ON A PROBLEM OF S. MAZUR

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Abstract. In this work a generalization of Mazur's problem concerning the continuity of linear functionals is given.

S. Mazur asked (about 1935) the following question [7, Problem 24]: In a Banach space $E$ an additive functional $f$ is given with the property that, for any continuous function $\varphi : [0,1] \rightarrow E$ the function $f \circ \varphi$ is Lebesgue-measurable. Is $f$ continuous? This question was answered affirmatively by I. Labuda and R. D. Mauldin in [3] by the following theorem:

Theorem 1. Let $E$ be a Banach space, $F$ a Hausdorff topological vector space, $f : E \rightarrow F$ an additive operator. If $f \circ \varphi$ is Lebesgue-measurable for any continuous function $\varphi : [0,1] \rightarrow E$, then $f$ is continuous.

The following more general theorem is due to Z. Lipecki [4].

Theorem 2. Let $G, H$ be Hausdorff topological abelian groups, $G$ is metrizable, complete, connected and locally arcwise connected, and let $f : G \rightarrow H$ be a homomorphism. If $f \circ \varphi$ is Lebesgue-measurable for any continuous function $\varphi : [0,1] \rightarrow G$, then $f$ is continuous.

Recently, R. Ger presented similar results concerning convex functionals [2].

The aim of this paper is to give another generalization of Mazur's problem. Namely, we show, that if in the original problem $f$ is an exponential polynomial, then the statement remains valid.

First we collect some necessary facts about polynomials and exponential polynomials on groups. Most of these results can be found in [5, 6]. Let $G$ be an abelian group, $H$ a complex linear space. The function $p : G \rightarrow H$ is called a polynomial if for some nonnegative integer $N$ we have

$$
\Delta^{N+1}_{y_1,\ldots,y_{N+1}}p(x) = 0
$$

for all $x, y_1, \ldots, y_{N+1}$ in $G$. The smallest integer $N$ with this property is called the degree of $p$ and is denoted by $\deg p$. It is well known [1] that any
function $p$ satisfying (1) can uniquely be represented in the form

$$p(x) = A_N(x, \ldots, x) + A_{N-1}(x, \ldots, x) + \cdots + A_1(x) + A_0$$

for all $x$ in $G$, where $A_k : G^k \to H$ is a $k$-additive and symmetric function ($k = 1, 2, \ldots, N$) and $A_0$ is in $H$. For the sake of simplicity we shall use the notation

$$A^{(k)}(x) = A_k(x, \ldots, x)$$

for all $x$ in $G$, that is $A^{(k)}$ is the diagonalization of the $k$-additive and symmetric function $A_k$, ($k = 1, 2, \ldots, N$).

Let $C$ denote the set of complex numbers. The function $m : G \to C$ is called an exponential if for all $x, y$ in $G$ we have

$$m(x + y) = m(x)m(y)$$

and $m$ is not identically zero. That is, exponentials are just the homomorphisms of $G$ into the multiplicative group of nonzero complex numbers.

The function $f : G \to H$ is called an exponential polynomial if it has a representation

$$(2) f(x) = \sum_{k=1}^{n} p_k(x) m_k(x)$$

for all $x$ in $G$, where $p_k : G \to H$ is a polynomial and $m_k : G \to C$ is an exponential ($k = 1, \ldots, n$). It is well known [6] that if in (2) we have $m_i \neq m_j$ for $i \neq j$ then the representation (2) for $f$ is unique.

In order to prove our main theorem for exponential polynomials we first consider the case of polynomials.

**Theorem 3.** Let $G$ be a metrizable topological abelian group which is complete, connected and locally arcwise connected, further let $H$ be a metrizable locally convex complex topological vector space and $p : G \to H$ a polynomial. If $p \circ \varphi$ is Lebesgue-measurable for any continuous function $\varphi : [0, 1] \to G$, then $p$ is continuous.

**Proof.** Let $p = A^{(N)} + q$, where $N \geq 1$ is an integer, $A_N : G^N \to H$ is $N$-additive and symmetric and $q : G \to H$ is a polynomial of degree at most $N - 1$. It is enough to show that $A_N$ is continuous, then by induction we have the statement. It is well known [1] that

$$A_N(x_1, x_2, \ldots, x_N) = \frac{1}{N!} \Delta^N_{x_1, x_2, \ldots, x_N} p(0)$$

$$= \frac{1}{N!} \sum_{1 \leq i_1 < \cdots < i_k} (-1)^{N-k} p(x_{i_1} + \cdots + x_{i_k})$$

which implies that the function $t \to A_N(\varphi(t), x_2, \ldots, x_N)$ is Lebesgue-measurable for any continuous function $\varphi : [0, 1] \to G$, and for any fixed $x_2, \ldots, x_N$ in $G$. Using the symmetry of $A_N$ and Theorem 2 we have that $A_N$ is continuous in each variable. From the theorem of Baire it follows that $A_N$ is
continuous at least at one point. Then, using the connectedness of $G$, it follows from Theorem 4.2 in [5] that $A_N$ is continuous.

**Theorem 4.** Let $G$ be a metrizable topological abelian group which is complete, connected and locally arcwise connected, further let $H$ be a metrizable locally convex complex topological vector space and $f: G \to H$ an exponential polynomial. If $f \circ \varphi$ is Lebesgue-measurable for any continuous function $\varphi: [0, 1] \to G$, then $f$ is continuous.

**Proof.** Let $f = \sum_{k=1}^{n} p_k m_k$, where $n \geq 1$ is an integer, $p_k: G \to H$ is a polynomial and $m_k: G \to C$ is an exponential $(k = 1, 2, \ldots, n)$, $m_i \neq m_j$ for $i \neq j$ and $p_k = A_k^{(N_k)} + q_k$, where $A_{k,N_k}: G^{N_k} \to H$ is $N_k$-additive and symmetric, $q_k: G \to H$ is a polynomial of degree at most $N_k - 1$, $A_{k,N_k} \neq 0$ $(k = 1, 2, \ldots, n)$. We show that $m_k, A_k^{(N_k)}$ is continuous $(k = 1, 2, \ldots, n)$.

By induction on $n$, first let $n = 1$, $f = p_1 m_1$. Here we use induction on the degree of $p_1$. If $\deg p_1 = 0$, then $p_1$ is constant and $p_1 \neq 0$. It is very easy to see, that in this case the property of $f$ implies that $m_1 \circ \varphi$ is Lebesgue-measurable for any continuous function $\varphi: [0, 1] \to G$, hence by Theorem 2, $m_1$ is continuous. Then $p_1 \circ \varphi$ is Lebesgue-measure for any continuous function $\varphi: [0, 1] \to G$, and by Theorem 3, $p_1$ is continuous and hence $f$ is continuous. If $\deg p_1 \geq 1$ then $p_1$ is nonconstant, hence there exists $y$ for which $\Delta y p_1$ is not identically zero and $\deg \Delta y p_1 < \deg p_1$. On the other hand

$$\Delta y p_1(x)m_1(x) = m_1(-y)p_1(x + y)m_1(x + y) - p_1(x)m_1(x) = m_1(-y)f(x + y) - f(x)$$

that is the function $(\Delta y p_1 m_1) \circ \varphi$ is Lebesgue-measurable for any continuous function $\varphi: [0, 1] \to G$ which implies the statement of the theorem for $n = 1$.

Let now $n \geq 2$. We show, that, for example, $m_2$ and $A_2^{(N_2)}$ are continuous. Let $y$ be an element for which $m_1(y) \neq m_2(y)$. Then we have

$$\Delta y^{N_1+1}(f m_1^{-1})(x) = m_1(x)^{-1} \sum_{j=0}^{N_1+1} \binom{N_1+1}{j} (-1)^{N_1+1-j} m_1(y)^{-j} f(x + jy)$$

by the definition of difference operators. From this equation we infer that the function $[m_1 \Delta y^{N_1+1}(f m_1^{-1})] \circ \varphi$ is Lebesgue-measurable for any continuous function $\varphi: [0, 1] \to G$, by the same property of $f$. On the other hand
holds for all $x$ in $G$. By taking differences we have
\[
\Delta_{j}^{N_{i}+1}(f m_{1}^{-1})(x) = \sum_{k=2}^{N_{i}+1} \Delta_{j}^{N_{i}+1}[(m_{k} m_{1}^{-1})(A_{k}^{(N_{i})} + q_{k})](x)
\]
\[
= \sum_{k=2}^{N_{i}+1} \left( \sum_{j=0}^{N_{i}+1} \left( \begin{array}{c} N_{i}+1 \\ j \end{array} \right) (-1)^{N_{i}+1-j} m_{k}(x) m_{1}(x)^{-1} m_{k}(y)^{j} m_{1}(y)^{-j} \right.
\times (A_{k}^{(N_{i})}(x+jy) + q_{k}(x+jy))
\]
\[
= m_{1}(x)^{-1} \sum_{k=2}^{n} m_{k}(x) \left[ \sum_{j=0}^{N_{i}+1} \left( \begin{array}{c} N_{i}+1 \\ j \end{array} \right) (-1)^{N_{i}+1-j} m_{k}(y)^{j} \right.
\times m_{1}(y)^{-j} (A_{k}^{(N_{i})}(x) + q_{k,j,y}(x)) \right]
\]
\[
= m_{1}(x)^{-1} \sum_{k=2}^{n} m_{k}(x) [(m_{k}(y)m_{1}(y)^{-1} - 1)^{N_{i}+1} A_{k}^{(N_{i})}(x) + q_{k,j,y}(x)],
\]
where $q_{k,j,y} : G \rightarrow H$ is a polynomial of degree at most $N_{k} - 1$ and $q_{k,j,y} : G \rightarrow H$ is a polynomial of degree at most $N_{k} - 1$ ($k = 2, \ldots, n$; $j = 0, 1, \ldots, N_{i}+1$). We have a representation for the exponential polynomial $m_{1}\Delta_{j}^{N_{i}+1}(f m_{1}^{-1})$ from which we infer—by the above consideration—that $m_{k}$ and its polynomial coefficient is continuous. It follows that $(m_{k}(y)m_{1}(y)^{-1} - 1)^{N_{i}+1} A_{k}^{(N_{i})}$ must be continuous, and especially—as $m_{2}(y) \neq m_{1}(y)$—the function $A_{2}^{(N_{2})}$ is continuous. Hence the theorem is proved.

References

1. D. Ž. Djoković, A representation theorem for $(X_{1}-1)(X_{2}-1)\ldots(X_{n}-1)$ and its applications, Ann. Polon. Math. 22 (1979), 189–198.

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