(v, Ra^{-1}(e^t f))| \leq C \left| v^1 \right|_{0,\eta} |e^t f|_{0,\eta} + |v^2|_{0,\eta} |e^t f|_{0,\eta}. \]

Similarly, \( R - R^* \) is an operator of order -1.

We have

\[ D_t v = a^{-1}(e^t f) - a^{-1}(D_t v) - \sum_{j=2}^{n} a^{-1} a_j D_j v. \]

Taking into account the definition of \( a^{-1} \) and of \( a_j \) and noting that \( t < 0 \), we obtain

\[ |\Re(v_1(R - R^*) D_t v)| \leq C \left| v^1 e^{-t/2} \right|_{0,\eta} \left\{ |f^1|_{0,\eta} + |v^1 e^{-t/2}|_{0,\eta} \right\} \\
\quad + C \left| v^2 \right|_{0,\eta} \left\{ |f^2 e^t|_{0,\eta} + |v^2|_{0,\eta} \right\} \\
\leq C((\langle v \rangle))_{0,\eta} \left\{ \left| f \right|_{0,\eta} + ((\langle v \rangle))_{0,\eta} \right\}. \]

Similarly, for \( (v, D_t R . v) \). Hence

\begin{align*}
(3.8) \quad & |\Re(v, Ra^{-1}(e^t f))| + |\Re(v, (R - R^*) D_t v)| + |\Re(v, D_t R . v)| \\
& \leq C((\langle v \rangle))_{0,\eta}^2 + C((\langle v \rangle))_{0,\eta} |f|_{0,\eta}.
\end{align*}

(4) We now estimate \( \langle v, R v \rangle \) by considering the pseudodifferential operator with symbol

\[ R(0, x', t; s, \tilde{\omega}) + \delta_*^{-1}\{\Pi(x', t; s, \tilde{\omega})B_1(x', t)\} * \Pi(x', t; s, \tilde{\omega})B_1(x', t). \]

With the properties of \( R \) we get

\begin{align*}
(3.9) \quad & \Re \{\langle v, R v \rangle + \delta_*^{-1}\{\langle v, \Pi B_1 \rangle * \Pi B_1(v)\} \} \geq \delta_* |v|_{0,\eta}^2.
\end{align*}

We estimate \( \Re\langle v, (\Pi B_1)^* \Pi B_1(v) \rangle \) by considering the boundary condition associated with the symbol \( \{\Pi(x', t; s, \tilde{\omega})B_1(x', t)\} \). Following [6, p. 60] and taking into account the results in [1, 4, 5], we have

\begin{align*}
(3.10) \quad & \Re\langle v, R v \rangle \geq \delta_* |v|_{0,\eta}^2 - C|g^1|_{0,\eta}^2 - C|e^{-t} \psi|_{0,\eta}^2.
\end{align*}

A computation as in [6, pp. 61–62] gives

\begin{align*}
(3.11) \quad & \mu |e^{-t} \psi|_{0,\eta}^2 + |e^{-t} D_t \psi|_{0,\eta}^2 + |D_x \psi|_{0,\eta}^2 \leq C \left( |g^1|_{0,\eta}^2 + |v|_{0,\eta}^2 \right).
\end{align*}
Using the formula for the composition as well as the adjoint formula together with the sharp Garding inequality, we have from (3.7)–(3.11) the estimate of the theorem for \( k = 0 \).

For estimates in \( H^k_0(\mathbb{R}^n \times (-\infty, 0)) \) we proceed as usual with the tangential derivatives first. The estimates for the normal derivatives are then obtained by using the noncharacteristic property.

Once the estimates are established, the existence is trivial to prove. Indeed, the adjoint problem has the same estimate, and by a Hilbert space argument, the problem has a unique solution in the considered space.

**Theorem 3.3.** Suppose all the hypotheses of Theorem 3.2 are satisfied. Then there exists a unique \( \{\hat{v}, \psi\} \) solution of the problem

\[
\begin{align*}
D_t \hat{v} + \hat{a} D_t \hat{v} + \sum_{j=2}^{n} \hat{a}_j D_j \hat{v} + M \hat{v} + N(\psi) &= \epsilon' \hat{f}, \\
b_0 D_t \psi + \epsilon' \sum_{j=2}^{n} b_j D_j \psi + \epsilon' B_1(\hat{v}) &= \epsilon' g^1 \quad \text{on } S_1.
\end{align*}
\]

Moreover, \( \{\hat{v}, \psi\} \) satisfies the energy estimate of Theorem 3.2. \( M, N \) are as in (3.1).

**Proof.** To simplify the notation we shall drop \( \hat{\cdot} \). Let \( 0 < \lambda < 1 \) and consider the problem

\[
\begin{align*}
D_t v + a D_t v + \sum_{j=2}^{n} a_j D_j v + \lambda(M v + N(\psi)) &= \epsilon' f, \\
b_0 D_t \psi + \epsilon' \sum_{j=2}^{n} b_j D_j \psi + \epsilon' B_1(v) &= \epsilon' g^1 \quad \text{on } S_1.
\end{align*}
\]

Denote by \( A \) the set \( \Lambda = \{\lambda: 0 < \lambda < 1, (3.12) \text{ has a unique solution satisfying the estimate of Theorem 3.2}\} \).

(i) \( \Lambda \) is nonempty. Indeed, by Theorem 3.2 we have \( \lambda = 0 \) in \( \Lambda \).

(ii) \( \Lambda \) is closed. Suppose that \( \lambda_n \) is in \( \Lambda \) and that \( \lambda_n \to \lambda \).

Corresponding to \( \lambda_n \), we have \( \{v_n, \psi_n\} \).

Set:

\[
F_n = \epsilon' f - \lambda_n \{M v_n + N(\psi_n)\}.
\]

Applying the estimate of Theorem 3.2 with \( F_n \) instead of \( \epsilon' f \), we obtain by taking \( \eta \) large and noting that \( t > 0 \)

\[
\eta \|v_n\|_{k, \eta}^2 + \|\nabla \psi_n\|_{k, \eta}^2 + \eta \|\psi_n\|_{k, \eta}^2 \leq C,
\]

\( C \) independent of \( n \).

From the weak compactness of the unit ball in a Hilbert space we get \( \{v, \psi\} \), solution of (3.12) with the desired estimate. Hence \( \Lambda \) is closed.

(iii) \( \Lambda \) is open. The proof is standard, using a Neumann series.

Therefore \( \Lambda = [0, 1] \) and with \( \lambda = 1 \), we get the theorem.

**Proof of Theorem 3.1.** We shall first establish the energy estimate. Let \( \{\Phi_n\} \) be a finite partition of unity and set \( v_n = \Psi_n v \) with \( v = (v^1, v^2, v^3) \). For \( n \) with \( \text{supp}(\Phi_n) \cap S_1 \neq \emptyset \) and \( \text{supp}(\Phi_n) \cap S_2 = \emptyset \), Theorem 3.3 gives the estimate for \( \{v^1_n, v^2_n\} \) and \( \psi^1_n \).

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For $q$ with $\text{supp}(\Phi_q) \cap S_2 \neq \emptyset$ and $\text{supp}(\Phi_q) \cap S_1 = \emptyset$, a proof exactly as that of Theorem 3.3 again gives us the estimate for $\{v^2_q, v^3_q\}$ and $\psi^2_q$.

We have only to establish the estimate for $q$ with $\text{supp}(\Phi_q) \cap S_1 \cap S_2 = \emptyset$ and $\text{supp}(\Phi_q) \subset \{x_1 : 0 < x_1 < 1\}$, i.e., interior estimates. And this is standard.

Combining everything and considering the adjoint problem, we obtain the stated result.

4

In this section we shall use the iteration method to show the existence of a unique solution $\{v^1, v^2, v^3, \psi^1, \psi^2\}$ of (2.15)–(2.17) in a neighborhood of $\tau = -\infty$, i.e., for small $t > 0$. As done in §3 we shall write $t$ for $\tau$ to simplify the notation

**Theorem 4.1.** Suppose that Assumptions (I) and (II) are satisfied. Then there exist:

1. $T_* \ll -1$;
2. a unique $(v, \psi)$, solution of (2.15), (2.17) in $(-\infty, T_*)$. Moreover,

$$\eta(((v)))^2_{k, \eta} + |v|^2_{k, \eta} + \eta |e^{-t} \psi|_{k, \eta}^2 + (\nabla \psi)^2_{k, \eta} \leq \epsilon_0.$$ 

For $k \geq k_0 > 1 + [(n + 1)/2]$, $\eta$ is sufficiently large. $(((\cdot)))_{k, \eta}$ and $(\cdot)_{k, \eta}$ are as in Theorem 3.1.

**Proof.** Let $\phi_\tau(t)$ be a $C^\infty(R)$-function with $\phi_\tau(t) = 1$ for $t \leq T$ and $\phi_\tau(t) = 0$ for $t \geq T + 1$. Consider the system

$$
\begin{aligned}
D_t v_k + a(v_{k-1}, \psi_{k-1})D_1 v_k + \sum_{j=2}^n a_j(v_{k-1})D_j v_k \\
+ N(v_{k-1}, \psi_{k-1}) \nabla \psi_k + M(v_{k-1}, \psi_{k-1}) v_k = e^t f \phi_\tau,
\end{aligned}
$$

$$
\begin{aligned}
b_0^i(v_{k-1}, \psi_{k-1})D_1 \psi_k^i + e^t \sum_{j=2}^n b_j(v_{k-1}, \psi_{k-1})D_j \psi_k^i \\
+ e^t B_i(v_k, \psi_{k-1}) = e^t g \phi_\tau,
\end{aligned}
$$

with $k = 1, 2, \ldots$.

1. Let $\{v_0, \phi_0\} = 0$; then it follows from Theorem 3.1 that there exists a unique $(v_1, \phi_1)$, solution of (4.1). Moreover,

$$\eta(((v_1)))^2_{k, \eta} + |v_1|^2_{k, \eta} + \eta |e^{-t} \psi_1|_{k, \eta}^2 + (\nabla \psi_1)^2_{k, \eta} \leq C \left\{ \sum_{i=1}^2 |\phi_T g^i|_{k, \eta}^2 + \eta^{-1} |\phi_T f|_{k, \eta}^2 \right\}.$$

$C$ depends on $\|A_j(\cdot)\|_{\mathcal{H}^k}$, $\|b_j(\cdot)\|_{\mathcal{H}^k}$ and is independent of $T$. Let $\epsilon_0 > 0$; then with $\phi_T$ as above, it is clear that for $T(\epsilon_0) \ll -1$, the right-hand side of (4.2) is less than $\epsilon_0$:

$$\eta(((v_1)))^2_{k, \eta} + |v_1|^2_{k, \eta} + \eta |e^{-t} \psi_1|_{k, \eta}^2 + (\nabla \psi_1)^2_{k, \eta} \leq \epsilon_0,$$

for $t \leq T(\epsilon_0)$.
(2) We reapply Theorem 3.1 with \( w = v_1, \chi = \psi_1 \). The constant \( C \) of Theorem 3.1 depends on
\[
\| A_j(\cdot; v_1, \psi_1) \|_{L^1_{\text{loc}}}^k, \quad \| b_j^i(\cdot; v_1) \|_{L^1_{\text{loc}}}^k.
\]
For \( k > 1 + [(n+1)/2] \), \( H^k_{\text{loc}} \) is an algebra. Since \( \{v_1, \psi_1\} \) satisfies (4.3), an easy computation gives
\[
C(\| A_j(\cdot; v_1, \psi_1) \|_{L^1_{\text{loc}}}^k, \| b_j^i(\cdot, v_1) \|_{L^1_{\text{loc}}}^k) = C_1(\| A_j(\cdot) \|_{L^1_{\text{loc}}}^k, \| b_j^i(\cdot) \|_{L^1_{\text{loc}}}^k, \varepsilon_0).
\]

Theorem 3.1 gives
\[
(4.4) \quad \eta(((v_2)))^2_{k,\eta} + |v_2^2|_{k,\eta} + \eta e^{-t}\psi_2 \|_{k,\eta}^2 + (\nabla \psi_2)^2_{k,\eta} \leq C_1 \{ \eta^{-1} |\phi_T f|^2_{k,\eta} + |\phi_T g|^2_{k,\eta} \}.
\]

With the definition of \( \phi_T \), there exists \( T_*(\varepsilon_0, \eta) \approx -1 \) such that
\[
C_1 \{ \eta^{-1} |\phi_T f|^2_{k,\eta} + |\phi_T g|^2_{k,\eta} \} \leq \varepsilon_0.
\]

Clearly \( T_* \) does not depend on \( v_2, \psi_2 \).

By induction we get \( \{v_s, \psi_s\} \), solution of (4.1) for all \( s = 1, 2, \ldots \).

(3) It remains to show that the sequence converges strongly in the appropriate norms. We have
\[
\begin{align*}
D_s(v_{s+1} - v_s) + a(v_s, \psi_s)D_1(v_{s+1} - v_s) \\
+ \sum_{j=2}^n a_j(v_s)D_j(v_{s+1} - v_s) + M(v_s, \psi_s)(v_{s+1} - v_s) \\
+ N(v_s, \psi_s)(\psi_{s+1} - \psi_s) \\
= e^t(\phi_T - \left\{ D_s v_s + a(v_s, \psi_s)D_1 v_s + \sum_{j=2}^n a_j(v_s)D_j v_s \\
+ M(v_s, \psi_s)v_s + N(v_s, \psi_s)\nabla \psi_s \right\}).
\end{align*}
\]

Replacing \( e^t(\phi_T) \) by (4.10), we obtain
\[
(4.5) \quad D_s(v_{s+1} - v_s) + a(v_s, \psi_s)D_1(v_{s+1} - v_s) + \sum_{j=2}^n a_j(v_s)D_j(v_{s+1} - v_s) \\
+ M(v_s, \psi_s)(v_{s+1} - v_s) + N(v_s, \psi_s)(\psi_{s+1} - \psi_s) = F_s
\]
where
\[
F_s = \left\{ a(v_{s-1}, \psi_{s-1}) - a(v_s, \psi_s) \right\}D_1 v_s + \sum_{j=2}^n \left\{ a_j(v_{s-1}) - a_j(v_s) \right\}D_j v_s \\
+ \left\{ M(v_{s-1}, \psi_{s-1}) - M(v_s, \psi_s) \right\}v_s + \left\{ N(v_{s-1}, \psi_{s-1}) - N(v_s, \psi_s) \right\}\nabla \psi_s.
\]
Similarly, for the boundary equations we get
\begin{equation}
 b_0^i(v_s)D_t(\psi_s^i - \psi_s^i) + e^t \sum_{j=2}^n b_j^i(v_s)D_j(\psi_{s+1}^i - \psi_s^i)
 + e^t B_j^i(v_{s+1}^i, \psi_{s+1}^i; \psi_s^i) - e^t B_j^i(v_s^i, v_{s+1}^i; \psi_s^i) = G_s^i.
\end{equation}

(4) We now estimate $F_s$ and $G_s^i$. Since $H^{k-1}$ is an algebra we have
\begin{equation}
 \eta^{-1} \left\| F_s \right\|_{k-1, \eta} + \left\| G_s^i \right\|_{k-1, \eta} \leq C_2 \left\{ \left( (v_s^i) \right)_{k, \eta} + \left| e^{-t} \psi_s^i \right|_{k, \eta}^2 \right\}
 \times \left\{ \left( (v_s - v_{s-1}) \right)_{k-1, \eta} + \left| e^{-t}(\psi_s - \psi_{s-1}) \right|_{k-1, \eta}^2 \right\}.
\end{equation}

$C_2$ is independent of $s$.

Applying the estimate of Theorem 3.1 we obtain
\begin{equation}
 \left( (v_s - v_{s+1}) \right)_{k-1, \eta} + \left| e^{-t}(\psi_s - \psi_{s+1}) \right|_{k-1, \eta}^2 \leq C_1 C_2 e_0 \left( \left( (v_s - v_{s-1}) \right)_{k-1, \eta}^2 + \left| e^{-t}(\psi_s - \psi_{s-1}) \right|_{k-1, \eta}^2 \right).
\end{equation}

With $C_1 C_2 e_0 \leq 1/2$, we have
\begin{equation}
 \{ v_s, \psi_s \} + \{ v, \psi \} \in H^{k-1}_1 \text{ and weakly in } H^k_1.
\end{equation}

It is not difficult to check that $\{ v, \psi \}$ is the unique solution of the problem.

5

In this section we shall give an example of a double shock wave arising from physical applications.

Consider the equations of isentropic gas in two-dimensional space:
\begin{equation}
 \left\{ \begin{array}{l}
 D_t \rho + \sum_{j=1}^2 D_j(\rho v_j) = 0, \\
 D_t(\rho v_i) + \sum_{j=1}^2 D_j(\rho v_j v_i + \delta_{ij}p(\rho)) = 0 \quad \text{in } \mathbb{R}^2 \times R^+_t,
 \end{array} \right.
\end{equation}

where $\rho$ is the density, $v = (v_1, v_2)$ is the velocity, $p(\rho)$ is the pressure, and $\delta_{ij}$ is the Kronecker delta function.

We take the initial jump surface $\Gamma_0$ to be the line $x_1 = 0$ and the initial conditions are
\begin{equation}
 \left\{ \begin{array}{l}
 \rho(x, 0) = \rho^+ \quad \text{if } x_1 > 0 \text{ or } x_1 < 0, \\
 v_i(x, 0) = v_i^\pm, \quad i = 1, 2,
 \end{array} \right.
\end{equation}

where $\rho^\pm$, $v_i^\pm$ are constants.

We look for double shock fronts of the form
\begin{equation}
 \Gamma^\pm(t) = \{(x, t): x_1 = t\lambda^\pm \}.
\end{equation}

Clearly $\Gamma^\pm(0) = \Gamma_0 = \{x_1: x_1 = 0\}$. 

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Let \( \{\bar{v}, \bar{\rho}\} \) be the constant state of Assumption (I). The Rankine-Hugoniot condition on \( \Gamma^- (t) \) may be written as

\[
\begin{align*}
-\lambda_- (\bar{\rho} - \rho^-) + (\bar{\rho} \bar{v}_1 - \rho^- v_1^-) &= 0, \\
\lambda_- (\bar{\rho} \bar{v}_1 - \rho^- v_1^-) + \{\bar{\rho} (\bar{v}_1)^2 + p(\bar{\rho}) - \rho^- (v_1^-)^2 - p(\rho^-)\} &= 0, \\
-\lambda_- (\bar{\rho} \bar{v}_2 - \rho^- v_2^-) + \{\bar{\rho} \bar{v}_1 \bar{v}_2 - \rho^- v_1^* v_2^-\} &= 0.
\end{align*}
\]

(5.3)

The shock front is noncharacteristic and hence \( \lambda_\neq v_1^- \), \( \lambda_- \neq \bar{v}_1 \). It follows that \( \bar{v}_2 = v_2^- \). Thus, on \( \Gamma^- (t) \) we have

\[
\begin{align*}
-\lambda_- (\bar{\rho} - \rho^-) + (\bar{\rho} \bar{v}_1 - \rho^- v_1^-) &= 0, \\
\lambda_- (\bar{\rho} \bar{v}_1 - \rho^- v_1^-) + \{\bar{\rho} (\bar{v}_1)^2 + p(\bar{\rho}) - \rho^- (v_1^-)^2 - p(\rho^-)\} &= 0, \\
\bar{v}_2 &= v_2^-.
\end{align*}
\]

(5.4)

Similarly, on \( \Gamma^+ (t) \) we get

\[
\begin{align*}
\lambda_+ (\bar{\rho} - \rho^+) + (\bar{\rho} \bar{v}_1 - \rho^+ \bar{v}_1^+) &= 0, \\
\lambda_+ (\bar{\rho} \bar{v}_1 - \rho^+ \bar{v}_1^+) + \{\bar{\rho} (\bar{v}_1)^2 + p(\bar{\rho}) - \rho^+ (\bar{v}_1^+)^2 - p(\rho^+)\} &= 0, \\
\bar{v}_2 &= v_2^+.
\end{align*}
\]

(5.5)

For polytropic gas, \( p(\rho) = A \rho^\gamma, \gamma > 1 \), it is known that the uniform stability condition of the shock front in the sense of Majda is equivalent to that of compressibility, i.e., \( \bar{\rho} > \rho_+, \bar{\rho} > \rho_- \). On the other hand, uniform stability is equivalent to the uniform well-posedness of the problem.

In view of Theorems 3.1, 4.1 and of Remark 3.1 it suffices to show the compressibility.

Let us consider the special symmetric case:

\[
\rho^+ = \rho^- = \rho, \quad v_1 = v_1^- = -v_1^+ > 0, \quad \lambda_+ = \lambda, \quad \lambda_- = -\lambda.
\]

Then (5.4), (5.5) become

\[
\begin{align*}
-\lambda (\bar{\rho} - \rho) + \rho v_1 &= 0, \\
\lambda (\rho v_1) + \rho v_1^2 + A \rho^\gamma - A (\bar{\rho})^\gamma &= 0.
\end{align*}
\]

(5.6)

Therefore

\[
(\bar{\rho} - \rho) \{A (\bar{\rho})^\gamma - A \rho^\gamma - \rho v_1^2\} = (\rho v_1)^2 > 0.
\]

(5.7)

Equation (5.7) has a unique solution \( \bar{\rho} > \rho \). With \( \bar{\rho} \) known, (5.6) gives \( \lambda \) and hence we obtain the two shock fronts \( \Gamma^\pm (t) \).

In the above example the constant state \( \{\bar{v}, \bar{\rho}\} \) of Assumption (I) is given by \( \bar{v}_1 = 0, \bar{v}_2 = v_2^- = v_2^+ \) and \( \bar{\rho} \) is the unique solution of (5.7). As for Assumption (II), we have two relations for the two unknowns \( (\lambda_+, \bar{\rho}) \) and hence the compatibility condition is always satisfied.
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