A NOTE ON LOCAL CHANGE OF DIFFEOMORPHISM

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Abstract. Let \( D(M) \) be the group of pseudo-isotopy classes of orientation preserving diffeomorphisms of a compact manifold \( M \) which restrict to the identity on \( \partial M \). If a compact manifold \( N \) of the same dimension as \( M \) is embedded in \( M \), then extending maps in \( D(N) \) as the identity on the exterior of \( N \) defines a homomorphism \( E: D(N) \to D(M) \). We ask if the kernel of \( E \) is finite and show that this is the case for special cases.

Introduction

Two diffeomorphisms \( f_0 \) and \( f_1 \) of a smooth manifold \( M \), which are the identity on the boundary \( \partial M \) of \( M \) if \( \partial M \) is nonempty, are called pseudo-isotopic (or concordant) if there is a diffeomorphism \( F \) of \( M \times I \) \((I = [0, 1])\) such that \( F(x, 0) = (f_0(x), 0), \ F(x, 1) = (f_1(x), 1) \) for all \( x \in M \) and that \( F \) is the identity on \( \partial M \times I \). The set \( D(M) \) of pseudo-isotopy classes of orientation preserving diffeomorphisms of \( M \) forms a group under composition of maps.

It is interesting by itself to compute \( D(M) \) and moreover it is sometimes related to geometrical problems, e.g. when \( M = D^n \) or \( S^n \) \([KM]\), \( S^p \times S^q \) \([B, L]\), \( CP^k \times D^q \) \([BP]\). The group \( D(M) \) is well understood for some \( M \), but in general it seems not so well understood. For instance, the author does not know a higher dimensional example of \( M \) such that \( D(M) \) is trivial.

Let \( N \) be a compact manifold embedded in \( M \) of the same dimension. Then there is defined a homomorphism \( E: D(N) \to D(M) \) by extending a map in \( D(N) \) as the identity on the exterior of \( N \). Then it is natural to ask

Question. Is \( \ker E \) trivial or finite?

An interesting case is when \( N \) is an \( n \)-disk \( D^n \). As is well known \( D(D^n) \) is isomorphic to the Kervaire-Milnor group \( \theta_{n+1} \) of oriented homotopy \( (n + 1) \)-spheres \((n \geq 4)\) and the group is nontrivial in general if \( n \geq 6 \). Thus if \( \ker E \) is trivial for \( N = D^n \), then one can conclude that \( D(M) \) is nontrivial in general.

In this paper we consider the case where \( N = CP^k \times D^q \). The group \( D(CP^k \times D^q) \) is fairly well understood by Browder-Petrie \([BP]\) in connection
with the study of semifree circle actions on homotopy spheres. The purpose of this paper is to prove the following, which gives an evidence supporting the above question.

**Theorem A.** Let $X$ be a closed orientable manifold of dimension $q \geq 2$ such that $H^1(X; \mathbb{Z}) = 0$. Then the kernel of

$$E = E_X : D(\mathbb{C}P^k \times D^q) \to D(\mathbb{C}P^k \times X)$$

is finite, where $D^q$ is any $q$-disk embedded in $X$.

**Remark.** The map $E_X$ depends on the choice of an embedding of $D^q$ into $X$ in general. However the disk theorem tells us that it only depends on a connected component of $X$ into which $D^q$ is embedded and on whether the embedding preserves orientation or not (we fix an orientation on $D^q$).

The group $D(\mathbb{C}P^k \times D^q)$ is finitely generated abelian. The rank $r_{k,q}$ of the free part is explicitly computed [BP]. In fact, if $q$ is even $r_{k,q} = 0$, i.e. $D(\mathbb{C}P^k \times D^q)$ is finite; so Theorem A is trivial in this case. If $q$ is odd, $r_{k,q}$ is nonzero in most cases. In fact, it is given by

$$r_{k,q} = [k/2] + a_{k,q} + b_{k,q}$$

where

$$a_{k,q} = \begin{cases} 1 & \text{if } k \text{ is odd and } q + 1 \equiv 0 \pmod{4}, \\ 0 & \text{otherwise}, \end{cases}$$

$$b_{k,q} = \begin{cases} 1 & \text{if } 3 \leq q \leq 2k + 1 \ (q \text{ : odd}), \\ 0 & \text{otherwise}. \end{cases}$$

Let $\text{Diff}_+ M$ be the group (with $C^\infty$ topology) of orientation preserving diffeomorphisms of $M$ which restrict to the identity on $\partial M$. The connected components $\pi_0(\text{Diff}_+ M)$ are nothing but the isotopy classes of those diffeomorphisms; so there is a natural epimorphism $\Pi : \pi_0(\text{Diff}_+ M) \to D(M)$.

Similarly to $E_X$, a homomorphism

$$E'_X : \pi_0(\text{Diff}_+ \mathbb{C}P^k \times D^q) \to \pi_0(\text{Diff}_+ \mathbb{C}P^k \times X)$$

is defined. Clearly $\Pi$ commutes with $E_X$ and $E'_X$. According to Cerf [C] $\Pi : \pi_0(\text{Diff}_+ \mathbb{C}P^k \times D^q) \to D(\mathbb{C}P^k \times D^q)$ is an isomorphism (when $\dim \mathbb{C}P^k \times D^q = 2k + q \geq 5$) because $\mathbb{C}P^k \times D^q$ is simply connected. Hence we have

**Corollary B.** Let $X$ be the same as in Theorem A. Then the kernel of $E'_X$ is finite when $2k + q \geq 5$.

The outline of the proof of Theorem A is as follows. First, by using the Atiyah-Singer invariant [AS, §7], we define an invariant $\sigma$ on a subgroup of $D(\mathbb{C}P^k \times X)$ containing the image of $E_X$ (§1). Next we see that the composition $\sigma \cdot E_X$ is independent of $X$ (Lemma 2.1). Thirdly we see that $\sigma \cdot E_X$ is a homomorphism (Lemma 3.1). Finally we check the finiteness of the kernel of
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\[ \sigma \cdot E_X \] for \( X = S^q \) (Theorem 4.1). The assumption \( H^1(X; \mathbb{Z}) = 0 \) is used for the well-definedness of the invariant \( \sigma \).

1. AN INVARIANT \( \sigma \)

As remarked in the introduction Theorem A is trivial when \( q = \dim X \) is even. Therefore \( q \) will be odd throughout this paper unless otherwise stated.

We orient \( X \). The disjoint union \( X \sqcup -X \) (\( - \) denotes the reversed orientation) is null-cobordant in the oriented cobordism. Extending maps in \( D(CP^k \times X) \) as the identity on \( CP^k \times (-X) \) induces a monomorphism

\[ D(CP^k \times X) \to D(CP^k \times (X \sqcup -X)). \]

Thus it suffices to prove Theorem A for \( X \) being null-cobordant.

Let \( \alpha \) be a generator of \( H^2(CP^k ; \mathbb{Z}) \) and we also regard \( \alpha \) as an element of \( H^2(CP^k \times X) \) via the projection map \( CP^k \times X \to CP^k \).

**Definition.** \( D_0(CP^k \times X) = \{[f] \in D(CP^k \times X) | f^* \alpha = \alpha \} \), where \([f]\) denotes the class of a diffeomorphism \( f \) of \( CP^k \times X \).

Clearly \( D_0(CP^k \times X) \) forms a subgroup of \( D(CP^k \times X) \). The image of \( E_X \) is contained in \( D_0(CP^k \times X) \). We shall define an invariant

\[ \sigma : D_0(CP^k \times X) \to F(S^1)/\mathbb{Z} \]

where \( F(S^1)/\mathbb{Z} \) is the quotient group of \( F(S^1) \), the fraction field of the complex character (or representation) ring \( R(S^1) \) of the circle group \( S^1 \), divided by the normal subgroup \( \mathbb{Z} \) consisting of integer valued constant functions on \( S^1 \). Since \( R(S^1) \) is the Laurent polynomial ring \( \mathbb{Z}[t, t^{-1}] \) as is well known, \( F(S^1) \) is the field of rational functions of \( t \).

Let \( Y \) and \( Y' \) be a pair of compact connected oriented manifolds which are bounded by \( X \) and have the same signature. Let \([f]\) be an element of \( D_0(CP^k \times X) \). We paste together \( CP^k \times Y \) and \( -CP^k \times Y' \) along the boundary by \( f \) to get a closed oriented manifold \( M \). The oriented diffeomorphism type of \( M \) does not depend on the choice of a representative \( f \) of \([f]\).

The \( S^1 \) bundle over \( CP^k \times X \) corresponding to \( \alpha \) is \( S^{2k+1} \times X \to CP^k \times X \) where \( S^1 \) acts on \( S^{2k+1} \) linearly and freely. Since \( f^* \alpha = \alpha \), \( f \) lifts to an \( S^1 \) equivariant diffeomorphism \( \tilde{f} \) of \( S^{2k+1} \times X \). The difference of two liftings of \( f \) is measured by a continuous map from \( X \) to \( S^1 \). The homotopy classes of such maps are exactly \( H^1(X; \mathbb{Z}) \) and we assume the group vanishes. Hence the lifting is unique up to \( S^1 \) equivariant isotopy.

We paste together \( S^{2k+1} \times Y \) and \( -S^{2k+1} \times Y' \) along the boundary by \( \tilde{f} \) to get a closed oriented manifold \( \tilde{M} = \tilde{M}([f], Y, Y') \) with a free \( S^1 \) action. Since \( q \) is odd, \( \tilde{M} \) is of odd dimension. Therefore the Atiyah-Singer invariant \( \sigma(g, \tilde{M}) \in \mathbb{C} \) is defined for \( g \neq 1 \in S^1 \). The function \( \sigma(\cdot, \tilde{M}) \) belongs to the fraction field \( F(S^1) \) (see [AS, §7]). Note that the function \( \sigma(\cdot, \tilde{M}) \) is
independent of the choice of a representative \( f \) because so is the oriented \( S^1 \)-diffeomorphism type of \( \tilde{M} \).

**Lemma 1.1.** The function \( \sigma(\cdot, \tilde{M}) \), regarded as an element of \( F(S^1)/\mathbb{Z} \), depends only on \([f]\), and not on the choice of \( Y \) and \( Y' \).

**Proof.** Let \( A(Y) \) be the closed \( S^1 \) manifold defined as
\[
A(Y) = D^{2k+2} \times X \cup S^{2k+1} \times Y
\]
where \( D^{2k+2} \times X \) and \( S^{2k+1} \times Y \) are pasted together along the boundary by \( \tilde{f} \). We consider the \( S^1 \) manifold defined as
\[
W = D^{2k+2} \times Y \cup A(Y) \times I \cup D^{2k+2} \times Y'
\]
where \( D^{2k+2} \times Y \) and \( D^{2k+2} \times Y' \) are attached to \( A(Y) \times I \) along \( D^{2k+2} \times X \) via the identity map at 0- and 1-levels respectively. We orient \( W \) suitably so that \( W \) is an oriented \( S^1 \) cobordism between \( M([f], Y, Y) \) and \( M([f], Y, Y') \).

By definition we have
\[
\sigma(g, \partial W) = L(g, W) - \text{Sign}(g, W) \quad \text{for } g \neq 1 \in S^1
\]
where \( L(g, W) \) is the number occurring on the right-hand side of the \( G \)-signature formula and \( \text{Sign}(g, W) \) is the equivariant signature of \( W \) evaluated at \( g \). Since \( S^1 \) is a connected group \( \text{Sign}(g, W) = \text{Sign} W \). As for \( L(g, W) \) the \( G \)-signature formula involves the characteristic classes of \( T^g W \) and the normal bundle of \( W^g \), so we have to investigate them. First we note that the \( S^1 \) action on \( W \) is semifree, so \( W^g = W^{S^1} \). The fixed point set \( W^{S^1} \) is \( Y \times I \cup (-Y') \). The normal bundle to \( W^{S^1} \) admits a complex structure induced from the \( S^1 \) action. As easily observed the bundle is trivial. Furthermore we have
\[
\text{Sign} W^{S^1} = \text{Sign} Y - \text{Sign} Y' = 0
\]
by the additivity property of signature (see [AS, §7]) and the assumption that \( \text{Sign} Y = \text{Sign} Y' \). Putting these into the \( G \)-signature formula, one can see that \( L(g, W) = 0 \). Thus we have
\[
\sigma(g, \tilde{M}([f], Y, Y')) - \sigma(g, \tilde{M}([f], Y, Y)) = \sigma(g, \partial W) = -\text{Sign} W
\]
and hence
\[
\sigma(\cdot, \tilde{M}([f], Y, Y')) = \sigma(\cdot, \tilde{M}([f], Y, Y)) \quad \text{in } F(S^1)/\mathbb{Z}.
\]
Applying the same argument to \( \tilde{M}([f^{-1}], Y', Y) \), we get
\[
\sigma(\cdot, \tilde{M}([f^{-1}], Y', Y)) = \sigma(\cdot, \tilde{M}([f^{-1}], Y', Y')) \quad \text{in } F(S^1)/\mathbb{Z}.
\]
Here we note that \( \tilde{M}([f^{-1}], Y', Y) = -\tilde{M}([f], Y, Y') \) and \( \tilde{M}([f^{-1}], Y', Y') = -\tilde{M}([f], Y', Y') \). Since the Atiyah-Singer invariant changes the sign if the orientation of the manifold is reversed, the above identities prove the lemma. Q.E.D.
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Definition. We set \( \sigma([f])(g) = \sigma(g, \widetilde{M}([f], Y, Y)) \) and regard \( \sigma([f]) \) as an element of \( F(S^1)/\mathbb{Z} \).

By Lemma 1.1 \( \sigma([f]) \in F(S^1)/\mathbb{Z} \) is an invariant of \( [f] \in D_0(CP^k \times X) \).

2. \( \sigma \cdot E_X \) IS INDEPENDENT OF \( X \)

As remarked before the image of \( E_X : D(CP^k \times D^q) \to D(CP^k \times X) \) is contained in \( D_0(CP^k \times X) \). Hence the composition \( \sigma \cdot E_X : D(CP^k \times D^q) \to F(S^1)/\mathbb{Z} \) is defined. The purpose of this section is to verify

**Lemma 2.1.** \( \sigma \cdot E_X \) is independent of \( X \).

*Proof.* It suffices to prove \( \sigma \cdot E_X = \sigma \cdot E_{S^q} \). Let \( Y \) be a connected compact oriented manifold bounded by \( X \). We may assume that \( H^1(Y;\mathbb{Z}) = 0 \), if necessary, by doing surgery to kill the fundamental group of \( Y \). Let \( \tilde{Y} \) be the cobordism between \( X \) and \( S^q \) obtained by removing a small open disk from \( Y \). We choose a smooth simple path in \( \tilde{Y} \) connecting \( X \) and \( S^q \) which is transverse to the boundary. The tubular neighborhood is of the form \( D^q \times I \), where \( I \) is the path direction and \( D^q \) is the normal direction.

Let \([f]\) be an element of \( D(CP^k \times D^q) \) and let id be the identity map of \( I \). The map \( f \times id \) restricts to the identity map on \( CP^k \times S^{q-1} \times I \); so it extends to a diffeomorphism of \( CP^k \times \tilde{Y} \), say \( F \), as the identity on the exterior of \( CP^k \times D^q \times I \). Since \( H^1(\tilde{Y};\mathbb{Z}) = 0 \), \( F \) lifts uniquely to an \( S^1 \) equivariant diffeomorphism \( \tilde{F} \) of \( S^{2k+1} \times \tilde{Y} \) up to \( S^1 \) equivariant isotopy.

We view \( \partial(Y \times I) \) as a triad consisting of three pieces \( Y \times \{0\} \cup X \times I \) (\( \cong Y \), \( \hat{Y} \), and \( D^{q+1} \) (\( \subset Y \times \{1\} \)) so that \( D^q \times I \) is embedded in the piece \( \hat{Y} \). We paste together two copies of \( S^{2k+1} \times Y \times I \) along \( S^{2k+1} \times \hat{Y} \) by \( \tilde{F} \). The resulting \( S^1 \) manifold \( V \) is an \( S^1 \) cobordism between \( \tilde{M}(E_X([f]), Y, Y) \) and \( \tilde{M}(E_{S^q}([f]), D^{q+1}, D^{q+1}) \). Since the \( S^1 \) action on \( V \) is free, we have

\[
\sigma(g, \tilde{M}(E_X([f]), Y, Y)) - \sigma(g, \tilde{M}(E_{S^q}([f]), D^{q+1}, D^{q+1})) = \text{Sign } V
\]

up to sign. This implies the lemma. Q.E.D.

3. ADDITIVITY

We shall denote \( \sigma \cdot E_X \) again by \( \sigma \). The purpose of this section is to verify

**Lemma 3.1.** \( \sigma : D(CP^k \times D^q) \to F(S^1)/\mathbb{Z} \) is a homomorphism.

*Proof.* By Lemma 2.1 we may assume \( X = S^q \). Let \([f_i] \) \((i = 1, 2)\) be elements of \( D(CP^k \times D^q) \). Since \( f_i \) is the identity on the boundary, one can deform \( f_i \) via isotopy so that \( f_i \) is the identity on the exterior of \( CP^k \times D_i \) where \( D_i \) is a small disk in \( D^q \). Furthermore we may assume that \( D_1 \) has no intersection with \( D_2 \).
Set \( E_{S^q}(\{f_i\}) = \{F_i\} \in D(CP^k \times D^q) \) and let \( \tilde{F}_i : S^{2k+1} \times S^q \to S^{2k+1} \times S^q \) be a lifting of \( F_i \), which is unique up to \( S^1 \) equivariant isotopy as \( H^1(S^q; \mathbb{Z}) = 0 \). We form a closed \( S^1 \) manifold

\[
(3.2) \quad \Sigma([f_i]) = D^{2k+2} \times S^q \cup S^{2k+1} \times D^{q+1}
\]

where \( D^{2k+2} \times S^q \) and \( S^{2k+1} \times D^{q+1} \) are pasted together by \( \tilde{F}_i \) along the boundary and \( S^1 \) acts on \( D^{2k+2} \) linearly extending the free \( S^1 \) action on \( S^{2k+1} \). The action on \( \Sigma([f_i]) \) is semifree and the fixed point set is \( \{0\} \times S^q \). It turns out that \( \Sigma([f_i]) \) is a homotopy \( (2k + q + 2) \)-sphere.

We regard \( \Sigma([f_i]) \) as a homotopy \( (2k + q + 2) \)-sphere with a semifree \( S^1 \) action and a decomposition as in (3.2). The connected sum \( \Sigma([f_i])#\Sigma([f_j]) \) can be done equivariantly around fixed points using the decompositions and it is also a homotopy \( (2k + q + 2) \)-sphere with a semifree \( S^1 \) action and a decomposition. Taking account of decompositions we have

\[
\Sigma([f_i])#\Sigma([f_j]) = \Sigma([f_1 \cdot f_2]) \quad (= \Sigma([f_i][f_j])).
\]

We abbreviate \( \tilde{M}(E_{S^q}(\{f_i\}), D^{q+1}, D^{q+1}) \) by \( \tilde{M}(\{f_i\}) \). We note that \( \tilde{M}([f_i]) \) agrees with the \( S^1 \) manifold obtained from \( \Sigma([f_i]) \) by doing surgery on the identity map: \( D^{2k+2} \times S^q \to D^{2k+2} \times S^q \subset \Sigma([f_i]) \). Therefore \( \tilde{M}([f_i][f_j]) \) is obtained from \( \Sigma([f_i])#\Sigma([f_j]) \) in this way.

Consider the equivariant boundary connected sum \( \Sigma([f_i]) \times I_1 \Sigma([f_j]) \times I \) at the l-level. This yields an \( S^1 \) cobordism between \( \Sigma([f_i]) \sqcup \Sigma([f_j]) \) and \( \Sigma([f_i])#\Sigma([f_j]) = \Sigma([f_1][f_2]) \). We attach three copies of the handle \( D^{2k+2} \times D^{q+1} \) to \( \Sigma([f_i]) \times I_1 \Sigma([f_j]) \times I \) via the identity maps from \( D^{2k+2} \times S^q \to D^{2k+2} \times S^q \) embedded in \( \Sigma([f_i]) \times \{1\} \# \Sigma([f_j]) \times \{1\} \), \( \Sigma([f_i]) \times \{0\} \), and \( \Sigma([f_i]) \times \{0\} \) respectively. This yields a semifree \( S^1 \) cobordism \( V \) between \( \tilde{M}([f_i]) \sqcup \tilde{M}([f_j]) \) and \( \tilde{M}([f_i][f_j]) \). As easily observed the complex normal bundle to \( V^{S^1} \) is trivial and \( \text{Sign} V^{S^1} = 0 \). Hence, similarly to the proof of Lemma 1.1, we have

\[
\sigma(, \tilde{M}([f_i])) + \sigma(, \tilde{M}([f_j])) = \sigma(, \tilde{M}([f_1][f_2])) \quad \text{in} \ F(S^1)/\mathbb{Z}.
\]

This proves the lemma as \( \sigma([f]) = \sigma(, \tilde{M}([f])) \). Q.E.D.

4. Review of the work of Browder-Petrie

In this section and the next section we prove the following theorem from which Theorem A follows immediately.

**Theorem 4.1.** The kernel of \( \sigma : D(CP^k \times D^q) \to F(S^1)/\mathbb{Z} \) is finite.

We need some knowledge about the group \( D(CP^k \times D^q) \) for the proof of Theorem 4.1. The group is analysed by Browder-Petrie [BP]. In this section we shall review their work.
There is a long exact sequence of groups:
\begin{equation}
\pi_{q+1}(GCP^k) \to hS(CP^k \times (D^{q+1}, S^q)) \to D(CP^k \times D^q) \to \pi_q(GCP^k).
\end{equation}

Here $GCP^k$ is the identity component of the space of self-maps of $CP^k$ and $hS(CP^k \times (D^{q+1}, S^q))$ is the set of equivalence classes of pairs $[Q, h]$ where $Q$ is an oriented smooth manifold and $h: [Q, \partial Q] \to CP^k \times (D^{q+1}, S^q)$ is a homotopy equivalence preserving orientation and $h|\partial Q: \partial Q \to CP^k \times S^q$ is a diffeomorphism. Two pairs $[Q_1, h_1]$ and $[Q_2, h_2]$ are equivalent if there is a diffeomorphism $d: Q_1 \to Q_2$ such that the composition $h_2 \cdot d$ is homotopic to $h_1$ relative boundary.

We shall explain the homomorphism $\partial$ and $\lambda$. First we note that $Q$ is diffeomorphic to $CP^k \times D^{q+1}$. This can be seen as follows. Since $h|\partial Q$ is a diffeomorphism, $CP$ can be embedded in the interior of $Q$ so that the embedding induces a homotopy equivalence and that the normal bundle is trivial. The complement $\hat{Q}$ of a small open tubular neighborhood of the embedded $CP^k$ in $Q$ turns out to be an $h$-cobordism between $CP^k \times S^q$ and $\partial Q = CP^k \times S^q$. Hence $\hat{Q}$ is diffeomorphic to $CP^k \times S^q \times I$ by the $h$-cobordism theorem and hence $Q$ is diffeomorphic to $CP^k \times D^{q+1}$. Thus any class in $hS(CP^k \times (D^{q+1}, S^q))$ can be represented by a pair $[CP^k \times D^{q+1}, h]$. Moreover, a similar argument shows that one can choose $h$ so that $h|CP^k \times D^q_-$ is the identity where $D^q_-$ is the lower hemisphere of $S^q$.

With this understood $\partial : hS(CP^k \times (D^{q+1}, S)) \to D(CP^k \times D^q)$ is defined by
\begin{equation}
\partial([CP^k \times D^{q+1}, h]) = [h|CP^k \times D^q_+].
\end{equation}

Note that the image of $\partial([CP^k \times D^{q+1}, h])$ through the homomorphism
\begin{equation}
E_S: D(CP^k \times D^q) \to D(CP^k \times S^q)
\end{equation}
is nothing but $[h|CP^k \times S^q]$.

Let $\rho: CP^k \times D^q \to CP^k$ be the projection. Then $\lambda: D(CP^k \times D^q) \to \pi_q(GCP^k)$ is defined by
\begin{equation}
(\lambda([f])(x))(u) = \rho(f(u, x))
\end{equation}
where $[f] \in D(CP^k \times D^q)$, $x \in D^q$, and $u \in CP^k$.

The linear action of the unitary group $U(k+1)$ on $CP^k$ induces a semihomomorphism $i: U(k+1)/\Delta \to GCP^k$, where $\Delta$ is the subgroup of $U(k+1)$ consisting of scalar multiples of the identity matrix. It is known that
\begin{equation}
i_\cdot: \pi_q(U(k+1)/\Delta) \otimes Q \to \pi_q(GCP^k) \otimes Q
\end{equation}
is an isomorphism (see [S] for example). In particular
\begin{equation}
\pi_q(GCP^k) \otimes Q = \begin{cases} Q & \text{if } 3 \leq q \leq 2k+1 \ (q: \text{odd}), \\ 0 & \text{otherwise}. \end{cases}
\end{equation}
There is another homomorphism

\[ \mu : \pi_\ast(U(k+1)/\Delta) \to D(CP^k \times D^q) \]

defined by

\[ \mu([\lambda])(u,x) = ((\lambda(x))(u),x) \]

where \( \lambda: (D^q, S^{q-1}) \to (U(k+1)/\Delta, Id) \). It is easy to see that \( \lambda \cdot \mu = i_\ast \). The exact sequence (4.2) together with the above observation shows

\[ \text{Lemma 4.3.} \]

The subgroup of \( D(CP^k \times D^q) \) generated by the subgroups

\[ \partial(hS(CP^k \times D^{q+1}, S^q)) \quad \text{and} \quad \mu(\pi_q(U(k+1)/\Delta)) \]

is of finite index in \( D(CP^k \times D^q) \).

5. Proof of Theorem 4.1

We abbreviate \( \tilde{M}(E_{S^q}([f]), D^{q+1}, D^q) \) by \( \tilde{M} \) and let \( M \) be the \( S^1 \) orbit space of \( \tilde{M} \). Since the \( S^1 \) action on \( \tilde{M} \) is free, the projection \( \tilde{M} \to M \) is an \( S^1 \) bundle. Let \( \alpha \) be the first Chern class of it. The associated disk bundle \( D_\alpha \) supports a semifree \( S^1 \) action rotating fibers, the zero section \( M \) being the fixed point set. Since \( \partial D_\alpha = \tilde{M} \), the Atiyah-Singer invariant \( \sigma(g, \tilde{M}) \) can be described using \( D_\alpha \). In fact, since the normal bundle to \( M \) in \( D_\alpha \) is the complex line bundle with \( \alpha \) being the first Chern class, we have

\[ \sigma(g, \tilde{M}) = 2^m \frac{g e_a^\alpha + 1}{g e_a^\alpha - 1} L(M)[M] - \text{Sign } D_\alpha \]

where \( 2m = 2k + q + 1 \) and \( L \) denotes the Atiyah-Singer L-class. Hence, since \( \sigma([f])(g) = \sigma(g, \tilde{M}) \) and \( \sigma([f]) \) is considered in \( F(S^1)/\mathbb{Z} \), we have

\[ \sigma([f]) = 2^m \frac{te^\alpha + 1}{te^\alpha - 1} L(M)[M] \]

where \( t \in F(S^1)/\mathbb{Z} \) is the image of the standard complex 1-dimensional \( S^1 \) representation to \( F(S^1)/\mathbb{Z} \).

Expanding \( \frac{te^\alpha + 1}{te^\alpha - 1} \) with respect to \( e^\alpha - 1 \), we have

\[ \frac{te^\alpha + 1}{te^\alpha - 1} = 1 - 2 \sum_{r \geq 0} \frac{t^r(e^\alpha - 1)^r}{(1-t)^{r+1}}. \]

It says that \( t = 1 \) is the only pole of \( \sigma([f]) \).

**Lemma 5.3.** If \([f] \in \partial(hS(CP^k \times (D^{q+1}, S^q)))\), then the highest degree of the pole of \( \sigma([f]) \) is at most \( k + 1 \).

**Proof.** Let \([f] = \partial([CP^k \times D^{q+1}, h])\). Then \( E_{S^q}([f]) = [h]CP^k \times S^q \) as remarked before. Since \( M \) is obtained by pasting together two copies of \( CP^k \times D^{q+1} \) along the boundary by \( h[CP^k \times S^q] \), \( M \) is homotopy equivalent
to $\mathbb{C}P^k \times S^{q+1}$. Therefore $\alpha^r = 0$ and hence $(e^\alpha - 1)^r = 0$ for $r > k$. This together with (5.1) and (5.2) implies the lemma. Q.E.D.

Since $\Delta$ is a circle group, the projection map $U(k+1) \to U(k+1)/\Delta$ induces an isomorphism $\pi_q(U(k+1)) \to \pi_q(U(k+1)/\Delta)$ for $q \geq 2$. Suppose $3 \leq q \leq 2k+1$ ($q$ : odd). Then $\pi_q(U(k+1))$ sits in the stable range; so it is infinite cyclic and is detected by the $(q+1)/2$th Chern classes of the complex vector bundles over $S^{q+1}$ corresponding to $\pi_q(U(k+1))$. We shall denote by $c([f]) \in H^{q+1}(S^{q+1};\mathbb{Z})$ the $(q+1)/2$th Chern class of

$$[f] \in \pi_q(U(k+1)) = \pi_q(U(k+1)/\Delta).$$

Hereafter we identify $\pi_q(U(k+1))$ with a subgroup of $D(CP^k \times D^q)$ via $\mu$ when $3 \leq q \leq 2k+1$.

**Lemma 5.4.** Let $3 \leq q \leq 2k+1$ and $[f] \in \pi_q(U(k+1))$. Then the highest degree of the pole of $\sigma([f])$ is at most $k+1 + (q+1)/2$ and the coefficient at the pole of degree $k+1 + (q+1)/2$ is $2^{m+1} c([f])[S^{q+1}]$ up to sign.

**Proof.** In this case $M$ is the total space of the complex projective bundle $\pi: M \to S^{q+1}$ associated with the complex vector bundle over $S^{q+1}$ corresponding to $[f]$. According to Borel-Hirzebruch [BH, p. 516] we have

$$\alpha^{k+1} + (-1)^{(q+1)/2} \pi^*(c([f])) \alpha^{k+1-(q+1)/2} = 0$$

and hence

$$(5.5) \quad \alpha^{k+(q+1)/2} = -(-1)^{(q+1)/2} \pi^*(c([f])) \alpha^k.$$  

On the other hand (5.1) and (5.2) show that the highest degree of the pole of $\sigma([f])$ is at most $k+1 + (q+1)/2$ and the coefficient at the pole of degree $k+1 + (q+1)/2$ is $2^{m+1} \alpha^{k+(q+1)/2}[M]$ up to sign. Here we have

$$\alpha^{k+(q+1)/2}[M] = \pm \pi^*(c([f])) \alpha^k[M] \quad \text{by (5.5))}$$

where the latter identity is the so-called integration along the fiber. This proves the lemma. Q.E.D.

To prove Theorem 4.1 it suffices to show that the kernel of $\sigma$ is finite when $\sigma$ is restricted to the subgroup of $D(CP^k \times D^q)$ which consists of all elements $[f]$ of the form $[f] = [f_1] + [f_2]$ where $[f_1] \in \partial(hS(CP^k \times D^{q+1}), S^q))$ and $[f_2] \in \mu(\pi_q(U(k+1)/\Delta))$. In fact, it suffices to show that each element of this group is of finite order because $D(CP^k \times D^q)$ is finitely generated abelian.

**Lemma 5.6.** If $\sigma([f_1] + [f_2]) = 0$, then $[f_2]$ is of finite order.

**Proof.** We may assume $3 \leq q \leq 2k+1$ ($q$ : odd) because otherwise $\pi_q(U(k+1)/\Delta)$ is a finite group. Hence we may view $[f_2]$ as an element of $\pi_q(U(k+1)) = \pi_q(U(k+1)/\Delta)$. By Lemma 3.1, we have $\sigma([f_1]) + \sigma([f_2]) = 0$.  

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Lemmas 5.3 and 5.4 tell us that the coefficient at the pole of degree $k + 1 + (q + 1)/2$ is $2^{n+1} c([f_2])[S^{q+1}]$ up to sign. Since it vanishes, $c([f_2]) = 0$ and hence $[f_2] = 0$. Q.E.D.

To show Theorem 4.1 we need to show only that $\sigma$ has a finite kernel when restricted to $\partial (hS(CP^k \times (D^{q+1}, S^q)))$, or any element in this kernel is of finite order. Remember that $M$ is then homotopy equivalent to $CP^k \times S^{q+1}$.

Lemma 5.7. Let $[f] \in \partial (hS(CP^k \times (D^{q+1}, S^q)))$. If $\sigma([f]) = 0$, then the total Pontrjagin class $p(M)$ of $M$ is of the same form as $CP^k \times S^{q+1}$, i.e. $p(M) = (1 + \alpha^2)^{k+1}$.

Proof. Since $M$ is homotopy equivalent to $CP^k \times S^{q+1}$, one can express

$$L(M) = u(\alpha) + v(\alpha)\beta$$

where $u(\alpha)$ and $v(\alpha)$ are polynomials of $\alpha$ of degree at most $k$ and $\beta$ is a generator of $H^{q+1}(M)$ corresponding to the factor $S^{q+1}$.

Remember that $M = CP^k \times D^{q+1} \cup CP^k \times D^{q+1}$; so $CP^k$ is naturally embedded in $M$ with the trivial normal bundle. This means that the restriction of $L(M)$ to the embedded $CP^k$ is of the same form as $CP^k$. Hence $u(\alpha)$ is determined. The identities (5.1) and (5.2) tell us that $\sigma([f])$ determines the values $\alpha^r L(M)[M]$ for $0 \leq r \leq k$, which determine $v(\alpha)$, in fact $v(\alpha) = 0$.

It is known that the total $L$-classes determine the total Pontrjagin classes and vice versa. Consequently $\sigma([f])$ determines $p(M)$. Since $p(M) = (1 + \alpha^2)^{k+1}$ is a solution of the equation $\sigma([f]) = 0$, the lemma follows. Q.E.D.

Let $[f] = \partial ([Q_f, h_f])$. Remember that we may assume $Q_f = CP^k \times D^{q+1}$. Since $\pi_{q+1}(GCP^k)$ is finite as $q + 1$ is even, the map $\partial$ has finite kernel by (4.2). Therefore to complete our proof of Theorem 4.1, it suffices to show that $[Q_f, h_f]$ is of finite order. This we shall do now. Petrie [P] defined a map

$$\gamma : hS(CP^k \times (D^{q+1}, S^q)) \to hS(CP^k \times S^{q+1})$$

as follows: $\gamma([Q, h]) = [\gamma(Q), \gamma(h)]$ where $\gamma(Q)$ is the manifold obtained by pasting together $Q$ and $CP^k \times D^{q+1}$ along the boundary by $h|\partial Q$, and $\gamma(h)|Q = h$, $\gamma(h)|CP^k \times D^{q+1}$ is the identity. We note that $M = \gamma(Q_f)$. By Lemma 5.7 $\gamma(Q_f)$ has the Pontrjagin classes of the same form as $CP^k \times S^{q+1}$. This fact seems to imply that $[\gamma(Q_f), \gamma(h_f)]$ is of finite order in $hS(CP^k \times S^{q+1})$. However this argument does not work because $hS(CP^k \times S^{q+1})$ does not admit a natural group structure.
To avoid this trouble we shall consider the following commutative diagram of surgery exact sequences:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & hS(\mathbb{C}P^k \times (D^{q+1}, S^q)) & \longrightarrow & [\mathbb{C}P^k \times D^{q+1}/\mathbb{C}P^k \times S^q, G/O] & \longrightarrow \\
& & \downarrow \gamma & & \downarrow \kappa & & \\
0 & \longrightarrow & hS(\mathbb{C}P^k \times S^{q+1}) & \longrightarrow & [\mathbb{C}P^k \times S^{q+1}, G/O] & .
\end{array}
\]

Here \(G/O\) is the homotopy fiber of the classifying map \(BO \to BG\) (\(BG\) is the classifying space of stable spherical fibrations), and \([A, G/O]\) is the set of homotopy classes of continuous maps from \(A\) to \(G/O\). In fact, induced from the \(H\)-space structure of \(G/O\), the set \([A, G/O]\) forms an abelian group. The vertical map \(\kappa^*\) is a homomorphism induced from the quotient map \(\kappa: \mathbb{C}P^k \times S^{q+1} \to \mathbb{C}P^k \times D^{q+1}/\mathbb{C}P^k \times S^q\). It is known that \(\eta\) is a homomorphism.

Let \(A\) be a finite CW-complex. The inclusion map \(j: G/O \to BO\) induces a homomorphism \(j_*: [A, G/O] \to [A, BO] = \tilde{K}(A)\) and there is a functor \(\text{ph}\) (called Pontrjagin character) from \(\tilde{K}(A)\) to \(\tilde{H}^4(A; \mathbb{Q})\). It is well known that

\[
(5.9) \quad j_* \text{ and } \text{ph} \text{ are both isomorphisms when tensored by } \mathbb{Q}.
\]

As easily checked \(\kappa^*: \tilde{H}^4(\mathbb{C}P^k \times D^{q+1}/\mathbb{C}P^k \times S^q; \mathbb{Q}) \to \tilde{H}^4(\mathbb{C}P^k \times S^{q+1}; \mathbb{Q})\) is injective. Hence it follows from (5.9) that \(\kappa^*\) in the diagram (5.8) has finite kernel. In the sequel it suffices to show that \(\eta^*([\gamma(Q_f), \gamma(h_f)])\) is of finite order since \(\eta\) is a monomorphism.

It is also known that

\[
(5.10) \quad \text{ph} \cdot j_* \cdot \eta^*([\gamma(Q_f), \gamma(h_f)]) = (\gamma(h_f)^*)^{-1} \text{ph}(\gamma(Q_f)) - \text{ph}(\mathbb{C}P^k \times S^{q+1})
\]

where \(\text{ph}(B)\) denotes the Pontrjagin character of the tangent bundle of a manifold \(B\). Since \(p(\gamma(Q_f)) = (1 + \alpha^2)^{k+1}\), \((\gamma(h_f)^*)^{-1} p(\gamma(Q_f)) = p(\mathbb{C}P^k \times S^{q+1})\) and hence the right-hand side of (5.10) is zero. Thus \(\eta^*([\gamma(Q_f), \gamma(h_f)])\) is of finite order by (5.9). \(\text{Q.E.D.}\)

References


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