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REPLACING HOMOTOPY ACTIONS BY TOPOLOGICAL ACTIONS. II

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ABSTRACT. A homotopy action of a group G on a space X is a homomorphism from G to the group $\mathrm{HAUT}(X)$ of homotopy classes of homotopy equivalences of X. George Cooke developed an obstruction theory to determine if a homotopy action is equivalent up to homotopy to a topological action. The question studied in this paper is: Given a diagram of spaces with homotopy actions of G and maps between them that are equivariant up to homotopy, when can the diagram be replaced by a homotopy equivalent diagram of G-spaces and G-equivariant maps? We find that the obstruction theory of Cooke has a natural extension to this context.

1. INTRODUCTION

A homotopy action of a group G on a space X is a homomorphism

$$\rho\colon G \to \mathrm{HAUT}(X),$$

where $\mathrm{HAUT}(X)$ is the group of homotopy classes of homotopy equivalences of X. (We abbreviate homotopy action to ho-action.) If G acts on the space in the usual sense then the induced homotopy action of G on X is called topological. A homotopy action ρ' of G on X' is equivalent to a homotopy action ρ'' on X'' iff there exists a homotopy equivalence $f\colon X' \to X''$ such that

$$\begin{tikzcd} & \mathrm{HAUT}(X') \arrow[d, "\mathrm{HAUT}(f)"] \\ G \arrow[ur, "\rho'"] \arrow[r, "\rho''"'] & \mathrm{HAUT}(X'') \end{tikzcd}$$

is commutative, where $\mathrm{HAUT}(f)$ is defined by $[g] \to [fgf^{\wedge}]$, where $f^{\wedge}$ is a homotopy inverse of $f^{\wedge}$.

In [3] George Cooke studied the problem of when a homotopy action is equivalent to a topological action. The present paper is a direct generalization of Cooke's ideas and methods, and extends Cooke's work to diagrams of spaces with a homotopy action and maps between them that are equivariant up to

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homotopy. For spaces X' and X'' with ho-actions of G, a map $f: X' \to X''$ is called homotopy equivariant (abbreviated ho-equivariant) iff for $g \in G$, the diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "\rho'(g)"'] & X'' \arrow[d, "\rho''(g)"] \\ X' \arrow[r, "f"] & X'' \end{tikzcd}$$

is homotopy commutative, where $\rho'(g)$, resp. $\rho''(g)$, is a map in the homotopy class $\rho'(g)$, resp. $\rho''(g)$. Put another way, if TOP denotes the category of spaces and maps, and ho-TOP the category of spaces and homotopy classes of maps, then a ho-action of G on X is just a G-object in ho-TOP, and f is ho-equivariant iff it is a morphism in ho-TOP of G-objects. A natural question to ask is: When is a homotopy equivariant map equivalent to a G-map, in the sense that there are homotopy equivalences

$$\phi': X' \to Y', \qquad \phi'': X'' \to Y'',$$

topological actions of G on Y' and Y'', and a G-equivariant map $h: Y' \to Y''$ such that the diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "\phi'"'] & X'' \arrow[d, "\phi''"] \\ Y' \arrow[r, "h"] & Y'' \end{tikzcd}$$

is homotopy commutative? More generally one can ask for a topological realization of a diagram of spaces with G-ho-action where the maps are ho-equivariant. The purpose of this paper is to provide a homotopy theoretic treatment of this problem extending the work of Cooke [3]. This is done in §3. To do so however requires an analogous extension of the work of Allaud [1] on classifying spaces for Hurewicz fibrations. This is accomplished in §2. In §4 we discuss some applications, and §5 is an attempt to relate our work to that of Dwyer and Kan [4, 5, 6].

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2. A CLASSIFICATION THEOREM FOR DIAGRAMS OF FIBRATIONS

By a homotopy commutative diagram we understand a functor $X: D \to$ ho-TOP where D is a kittygory ($:=$ small category). If B is a fixed space we denote by TOP$/B$ the category with objects $\gamma: X \to B$, $\rho \in$ Morph(TOP), and morphisms the strictly commutative triangles (in TOP):

$$\begin{tikzcd} X' \arrow[rr, "F"] \arrow[dr, "\gamma'"'] & & X'' \arrow[dl, "\gamma''"] \\ & B & \end{tikzcd}$$

Homotopies of morphisms are fibrewise homotopies, and the associated category with the same objects and fibrewise homotopy classes of maps will be denoted by ho-TOP$/B$. A homotopy commutative diagram of Hurewicz fibrations over the pointed space B consists of:

- a functor $E: D: \to$ ho-TOP$/B$ where $d \in \mathrm{Obj}(D)$, $\gamma(d): E(d) \to B$ is a Hurewicz fibration, and
- a natural transformation $\nu: X \to o(E)$, where o: ho-TOP$/B \to$ ho-TOP is the forgetful functor, such that for every $d \in \mathrm{Obj}(D)$, $\nu(d): X(d) \to E(d)$ is a homotopy equivalence onto the fibre $F(*) = \gamma(d)^{-1}(*)$, $* \in B$ the basepoint.

Example. If D is the category with two objects $\{0, 1\}$ and one morphism $0 \to 1$ then a homotopy commutative diagram of Hurewicz fibrations of type D consists of two Hurewicz fibrations $E(0) \downarrow B$, $E(1) \downarrow B$ and a homotopy class of fibrewise maps

$$E(0) \to E(1) \downarrow B$$

together with homotopy equivalences $X(i) = \pi(i)^{-1}(*)$ for spaces $X(i)$, $i = 0, 1$.

For a fixed kittygory D and pointed space B we denote (following [1]) by $H_D(B, X)$ the set of equivalence classes of homotopy commutative diagrams of Hurewicz fibre spaces over B where the fibres are of type $X: D \to$ ho-TOP. For a fixed functor $X: D \to$ ho-TOP, $H_D(-, X)$ defines a cofunctor $\mathrm{TOP}_* \to \mathrm{ENS}$, where ENS is the category of sets.

Theorem 2.1. *Let D be a kittygory and $X: D \to$ ho-TOP a CW-valued functor. Then the functor:*

$$H_D(-, X): \mathrm{TOP}_* \to \mathrm{ENS}$$

is a homotopy functor (in the sense of E. H. Brown).

Proof. Introduce the category

$$\mathrm{FUN} := \mathrm{FUN}(D, \text{ho-TOP}/\mathrm{TOP}_*)$$

of functors from D to the homotopy category of spaces over pointed spaces. This is a closed model category (see e.g. [2]) in a natural way. Note that a homotopy commutative diagram of Hurewicz fibrations of type D and fibres of type $X: D \to$ ho-TOP is a fibration (= fibrant object) in FUN. The argument of Allaud [1, III] being "formal homotopy theory" goes through in this context to yield the desired conclusion. □

Corollary 2.2. *Let D be a kittygory and $X: D \to$ ho-TOP a CW-valued functor. Then there exists a pointed space B_X, and a homotopy commutative diagram $U: D \to$ ho-TOP$/B_X$ of Hurewicz fibrations, such that the natural transformation*

$$[-, B_X] \to H_D(-, X)$$

given by pullback is an equivalence of functors.

Proof. This follows from Brown's theorem [7]. □

The space B_X has an alternate description that will be of use in the next section. We turn to this now.

Let $X: D \to$ ho-TOP be a homotopy commutative diagram. Denote by $HE(X)$ the space of all self-homotopy equivalences of X, i.e., an element of $HE(X)$ is an assignment to each $d \in \mathrm{Obj}(D)$ a self-homotopy equivalence $\Theta(d): X(d) \to X(d)$ such that upon passing to homotopy classes Θ defines a natural equivalence of the functor X with itself. Note that $HE(X)$ is an associative H-space with unit, and $\pi_0(HE(X)) \simeq \mathrm{HAUT}(X)$ is the group of self-equivalences of the functor X. Since $HE(X)$ is associative it has a classifying space $BHE(X)$. Over $BHE(X)$ there is a homotopy commutative diagram of principal fibrations where for each $d \in \mathrm{Obj}(D)$ the fibre ($= H$-space of the bundle) is $HE(X(d))$, the H-space of self-homotopy equivalences of $X(d)$. Passing to the associated fibrations with fibre $X(d)$ we obtain a functor $V: D \to$ ho-TOP$/BHE(X)$ which has a natural structure of a homotopy commutative diagram of Hurewicz fibrations over $BHE(X)$ with fibres of type X. Hence we obtain a classifying map

$$v: BHE(X) \to B_X$$

for $V: D \to$ ho-TOP$(BHE(X))$ from (2.2). The following is then clear (see e.g. [1]).

Theorem 2.3. *Let D be a kittygory and $X: D \to$ ho-TOP a CW-valued functor. Then $v: BHE(X) \to B_X$ is a homotopy equivalence.* □

3. Homotopy actions on homotopy commutative diagrams

Let D be a kittygory and $X: D \to$ ho-TOP a homotopy commutative diagram. By a homotopy action of the group G on X we understand a homomorphism $\rho: G \to \mathrm{HAUT}(X)$ where $\mathrm{HAUT}(X)$ is the group of automorphisms of the functor X. If $X', X'': D \to$ ho-TOP and ρ', ρ'' are ho-actions of G on X', X'', then a natural transformation $F: X' \to X''$ is an equivalence iff the diagram

$$\begin{tikzcd} & \mathrm{HAUT}(X') \arrow[d, "\mathrm{HAUT}(F)"] \\ G \arrow[ur, "\rho'"] \arrow[r, "\rho''"'] & \mathrm{HAUT}(X') \end{tikzcd}$$

commutes.

For a homotopy commutative diagram $X: D \to$ ho-TOP, an action of G on X consists of a homomorphism $d \in \mathrm{Obj}(D)$ such that

$$\rho(d): G \to HE(X(d))$$

and a G-equivariant map $\phi \in \mathrm{Morph}(D)$ such that

$$f(\phi): X(\mathrm{domain}(\phi)) \to X(\mathrm{range}(\phi));$$

passing to homotopy classes gives a homotopy action of G on X.

Such an action is called topologically realizable, or simply topological. (N.B. The group G actually acts on X, the maps of the diagram are G-maps, but the diagram commutes only up to homotopy. See §5 for a further discussion of this point.)

For a diagram $X: D \to \text{ho-TOP}$ we denote by $HE(X)$ the associative H-space of all tuples $h(d) \in HE(X(d))$ such that passing to homotopy classes gives an automorphism of the functor X. The passage to homotopy classes defines an H-map

$$HE(X) \to \mathrm{HAUT}(X) = \pi_0(HE(X))$$

and induces a fibration

$$\pi: BHE(X) \to B\,\mathrm{HAUT}(X)$$

whose fibre is $BHE_{\mathrm{id}}(X)$, where $HE_{\mathrm{id}}(X)$ is the sub-H-space of tuples $h(d)$, such that $h(d)$ is homotopic to $\mathrm{id}_{X(d)}$.

Theorem 3.1. *Let D be a kittygory, $X: D \to \text{ho-TOP}$ a homotopy commutative diagram and $\rho: G \to \mathrm{HAUT}(X)$ a ho-action of G on X. Then ρ is topologically realizable iff the lifting problem*

$$(*) \qquad \begin{tikzcd} & BHE(X) \arrow[d, "\pi"] \\ BG \arrow[ur, dashed, "\Phi"] \arrow[r, "B\rho"] & B\,\mathrm{HAUT}(X) \end{tikzcd}$$

has a solution.

Proof (after Cooke). Suppose that ρ is topologically realizable by $\mu: G \to HE(Y)$. Set $F: X \to Y$ be an equivalence. WLOG we may suppose that G acts freely on each $Y(d)$ for $d \in \mathrm{Obj}(D)$. We then have principal G-bundles

$$Y(d) \to Y(d)/G, \qquad d \in \mathrm{Obj}(D).$$

Let these be classified by

$$\phi(d): Y(d) \to BG.$$

Since the maps in the diagram Y are equivariant, we have induced maps

$$f/G: Y(\mathrm{domain}(f))/G \to Y(\mathrm{range}(f))/G$$

and a functor

$$Y/G: D \to \text{ho-TOP}.$$

Convert each of the maps $\phi(d)$ into a fibration (see [2]). The fibre of $\phi(d)$ is then homotopy equivalent to $Y(d)$, which via $F(d)$ is homotopy equivalent to $X(d)$. Thus we have described a functor

$$Y/G: D \to \text{ho-TOP}/BG$$

which is a homotopy commutative diagram of Hurewicz fibrations of type $X: D \to \text{ho-TOP}$. By (2.1) and (2.2) we therefore obtain a classifying map

$$\Theta: BG \to BHE(X)$$

for Y/G. To check that the diagram

$$\begin{tikzcd} & BHE(X) \arrow[d, "\pi"] \\ BG \arrow[ur, "\Theta"] \arrow[r, "B\rho"'] & B\,\mathrm{HAUT}(X) \end{tikzcd}$$

homotopy commutes it suffices to check commutativity on π_1, where it is clear.

Conversely given a solution to the lifting problem $(*)$ we obtain an induced diagram of Hurewicz fibrations

$$W: D \to \text{ho-TOP}/BG.$$

Let $EG \downarrow^u BG$ be the universal covering and form the pullback functor

$$u^*W: D \to \text{ho-TOP}/EG.$$

Note that G acts on each $u^*W(d)$ and each map $u^*W(\phi)$ is G-equivariant. Moreover, EG being contractible, u^*W is equivalent to $u^*W|_*$ ($=$ restrict u^*W to a point) which is just X. Thus u^*W provides a topological realization of X. □

Thus as in Cooke [3] the realization problem is reduced to an obstruction theory problem. Specifically:

Corollary 3.2. *Let D be a kittygory and $\rho: G \to \mathrm{HAUT}(X)$ a ho-action of G on the homotopy commutative diagram $X: D \to$ ho-TOP. Then ρ is realizable topologically iff a sequence of obstructions*

$$\Theta^n[\rho] \in H^n(G; \pi_{n-2}(HE_{\mathrm{id}}(X)))|n \geq 3$$

all vanish.

Proof. This follows by noting that

$$\pi_1(BHE(X)) \to \pi_1(B\,\mathrm{HAUT}(X))$$

is an isomorphism, and that $BHE_{\mathrm{id}}(X)$ is the fibre of π. □

4. Applications

The main results of the previous section show that a homotopy action of G on a homotopy commutative diagram $X: D \to$ ho-TOP is equivalent to a topological action iff a sequence of obstructions

$$\Theta^n \in H^n(G; \pi_{n-2}(HE_{\mathrm{id}}(X)))|n \geq 3$$

all vanish. Note that $HE_{\mathrm{id}}(X)$ is simply the component of the identity map in $\mathrm{map}(X, X)$. If X is p-complete for a prime p in the sense that $X(d)$ is p-complete for every $d \in \mathrm{Obj}(D)$, then the space $\mathrm{map}(X, X)$ [2] is also p-complete and hence a fortiori the component of the identity. This leads to the following result.

Proposition 4.1. *Let G be a finite group of order prime to p, p a prime, and $X: D \to$ ho-TOP a p-complete homotopy commutative diagram. Then any ho-action of G on X is topologically realizable.*

Proof. Since X is p-complete so is $HE_{\mathrm{id}}(X)$. Therefore $\pi_m(HE_{\mathrm{id}}(X))$ is p-complete. Since $p \mid |G|$ it follows that the groups $H^n(G; \pi_{n-2}(HE_{\mathrm{id}}(X)))$ all vanish. □

The groups $G = Z$ and $G = Z \times Z$ are special in that $BG = S^1$ or $S^1 \times S^1$ have dimension at most 2. Since the first obstruction to realizing a ho-action by a topological action occurs in dimension 3 we obtain:

Proposition 4.2. *Let $X: D \to$ ho-TOP be a homotopy commutative diagram. Then any ho-action of Z or $Z \times Z$ can be realized topologically.* □

In particular this says that a self-homotopy equivalence $f: X \to X$ of a space X is realizable up to homotopy by a homeomorphism $h: Y \to Y$ of a homotopy equivalent space.

Finally we note the analog of [3, 2.4] for maps.

Proposition 4.3. *Let R be a commutative ring and G a finite group whose order is invertible in R. If ρ is a ho-action of G on a homotopy commutative diagram $X: D \to$ ho-TOP then there exists a homotopy commutative diagram $X/G: D \to$ ho-TOP and a morphism $\pi: X \to X/G$ such that for all $d \in \mathrm{Obj}(D)$*

$$\pi(d): H^*(X(d); R) \to H^*(X(d); R)^G.$$

Proof. The homotopy action ρ induces a homotopy action on the R-completion $X_R: D \to$ ho-TOP of X. By (4.1) ρ_R is topologically realizable. Let

$$\phi: X \to Y: D \to \text{ho-TOP}$$

to be a topological realization of ρ on Y. WLOG we may suppose G acts freely on Y. A standard transfer argument now shows the composite $X \to Y \to Y/G$, where Y/G is the diagram of orbit spaces, satisfies the conclusion. □

5. Closing remarks

Suppose that D is a kittygory and that the group G acts on the homotopy commutative diagram $X: D \to$ ho-TOP. A natural question to ask is: Does there exist a strictly commutative diagram $Y: D \to$ TOP, an action of G on Y (so all spaces $Y(d)$ are G-spaces and all maps $Y(\phi)$ are G-equivariant) and a homotopy equivalence $X \to$ ho-Y, where ho-Y is the composite $Y: D \to$ TOP $\to$ ho-TOP? One method to approach this problem would be to bring in the work of Dwyer and Kan [4, 5, 6]. These authors study the question: Given a homotopy commutative diagram $X: D \to$ ho-TOP, when does there exist a commutative diagram $Y: D \to$ TOP and a homotopy equivalence $X \to$ ho-Y? They prove [5] there is an obstruction theory for this problem analogous to that

of Cooke [3]. In particular they state that when $D = G$ (where G denotes the category with one object and endomorphism set G), their obstruction theory reduces to that of Cooke. It would be interesting to see the relations between all these results spelled out in detail.

N.B. If D contains no closed loops (e.g. D is a tree) then there is no distinction between homotopy commutativity and strict commutativity. This suggests that perhaps the fundamental group of the components of the topological realization of the category D should enter in some way. The higher homotopy groups enter into the situation when D is endowed with commuting homotopies etc.

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